Testing for Vertical Inequity in Property Tax Systems

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Abstract. **Models for testing assessor performance have been widely discussed in the literature. Many have been used in practice. The purposes of this study are to evaluate the performance of existing models and to propose two new models. We find that existing models can be used correctly to test for inequity when their functional form is consistent with the pattern of the assessment-sales ratio data. Results from the application of different models show inconsistencies since the appropriate functional form may vary for different data sets. The new models have the ability to emulate the forms of the existing models as well as handle more complex relationships.**

Introduction

The uniformity of property tax assessments is an important issue for taxpayers, assessors, and others involved in or affected by the assessment process. Ideally for a tax jurisdiction, the assessed value should be the same fraction of the market value for each property within a given class. One key question, then, is just how to test to see if such uniformity is present in a statistically significant sense.

Many statistical models have been proposed to test for assessment uniformity. One group of models includes the relatively simple coefficients of dispersion and variation. These are especially helpful as measures of horizontal inequity, which is discrimination in the tax base between similarly valued real property. A second set of models is used to test for the presence of vertical inequity, that is, progressivity or regressivity in the tax base between different value levels of real property.[1] These may be characterized as regression models and include some fairly complex multiplicative functions. Whichever statistical model is chosen, it must be able to accurately test for inequity. Therefore, correct statistical modelling is critical in judging the equity of a property tax system.

One purpose of this study is to show that current models are not always able to properly test for vertical inequity. The other purpose is to introduce two alternate models. These alternate models allow for more flexibility in the functional form that may be needed to reflect the patterns in assessment-sales (*a–s*) ratio data for many jurisdictions.

Various statistical models from the literature are described briefly in the next section, along with a justification for an alternative type of model. The two proposed models are

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introduced in section three. Section four contains an application of the existing and proposed models to real and simulated data sets. The final section includes a discussion of the results and some conclusions about existing and proposed models.

Vertical Equity Models

In the property tax system there are two types of inequity. The first type is horizontal. This occurs when equal-valued property of a certain kind is assessed at more than one rate. Horizontal variation in *a–s* ratios is always assumed to be present and is acceptable within limits. Some variation is acceptable because buyer and seller motivations differ from sale to sale causing transaction prices to vary around the theoretically true assessed values.

The second form of inequity is vertical. This occurs when property tax rates are systematically different for properties of different value. Two examples of vertical inequity are progressivity and regressivity in the property tax structure. Regressivity in the tax structure occurs when these *a–s* ratios tend to decline with increasing property value. Progressivity occurs when *a–s* ratios increase for higher property values. These relations arise for any class of property when *a–s* ratios are correlated with property values. In general, statistically significant vertical inequity is not acceptable.

The existing vertical equity models assume these relations are linear, linear transformable or of the simple quadratic type. But a more complex form of vertical inequity can exist as we shall see.

Existing Models

Exhibit 1 contains a summary of models designed to test for the presence of vertical inequity. Recent works by Bell [2], Cheng [6], Kochin and Parks [14], and Reinmuth [18] point out that the use of a linear regression model, such as the Paglin and Fogarty or IAAO model, does not allow for an accurate test of nonlinear conditions. The Cheng model, like

Exhibit 1
Tests of Vertical Inequity

Model	Null Hypothesis	Proposed by
$AV = \beta_0 + \beta_1 SP$	$\beta_0 = 0$	Paglin & Fogarty [15]
$AV/SP = \alpha_0 + \alpha_1 SP$	$\alpha_1 = 0$	IAAO [11]
$AV = \phi_0 SP^{\phi_1}$	$\phi_1 = 1$	Cheng [6]
$AV = \tau_0 + \tau_1 SP + \tau_2 SP^2$	$\tau_0 = \tau_2 = 0$	Bell [2]
$SP = \Omega_0 AV^{\Omega_1}$	$\Omega_1 = 1$	Kochin & Parks [14]

where:

AV = assessed value of the property,
SP = sales price (market value) of the property,
AV/SP = assessed value to sales price ratio of the property,
$\beta, \alpha, \phi, \tau, \Omega$ = coefficient estimators.

the Kochin and Parks model, has the capability to analyze data whose *a–s* ratios are not linear over the sales price range. Their major distinction is that they use long-transformed data to test for vertical inequity. The Cheng model uses assessed value as the dependent variable and sales price as the explanatory variable. Kochin and Parks reverse the dependent and explanatory variables.[2] They justify reversing the regression model by assuming that assessment error is less than market-driven error; however, Bell [2, pp. 127, 128] argues that assessments are based on incomplete information and assessors are unable to be completely objective, resulting in more error in these assessed values than in sales prices.

Another extension of vertical equity models comes from Bell, who used a quadratic functional form. His expansion of earlier models allows the assessor more flexibility in identifying a progressive or regressive tax structure. By comparing the sign of the quadratic and intercept terms, the assessor's performance can be determined to be quadratic-progressive (quadratic term is positive with a zero intercept) or quadratic-regressive (quadratic term is negative with a zero intercept) when the data show a statistically significant quadratic term. It also allows for the possibility that the assessor's performance is linear-progressive (intercept term is negative) or linear-regressive (intercept term is positive) assessment process, should the quadratic term be statistically insignificant [2, p. 130]. Bell concludes that linear models are not sufficient to capture the true nature of some data, and a model that includes a provision for nonlinearity is superior to one that does not [2, p. 131].

