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ABSTRACT

A Microfoundation for Production Functions: Assignment of Heterogenous Workers to Heterogenous Jobs*

In very different fields of economics, economic inference and policy evaluation require economists to parametrize a production function that links measures of input factors to measures of output. While doing so, strong assumptions are implicitly made about microeconomic variables governing the shape of the aggregate production function. In this paper, I develop an assignment model that provides a microeconomic foundation for aggregate production functions. The shape of the production function depends crucially on the distribution of workers and jobs and the type of technological changes depends crucially on the evolution of these distributions. Sufficient and necessary conditions are provided for the production function to be of the Constant Ratio of Elasticities of Substitution form, a form nesting the broadly used Constant Elasticity of Substitution form. This model provides a way to evaluate how stringent assumptions about the type of production function or technological change are by comparing the implied distribution of jobs and its evolution over time to observations of the distribution of jobs and its evolution over time.

JEL Classification: D2, D3, J3, O3

Keywords: microfoundations of production function, tasks assignment, technical change

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1 Introduction

In very different fields of economics, economic inference and policy evaluation require economists to parametrize a production function that links measures of input factors to measures of output. While doing so, strong assumptions are implicitly made about microeconomic variables such as the distribution of factors (see Houthakker (1955-1956)), the distribution of ideas (see Jones (2005)), the distribution of productivity (see Rosen (1978)) or the distribution of jobs (this paper). For instance, in the literature about skill-biased technological change and rising skill-premium, a standard assumption is that aggregate production is of the Constant Elasticity of Substitution (CES) type and technical change is skilled-labor augmenting.[1] What the CES assumption implies for the distribution of jobs or what the skilled-labor augmenting technical change implies for the evolution of the distribution of jobs over time is not known. A general drawback of using production functions is that these functions lack micro-economic foundations that would enable us to justify our parametrization. Without identifying the microeconomic forces that govern the structure of production, economic inference and policy implications based on reduced form aggregate production functions may be highly hazardous.

Surprisingly enough, very little is known about the microeconomic foundations of production functions. To my knowledge, only Houthakker (1955-1956), Levhari (1968), Rosen (1978) and more recently Jones (2005) and Lagos (2007) have viewed production functions as reduced form of micro-founded models. In Houthakker's (1955-1956) original work,[2] the aggregate production function is derived from the distribution of inputs across productive cells (e.g. firms, machines or workers). Houthakker shows that when the distribution of inputs is of the generalized Pareto type, the aggregate production function then has the Cobb-Douglas form.[3] A shortcoming of this approach is that inputs are randomly distributed across productive cells whereas inputs mix are more likely to be the result of a matching process at the level of the productive cell. Rosen's (1978) tasks assignment model accounts for the non random assignment of inputs. In his model, the prob-

[1] See Katz and Murphy (1992) and Acemoglu (2002) and references therein.

[2] See also Johansen (1972), Sato (1975) and Hildenbrand (1981) among others.

[3] Levhari (1968) showed that this result could be extended to CES production function when inputs are distributed according to a Beta distribution. Jones (2005) builds on this model and derives the shape of the production function and the direction of technical change from the distribution of ideas. Lagos (2007) also builds on Houthakker's model and derive the shape of the aggregate production function in a model with search frictions in the labor market.

lem is to assign workers of different productive types to heterogenous tasks so as to maximize aggregate output. Rosen (1978) then derives the shape of the aggregate production function from the assignment of workers to tasks. When tasks are uniformly distributed and under additional assumptions on the productivity of workers in the various tasks, the aggregate production function is of the CES type with constant returns to scale.

In this paper, I consider a tasks assignment model with two types of workers and heterogeneity of workers within types and, general distributional form of tasks. In contrast to Rosen's (1978) model that ignores capital, each task is associated with a unit of capital, a machine for the sake of the argument, and output is therefore produced using both capital and labor inputs. The assignment problem is solved using Ricardo's principles of comparative advantage and differential rents. In the economy considered, each machine can only be operated by one worker at a time. As a result, the shape of the aggregate production function is governed principally by the distribution of tasks in contrast to Rosen's (1978) model in which the shape of the production function is driven by the productivity of the worker-task matches. This makes the model of this paper very convenient to check the validity of assumptions made when using macro production functions. While information on the physical productivity of worker-task pairs is rarely available in data sets, information about the distribution of jobs can readily be derived in most data sets using occupational codes for instance.

Closed form solutions for the shape of the aggregate production function can be recovered in this general assignment model. In particular, I will show that the aggregate production function has the Constant Ratio of Elasticities of Substitution (CRES) type when tasks are distributed according to the Beta distribution and the productivity of a match worker-task is Cobb-Douglas and function of the output level. With the additional restriction that the Beta distribution is symmetric, the aggregate production function degenerates to the CES type with general returns to scale.

The remaining structure of the paper is as follows. In the next section, I present the model of tasks assignment with multidimensional skills. In section 3, a parametrization of the model is proposed so that the aggregate production function is of the CES type. An extension of the model to derive the CRES production function type is proposed and necessary conditions for the aggregate production function to be of the C(R)ES type are derived. This parametrization enables us to reveal what microeconomic assumptions on the distribution of tasks are made when imposing the aggregate production function to be of the C(R)ES type. In section 4, using the results of the model, I argue that the type of changes in the distribution of tasks implicitly imposed in the skill-biased technical change literature (see Katz and Murphy

(1992) or Acemoglu (2002)), by using a CES production function with skill-labor augmenting technical change, does *not* fit well with recent empirical evidence about job polarization (see Autor et al. (2006)) and the density increase in the upper tail of the distribution of firm size (see Gabaix and Landier (2008) and Terviö (2008)).

2 Tasks assignment model with two types of heterogenous workers and a continuum of tasks

2.1 Setting of the model

Consider an economy where workers are heterogenous in terms of their type and level of skills. By analogy to product differentiation, the types of skills correspond to horizontal differentiation between workers and the level of skills corresponds to vertical differentiation between workers within types. For simplicity, let there be two types of skills, say type 1 and 2. To fix ideas, think of type 1 skills as manual skills and type 2 skills as intellectual skills. The mass of type k workers is S_k and without loss of generality, the total mass of workers is assumed to be unity, i.e. $S_1 + S_2 = 1$. Within types, workers are heterogenous with respect to their skills level. Let $t_k \in [\underline{t}_k, \infty)$, indicate the level of skills of a type k worker. The distribution of workers within types is then characterized by the probability density function $s_k(t_k)$ for $k = 1, 2$ and the cumulative density function $S_k(t_k)$, with $\lim_{t_k \to \infty} S_k(t_k) \equiv S_k$.

This economy produces a composite commodity by means of the input of an infinite number of different tasks. Each task is associated with a unit of capital, a machine for the sake of the argument, and the various tasks correspond to machines with different characteristics.[4] To produce output, each machine needs to be operated by a fixed proportion of workers, i.e. one and only one worker. Aggregate output Y is obtained by summing up the production in each single task.[5]

[4] This part of the model is to a large extent similar to the differential rents models described in Sattinger (1979 and 1993). The terminology "task" and "machine" are interchangeable throughout the paper. In general I will use "task" for the sake of simplicity but when needed I will refer explicitly to machines.

[5] This contrasts with Rosen's (1978) model that ignores capital. In Rosen's model, tasks from different jobs are needed in fixed proportions to produce aggregate output, there is no substitution possibilities between tasks, but each job can be filled with more than one worker. This also contrasts with assignment models proposed by Lucas (1978), Rosen (1981, 1982), Becker (1981), Kremer (1993), Kremer and Maskin (1996), Garicano (2000),

Output at each task can be produced by workers with different types and levels of skills but workers of different types and levels of skills differ in their productivity. For instance, while an intellectual worker could operate a circular saw with some productivity, a manual worker would probably be more productive. Similarly, while a manual worker could use a computer productively, an intellectual worker would probably make better use of the same computer. These examples suggest that, under certain assumptions, machines could be ranked on a one dimensional support. The position of machines on this support would indicate a gradual change from extremely manual machines to extremely intellectual machines as we move from the left to the right.

