Option Theory and Defaultable Mortgage Pricing

*Wai K. Leung**

Abstract. **The existing mortgage pricing literature either fails to consider the default option or gives numerical results only. Solutions using numerical methods not only do not provide the intuition of analytic solutions, but also are very expensive in computation time, since a supercomputer is frequently required. We, therefore, have employed the Cox-Ross [1976] approach to price a fixed-rate mortgage with a default option. We are able to provide analytic solutions, comparative statics and more simulation results not available in existing models.**

Introduction

Considerable progress has been made in the mortgage pricing literature area. Dunn and McConnell [1981], Cunningham and Hendershott [1984], Foster and Van Order [1984], Epperson et al. [1985] and Kau et al. [1986] have discussed the pricing of fixed-rate mortgages. But the existing mortgage pricing literature either fails to consider the default option in a defaultable mortgage or provides numerical solutions only. Default risk is very important because 73% of the mortgage loans made in 1987 are not guaranteed. Solutions using numerical methods cannot give complete comparative statics results, nor do they provide the intuition of analytical solutions. The finite difference method, used frequently in numerical solutions, also requires substantial computing time. Our paper takes into account the default option in a defaultable mortgage. We are able to provide analytical solutions and comparative statics results. Simulation results using our solutions also need substantially less computing time than the finite difference approach.[1] Moreover, our model also provides more simulation results than available from existing models.[2]

In section two, a model of the fixed-rate mortgage is developed. We then show that the Cox–Ross [CR] approach [1976] can be used to price a mortgage with a default option. An analytical solution is obtained for the more general defaultable mortgage. Comparative statics results are given in the third section. In the fourth section we give some simulated results. The conclusions are in section five.

*Department of Finance, Louisiana State University, Baton Rouge, Louisiana 70803.
Date Revised—April 1989; Accepted—April 1989.

A Model of Mortgage Pricing

The Basic Model

We assume that the value of a mortgage $M = M(B, H, t)$ depends on the current default-free discount bond price B, house price H and time t. The discount bond price B and the house price H are random variables that develop according to the following process:

$$dB = a(B, t)dt + b(B, t)dz_1 \tag{1}$$

$$dH = e(H, t)dt + f(H, t)dz_2 \tag{2}$$

where z is a Wiener process. Hereafter, the arguments of $a(B, t)$, $b(B, t)$, $e(H, t)$ and $f(H, t)$ are suppressed for convenience; a, b, e, f are deterministic functions; dt is the time increment; and dz is the increment of the Wiener process.

M is a contingent claim of bond and house prices. The stochastic equation for development of the mortgage price is:

$$dM = [g(B, H, t)M - p(B, H, t)]dt + Mk(B, H, t)dz_1 + Mq(B, H, t)dz_2 \tag{3}$$

When a hedging portfolio with the mortgage, house and default-free bond is formed, riskless hedging can be assured by continuous trading. So we can apply the CR expected value method to price our mortgage.

General Solution of the Model

When actually solving equation (3) for a defaultable mortgage, we represent the defaultable mortgage as the discounted value of the cash flow to be obtained. An advantage of this method, an application of the CR approach, is that we do not have to solve equation (3) directly, which would require transforming the stochastic equation into a deterministic partial differential equation. Moreover, a partial differential equation usually does not have a closed-form or analytical solution. Furthermore, the discounted value approach provides more economic intuition than does a partial differential equation.

Let's consider a mortgage at time t with maturity time T. The time-to-maturity $T - t$ is divided into n equal intervals, each of length dt. The future spot house price $H(i)$ is received by the mortgagor at idt (i is an integer) from initial time t when the mortgage is in default, if the mortgage has not been in default at all previous time periods. When the mortgage is not in default, the cash flow of P_n, one of the n fixed mortgage payments, is realized.