Justification for an Alternative Type of Model

The basis for a different, more complex model is found in the works of Bell [2] and Gaston [10]. First, Bell points out that Kochin and Parks [13] indicate the possible need for a nonlinear approach [2, p. 129]. In addition, Bell draws on the fact that the regression of *a–s ratios* on sales prices gives rise to estimates of parameters for a homogeneous nonlinear (quadratic) model for the regression of *assessment values* on sales prices. Bell uses the notion that these estimates are biased, along with the reference to nonlinearity by Kochin and Parks, to introduce an orthodox quadratic type approach.

Bell's observations on the possible nonlinear nature of the relationship between assessments and sales is confirmed by Gaston's quadrature test [10, p. 189]. Gaston's test revealed a nonlinear relationship between these variables. In fact, when the data were plotted they revealed a serpentine or S-shaped relationship. Such an S-shape would be concave from above for low- to medium-priced sales and convex from above for medium- to high-priced sales. Should this S-shaped relationship exist, then the quadratic model proposed by Bell would have to compromise one of the two curves in the relationship because it is incapable of modelling a third derivative.

If the cost approach is used in a mass appraisal system for arriving at assessed values there are several reasons why vertical inequity and the S-shaped relationship may exist. In the cost approach, the difference between neighborhoods is reflected in land value. If the land value attributed by the assessor to a neighborhood is not the same as that placed on it by the market, vertical inequity may be the result. For example, if the assessor avoids extreme ranges of land value (i.e., places too high a value on land in a low-priced neighborhood and too low a value on land in a high-priced neighborhood), this will result in a regressive tax structure and an S-shaped relationship.

Cost manuals are often used in the cost approach. One shortcoming of the manuals is that they are designed for the "average" house within each style and quality category. With an appraiser valuing individual homes, this shortcoming can be taken into account. In computer-assisted mass appraisal, however, the houses in extreme disrepair or the very expensive custom houses may be improperly assessed. For example, in expensive houses, the cost manual may substantially understate the value of customized, special order items, or might not even provide for such items. This would tend to result in an underassessment of such residences. Further, it is likely that the cost manual will not properly price certain housing attributes. It may, for instance, understate the loss in value resulting from depreciation, and older property may thus be assessed at too high a level. Since it is likely that older properties will be in the lower priced range, this error will result in vertical inequity. Another possibility is that with increasing real income, a general drift over the long run toward higher priced homes and away from the lowest priced homes would increase relative depreciation of older homes. Depreciation schedules may not capture this effect and would thus tend to overstate the true values of these lowest priced properties. Even though vertical inequity may result from each of these examples, existing models may not detect a problem should an S-shaped relationship between assessed value and sales price be present.

Each of the existing models allows for one particular functional relationship between sales price and assessed value. The testing procedure is accurate as long as the assumed relationship is appropriate. If the relation is incorrect, inconsistent results across different models are to be expected (Cannaday, et al. [5]). There is a likely functional misspecification for some proportion of data sets when any single existing model is chosen.

Recognizing this problem, an alternative known as spline regression modelling is proposed. This approach is capable of covering all the situations that can be handled by existing linear or nonlinear models. In addition, the spline model is capable of accurately testing for the conditions that Bell and Gaston point out.[3] The flexibility from using spline models reduces errors in conclusions about vertical inequity that are bound to occur with the existing models' limited functional forms. The proposed new approach uses a segmented polynomial that, in effect, allows multiple relationships to exist among low-, average- and high-valued properties when comparing assessed values to sales prices. More specifically, spline models have the flexibility to handle S-shaped relationships in which the midrange of properties are correctly assessed, the very low-valued properties are overassessed, and the very high-valued properties are underassessed. None of the existing models can accurately test for the presence of vertical inequity in this situation.

Spline Regression Modeling: An Alternate Approach

Spline regression models have been used to analyze data sets in various academic and professional fields when those sets do not conform to computationally simple mathematical equations. In other words, spline regression techniques are useful when different regions of data are explained by different functions. Smith [20, p. 57] describes splines as follows:

> Splines are generally defined to be piecewise polynomials of degree n whose function values and first n–1 derivatives agree at points where they join. The abscissas of these joint points are called knots. Polynomials may be considered

a special case of splines with no knots, and piecewise (also known as grafted or segmented) polynomials with fewer than the maximum number of continuity restrictions may also be considered splines. The number and degrees of polynomial pieces and the number and position of knots may vary in different situations.

Both the cubic and the piecewise spline regression models that are proposed in this paper allow for a series of three jointly determined functional relationships to be tested. Each relationship corresponds to a segment of the overall data range.

Significance tests of truncated variables can be biased due to the way knot locations are chosen. Generally, knot locations should be chosen to correspond to the overall behavior of the data; however, this introduces bias into the analysis. Such bias would require a lower than conventional *alpha* value for testing purposes. One way to eliminate the problem, should the data set be sufficiently large, would be to use a random sample of the data for knot estimation and to use the remaining observations to test for vertical inequity. This approach would not require a lower *alpha* value. In this analysis we are using splines to test for inequity in the same data set that was used to estimate knot locations. Therefore a lower than conventional *alpha* value was used.[4]

We should also indicate that smaller sample sizes may result in estimators that are not very robust. An unusual datapoint might distort the estimators substantially.[5] Care that the data are free of recording errors prior to carrying out the analysis may be the best way to reduce the importance of the issue. In general, larger samples are likely to generate more robust estimators.

Wegman and Wright [23] state that the statistical properties of splines are not well developed. In light of this, we reiterate the importance of having underlying reasons for believing that splines exist. We earlier gave some reasons for such a view in favor of splines for appraisal-sales regressions. The basis for using splines is thus founded, in part, on the a priori expectation for their presence. It is also based on the strength of the statistical fit that may be found in any given instance. A very good fit indicates a useful model has been found. In any event, the statistical fitting and testing of splines should be done with these matters in mind.