The required assumptions for the support of tasks to be unidimensional are better understood with the following experiment in mind. Suppose we assign a randomly chosen manual worker successively to each task of this economy and rank these tasks by *decreasing* productivity. Similarly, we assign a randomly chosen intellectual worker successively to each task and rank the tasks by *increasing* productivity. The first fundamental assumption about the support of tasks is that the ranking of tasks would be exactly the same in both cases, meaning that the most manual tasks are also the least intellectual ones and vice versa as caricatured in Figure 1. Note that this assumption is a sufficient condition for comparative advantage of *types* of workers to arise. Manual workers have a comparative advantage in low ranked, manual, tasks and intellectual workers have a comparative advantage in high ranked, intellectual, tasks.

The second fundamental assumption about the support of tasks is that the ranking of tasks is the same for all workers, or stated otherwise, does not depend on the *level* of skills. A sufficient condition for the rank of tasks to be the same for all workers is the complementarity between machines and skills. The complementarity assumption stipulates that the change in productivity associated with an increase in the level of manual (intellectual) skills is larger in more manual (respectively intellectual) tasks. Hence, increasing the level of manual (intellectual) skills in Figure 1 would level up the bars in job titles with low (respectively high) rank compared to job titles with high (low) rank but would not affect the rank of job titles.

Given these two assumptions we can define the support of tasks as follows. Let v denote a task and, without loss of generality, let the support of task be the unit interval $(0, 1)$, with tasks v increasing from 0 to 1 as the rank of tasks defined above increases. To fix ideas, tasks close to 0 are for instance

Garicano and Rossi-Hansberg (2004, 2006) and Fox (2006) where the output of several tasks are complementary.

the tasks of a carpenter and tasks close to 1 are the tasks of a rocket scientist. Similarly, machines close to 0 could be circular saws and machines close to 1 could be computers. Tasks in the middle of the support are the "anybody can do it as efficiently" tasks.

Once the support of tasks is defined, the productivity of a worker with t_k units of skills of type k when assigned at task v can be defined by the function $p_k(t_k, v)$. By definition of the support of tasks, the comparative advantage assumptions implies $\frac{\partial p_1(t_1,v)}{\partial v} < 0$ and $\frac{\partial p_2(t_2,v)}{\partial v} > 0$ $\forall v, t_k$ and the complementarity assumption implies $\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v} \leq 0$ and $\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v} \geq 0$ $\forall v, t_k$. To these two assumptions, it seems reasonable to add an absolute advantage assumption indicating that more skilled workers are more productive at all tasks. As it will be shown below, this assumption guarantees that equilibrium wages increase with the level of skills. The absolute advantage assumption implies $\frac{\partial p_k(t_k,v)}{\partial t_k} > 0$ $\forall v, t_k$. These assumptions are summarized in Assumption A.

Assumption A: i) Comparative advantage of skills types, i.e. $\frac{\partial p_1(t_1,v)}{\partial v} < 0$ and $\frac{\partial p_2(t_2,v)}{\partial v} > 0$ $\forall v, t_k$, *ii)* complementarity of skills and machines $\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v} \leq 0$ and $\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v} \geq 0$ $\forall v, t_k$, *iii)* absolute advantage of skilled workers, $\frac{\partial p_k(t_k,v)}{\partial t_k} > 0$ $\forall v, t_k$,

The distribution of tasks is characterized by the probability density function $f(v)$ and cumulative density function $F(v)$ and assumed exogenous. For the sake of simplicity, it is further assumed that the mass of tasks equals the mass of workers. Finally, this economy is perfectly competitive so that no worker and no owner of capital can affect the wage and rental rates.

2.2 Tasks assignment and equilibrium

Following Sattinger (1979 and 1993), the general equilibrium of this model is derived in three steps once we assume that the distribution of tasks does not depend on wages. In the first step, we make a tentative assumption about the assignment of workers to tasks in equilibrium. The second step consists to derive the associated equilibrium wages for this assignment. Finally, in the third step, we check whether the second order conditions for equilibrium are satisfied by the equilibrium wages derived in step 2.

Step 1: *Tentative tasks assignment*

Given assumption *A i)*, an efficient assignment of workers to tasks will maximize output by assigning workers with skills of type 1 to tasks close to

0 and workers with skills of type 2 to tasks close to 1. Given assumption *A ii)* and *iii)*, task 0 will be assigned to those with the highest level of type 1 skills, from the remaining task, the nearest to zero will be assigned to those workers with the second highest level of type 1 skills and so on until either all workers with type 1 skills have been assigned to a task or no more type 1 workers are willing to supply their skills at the outgoing wage. Let the level of skills of the last type 1 worker willing to supply her skills be $t_{1,\varepsilon_1} \geq \underline{t}_1$ and the task to which this worker is assigned be $\varepsilon_1 = F^{-1}(S_1 - S_1(t_{1,\varepsilon_1}))$,[6] so that $t_{1,\varepsilon_1} = \underline{t}_1$ and $\varepsilon_1 = F^{-1}(S_1)$ when all type 1 workers find it profitable to supply their skills. By symmetry, among workers with skills of type 2, those with the highest level of type 2 skills will be assigned to task 1 and so on until either all type 2 workers have been assigned to a task or no more type 2 workers are willing to supply their skills at the outgoing wage. Let this worker have skills $t_{2,\varepsilon_2} \geq \underline{t}_2$ and the task to which this worker is assigned be given by $\varepsilon_2 = 1 - F^{-1}(S_2 - S_2(t_{2,\varepsilon_2}))$, so that $t_{2,\varepsilon_2} = \underline{t}_2$ and $\varepsilon_2 = 1 - F^{-1}(S_2)$ when all type 2 workers find it profitable to supply their skills. Hence, when equilibrium wages are so that all workers are willing to supply their skills, we have $\varepsilon_1 = \varepsilon_2 = \varepsilon$.[7] The marginal task ε is simply derived by taking the inverse of the cumulative density function of tasks evaluated at S_1, that is $\varepsilon = F^{-1}(S_1)$.

This efficient tasks assignment results in a mapping function v_1 that associates to each value of skills t_1 a single value of task $v \in (0, \varepsilon_1)$, i.e. $v_1 = v_1(t_1)$ with $v_1' < 0$, and a mapping function v_2 that associates to each value of skills t_2 a single value of task $v \in (\varepsilon_2, 1)$, i.e. $v_2 = v_2(t_2)$ and $v_2' > 0$.[8] The

[6] Assuming that all workers with skills higher than t_{1,ε_1} are willing to work given that worker with skills t_{1,ε_1} is willing to work, requires two assumptions. First, equilibrium wages increase with skills. I will show below that equilibrium wages do increase with skills as long as the absolute advantage assumption *A iii)* holds, that is, as long as productivity increases with skills. Second, either all workers have the same reservation wage or reservation wages increase with skills but at a lower rate than equilibrium wages.

[7] Note two things. First, firms are indifferent between assigning a worker with type 1 or type 2 at task ε. Second, I herewith implicitly assume that firms owning the marginal machines are at least indifferent between supplying the machine to the market or withholding the machine from the market and, workers with the lowest skills of each type are at least indifferent between being assigned to machine ε or remaining unemployed. Hence, I assume that the equlibrium pricing functions, wages and rents, are so that both workers and machines are rewarded at least their reservation prices. This assumption is discussed below.

[8] See Sattinger (1975). Functions v_i, $i = 1, 2$ play the same role as the function $h(g)$ in Sattinger (1975) p. 356, where g is workers' ability (single scale) and $h(g)$ the difficulty (single scale) of the task performed by workers with ability g in equilibrium and, $c(u)$ in Teulings (1995a), (1995b) and (2005) where u is the normalized level of skills and $c(u)$ the associated job complexity in equilibrium.

functions v_1 and v_2 are monotonic, decreasing and increasing respectively on $t_1 \in [t_{1,\varepsilon_1}, \infty)$ and $t_2 \in [t_{2,\varepsilon_2}, \infty)$ since within types of skills more skilled workers are assigned to more productive machines in equilibrium.