$DM(r, H, t, T, n)$ is a defaultable mortgage which can be in default for n periods, at time $t + dt, t + 2dt, \ldots, t + ndt$ $(t + ndt = T)$. Similarly, $DM(r, H, t, T, n - i)$ can be in default only for $n - i$ periods at $t + dt, t + 2dt, \ldots, t + (n - i)dt(t + \{n - i\}dt = T)$, etc. Hereafter, $DM(r, H, t, T, n)$ and $DM(t, T, n)$ are interchangeable.

$$DM(r, H, t, T, n) =$$

$$\sum_{i=1}^{n} B(r, t, i) \int_{DM(t+dt, T, n-1)}^{\infty} \cdots \int_{0}^{DM(t+idt, T, n-i)} H(i)dF[H(i)|H(1) \ldots H(i-1)]$$

$$+ \sum_{i=1}^{n} B(r, t, i) \int_{DM(t+dt, T, n-1)}^{\infty} \cdots \int_{DM(t+idt, T, n-1)}^{\infty} p_n df[\mathrm{H}(i)|H(1) \ldots H(i-1)] \tag{4}$$

The first term on the right-hand side of equation (4) is the sum of the discounted values of all house prices at the time of default. (When the mortgage is in default, the lender gets the house.) The second term on the right-hand side of equation (4) is the sum of the discounted values of mortgage payments when the mortgage is not in default and the lender receives a mortgage payment. $B(r, t, i)$ is a default-free bond at current time t with stochastic default-free interest rates and maturity i time intervals later, each of length dt. $DM(t + ndt, T, n - i)$ is the last mortgage payment P_n. $F\{H(i)|H(1) \ldots H(i-1)\}$ is an i-variate cumulative lognormal distribution function where $H(i)$ is the future spot house price idt time periods later. Equation (4) essentially expresses a mortgage price as the sum of the discounted value of two streams of cash flows, one from the house and the other from the mortgage payments. Equation (4), the solution to equation (3), does not involve any approximation. The major assumption here is that each cash flow comes in a discrete manner.

Risk-free discount bond prices $B(r, t, i)$ can be derived from the equation for the default-free interest rate r. (The derivation of bond prices is available from the author.) Simplifying the right-hand side of the equation (4) by using the technique for solving an equation on lognormal distribution discussed in equation 22 of Smith [1976], we get:

$$
\begin{aligned}
&DM(r, H, t, T, n) \\
&= H(N_1\{-d_1(DM[T-dt, n-1], dt)\} \\
&+ N_2\{d_1(DM[T-dt, n-1], dt), -d_1(DM[T-2dt, n-2], 2dt); -\rho_{12}\} \\
&+ N_3\{d_1(DM[T-dt, n-1], dt), d_1(DM[T-2dt, n-2], 2dt), \\
&- d_1(DM[T-3dt, n-3], 3dt); \rho_{12}, -\rho_{13}, -\rho_{23}\} + \ldots\ldots) \\
&+ P_n(B(r, t, 1)N_1\{d_2(DM[T-dt, n-1], dt)\} \\
&+ B(r, t, 2)N_2\{d_2(DM[T-dt, n-1], dt\}, d_2(DM[T-2dt, n-2], 2dt); \rho_{12}\} \\
&+ B(r, t, 3)N_3\{d_2(DM[T-dt, n-1], dt), d_2(DM[T-2dt, n-2], 2dt), \\
&d_2(DM[T-3dt, n-3], 3dt); \rho_{12}, \rho_{13}, \rho_{23}\} + \ldots\ldots)
\end{aligned}
\tag{5}
$$

Here N_1, N_2, N_3 are a univariate, bivariate, and trivariate, cumulative normal distribution, respectively, and H is the house price. The first term on the right-hand side of equation 5, N_1, is the probability that the house price is below the defaultable mortgage price in the first instant dt and that the lender gets the house because of default. The second term N_2 is the probability that the mortgage is in default at the second instant $2dt$, given that the house is not in default at time dt and so on. The first term after $B(1)$, $N_1(d_2)$ is the probability that the mortgage is not in default at dt, so the lender gets the mortgage payment P_n. The second term $N_2(d_2)$ is the probability that the mortgage is not in default at either dt or $2dt$, and so the lender gets another payment P_n, etc.