Cubic Spline Model

A cubic spline regression model is a special case of spline modeling [4, pp. 64, 65]. The use of a cubic spline model will allow data to fit a cubic equation, without restricting its ability to take on lower powered equation forms. In other words, the cubic spline model can accurately portray linear, quadratic, and cubic relationships. If a cubic model is applied to linear data, the higher powered estimators will be statistically unimportant, and only the linear term will be significant. The same conditions hold true for the cubic term in the case of quadratic relationships.

The cubic spline regression model is built from a nonlinear regression equation, represented by:

$$\begin{aligned} AV_j = {} & \beta_{00} + \beta_{01} SP_j + \beta_{02}[SP_j^2 - (SP_j - t_2)_+^{\,2}] \\ & + \beta_{03}[SP_j^3 - (SP_j - t_1)_+^{\,3} - 3t_1(SP_j - t_2)_+^{\,2}] \\ & + \beta_{11}(SP_j - t_1)_+ + \beta_{12}[(SP_j - t_1)_+^{\,2} - (SP_j - t_2)_+^{\,2}] \\ & + \beta_{21}(SP_j - t_2)_+ + \varepsilon_j \end{aligned} \qquad (1)$$

where:

AV_j = assessed value of the j^{th} property,
SP_j = sales price of the j^{th} property,
t_1 = the first knot (the break-point between the first and second functions),
t_2 = the second knot (the break-point between the second and third functions),
ε_j = error term for the j^{th} property,
$(x)_+$ = any variable in the model whose value is limited to the range bounded by zero and the true positive mathematical expression contained within the parentheses, and
β_{00} through β_{21} are parameters.

The notation "$(x)_+$" represents the variables within the cubic spline model where truncated values are used. A truncated value will equal the mathematical value of the expression contained within the parentheses when that value is non-negative, and it will equal zero when the mathematical value of the expression contained within the parentheses is negative. For example, if SP_j is greater than t_2 and $(SP_j - t_2)$ is 12564.90, then $(SP_j - t_2)_+$ is this same 12564.90. If, however, SP_j is less than t_2 and $(SP_j - t_2)$ is -123.25, then $(SP_j - t_2)_+$ is zero.

When this model reflects a linear relationship between assessed values and corresponding sales prices, and when neither regressivity nor progressivity is present, $\beta_{00}=0$, $\beta_{02}=0$, $\beta_{03}=0$, $\beta_{11}=0$, $\beta_{12}=0$, *and* $\beta_{21}=0$. The use of truncated polynomials, or "$(x)_+$" functions, allows data to be fit by ordinary least squares, while still allowing hypotheses to be tested [20, p. 62].

The procedure used in this paper for finding the knot locations is the SAS routine known as DUD. The procedure estimates where structural changes (knot locations) occur in the data set. The routine calculates knot locations such that the three segments fit better, in a sums of squares sense, than any other three segments over the data range. The procedure iterates a series of knot location estimates, such that the error sums of squares of the final iteration converge to a minimum [19, pp. 22–24]. When the process is complete, the parameter estimates for these knot locations can be entered into the cubic spline regression equation's truncated values (as t_1 and t_2) prior to estimation of the cubic spline's regression coefficients.

This estimation process requires the investigator to enter knot starting values that the routine uses to search for the optimal knot locations. It is possible that the initial estimates could yield locally rather than globally optimal knot locations. To avoid this problem, running the DUD procedure several times with different initial knot starting values is recommended. Identical results with different starting values indicate that global optimum values are more likely to have been found.[6]

Piecewise Linear Spline Model

The piecewise linear spline regression model proposed here is a simplification of the cubic spline regression model described above [17, p. 520]. The primary reason for introducing the piecewise spline model is to have, as an option, a model that is easier to use and understand. The same DUD iteration process is used for the piecewise spline as for the cubic spline. The proposed piecewise spline regression model may be written in the form:

$$AV_j = \alpha_{00} + \alpha_{10}SP_j + \alpha_{01}LOW_j + \alpha_{02}HIGH_j + \alpha_{11}LOWSP_j + \alpha_{12}HISP_j + \varepsilon_j \qquad (2)$$

where:

LOW_j = dummy variable equaling one if the j^{th} unit sale price is lower than the first knot. If higher, this equals zero;

$HIGH_j$ = dummy variable equaling one if the j^{th} unit sale price is higher than the second knot. If lower, this equals zero;

$LOWSP_j$ = sale price of the j^{th} unit if the sale price is lower than the first knot. If higher, this equals zero;

$HISP_j$ = sale price of the j^{th} unit if the sale price is higher than the second knot. If lower, this equals zero;

α_{00} through α_{12} are parameters; and the other variables are as before.

When this model reflects a linear relationship between assessed values and sales prices, the slope condition $\alpha_{10} = \alpha_{11} = \alpha_{12}$ holds true. When this model reflects a non-progressive or non-regressive relationship between the assessed values and sales prices, the intercept condition $\alpha_{00} = \alpha_{01} = \alpha_{02} = 0$ also holds true. This simplified spline model, when compared to the cubic version, results in a less complex explanation for vertical inequity, should it exist. In other words, if the problem is associated with a lack of linearity, then it is not true that $\alpha_{10} = \alpha_{11} = \alpha_{12}$. Also if the problem lies in a linearly progressive or regressive assessed value to sales price relationship, then it is not true that $\alpha_{00} = \alpha_{01} = \alpha_{02} = 0$.