To show how the assignment equilibrates the supply of and demand for skills, I split the interval of tasks assigned to type 1 workers, $v \in (0, \varepsilon_1)$ into N intervals of equal length $\Delta_i = \Delta = \frac{\varepsilon_1}{N}$ for $i = 1, ..., N$. Labor demand in interval i, that is the number of tasks to be filled in interval i, is $\int_{(i-1)\Delta}^{i\Delta} f(v)dv$. To equilibrate each tasks interval firms will assign the $\int_0^{\Delta} f(v)dv$ most skilled workers to the first interval, the following $\int_{\Delta}^{2\Delta} f(v)dv$ most skilled workers to the second interval and so on and so forth until the last interval is filled with the $\int_{(N-1)\Delta}^{N\Delta} f(v)dv$ least skilled workers. Note however that, since the density distribution of workers by level of skills needs not correspond to the density distribution of tasks, the skill differences between the most and least skilled workers in each interval needs not be the same. For instance, suppose that skills are normally distributed among workers of type 1 and tasks are uniformly distributed on $(0, 1)$. The skill differential will first decrease (upper tail of a normal distribution is thinner than the upper tail of a uniform distribution) and increase once the median skilled worker has been assigned. The assignment of tasks to workers deforms (stretches) the density distribution of skills of each type such as to make it fit the distribution of tasks in equilibrium.

The assignment of workers can therefore directly be derived from the density of tasks by performing the transformation of variables $v = v_k(t_k)$ and noting that $dv = v_k'(t_k)dt_k$. This yields:

$$\int_{v_1(t_{1,\varepsilon_1})}^{v_1(\infty)} f(v_1(t_1))v_1'(t_1)dt_1 = \int_{t_{1,\varepsilon_1}}^{\infty} s_1(t_1)dt_1 \equiv S_1(t_{1,\varepsilon_1}) \text{ for } v_1(t_{1,\varepsilon_1}) \leq \varepsilon_1 \quad (1)$$

$$\int_{v_2(t_{2,\varepsilon_2})}^{v_2(\infty)} f(v_2(t_2))v_2'(t_2)dt_2 = \int_{t_{2,\varepsilon_2}}^{\infty} s_2(t_2)dt_2 \equiv S_2(t_{2,\varepsilon_2}) \text{ for } v_2(t_{2,\varepsilon_2}) \geq \varepsilon_2 \quad (2)$$

The mapping functions therefore equilibrate supply and demand everywhere on t_k, $k = 1, 2$, so that we have $s_k(t_k) = f(v_k(t_k))v_k'(t_k)$ for all $t_k \geq t_{k,\varepsilon_k}$, $k = 1, 2$. These are first order nonlinear nonautonomous differential equations.[9]

Step 2: *Equilibrium wages*

[9]Examples of closed form solutions are available in the one skill dimension case in Sattinger (1975), (1979) and (1993). Teulings (1995a), (1995b) and (2005) solves a second order differential equation in the one skill scale case.

The owner of machine v seeks to maximize the profits derived from its machine. The profits from assigning a worker with skills t_k are $r(v) = p_k(t_k, v) - w_k(t_k)$. The owner will therefore compare the productivity increase to the wage increase associated to a worker with higher skills t_k for all k. This yields the following first order condition:

$$\frac{\partial p_k(t_k, v)}{\partial t_k} = w_k'(t_k) \ \forall k = 1, 2 \tag{3}$$

Note that from assumptions A iii), we therefore have that equilibrium wages are increasing with the level of skills, $w_k'(t_k) > 0 \ \forall k = 1, 2$.

The equilibrium rents are obtained in a similar fashion by noting that earnings are given by $w_k(t_k) = p_k(t_k, v) - r(v)$. Earnings maximization leads workers with skills t_k to compare the productivity increase to the rent increase associated to a machine ranked to the left or the right of v. Hence, the first order conditions to earnings maximization are given by:

$$\frac{\partial p_k(t_k, v)}{\partial v} = r'(v) \ \forall k = 1, 2 \tag{4}$$

Moreover, in the case where all workers supply their skills, the owners of machine ε are indifferent between employing the worker supplying the lowest level of skills of type 1, $\underline{t}_1$, or the worker supplying the lowest level of skills of type 2, $\underline{t}_2$. Stated otherwise, the rents of the owners of machines ε are equal whether worker $\underline{t}_1$ or $\underline{t}_2$ are assigned to machine ε: $p_1(\underline{t}_1, \varepsilon) - w_1(\underline{t}_1) = p_2(\underline{t}_2, \varepsilon) - w_2(\underline{t}_2)$.

Step 3: *Second order conditions*

The equilibrium assignment defined by the mapping functions v_k, is a valid one only when the firm's second order condition to profits maximization, that is profits are concave in t_k, is satisfied. Put in equation:

$$\begin{aligned} \left[\frac{\partial^2 p_k(t_k, v)}{\partial t_k^2}\right]_{v=v_k(t_k)} - w_k^{"}(t_k) \quad &< \quad 0 \ \forall k = 1, 2 \\ &\Leftrightarrow \\ -\left[\frac{\partial^2 p_k(t_k, v)}{\partial t_k \partial v} v_k'\right]_{v=v_k(t_k)} \quad &< \quad 0 \end{aligned}$$

since $w_k^{''}(t_k) = \left[\frac{\partial^2 p_k(t_k,v)}{\partial t_k^2}\right]_{v=v_k(t_k)} + \left[\frac{\partial^2 p_k(t_k,v)}{\partial t_k \partial v} v_k'(t_k)\right]_{v=v_k(t_k)}$.

The second order conditions for earnings maximization read as:

$$
\begin{aligned}
\left[\frac{\partial^2 p_k(t_k,v)}{\partial v^2}\right]_{t_k=v_k^{-1}(v)} - r''(v) &< 0 \; \forall k=1,2 \\
&\Leftrightarrow \\
-\left[\frac{\partial^2 p_k(t_k,v)}{\partial t_k \partial v}\frac{1}{v_k'}\right]_{t_k=v_k^{-1}(v)} &< 0
\end{aligned}
$$

since $r''(v) = \left[\frac{\partial^2 p_k(t_k,v)}{\partial v_k^2}\right]_{t_k=v_k^{-1}(v)} + \left[\frac{\partial^2 p_k(t_k,v)}{\partial t_k \partial v}\frac{1}{v_k'}\right]_{t_k=v_k^{-1}(v)}$.

Since $v_1' < 0$ and $v_2' > 0$ these second order conditions therefore imply that $\left[\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v}\right]_{v=v_1(t_1)} < 0$ and $\left[\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v}\right]_{v=v_2(t_2)} > 0$. Hence, as long as the cross derivative $\frac{\partial^2 p_1(t_1,v)}{\partial t_1 \partial v}$ is negative and the cross derivative $\frac{\partial^2 p_2(t_2,v)}{\partial t_2 \partial v}$ is positive, that is as long as assumption A *ii)* holds, an assignment where within types of skills more skilled workers get more productive machines, i.e. $v_1' < 0$ and $v_2' > 0$, is valid.

Equilibrium pricing functions

Evaluating the differential equation 3 at $v = v_k(t_k)$ and integrating over t_k yields the wage function for workers with skills of type k.

$$
w_k(t_k^*) = w_{k0} + \int_{\underline{t}_k}^{t_k^*} \left[\frac{\partial p_k(t_k,v)}{\partial t_k}\right]_{v=v_k(t_k)} dt_k \tag{5}
$$

where w_{k0} is a constant of integration.

Similarly, evaluating the differential equation 4 at $t_k = v_k^{-1}(v)$ and integrating over v yields the rent function as follows:

$$
r(v^*) = r_0 + \int_{\varepsilon_k}^{v^*} \left[\frac{\partial p_k(t_k,v)}{\partial v}\right]_{t_k=v_k^{-1}(v)} dv \tag{6}
$$

where r_0 is a constant of integration.

