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A New Approach to Detect Spurious Regressions using Wavelets

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Abstract

In this paper, we propose the use of wavelet covariance and correlation to detect spurious regression. Based on Monte Carlo simulation results and experiments with real exchange rate data, it is shown that the wavelet approach is able to detect spurious relationship in a bivariate time series more directly. Using the wavelet approach, it is sufficient to detect a spurious regression between bivariate time series if the wavelet covariance and correlation for the two series are significantly equal to zero. The wavelet approach does not rely on restrictive assumptions which are critical to the Durbin Watson test. Another distinct advantage of the graphical wavelet analysis of wavelet covariance and correlation to detect spurious regression is the simplicity and efficiency of the decision rule compared to the complicated Durbin-Watson decision rules.

JEL Classifications: C19, C65

Keywords: Wavelet analysis, spurious regression

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1. Introduction

Critical to the properties of standard estimation and testing procedure in bivariate regression is the assumption that the y_t and x_t series are stationary. When the two series are nonstationary, the population covariance are ill-defined because the series are not fluctuating around a constant mean.

Following Granger and Newbold (1974), suppose two variables y_t and x_t are generated by two independent random walks,

$$y_t = y_{t-1} + \varepsilon_{1t}, \quad \varepsilon_{1t} \sim IID(0,\sigma^2) \tag{1}$$

$$x_t = x_{t-1} + \varepsilon_{2t}, \quad \varepsilon_{2t} \sim IID(0,\sigma^2) \tag{2}$$

where ε_{1t} and ε_{2t} are mutually independent. Nothing in this data generating process(DGP) suggests a relationship between y_t and x_t but regressing y_t on x_t using the standard linear regression model $y_t = \alpha + \beta x_t + \varepsilon_t$ produces a regression results characterized by fairly high R^2 statistics and highly autocorrelated residuals and a significant value for β .

This is the notorious phenomenon of spurious regressions where two independent nonstationary series are spuriously related due to the fact they are both trended. As Phillips (1986) demonstrated, the OLS estimate do not converge in probability as the sample size increases, the t- and F-statistics lack well-defined asymptotic distributions and the Durbin-Watson statistic converges to zero.

Typically, Granger and Newbold (1974) argued that spurious regression may be detected by checking for a high R^2 statistics and a low Durbin-Watson statistic. This is not without its problems. Logically, this is fallacious since it is affirmation of the consequent (Restall, 2006; Copi and Cohen, 2005). In other words, even if spurious regression is characterized by high R^2 statistics and a low Durbin-Watson statistic, the manifestation of a high R^2 statistics and a low Durbin-Watson statistic may not be conclusive proof of a spurious regression.

The Durbin-Watson d statistic (Durbin and Watson, 1951) can only detect first order autocorrelation since it requires that the disturbances be generated by a first order autoregressive scheme, $u_t = \rho u_{t-1} + \varepsilon_t$. Hence, it cannot be used to detect higher order schemes. Other assumptions which may not be fulfilled in practice are also required. Ideally, the regression model should include the intercept term but should not include the lagged values of the dependent variable as one of the explanatory variables, the explanatory variables should be non-stochastic or fixed in repeated sampling, the disturbances should be normally distributed and generated. Last but not least, there should be no missing observations. Strictly speaking, a low Durbin Watson d statistic does not necessarily imply no autocorrelation since the test requires an extensive and careful checking of a set of decision rules. In practice, how carefully researchers checked their Durbin-Watson statistic against these decision rules remains an open question.

In this paper, we propose an alternative approach to detecting spurious regression in bivariate time series based on wavelets.

The wavelet approach is a relatively new innovation in the area of applied econometrics although it has received widespread attention among statisticians (Donoho and Johnstone, 1994, 1995, 1996; Donoho et al., 1995; Percival, 1995; McCoy and Walden, 1996).