The piecewise spline regression model gives results similar to that of the cubic spline model even though the piecewise model is simpler. The piecewise model requires the use of a Chow *F*-test for determining statistically significant conclusions that can be reached in any application (see Chow [7] and Kennedy [12]). We recommend the use of the piecewise model because of its relative simplicity and because cubic splines are generally not required to show vertical inequity.

Analysis of Results and Findings

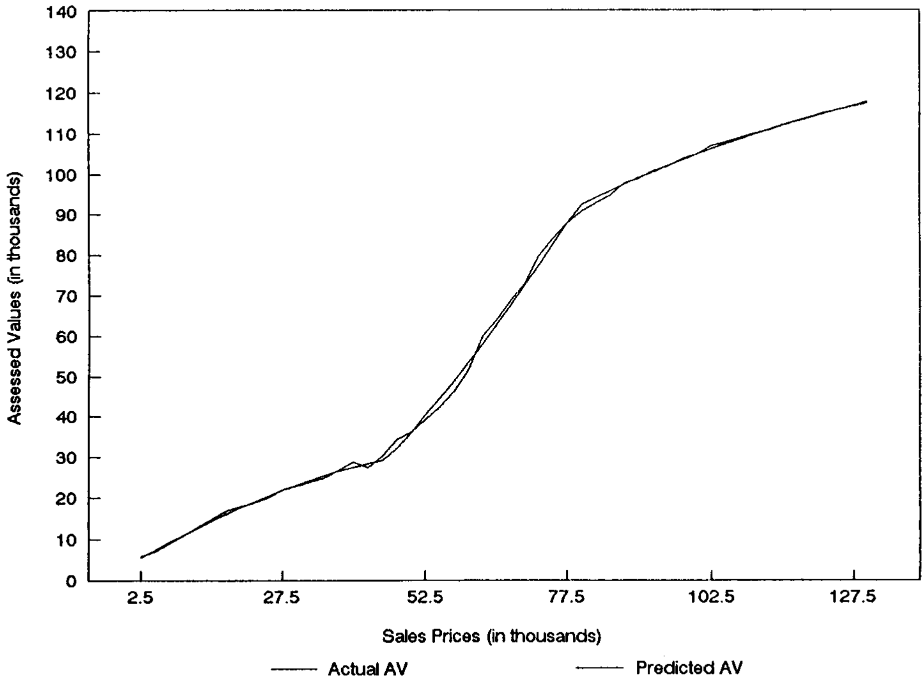
This section is divided into two parts. The first part uses data that has been contrived to show clearly how existing models could give erroneous results. The second part uses actual condominium sales data taken from an area near the Chicago central business district.

Contrived Data

A data set was constructed to demonstrate the usefulness of the cubic and piecewise spline models. Data plots and the two spline models are shown in Exhibit 2. Plots of the Kochin and Parks and Paglin and Fogarty models, along with the data, are shown in Exhibit 3. The numerical results are in Exhibit 4.

As may be seen in these graphs the data set is designed to have an S-shape. The graphs show that both spline models fit very well. Based on the statistical results shown in Exhibit 4, we may conclude that both spline models indicate clearly a nonlinear S-shaped relationship. This may be seen by examining the last two columns of Exhibit 4, which show probability (prob) levels in parentheses less than 0.05 for the two spline models. A probability level less than 0.05 means there is less than a 5% chance that the corresponding coefficient or *F*-statistic is zero. In the case of the cubic spline, if any one of the probabilities is less than 0.05, there is strong evidence of vertical inequity.