The wage and rent functions are identified up to constants of integration. Following Sattinger (1979),[10] the model is closed by specifying exogenous reserve prices for the marginal workers and machine. For the least skilled workers in both groups to be indifferent between being assigned to machine ε or remaining unemployed we need $w_k(\underline{t}_k) = w_{k0} = \widetilde{w} > 0$ where $\widetilde{w}$ is the reservation wage. Since firms owing machines ε are indifferent between employing the least skilled worker of each type, we have $r(\varepsilon) = r_0 = p_k(v_k^{-1}(\varepsilon), \varepsilon) - \widetilde{w} \geq 0$ for $k = 1, 2$ and $\forall \varepsilon$. Hence, for the firms owing machines ε to be indifferent between supplying the machine to the market or withholding the machine from the market we need $r_0 = p_k(v_k^{-1}(\varepsilon), \varepsilon) - \widetilde{w} = \widetilde{r}$ for all k where $\widetilde{r}$ is the reserve price for the owner of capital. So, as long as $r_0 = \widetilde{r}$ and $w_{k0} = \widetilde{w}$, there will be full employment in the economy.

2.3 The shape of the aggregate production function

Using the mapping functions v_k, the aggregate output level is given by:[11]

$$Y(\varepsilon) = \int_0^{\varepsilon} p_1(v_1^{-1}(v), v) f(v) dv + \int_{\varepsilon}^{1} p_2(v_2^{-1}(v), v) f(v) dv \tag{7}$$

To retrieve the shape of the aggregate production function associated to this economy, we can derive the demand for type k workers by integrating the number of workers per unit of output for each skills type (the inverse of the worker's productivity) over the spectrum of tasks. This is given by:

$$\frac{D_1(\varepsilon)}{Y(\varepsilon)} = \int_0^{\varepsilon} \frac{f(v)}{p_1(v_1^{-1}(v), v)} dv \tag{8}$$

$$\frac{D_2(\varepsilon)}{Y(\varepsilon)} = \int_{\varepsilon}^{1} \frac{f(v)}{p_2(v_2^{-1}(v), v)} dv \tag{9}$$

Solving equation 8 for the marginal task ε and plugging this result into equation 9, one can derive, under invertible conditions, the shape of the production function as $Y(\varepsilon) = h(D_1(\varepsilon), D_2(\varepsilon))$. In the general case, analytical solutions for $h(.,.)$ will not be possible. However, as I will show in the next

[10] Costrell and Loury (2004) consider a continuum of tasks in a single enterprise, not in the whole economy, and therefore close the model by a free entry condition that drives profits of each enterprise down to 0.

[11] Note that using the mapping functions to perform the change of variables $v_k = v_k(t_k)$, one can also express total output as a function of the distribution of skills replacing v by $v_k(t_k)$ and $f(v)dv$ by $s(t_k)dt_k$.

section, under certain restrictions, h could take the general form of a CRES production function which admits the CES production function as a special case.

3 Parametric specification

3.1 Tasks distribution

Suppose tasks follow a Beta distribution. The probability density function of tasks is then given by:

$$f(v|d_1, d_2) = \frac{1}{B(d_1+1, d_2+1)} v^{d_1}(1-v)^{d_2}$$

with $d_j > -1$ and $B(.)$ is the Beta function and cumulative distribution $F(v^*|d_1, d_2) = \int_0^{v^*} f(v|d_1, d_2)dv$.

The mean task is given by $E[v] = \frac{d_1+1}{2+d_1+d_2}$ ($E[v] = \frac{1}{2}$ when $d_1 = d_2$) and the variance by $Var[v] = \frac{(d_1+1)(d_2+1)}{(2+d_1+d_2)(3+d_1+d_2)}$, with $\frac{\partial Var[v]}{\partial d_k} < 0$. Moreover, the distribution is skewed toward 0 when $d_1 > d_2$ and vice versa.

The Beta distribution is appealing because its support ranges from 0 to 1, it has only two parameters, and its shape is extremely flexible. If $d_j < 0$ for all j, the distribution is U-shaped. If $d_1 < 0$ and $d_2 > 0$, the distribution has a inverted J-shape and if $d_1 > 0$ and $d_2 < 0$, the distribution has a J-shape. If $d_1 > 0$ and $d_2 > 0$ the distribution is unimodal. If $d_1 = d_2 = d$ and $d = 0$ tasks are uniformly distributed. Moreover, for $d > 1$ the Beta distribution and the normal distribution with average $\frac{1}{2}$ and variance equal to $\frac{(d_1+1)(d_2+1)}{(2+d_1+d_2)(3+d_1+d_2)}$ look alike.

3.2 Productivity of a match worker-machine

Suppose further that the productivity of workers with skills k assigned to machine v is Cobb-Douglas as follows:

$$p_1(t_1, v) = b_1 t_1^{m_1}(1-v)^{n_1} \quad (10)$$
$$p_2(t_2, v) = b_2 t_2^{m_2} v^{n_2} \quad (11)$$

The parameters b_k are strictly positive and indicate the efficiency units of workers with skills k. The parameters m_k indicates the elasticity of output with respect to skills, with $m_k > 0$ to satisfy assumption A $iii)$, i.e. $\partial p_k/\partial t_k > 0$. n_k indicates the elasticity of output with respect to tasks, with $n_k > 0$ to satisfy assumptions A $i)$ and $ii)$.

3.3 Mapping functions

As shown above, the resolution of the assignment problem requires to solve first order nonlinear nonautonomous differential equations. Closed form solutions are unlikely to exist. One way to circumvent this problem is to impose the shape of the mapping functions $v_k(t_k)$ and solve the equations for the density functions $s_k(t_k)$. Then, using individual data containing information on skills, one could calibrate the parameters of the mapping functions so as to fit as close as possible (non)parametric estimations of the density functions.

This solution is particularly adequate in our context since the aim of this paper is to infer on the shape of the mapping function that combined with the above assumptions on the tasks density function and on the productivity of worker-machine pairs yields a CES production function at the aggregate output level.

Suppose that the function assigning skills of type 1 to tasks in equilibrium is such that $v = v_1(t_1) = \varepsilon \left(\frac{\underline{t}_1}{t_1}\right)^{1/a_1}$ with $a_1 \geq 0$ to satisfy $v_1' < 0$ and the function assigning skills of type 2 to tasks is such that $v = v_2(t_2) = 1 - (1-\varepsilon)\left(\frac{\underline{t}_2}{t_2}\right)^{1/a_2}$ with $a_2 \geq 0$ to satisfy $v_2' > 0$. These specifications for the mapping functions imply full employment. Workers with the highest level of type 1 and type 2 skills, $t_j \to \infty$, are assigned to task 0 and 1 respectively and workers with the lowest levels of both types of skills, $t_j = \underline{t}_j$, are assigned to the marginal task ε.[12]

Using the expression of the mapping functions and the density of tasks, the density of workers with t_1 skills of type 1 is given by:

$$s_1(t_1) = \frac{A\underline{t}_1}{a_1}\varepsilon^{d_1+1}\left(1-\varepsilon\left(\frac{\underline{t}_1}{t_1}\right)^{1/a_1}\right)^{d_2}\left(\frac{\underline{t}_1}{t_1}\right)^{d_1/a_1+1/a_1-1} \tag{12}$$

for $0 < \underline{t}_1 < t_1$.

The density of workers with t_2 skills of type 2 is given by:

[12]Unemployment of type j workers could be derived by replacing $\underline{t}_j$ in $v_j(.)$ by t_j^* with $t_{j,\varepsilon_j} > \underline{t}_j$.

$$s_2(t_2) = \frac{A\underline{t}_2}{a_2}(1-\varepsilon)^{d_2+1}\left(1-(1-\varepsilon)\left(\frac{\underline{t}_2}{t_2}\right)^{1/a_2}\right)^{d_1}\left(\frac{\underline{t}_2}{t_2}\right)^{d_2/a_2+1/a_2-1} \quad (13)$$

for $0 < \underline{t}_2 < t_2$.