Comparative Statics Results and an Approximation of General Solution

Comparative Statics

Equation 5 is an analytical solution for a defaultable mortgage. The comparative statics results are also similar to those in Geske and Johnson [1984]. They are:

$\frac{\partial DM}{\partial H} > 0$, $\frac{\partial DM}{\partial P_n} > 0$, $\frac{\partial DM}{\partial \lambda^2} > 0$ and $\frac{\partial DM}{\partial r}$ and $\frac{\partial DM}{\partial T}$ are indeterminate, where H, P_n, λ^2, r and T are the house price, mortgage payment, interest-rate variance, interest rate and time-to-maturity respectively.

An Approximation of General Solution

We denote $DM(t, T, n)$ by $DM(u, n)$ with the understanding that all mortgages are evaluated at current time t and $u = T - t$, the time to maturity. $DM(u, 1)$, $DM(u, 2)$, . . . , $DM(u, n)$, form a series. The defaultable mortgage price, which can be in default at any time during the life of the mortgage, is the limit of this series.

$DM(u, 1)$ is a mortgage which can be in default only at maturity.

$$DM(u, 1) = \sum_{1}^{n-1} PB(r, t, t) + HN_1(-d_1(P, u)) \quad + PB(r, t, n)d_2(P, u)) \tag{6}$$

The first term for $DM(u, 1)$ is the $n - 1$ installments of mortgage payments the lender gets, because the mortgage cannot be in default before time T. The borrower will default if the house price is below P, the mortgage payment. The second is what the lender gets when the mortgage is in default at time T. When the mortgage is not in default at time T, the third term of equation (6) is received. The probability of default is $N_1(-d_1)$, which is the probability that the house price at that time will be below the mortgage payment P. The derivations for $DM(u, 2)$ and $DM(u, 3)$ are similar and are available from the author.

When n tends to infinity, the defaultable mortgage becomes one with continuous mortgage payments. An approximation formula for the mortgage with continuous payments, similar to the one in Geske Johnson [1984], is:

$$DM(r, H, t, T, \infty) = DM(t, T, 3) + 7/2\{DM(t, T, 3) - DM(t, T, 2)\} \\ - 1/2\{DM(t, T, 2) - DM(t, T, 1)\} \tag{7}$$

Simulations of Mortgage Prices

In order to analyze or evaluate our model, we provide the following simulation results. The values of parameters used in our simulation are:

$k = 0.25$
$a = 0.08$ (decreasing yield curve), 0.1 (level yield curve) and 0.12 (increasing yield curve)
$r(\infty)$, the limiting yield on a discount bond
$= 0.125$
$f(H, t) = \Lambda H$ and $\Lambda = 0.1$ to 0.2
$\lambda = -0.1$ to 0.1
$r(0)$, initial interest rate
$= 0.1$
$H(0)$, initial house price
$= \$100,000$
ltv, loan-to-value ratios
$= 0.8$ to 1.0
$u = T - t$, time-to-maturity
$= 15$ to 30 years

In Exhibit 1, Panel A, are the values of defaultable mortgages for $\gamma = 0$. When $\sigma = 0.2$, $\Lambda = 0{,}1$, $ltv = 0.8$, $T = 15$ years, the price of a defaultable mortgage is \$79,314.

Exhibit 1
Values of Mortgages and Default Options

Panel A: Values of Defaultable Mortgages (Level Yield Curve, $\gamma = 0.00$)
Time to Maturity (Years)