Exhibit 2
Contrived Data: Cubic Spline Regression Model



Contrived Data: Piecewise Spline Regression Model

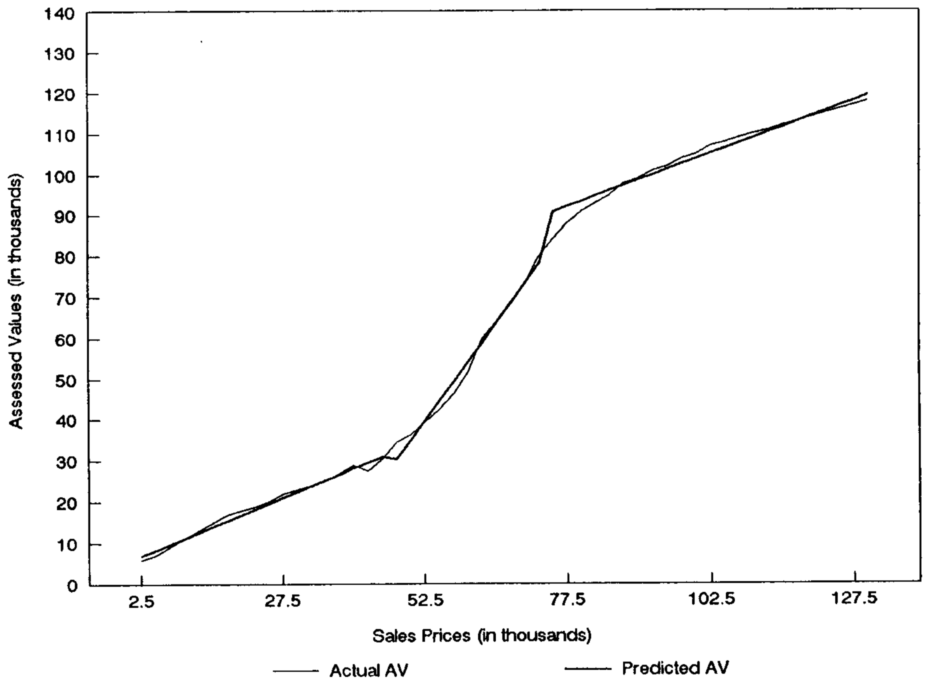
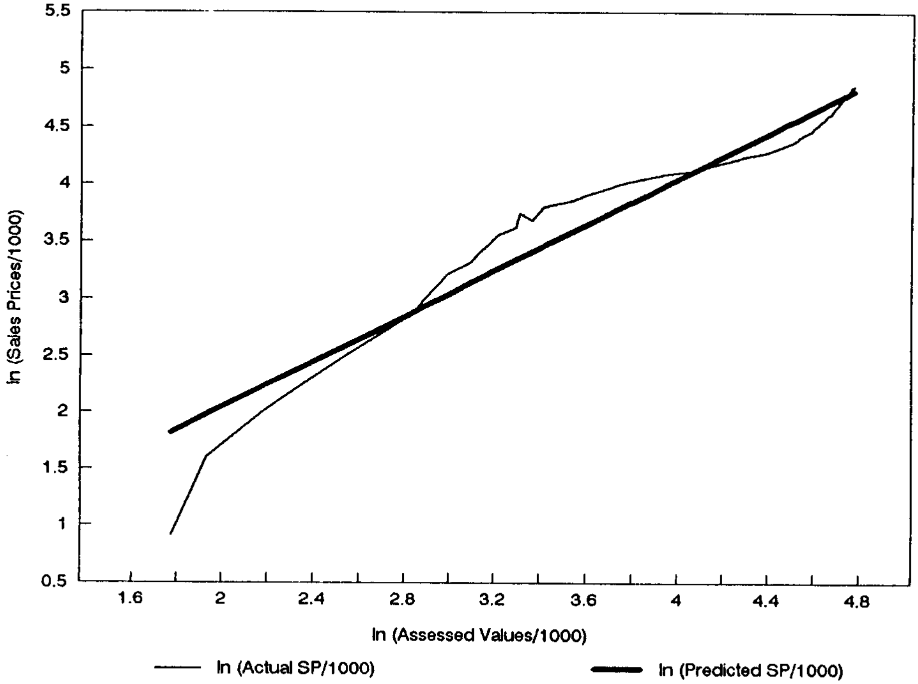


Exhibit 3
Contrived Data: Kochin and Parks Regression Model



Contrived Data: Paglin and Fogarty Regression Model

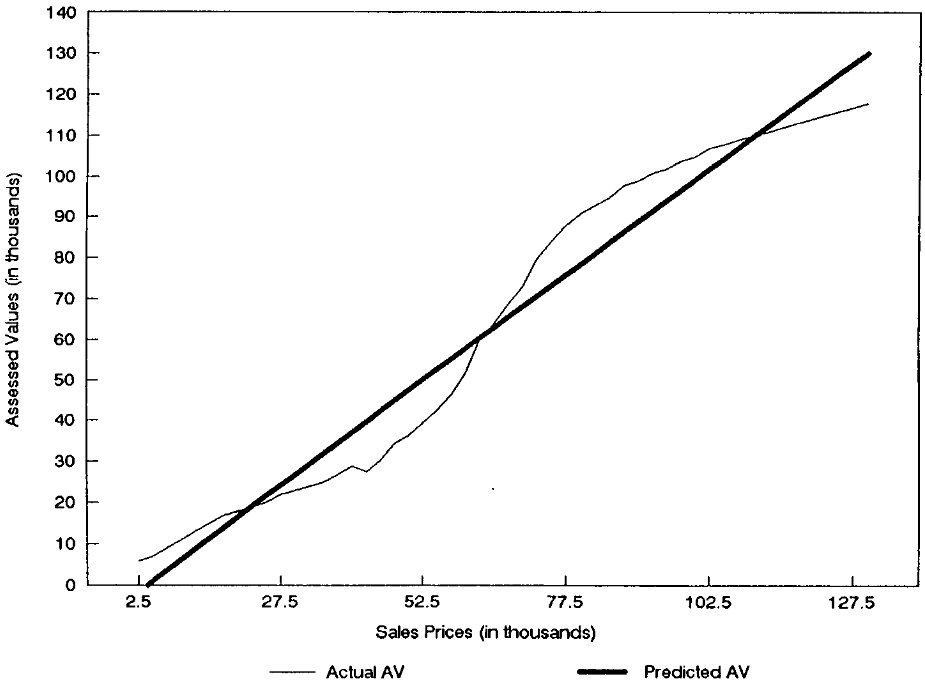


Exhibit 4
Contrived Data: Results

MODEL Test*	Paglin/ Fogarty	IAAO	Bell	Kochin/ Parks	Cheng	Cubic Spline	Piecewise Spline
$\beta_0=0$ $\beta_{00}=0$	−4.132 (0.0665)	na	na	na	na	3.685 (0.0001)	na
$\alpha_1=0$, $\phi_1=1$ $\Omega_1=1$	na	−0.0003 (0.7524)	na	1.0007 (0.9848)	0.941 (0.0829)	na	na
$\tau_0=\tau_2=0$	na	na	F=2.1357 (0.1290)	na	na	na	na
$\beta_{02}=0$	na	na	na	na	na	−0.003 (0.0476)	na
$\beta_{03}=0$	na	na	na	na	na	−0.00003 (0.0003)	na
$\beta_{11}=0$	na	na	na	na	na	1.2590 (0.0001)	na
$\beta_{12}=0$	na	na	na	na	na	0.0161 (0.0001)	na
$\beta_{21}=0$	na	na	na	na	na	−1.4573 (0.0001)	na
$\alpha_{00}=\alpha_{01}=$ $\alpha_{02}=0$	na	na	na	na	na	na	F=449.03 (0.0001)
H_0: No Vertical Inequity	Accept	Accept	Accept	Accept	Accept	Reject	Reject

*The null test values given for the different models, as shown in Exhibit 1, are for the β_x, α_x, τ_x, Ω_x, ϕ_x, β_{xx}, and α_{xx} parameters of the Paglin and Fogarty, IAAO, Bell, Kochin and Parks, Cheng, cubic spline, and piecewise spline models, respectively. Nonparenthetic numerical values in the first eight rows indicate coefficient estimators. The piecewise and Bell models require an F-statistic to test the hypotheses. The prob values for the F-statistics and coefficient estimators are given in parentheses.