The aggregate output level, obtained by summing up the product of each worker or equivalently the product at each task in equilibrium reads as:

$$\begin{aligned} Y(\varepsilon) &= \int_0^\varepsilon p_1(v_1^{-1}(v), v) f(v) dv + \int_\varepsilon^1 p_2(v_2^{-1}(v), v) f(v) dv \\ &= b_1 \underline{t}_1^{m_1} \varepsilon^{a_1 m_1} A \int_0^\varepsilon v^{d_1 - a_1 m_1}(1-v)^{d_2+n_1} dv + \\ &\quad b_2 \underline{t}_2^{m_2}(1-\varepsilon)^{a_2 m_2} A \int_\varepsilon^1 v^{d_1+n_2}(1-v)^{d_2 - a_2 m_2} dv \end{aligned}$$

After recognizing that $A_1 v^{d_1 - a_1 m_1}(1-v)^{d_2+n_1}$ is the density of the Beta distribution with parameter $d_1 - a_1 m_1$ and $d_2 + n_1$ with $A_1 = \frac{1}{B(d_1 - a_1 m_1, d_2 + n_1)}$ and $A_2 v^{d_1+n_2}(1-v)^{d_2 - a_2 m_2}$ is the density of the Beta distribution with parameters $d_1 + n_2$ and $d_2 - a_2 m_2$ with $A_2 = \frac{1}{B(d_1+n_2, d_2 - a_2 m_2)}$, the aggregate output level reads as:

$$\begin{aligned} Y(\varepsilon) &= b_1 \underline{t}_1^{m_1} \varepsilon^{a_1 m_1} \frac{A}{A_1} F(\varepsilon | d_1 - a_1 m_1, d_2 + n_1) + \\ &\quad b_2 \underline{t}_2^{m_2}(1-\varepsilon)^{a_2 m_2} \frac{A}{A_2}(1 - F(\varepsilon | d_1 + n_2, d_2 - a_2 m_2)) \end{aligned}$$

3.4 The shape of the aggregate production function

Given the structural form in equation 10 and 11, the employment of type k workers per unit of output in equilibrium reads as:

$$
\begin{aligned}
\frac{D_1(\varepsilon)}{Y(\varepsilon)} &= \int_0^{\varepsilon} \frac{f(v|d_1,d_2)}{p_1(v_1^{-1}(v),v)} dv && (14)\\
&= \frac{A}{\varepsilon^{a_1 m_1}} \frac{1}{b_1 \underline{t}_1^{m_1}} \int_0^{\varepsilon} v^{d_1+a_1 m_1}(1-v)^{d_2-n_1} dv \\
\frac{D_2(\varepsilon)}{Y(\varepsilon)} &= \int_{\varepsilon}^{1} \frac{f(v|d_1,d_2)}{p_2(v_2^{-1}(v);v)} dv && (15)\\
&= \frac{A}{(1-\varepsilon)^{a_2 m_2}} \frac{1}{b_2 \underline{t}_2^{m_2}} \int_{\varepsilon}^{1} v^{d_1-n_2}(1-v)^{d_2+a_2 m_2} dv
\end{aligned}
$$

Analytical solutions for these integrals exist. An interesting special case of which is met when $d_j = n_k$, $j \neq k$.[13] The solutions of these integrals are then:

$$
\begin{aligned}
\frac{D_1(\varepsilon)}{Y(\varepsilon)} &= \frac{A}{b_1 \underline{t}_1^{m_1}} \frac{1}{d_1 + a_1 m_1 + 1} \varepsilon^{d_1+1} && (16)\\
\frac{D_2(\varepsilon)}{Y(\varepsilon)} &= \frac{A}{b_2 \underline{t}_2^{m_2}} \frac{1}{d_2 + a_2 m_2 + 1} (1-\varepsilon)^{d_2+1} && (17)
\end{aligned}
$$

Note that, for a symmetric distribution of tasks, i.e. $d_1 = d_2 = \frac{1}{\theta} - 1$, solving the system for the marginal task ε yields:

$$
\begin{aligned}
\varepsilon &= \left(\frac{D_1(\varepsilon)}{Y} \frac{b_1 \underline{t}_1^{m_1}(d_1 + a_1 m_1 + 1)}{A} \right)^{\theta} && (18)\\
&= 1 - \left(\frac{D_2(\varepsilon)}{Y} \frac{b_2 \underline{t}_2^{m_2}(d_2 + a_2 m_2 + 1)}{A} \right)^{\theta} && (19)
\end{aligned}
$$

Moreover, equating the left hand side of both equation 18 and equation 19 and solving for Y yields:

[13] The restriction $d_j = n_k > 0$ implies that the distribution of tasks must be unimodal. excluding $U-$*shaped* distributions met for $-1 < d_j < 0$ for all j, or $J-$*shaped* distributions met when $-1 < d_j < 0$ and $d_k > 0$. I will show in the next section that changing slightly the shape of the production function, within the Cobb-Douglas family, CES production functions can still be recorved for $U-$*shaped*, $J-$*shaped* or uniform distribution of tasks.

$$Y(\varepsilon) = \frac{1}{A}\left[\left(\chi_1 D_1(\varepsilon)\right)^{\theta} + \left(\chi_2 D_2(\varepsilon)\right)^{\theta}\right]^{\frac{1}{\theta}} \tag{20}$$

where $\chi_j = b_j \underline{t}_j^{m_j}\left(d_j + a_j m_j + 1\right)$

Equation 20 reads as a CES production function with elasticity of substitution parameter $\sigma = \frac{1}{1-\theta}$. Since the parameter θ is related to the distribution of tasks, the elasticity of substitution parameter σ is directly linked to the distribution of tasks.

Result R1: When i) the productivity of a match worker-task is Cobb-Douglas, i.e. as defined as in equations 10 and 11, ii) the mapping functions are given by $v = v_1(t_1) = \varepsilon\left(\frac{\underline{t}_1}{t_1}\right)^{1/a_1}$ and $v = v_2(t_2) = 1 - (1-\varepsilon)\left(\frac{\underline{t}_2}{t_2}\right)^{1/a_2}$, iii) $d_j = n_k$, $j \neq k$ and iv) the distribution of tasks follows a symmetric Beta distribution, then the aggregate production function resulting from the assignment of multi-skilled heterogenous workers to heterogenous tasks has the CES shape.

3.5 Extension

Result R1 is very convenient as it links the assignment literature to the empirical literature of wage inequality in which the CES production function has been extensively used. However, it is restrictive in the sense that only symmetric distributions of machines are covered, though these distributions can range from the uniform distribution when $\theta = 1$ ($\sigma \to \infty$ linear production function), inverted-U shape for $2 > \frac{1}{\theta} > 1$ ($\sigma \in (2, \infty)$) and normal look alike distributions for $\frac{1}{\theta} > 2$, ($\sigma \in (1, 2)$).

Nevertheless, one can generalize this finding and solve for more general shapes of the production function assuming that the productivity of workers in the various tasks depends on how many units of aggregate output are produced. Assume for instance that producing an extra unit of output affects the productivity of all workers in every tasks. Let $b_k \equiv b_k(Y) = r_k Y^{1-\frac{\theta}{\theta_k}}$ with $r_k > 0$, $\theta > 0$ and $\theta_k = \frac{1}{d_k+1}$, so that employing more workers in each task reduces (increases) the productivity per unit of output of each worker in every tasks, i.e. $\frac{\partial p_k}{\partial Y} < 0$ for $1 - \frac{\theta}{\theta_k} < 0$ (respectively $\frac{\partial p_k}{\partial Y} > 0$ for $1 - \frac{\theta}{\theta_k} > 0$).