The discrete wavelet transform (DWT) is a useful method for time series as it can exhibit features that vary in time and frequency. The wavelet transform performs well in adapting to local features in a time series because it efficiently partitions the time-frequency plane, using short basis functions for high frequency oscillations and long basis functions for low-frequency oscillations.

Another advantage of the DWT is that economic and financial data are intrinsically discrete, so the DWT offers a parsimonious and cost effective discrete transformation and computation of massive amounts of these data.

Initially, the DWT was developed for geophysical applications (Gencay et al, 2003) but has been found applications in signal processing (Mallat, 1989). The need for time-frequency analysis of economic time series has been discussed in Brock (2000). Wavelet analysis found its way into economics via the works of J. B Ramsey and his co-authors. Ramsey and Zhang (1997)'s time-frequency analysis of foreign exchange rate using wavelets demonstrated that wavelet analysis can capture parsimoniously non-stationary events in the series by differentiating between short(high-frequency) shocks and longer (low-frequency) shocks. Ramsey and Lampart (1998a, 1998b) performed wavelet decompositions to identify different relationships between money and income and between consumption and income. Statistical introduction to wavelet analysis include Vidakovic (1999) and Percival and Walden (2000). Ramsey (2002) is recent literature review of wavelets in economics and finance while an intuitive introduction for economists is provided by Schleicher (2002) and Crowley (2005). A collection of key papers on wavelets can be found in Heil and Walnut (2006).

Although its usefulness in univariate time series analysis is widely acknowledged (Gencay et al 2003), wavelet analysis can be easily extended to deal with bivariate time series. This paper contributes to the literature by demonstrating one such application to bivariate time series. We demonstrate that the wavelet approach can provide fresh perspectives on the phenomenon of spurious regression.

One key advantage of the wavelet approach to detect spurious regression is that instead of relying on the conventional and logically shaky practice of examining the R^2 statistics and the Durbin-Watson statistic, we can examine more directly and objectively the relationship between two time series on the basis of the properties of the bivariate time series.

In addition, unlike a residual test like the Durbin-Watson d statistic, the spurious relationship between a bivariate time series is detected more directly and without the need to rely on assumptions mentioned earlier which are critical to the Durbin Watson test but which may not be fulfilled in practice. For instance, the wavelet approach can deal with higher order correlation, deal with non-stochastic explanatory variables and need not require that the disturbances should be normally distributed and generated. Last but not least, missing observations are not a bane to the wavelet approach but are liabilities to the Durbin-Watson approach.

A third distinct advantage of the wavelet approach is its easy interpretation and detection of spurious regression via a graphical analysis. Not only does graphical wavelet analysis of wavelet covariance and correlation allow us to examine the data as an aggregate but also their behavior over time and scale, they can provide us with an easy-to-understand summary of a complex problem. Unlike the Durbin-Watson decision rules, which are complicated to apply, the wavelet approach allows a decision to make more efficiently via a graphical inspection of the wavelet covariance and correlation.

This paper is organized as follows: Section 2 introduces key ideas in the wavelet approach to detect spurious regression. Our focus is on wavelet covariance and correlation. In section 3, we perform a Monte Carlo simulation of a spurious regression a la Granger and Newbold and analyze visually the properties of the wavelet covariance and correlation. This is compared against those exhibited by a non-spurious regression. Section 4 concludes and identifies potential areas for future research.

2. Wavelet Covariance and Correlation

Let (y_t, x_t) be a bivariate stochastic process with univariate spectra $S_x(f)$ and $S_y(f)$ respectively and let $W_{j,t} = (w_{y,j,t}, w_{x,j,t})$ be the scale λ_j wavelet coefficients computed from (y_t, x_t). The wavelet transform is applied to each process in (y_t, x_t) independently to obtain the respect wavelet coefficient process.