		15			20			25			30		
LTV	$\sigma_r \diagdown \Lambda_H$	0.10	0.15	0.20	0.10	0.15	0.20	0.10	0.15	0.20	0.10	0.15	0.20
0.80	.10	75899	75837	75413	74802	74716	74225	73982	73886	73386	73389	73294	72820
	.15	77443	77290	76672	76954	76700	75903	76591	76263	75371	76327	75961	75039
	.20	79314	78979	78090	79445	78872	77674	79497	78753	77367	79518	78680	77202
0.85	.10	80641	80529	79917	79475	79335	78665	78603	78456	77796	77972	77834	77223
	.15	82270	82019	81165	81735	81361	80327	81336	80884	79768	81049	80565	79439
	.20	84211	83712	82544	84282	83516	82037	84286	83347	81693	84283	83259	81533
0.90	.10	85381	85193	84343	84144	83927	83041	83220	83006	82158	82552	82358	81588
	.15	87084	86695	85560	86497	85970	84666	86060	85459	84094	85751	85130	83780
	.20	89068	88360	86875	89064	88073	86288	89015	87859	85922	88992	87767	85780
0.95	.10	90116	89815	88679	88808	88488	87348	87833	87530	86466	87130	86863	85912
	.15	91879	91304	89843	91233	90517	88911	90758	89981	88344	90429	89651	88059
	.20	93870	92906	91069	93779	92531	90422	93678	92284	90051	93640	92198	89943
1.00	.10	94844	94385	92910	93465	93007	91577	92439	92024	90716	91702	91346	90193
	.15	96644	95831	94002	95936	94993	93057	95424	94443	92514	95079	94123	92274
	.20	98600	97333	95118	98414	96882	94434	98266	96616	94079	98222	96552	94021

Panel B: Values of Default Options (Level Yield Curve, $\gamma = 0.00$)
Time to Maturity (Years)

		15			20			25			30		
LTV	$\sigma_r \diagdown \Lambda_H$	0.10	0.15	0.20	0.10	0.15	0.20	0.10	0.15	0.20	0.10	0.15	0.20
0.80	.10	0	63	487	2	88	579	3	99	600	4	99	573
	.15	10	163	781	36	291	1087	73	402	1294	107	473	1395
	.20	69	404	1292	243	816	2014	478	1221	2608	693	1531	3009
0.85	.10	2	114	726	5	145	815	7	154	814	8	146	757
	.15	24	275	1128	67	442	1475	120	572	1688	163	646	1772
	.20	133	633	1800	386	1152	2631	687	1626	3280	942	1965	3692
0.90	.10	6	194	1044	11	227	1113	14	228	1076	15	210	980
	.15	50	439	1574	117	645	1949	187	788	2153	238	858	2208
	.20	238	946	2431	585	1576	3361	956	2112	4049	1246	2471	4458
0.95	.10	14	315	1452	22	342	1482	25	328	1392	25	292	1242
	.15	96	671	2132	193	909	2515	281	1058	2695	337	1115	2707
	.20	397	1362	3198	851	2099	4207	1291	2686	4918	1611	3053	5308
1.00	.10	31	490	1964	41	498	1929	43	458	1767	40	395	1548
	.15	172	985	2814	302	1245	3181	407	1388	3317	464	1420	3269
	.20	628	1896	4111	1196	2728	5176	1702	3352	5889	2042	3712	6243

Values of defaultable mortgages decrease with respect to time-to-maturity and standard deviation of house price. This is probably due to the fact that probabilities of default are higher for longer times to maturity and greater fluctuations of house prices. Defaultable mortgage values also decrease with respect to the correlation coefficient γ betwen bond and house prices. Values of defaultable mortgages increase with respect to loan-to-value ratios and standard deviations of bond prices. The first relationship is obvious. The second occurs because bond prices are an increasing function of interest-rate standard deviations such that defaultable mortgage prices should be higher when discount bond prices are higher.

In Exhibit 1, Panel B contains the values of default options for $\gamma = 0$. When $ltv = 0.8$, $u = 30$ years, $\Lambda = 0.2$, $\sigma = 0.15$, the default option value is \$1395. The values of default options are extremely sensitive to the values of time-to-maturity, standard deviations of interest rates and house prices, and loan-to-value ratios. When $u = 20$ years, $ltv = 0.8$, $\sigma = 0.2$, values of default options change from \$243 for $\Lambda = 0.01$, to \$2014 for $\Lambda = 0.2$. And when $\Lambda = 0.2$, values of the default options change from \$579 for $\sigma = 0.1$ to \$2014 for $\sigma = 0.2$. Values of the default option increase with respect to standard deviations of interest rates and house rates, as well as loan-to-value ratios. The reason why higher loan-to-value ratios result in higher default option values is trivial. In the Black-Scholes option model, higher option values correspond to higher variances of the underlying asset. This is also what we find in our simulations. One possible explanation is that default options have very limited downside risk and much higher upside potential for higher interest-rate and house price variances. Higher interest-rate and house price risks would increase the probabilities of obtaining higher future cash flows from exercising the options, which in turn increase the option values. There are, of course, higher probabilities of not exercising the option, thus reducing the values of the options. The downside risk is lower than the upside potential because of the asymmetry of the distribution of future cash flows from exercising the options.