Under this assumption, the demand for type k workers reads as:

$$\frac{D_1(\varepsilon)}{Y(\varepsilon)} = \frac{A}{r_1 Y(\varepsilon)^{1-\frac{\theta}{\theta_1}} \underline{t}_1^{m_1}} \frac{1}{d_1 + a_1 m_1 + 1} \varepsilon^{d_1+1} \tag{21}$$

$$\frac{D_2(\varepsilon)}{Y(\varepsilon)} = \frac{A}{r_2 Y(\varepsilon)^{1-\frac{\theta}{\theta_2}} \underline{t}_2^{m_2}} \frac{1}{d_2 + a_2 m_2 + 1} (1-\varepsilon)^{d_2+1} \tag{22}$$

Which solving for the marginal task yields:

$$\varepsilon = \frac{1}{Y(\varepsilon)^{\theta}} \left(\frac{r_1 \underline{t}_1^{m_1} (d_1 + a_1 m_1 + 1)}{A} D_1(\varepsilon) \right)^{\theta_1} \tag{23}$$

$$= 1 - \frac{1}{Y(\varepsilon)^{\theta}} \left(\frac{r_2 \underline{t}_2^{m_2} (d_2 + a_2 m_2 + 1)}{A} D_2(\varepsilon) \right)^{\theta_2} \tag{24}$$

$$\Leftrightarrow$$

$$Y(\varepsilon) = \left[(\psi_1 D_1(\varepsilon))^{\theta_1} + (\psi_2 D_2(\varepsilon))^{\theta_2} \right]^{\frac{1}{\theta}} \tag{25}$$

where $\psi_j = \frac{r_j \underline{t}_j^{m_j} (d_j + a_j m_j + 1)}{A}$.

Equation 25 reads as the CRES production function (see Houthakker (1960) and Dick and Medoff (1975)). This function degenerates to a CES production function with returns to scale $\frac{\theta_1}{\theta}$ when $\theta_1 = \theta_2$ that is for symmetric tasks distributions. When $\theta_1 = \theta_2 = \theta$, the function degenerates to the constant returns to scale CES production function discussed above.

Result R2: When i) the productivity of a match worker-task is Cobb-Douglas and is function of the output level, i.e. as defined as in equations 10 and 11, ii) the mapping functions are given by $v = v_1(t_1) = \varepsilon \left(\frac{t_1}{\underline{t}_1}\right)^{1/a_1}$ and $v = v_2(t_2) = 1 - (1-\varepsilon)\left(\frac{t_2}{\underline{t}_2}\right)^{1/a_2}$, iii) $d_j = n_k$, $j \neq k$, and iv) the distribution of tasks follows a Beta distribution, then the aggregate production function resulting from the assignment of multi-skilled heterogenous workers to heterogenous tasks is of the CRES shape.

3.6 Necessary conditions

The previous section has shown sufficient conditions for the distribution of tasks, the shape of the productivity of worker-task pairs and the mapping functions to derive a CRES production function. In this section, I derive the necessary conditions to yield a CRES production function. From equations 23 and 24 it is easy to see that in order to derive a CRES production function, we need the following relationship between the marginal task and the demand for workers of each type:

$$\begin{aligned}\varepsilon &= \left(\frac{D_1}{Y}\Lambda_1^{-1}\right)^{\theta_1}\\ 1-\varepsilon &= \left(\frac{D_2}{Y}\Lambda_2^{-1}\right)^{\theta_2}\end{aligned}$$

where Λ_j are parameters.

Rearranging and using the definition of the demand for workers per unit of outputs, we have the following expression for the demand for workers of each type per unit of output:

$$\frac{D_1}{Y} \equiv \int_0^{\varepsilon} \frac{f(v)}{p_1(t_1(v),v)}dv = \Lambda_1\left[v^{\beta_1}\varepsilon^{\gamma_1}\right]_0^{\varepsilon} = \Lambda_1\varepsilon^{1/\theta_1} \tag{26}$$

$$\frac{D_2}{Y} \equiv \int_{\varepsilon}^{1} \frac{f(v)}{p_2(t_2(v),v)}dv = \Lambda_2\left[(1-v)^{\beta_2}(1-\varepsilon)^{\gamma_2}\right]_{\varepsilon}^{1} = \Lambda_2(1-\varepsilon)^{1/\theta_2} \tag{27}$$

where $\beta_j + \gamma_j = \frac{1}{\theta_j}$ and $t_j(v) \equiv v_j^{-1}(v)$.

Note that at this point, it is convenient to write the inverse mapping functions[14] $t_j(v) \equiv v_j^{-1}(v)$ as functions of two arguments, v and ε, i.e. $t_j(v,\varepsilon)$. Equations 26 and 27 imply that:

$$\frac{f(v)}{p_1(t_1(v,\varepsilon),v)} = \frac{\Lambda_1}{\beta_1}v^{\beta_1-1}\varepsilon^{\gamma_1} \tag{28}$$

$$\frac{f(v)}{p_2(t_2(v,\varepsilon),v)} = \frac{\Lambda_2}{\beta_2}(1-v)^{\beta_2-1}(1-\varepsilon)^{\gamma_2} \tag{29}$$

Since the right hand sides of equations 28 and 29 are multiplicatively separable in v and ε and $(1-v)$ and $(1-\varepsilon)$ respectively, the left hand sides of these equations must also be multiplicatively separable in v and ε and, $(1-v)$

[14] The mapping functions are strictly monotonic on v so that their inverse are well-defined and also strictly monotonic.

and $(1-\varepsilon)$ respectively. For this to be true, i) the functions $p_j(t_j(v,\varepsilon),v)$ must be multiplicatively separable in t_j and ε so that $p_j(t_j(v,\varepsilon),v) = p_j(t_j(v,\varepsilon)) \cdot g_j(v)$, ii) the function $t_j(v,\varepsilon)$ must be multiplicatively separable in v and ε so that $t_j(v,\varepsilon) = t_j(v) \cdot h_j(\varepsilon)$ and iii) the functions $p_j(.)$ must be multiplicatively separable so that $p_j(t_j(v) \cdot h_j(\varepsilon)) = p_j(t_j(v)) \cdot p_j(h_j(\varepsilon))$. When *i)*, *ii)* and *iii)* are satisfied, we have:

$$\frac{f(v)}{p_1(t_1(v)) \cdot p_1(h_1(\varepsilon)) \cdot g_1(v)} = \frac{\Lambda_1}{\beta_1} v^{\beta_1 - 1} \varepsilon^{\gamma_1} \tag{30}$$

$$\frac{f(v)}{p_2(t_2(v)) \cdot p_2(h_2(\varepsilon)) \cdot g_2(v)} = \frac{\Lambda_2}{\beta_2} (1-v)^{\beta_2 - 1} (1-\varepsilon)^{\gamma_2} \tag{31}$$

It is easy to see that $p_1(h_1(\varepsilon))$ must be of the form $c_1 \varepsilon^{-\gamma_1}$ and $p_1(h_1(\varepsilon))$ of the form $c_2(1-\varepsilon)^{-\gamma_2}$.

Hence, necessary conditions to obtain a CRES aggregate production are:

1. The productivity of worker-task is multiplicatively separable in worker and task characteristics, i.e. $p_j(t_j, v) = p_j(t_j) \cdot g_j(v)$,

2. the mapping functions are multiplicatively separable in task characteristics and the marginal task, $t_j(v,\varepsilon) = t_j(v) \cdot h_j(\varepsilon)$ and,

3. the contribution of worker's characteristics to the output of worker-task pairs is itself multiplicatively separable, i.e. $p_j(t_j \cdot h_j) = p_j(t_j) \cdot p_j(h_j)$

Necessary conditions 1 and 3 are about the production function while necessary condition 2 is about the mapping functions.

We can even go further and show that if the p_j is Cobb-Douglas, the most popular multiplicative separable production function in the literature, then the mapping functions must be power functions and the distribution of tasks must follow a beta distribution, which is the example given above.