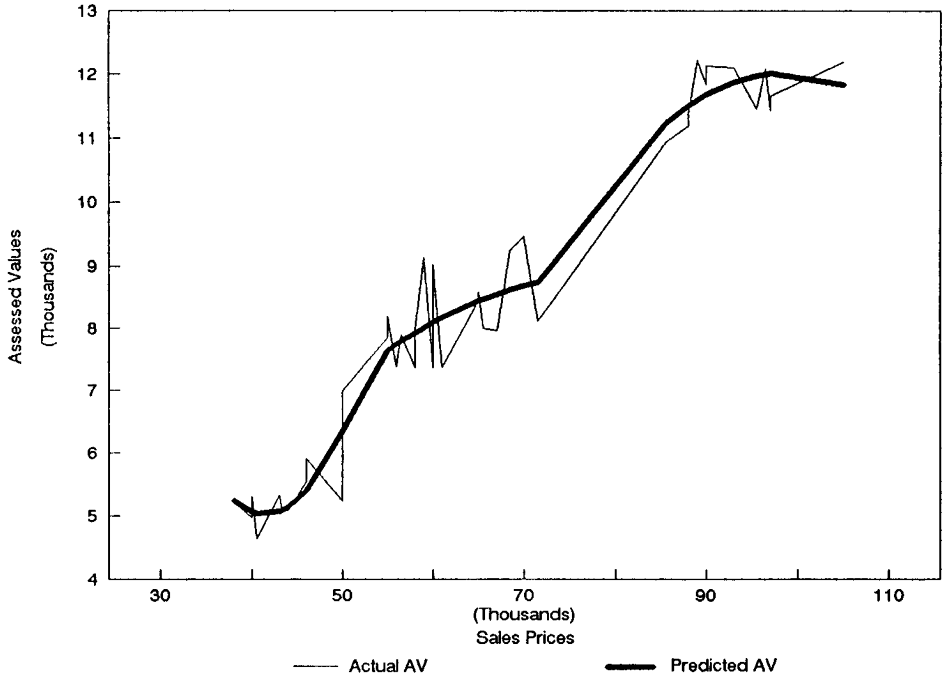
All of the other models yield incorrect results when testing for the existence of vertical inequity. This may be seen by inspecting the first five columns and the top three rows of the exhibit. The probability levels shown there are all greater than 0.05. In essence these other models do not detect the regressive and progressive structures designed into the data set. These non-spline models lead to incorrect conclusions because they have linear, simple nonlinear, or quadratic functional forms and are thus incapable of handling S-shaped forms of varying segment length.

Chicago Condominium Data

Data from the Sandburg Village area in Chicago is used to show how the existing tests can provide conflicting results.[7] Plots of these figures are included in Exhibits 5 and 6.