Option values increase with respect to time-to-maturity in nearly all cases, especially when standard deviations of interest rates have the values of 0.15 and 0.2. We find that this is also basically consistent with results in the option literature, which state that option values increase with respect to increases in time-to-maturity.

Conclusions

We first assume that discount bonds and house prices follow a Brownian motion process. The current price of the defaultable mortgage, which is a contingent claim of the discount bond and house prices, depends on bond and house prices. The mortgage price, then, follows a Brownian motion process. The mortgage price equation is solved with the stochastic-differential-equation method by taking expectations, as opposed to the finite-difference method popular in mortgage-pricing literature. Default is defined as the situation where the house price is lower than the value of the defaultable mortgage. An analytical expression is derived for the defaultable mortgage which can be in default for n periods. As n approaches infinity, the limit of the derived equation provides the price of a mortgage with continuous mortgage payments, which are defaultable at any time. We also provide analytical comparative statics results which are not available in existing literature. Because our model is more efficient than the numerical method model, we can also present more simulation results for wider ranges of parameters.

Notes

[1]For a fifteen-year mortgage, an interaction in Kau et al.'s 1988 paper takes 200 seconds of CPU time in the cyber 205 supercomputer. Our model uses about one second of IBM 3084 CPU time for getting the values of default option, defaultable mortgage, and default-free discount-bond price in one interaction. Even a personal computer could be used to run the model.

[2]Because our simulation results take considerably less computing time, we are able to provide more simulation results under different times-to-maturity and correlation coefficient between bond price and house price. We also provide more results for increasing and decreasing yield curves, in addition to the level one. As a comparison, Kau et al. provides simulation results for mortgages with time-to-maturity of fifteen years only. We provide results for fifteen, twenty, twenty-five and thirty years, respectively.

References

[1] J. Cox and R. Ross. The Valuation of Options for Alternative Stochastic Processes. *Journal of Financial Economics* 3 (Jan. 1976).

[2] Donald F. Cunningham and Patric H. Hendershott. Pricing FHA Mortgage Default Insurance. *Housing Finance Review* 3 (1984), 373-92.

[3] K. Dunn and J. McConnell. Valuation of GNMA Mortgage-backed Security. *Journal of Finance* (June 1981), 599-616.

[4] James F. Epperson, James B. Kau, Donald C. Keenan and Walter J. Muller, III. Pricing Default Risk in Mortgages. *AREUEA Journal* 13 (1985), 261-72.

[5] Chester Foster and Robert Van Order. An Option-Based Model of Mortgage Default. *Housing Finance Review* 3 (1984), 351-72.

[6] R. Geske and H. E. Johnson. The American Put Option Valued Analytically. *Journal of Finance* 39 (Dec. 1984).

[7] J. Kau et al. Option Theory and Fixed Rate Mortgages. University of Georgia working paper, University of Georgia, 1988.

[8] C. Smith. Option Pricing: A Review. *Journal of Financial Economics* 3 (1976), 3-51.

This paper is adapted from chapter 3 of my Ph.D. dissertation at the University of Texas-Austin. I would like to thank members of my dissertation committee, Samuel H. Cox (chairman), Niles Hansen, Richard D. MacMinn, Francis B. May and Leslie Young, for their guidance. I also would like to thank participants of the LSU Finance Seminar, Louis O. Scott and Christopher G. Lamoureux, for their comments. Comments of two anonymous referees and the Editor, Don R. Epley, are gratefully acknowledged as well. All remaining errors are mine.