In what follows I assume that the product of a worker-task pair is Cobb-Douglas. In fact I distinguish between the following four specifications that satisfy assumption A.

$$A \ : \ \begin{cases} p_1(t_1, v) = b_1 t_1^{m_1} v^{n_1} \text{ and } b_1, m_1 > 0 \text{ and } n_1 < 0 \\ p_2(t_2, v) = b_2 t_2^{m_2} v^{n_2} \text{ and } b_2, m_2, n_2 > 0 \end{cases}$$

$$B \ : \ \begin{cases} p_1(t_1, v) = b_1 t_1^{m_1} (1-v)^{n_1} \text{ and } b_1, m_1, n_1 > 0 \\ p_2(t_2, v) = b_2 t_2^{m_2} v^{n_2} \text{ and } b_2, m_2, n_2 > 0 \end{cases}$$

$$C \ : \ \begin{cases} p_1(t_1, v) = b_1 t_1^{m_1} v^{n_1} \text{ and } b_1, m_1 > 0 \text{ and } n_1 < 0 \\ p_2(t_2, v) = b_2 t_2^{m_2} (1-v)^{n_2} \text{ and } b_2, m_2 > 0 \text{ and } n_2 < 0 \end{cases}$$

$$D \ : \ \begin{cases} p_1(t_1, v) = b_1 t_1^{m_1} (1-v)^{n_1} \text{ and } b_1, m_1 > 0 \text{ and } n_1 < 0 \\ p_2(t_2, v) = b_2 t_2^{m_2} (1-v)^{n_2} \text{ and } b_2, m_2 > 0 \text{ and } n_2 < 0 \end{cases}$$

For all four specifications, the assumption on the production function implies that $p_j(t_j(v, \varepsilon)) = b_j \left(t_j(v, \varepsilon)\right)^{m_j}$ and hence, using necessary condition *ii*), that $p_j(h_j(\varepsilon)) = b_j h_j^{m_j}(\varepsilon)$ and $p_j(t_j(v)) = b_j t_j^{m_j}(v)$. From the necessary conditions we also know that $p_1(h_1(\varepsilon))$ is of the form $c_1 \varepsilon^{-\gamma_1}$ and $p_2(h_2(\varepsilon))$ is of the form $c_2(1-\varepsilon)^{-\gamma_2}$. Hence, the shape of the functions $h_j(\varepsilon)$ must be:

$$h_1(\varepsilon) = \left(\frac{c_1}{b_1}\right)^{1/m_1} \varepsilon^{-\gamma_1/m_1} \tag{32}$$

$$h_2(\varepsilon) = \left(\frac{c_2}{b_2}\right)^{1/m_2} (1-\varepsilon)^{-\gamma_2/m_2} \tag{33}$$

Moreover, by definition of the mapping functions and using necessary condition *ii*), we have $t_j(\varepsilon, \varepsilon) = t_j(\varepsilon) h_j(\varepsilon) = \underline{t}_j$. Replacing h_j by its expression from equations 32 and 33, this yields the shape of the $t_j(v)$ functions as:

$$t_1(v) = \underline{t}_1 \left(\frac{b_1}{c_1}\right)^{1/m_1} v^{\gamma_1/m_1} \tag{34}$$

$$t_2(v) = \underline{t}_2 \left(\frac{b_2}{c_2}\right)^{1/m_2} (1-v)^{\gamma_2/m_2} \tag{35}$$

Note that for the mapping function for type 1 skills to be decreasing and the mapping function for type 2 skills to be increasing we need $\gamma_j < 0$ for $j = 1, 2$. Plugging equations 34 and 35 in $p_j(t_j(v)) = b_j t_j^{m_j}(v)$ we obtain the following expression for $p_j(t_j(v))$:

$$p_1(t_1(v)) = b_1\underline{t}_1^{m_1}\frac{b_1}{c_1}v^{\gamma_1} \quad (36)$$

$$p_2(t_2(v)) = b_2\underline{t}_2^{m_2}\frac{b_2}{c_2}(1-v)^{\gamma_2} \quad (37)$$

Plugging the expression of $p_j(t_j(v))$ and $p_j(h_j(\varepsilon))$ into equations 30 and 31 and rearranging yields:

$$\frac{f(v)}{g_1(v)} = b_1^2\underline{t}_1^{m_1}\frac{\Lambda_1}{\beta_1}v^{\beta_1-1+\gamma_1} \quad (38)$$

$$\frac{f(v)}{g_2(v)} = b_2^2\underline{t}_2^{m_2}\frac{\Lambda_2}{\beta_2}(1-v)^{\beta_2-1+\gamma_2} \quad (39)$$

Although the results so far are generic to all four specifications of the production function, the shapes of the functions $g_j(v)$ differ for each of the four specifications of the production functions presented above. Hence, the density function $f(v)$ depends on the choice of the specification of the production function.

Case A :

The function $g_j(v) = v^{n_j}$ which yields:

$$f(v) = b_1^2\underline{t}_1^{m_1}\frac{\Lambda_1}{\beta_1}v^{\beta_1-1+\gamma_1-n_1} \quad (40)$$

$$f(v) = b_2^2\underline{t}_2^{m_2}\frac{\Lambda_2}{\beta_2}(1-v)^{\beta_2-1+\gamma_2}v^{n_2} \quad (41)$$

Noting that $\beta_j + \gamma_j = \frac{1}{\theta_j}$, for both equations to be true for all v, we need $n_2 = \frac{1}{\theta_1} - 1 - n_1$, $\theta_2 = 1$ and $b_1^2\underline{t}_1^{m_1}\frac{\Lambda_1}{\beta_1} = b_2^2\underline{t}_2^{m_2}\frac{\Lambda_2}{\beta_2} = \frac{1}{B(n_2+1,1)}$. The density function is then $f(v) = \frac{1}{B(n_2+1,1)}v^{n_2}$ which is a special case of the Beta distribution, namely the strictly increasing density family.

Case B :(example of the sufficient conditions derive in the previous section)

The functions $g_1(v) = (1-v)^{n_1}$ and $g_2(v) = v^{n_2}$ which yields:

$$f(v) = b_1^2 \underline{t}_1^{m_1} \frac{\Lambda_1}{\beta_1} v^{\beta_1 - 1 + \gamma_1} (1-v)^{n_1} \quad (42)$$

$$f(v) = b_2^2 \underline{t}_2^{m_2} \frac{\Lambda_2}{\beta_2} (1-v)^{\beta_2 - 1 + \gamma_2} v^{n_2} \quad (43)$$

For both equations to be true for all v, we need $\theta_j = \frac{1}{1+n_k}$ $j \neq k$ and $b_1^2 \underline{t}_1^{m_1} \frac{\Lambda_1}{\beta_1} = b_2^2 \underline{t}_2^{m_2} \frac{\Lambda_2}{\beta_2} = \frac{1}{B(n_1+1, n_1+1)}$. The density function is then $f(v) = \frac{1}{B(n_1+1, n_1+1)} v^{n_1}(1-v)^{n_1}$ with $n_j > 0$ which is a special case of the Beta distribution, namely the unimodal family.

Case C :

The functions $g_1(v) = v^{n_1}$ and $g_2(v) = (1-v)^{n_2}$ which yields:

$$f(v) = b_1^2 \underline{t}_1^{m_1} \frac{\Lambda_1}{\beta_1} v^{\beta_1 - 1 + \gamma_1 + n_1} \quad (44)$$

$$f(v) = b_2^2 \underline{t}_2^{m_2} \frac{\Lambda_2}{\beta_2} (1-v)^{\beta_2 - 1 + \gamma_2 + n_2} \quad (45)$$

For both equations to be true for all v, we need $\theta_j = \frac{1}{1-n_j}$, $0 < n_j < 1$ and $b_1^2 \underline{t}_1^{m_1} \frac{\Lambda_1}{\beta_1} = b_2^2 \underline{t}_2^{m_2} \frac{\Lambda_2}{\beta_2} = 1$. The density function is then $f(v) = 1$ which is a special case of the Beta distribution, namely the uniform distribution.