Exhibit 5

Sandburg Village Data: Cubic Spline Regression Model



Sandburg Village Data: Piecewise Spline Regression Model

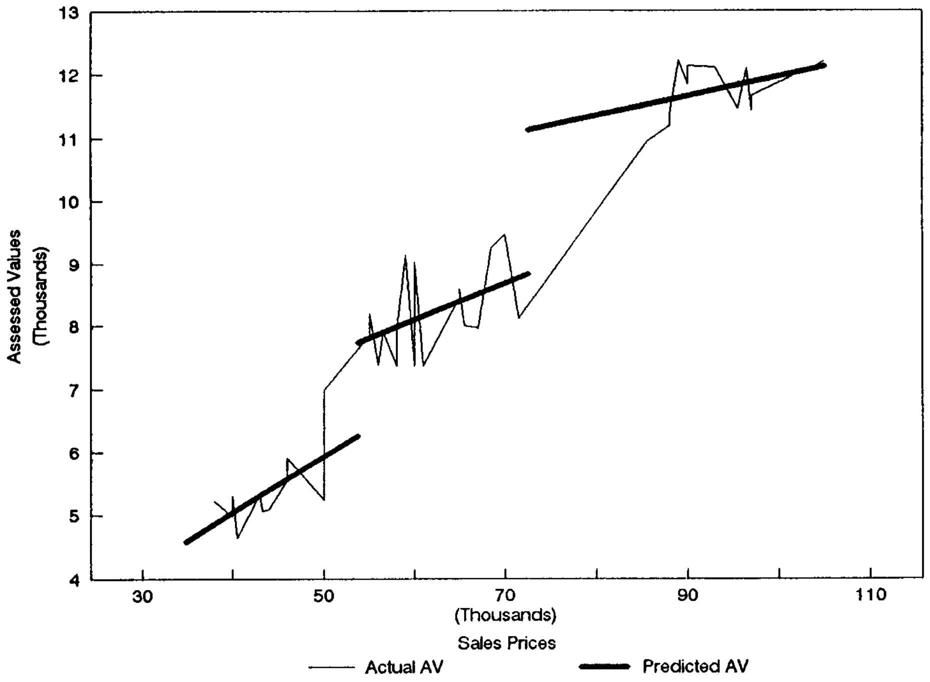
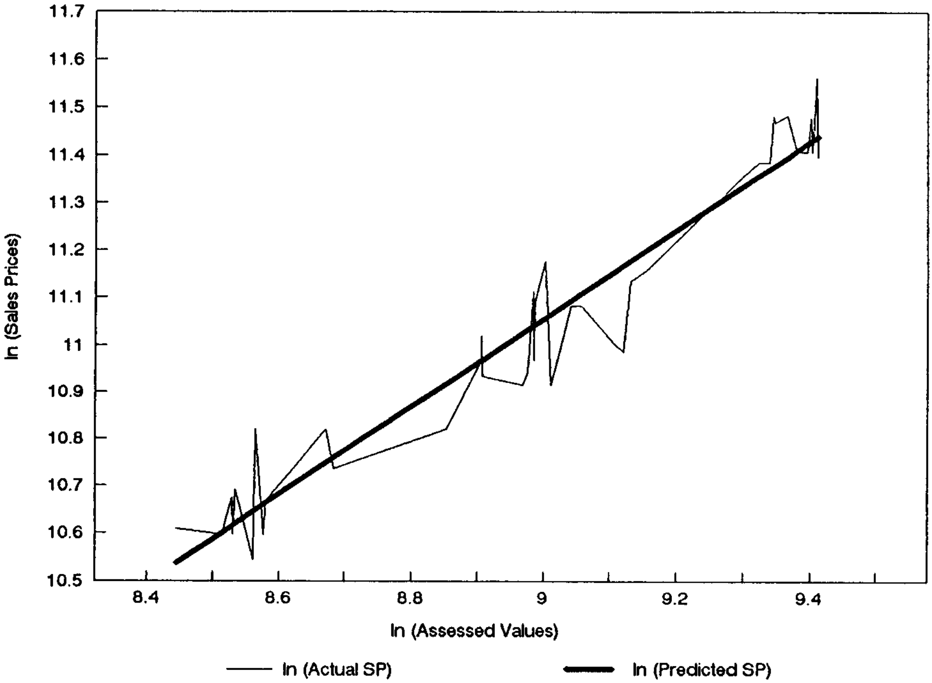
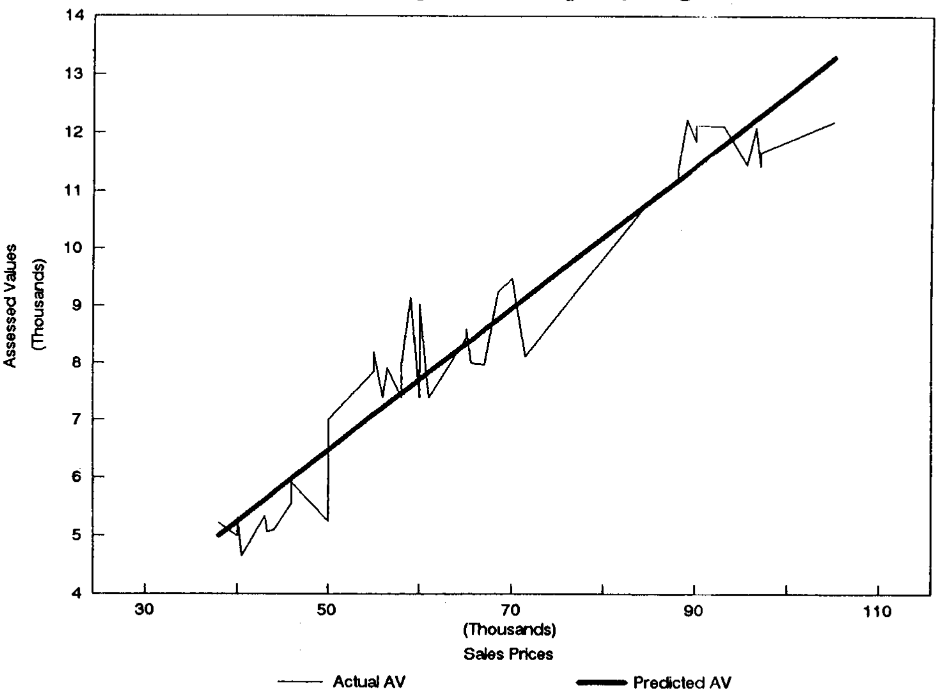


Exhibit 6

Sandburg Village Data: Kochin and Parks Regression Model



Sandburg Village Data: Paglin and Fogarty Regression Model



The data set comes from the Sandburg Village area located just north of the Chicago central business district. The data consists of forty-three condominium sales from January 1984 through December 1984.[8] In the modelling process, the 1984 assessed values are fitted on the corresponding sales prices.

In examining plots in Exhibit 5, even though the data for Sandburg Village is not as smooth as the contrived data, the relationship does not appear to be linear. The graph shows that there are three distinct relationships; i.e., one each for the low-priced properties, the intermediate-priced properties, and the high-priced properties. The piecewise spline model shows that each of these groupings can be modelled with line segments of different intercept and slope.[9] Exhibit 6 shows the model fits of Paglin and Fogarty, and of Kochin and Parks.