Case D :

The functions $g_j(v) = (1-v)^{n_j}$ which yields:

$$f(v) = b_1^2 \underline{t}_1^{m_1} \frac{\Lambda_1}{\beta_1} v^{\beta_1 - 1 + \gamma_1} (1-v)^{n_1} \quad (46)$$

$$f(v) = b_2^2 \underline{t}_2^{m_2} \frac{\Lambda_2}{\beta_2} (1-v)^{\beta_2 - 1 + \gamma_2 + n_2} \quad (47)$$

For both equations to be true for all v, we need $n_1 = \frac{1}{\theta_2} - 1 + n_2$ and $\theta_1 = 1$ and $b_1^2 \underline{t}_1^{m_1} \frac{\Lambda_1}{\beta_1} = b_2^2 \underline{t}_2^{m_2} \frac{\Lambda_2}{\beta_2} = \frac{1}{B(1, n_1+1)}$. The density function is then $f(v) = \frac{1}{B(1, n_1+1)}(1-v)^{n_1}$ which is a special case of the Beta distribution, namely the strictly decreasing density family.

The results obtained for the four separate cases are summarized in Table 1.

4 Implications

The assignment model proposed in this paper provides microfoundations to production functions. By deriving the shape of the aggregate production function from the assignment of heterogenous workers to heterogenous tasks, the model provides a way to evaluate how stringent assumptions about the type of production functions or technological change are by comparing the implied distribution of jobs and its evolution over time to observations of the distribution of jobs and its evolution over time.

For instance, the workhorse model in the skill-biased technical change literature (see Katz and Murphy (1992) or Acemoglu (2002)) assumes that i) aggregate output is produced with a CES technology and ii) technical change can only alter the efficiencies of skilled and unskilled labor over time, i.e. σ, the elasticity of substitution parameter, is constant over time. The assignment model developed in this paper can be used to evaluate how realistic these assumptions are. Assumption i) implies that $d_1 = d_2 = \frac{1}{\theta} - 1$, and therefore that the distribution of jobs is symmetric around task 0.5, and $d_j = n_k$. Assumption ii) implies that only the efficiency parameters χ_j may change over time and hence that d_1, d_2, n_1, n_2 and θ are constant over time.

Assumption ii) implies that the distribution of tasks does not change over time. This assumption seems to be at odds with recent evidence provided by Autor et al. (2006) about the job polarization in the US economy. It also contradicts recent empirical evidence by Gabaix and Landier (2008) and Terviö (2008) that show that the rising wage inequality at the top of the wage distribution (CEO pay) is merely explained by the density increase in the upper tail of the distribution of firm size.[15]

In the light of the recent empirical evidence shown by Autor et al. (2006), Gabaix and Landier (2008) and Terviö (2008), the workhorse model used in the SBTC literature should be replaced by a model with i) a more general production function of for instance the CRES type, and ii) technical change that affect the distribution of capital (either firm size, tasks or machines) over time and therefore allow the ease to substitute between skill types of workers to vary over time.

[15] In Gabaix and Landier (2006) and Terviö (2007), manargers of different skills are assigned to firms of different size. In their models, firm size plays the same role as machines in the model developed in this paper.

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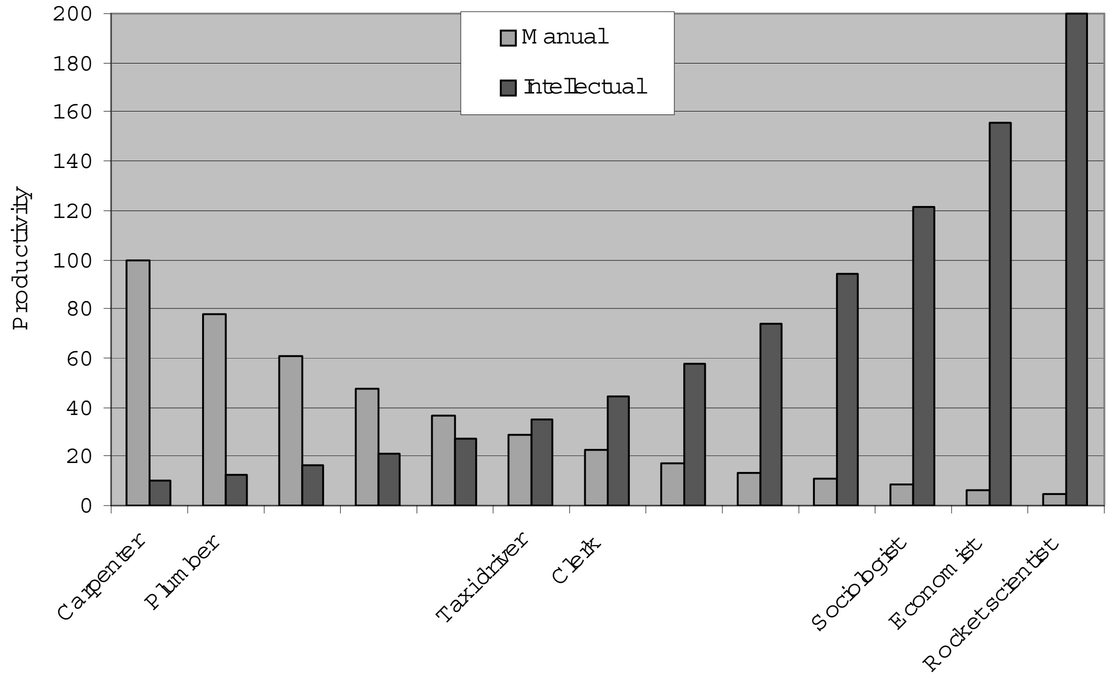

Figure 1: Job titles and the support of tasks. The support of tasks is derived from the assumption that i) ranking job titles by decreasing productivity when performed with manual skills is equivelent to ranking tasks by increasing productivity when performed with intellectual skills (comparative advantage of skills types) and ii) the ranking of job titles is indepedent of the level of skills (complementarity of skills and tasks).

Table 1: Cobb-Douglas productivity and the implied shape of the mapping and tasks density functions to obtain a CRES production function.

Case	Assumptions Production $p_1(t_1, v)$	$p_2(t_2, v)$	Implications Mapping functions $t_1(v)$	$t_2(v)$	Density function $f(v)$
A	$b_1 t_1^{m_1} v^{n_1}$	$b_2 t_2^{m_2} v^{n_2}$	$\underline{t}_1 \left(\frac{b_1}{c_1}\right)^{\frac{1}{m_1}\,1} v^{\frac{\gamma_1}{m_1}}$	$\underline{t}_2 \left(\frac{b_2}{c_2}\right)^{\frac{1}{m_2}} (1-v)^{\frac{\gamma_2}{m_2}}$	$\frac{1}{B(n_2+1,1)} v^{n_2}$
B	$b_1 t_1^{m_1} (1-v)^{n_1}$	$b_2 t_2^{m_2} v^{n_2}$	$\underline{t}_1 \left(\frac{b_1}{c_1}\right)^{\frac{1}{m_1}\,1} v^{\frac{\gamma_1}{m_1}}$	$\underline{t}_2 \left(\frac{b_2}{c_2}\right)^{\frac{1}{m_2}} (1-v)^{\frac{\gamma_2}{m_2}}$	$\frac{1}{B(n_1+1,n_1+1)} v^{n_1} (1-v)^{n_1}$
C	$b_1 t_1^{m_1} v^{n_1}$	$b_2 t_2^{m_2} (1-v)^{n_2}$	$\underline{t}_1 \left(\frac{b_1}{c_1}\right)^{\frac{1}{m_1}\,1} v^{\frac{\gamma_1}{m_1}}$	$\underline{t}_2 \left(\frac{b_2}{c_2}\right)^{\frac{1}{m_2}} (1-v)^{\frac{\gamma_2}{m_2}}$	1
D	$b_1 t_1^{m_1} (1-v)^{n_1}$	$b_2 t_2^{m_2} (1-v)^{n_2}$	$\underline{t}_1 \left(\frac{b_1}{c_1}\right)^{\frac{1}{m_1}\,1} v^{\frac{\gamma_1}{m_1}}$	$\underline{t}_2 \left(\frac{b_2}{c_2}\right)^{\frac{1}{m_2}} (1-v)^{\frac{\gamma_2}{m_2}}$	$\frac{1}{B(1,n_1+1)} (1-v)^{n_1}$