The empirical results for Sandburg Village are in Exhibit 7. Most of the existing models show no vertical inequity is present. This can be seen by looking at probability levels in the

Exhibit 7
Sandburg Village: Results

MODEL Test*	Paglin/ Fogarty	IAAO	Bell	Kochin/ Parks	Cheng	Cubic Spline	Piecewise Spline
$\beta_0=0$ $\beta_{00}=0$	285.36 (0.4151)	na	na	na	na	38259 (0.0192)	na
$\alpha_1=0$, $\phi_1=1$ $\Omega_1=1$	na	−2.23E-8 (0.7871)	na	0.9317 (0.0852)	1.0023 (0.9561)	na	na
$\tau_0=\tau_2=0$	na	na	F=3.8870 (0.0287)	na	na	na	na
$\beta_{02}=0$	na	na	na	na	na	2.9E-5 (0.0069)	na
$\beta_{03}=0$	na	na	na	na	na	−2.8E-11 (0.0038)	na
$\beta_{11}=0$	na	na	na	na	na	−0.3038 (0.1404)	na
$\beta_{12}=0$	na	na	na	na	na	−1.9E-5 (0.0057)	na
$\beta_{21}=0$	na	na	na	na	na	0.2102 (0.1127)	na
$\alpha_{00}=\alpha_{01}=$ $\alpha_{02}=0$	na	na	na	na	na	na	F=7.101 (0.0007)
H_0: No Vertical Inequity	Accept	Accept	Reject	Accept	Accept	Reject	Reject

*The null test values given for the different models, as shown in Exhibit 1, are for the β_x, α_x, τ_x, Ω_x, ϕ_x, β_{xx} and α_{xx} parameters of the Paglin and Fogarty, IAAO, Bell, Kochin and Parks, Cheng, cubic spline, and piecewise spline models, respectively. Nonparenthetic numerical values in the first eight rows indicate coefficient estimators. The piecewise and Bell models require an F-statistic to test the hypotheses. The prob values for the F-statistics and coefficient estimators are given in parentheses.

first five columns and first three rows of the exhibit. The Bell model gives weak rejection, with a probability level of 0.0287. The spline models both gave strong rejection of the hypothesis of no vertical inequity. The cubic spline model rejects the hypothesis, as suggested by probability levels much below 0.05 for the coefficients β_{00}, β_{02}, β_{03}, and β_{12}. Recall that only one of the probabilities in this column need be less than 0.05 for rejection of the hypothesis. For the piecewise spline model, the very high F-statistic and its corresponding low probability level show that there is a difference in treatment between the assessment percentage of low-, middle- and high-valued property. Apparently vertical inequity is present.

Findings

The existing models that are used to test for vertical inequity are sufficient for cases where the *a–s* ratios within the assessment district are either linear or nonlinear with a continuous first derivative (the Bell, Kochin and Parks, and Cheng models). The existing models, however, are incapable of properly testing for vertical inequity in a situation where the relationship is nonlinear with a discontinuous first derivative, as is shown with the contrived data. This also appears to be the case for the actual data. The cubic spline and the piecewise spline models appear to be superior to existing models in detecting vertical inequity.

Conclusion

The issue of vertical inequity within jurisdictions will continue to be an important topic for as long as states and municipalities use property taxation based on assessed values. Depending on the models used to test for vertical inequity, inconsistent results will often be obtained. The question of which model to use depends on which end of the complexity/accuracy continuum the assessing district chooses.

The spline models introduced in this paper can be used to handle situations and give results that encompass those for existing models, as well as handle more complex situations. Therefore, the spline models are superior in testing for vertical inequity. The increased complexity of spline modelling must be compared with the likely increased accuracy in discerning the existence of vertical inequity. The choice depends on the extent to which it is felt that S-shaped vertical inequity may be present in the jurisdiction and how important it is to test for such inequity. The potential inequity in a tax system could be extensive. Such extensive inequity would remain undetected if only previously proposed models were to be used.

Notes

[1]See Bell [2], Cheng [6], Kochin and Parks [13 and 14], Paglin and Fogarty [15], and Reinmuth [18].

[2]An extension of the Cheng and the Kochin and Parks models would be one using Box-Cox estimation. This would improve the statistical fit for the family of one independent variable models. The Box-Cox procedure is designed to provide a best model form within a *family* of models. Such a transformation for data fitted with the Cheng or Kochin and Parks models would not, however, provide a good functional form for the nonlinear cases discussed on pages 321 through 322. See Kennedy [12, p. 83] and Fomby, et al. [9, pp. 424ff].

[3]An extension of Bell's model by adding a cubic term is insufficiently flexible. Such a model implies a constant third derivative, which may not be the case at all.
[4]For an exposition of the proper use of splines, see Suits, et al. [21] and Wold [24].
[5]Wegman and Wright [23, p. 362] have indicated the development of the underlying statistical properties for robustness of small sample estimators is only at its beginning stages. It has been suggested that estimating with other than a sum of squares method can improve matters (see Anderssen, et al. [1]).
[6]Connor [8] has conducted research on uniform distributions regarding optimal knot locations with different numbers of segments. He found that for a model with three segments, the optimal knot locations are the 27th and 73rd percentile observations. These locations could be used by the investigator as initial starting points for the DUD procedure.
[7]Since the purpose of this paper is to show how the existing tests may give conflicting results, these results do not measure the amount of inequity, examine the probable cause(s) of the inequity, or discuss how to correct for the inequity. To explore these other areas further, see Sunderman, et al. [22] and Birch, et al. [3].
[8]Wold [24] determined the minimum number of observations required per interval could be as few as four or five. This would suggest a three-segment model could be run with as few as twelve to fifteen observations. This point is further substantiated by Poirier [16], who states that the minimum number of observations required to test a cubic model is fifteen. Therefore, our sample size is sufficient with forty-three observations, in total, including as few as eleven in one of the three segments.
[9]The Sandburg data were plotted by connecting the observations with line segments. It should be noted also that there are no datapoints between sales prices of approximately $73,000 and $85,000. This is why there is such a large difference between the cubic and piecewise spline graphs.

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