Definition 1: The wavelet covariance of (y_t, x_t) for scale λ_j is defined as

$$\gamma_{(y,x)}(\lambda_j) = \frac{1}{2\lambda_j} Cov(w_{y,j,t}, w_{x,j,t}) \tag{3}$$

The wavelet covariance decomposes the covariance between two stochastic processes on a scale-by-scale basis (Whitcher, 1998), hence

$$\sum_{j=1}^{\infty} \gamma_{(y,x)}(\lambda_j) = Cov(y_{j,t}, x_{j,t}) \tag{4}$$

We use the maximal overlap discrete wavelet packet transform (MODWT) as in Gencay et al (2003) to construct $\gamma_{(x,y)}(\lambda_j)$.

Suppose a length N realization of the bivariate process (y_t, x_t) is given by $(\mathbf{y}, \mathbf{x}) = ((y_0, x_0), (y_1, x_1), ..., (y_{N-1}, x_{N-1}))$ and a partial MODWT of order $J_p < \log_2(T)$ to each univariate process y_t and x_t. This produces J length N vectors of MODWT coefficients

$$\begin{aligned}\widetilde{\mathbf{W}}_{\mathbf{j}} &= (\widetilde{W}_{j,0}, \widetilde{W}_{j,1}, ..., \widetilde{W}_{j,N-1}) \\ &= ((\widetilde{w}_{y,j,0}, \widetilde{w}_{x,j,0}), (\widetilde{w}_{y,j,1}, \widetilde{w}_{x,j,1}), ..., (\widetilde{w}_{y,j,N/2^j-1}, \widetilde{w}_{x,j,N/2^j-1}))\end{aligned}$$

and the length N vectors of MODWT scaling coefficients

$$\begin{aligned}\widetilde{\mathbf{V}}_{\mathbf{j}} &= (\widetilde{V}_{j,0}, \widetilde{V}_{j,1}, ..., \widetilde{V}_{j,N-1}) \\ &= ((\widetilde{v}_{y,j,0}, \widetilde{v}_{x,j,0}), (\widetilde{v}_{y,j,1}, \widetilde{v}_{x,j,1}), ..., (\widetilde{v}_{y,j,N/2^j-1}, \widetilde{v}_{x,j,N/2^j-1}))\end{aligned}$$

Based on the MODWT, an unbiased estimator of the wavelet covariance can be derived:

$$\widetilde{\gamma}_{(y,x)}(\lambda_j) = \frac{1}{\widetilde{N}_j} \sum_{l=L_j-1}^{N-1} \widetilde{w}_{y,j,l}, \widetilde{w}_{x,j,l} \tag{5}$$

where $\widetilde{N}_j = N - L_j + 1$.

The large sample properties of $\widetilde{\gamma}_{(x,y)}(\lambda_j)$ are fairly well established in Whitcher (1998) and Serroukh and Walden(2000). For both Gaussian and non-Gaussian bivariate processes, the estimator $\widetilde{\gamma}_{(y,x)}(\lambda_j)$ is asymptotically Gaussian distributed with mean

$\gamma_{(y,x)}(\lambda_j)$ and variance $S_{\overline{w},j}(0)/\widetilde{N}_j$, where $S_{\overline{w},j}(0)$ is the spectral density function (SDF) of the product of the λ_j scale wavelet coefficients $\overline{w}_{j,t} = w_{y,j,t}w_{x,j,t}$ evaluated at zero frequency.

The derivation of an approximate $(1-\alpha)$ confidence interval for $\widetilde{\gamma}_{(y,x)}(\lambda_j)$ is straightforward (see appendix):

$$\widetilde{\gamma}_{(y,x)}(\lambda_j) \pm \xi_{\frac{\alpha}{2}}\left[\frac{\hat{S}_{\overline{w},j}(0)}{\widetilde{N}_j}\right]^{1/2} \qquad (6)$$

Wavelet Correlation

Normalizing the wavelet covariance by the variability in the observed wavelet coefficients defines the **wavelet correlation.**

Definition 3: The wavelet correlation is defined as

$$\rho_{(y,x)}(\lambda_j) = \frac{\gamma_{(y,x)}(\lambda_j)}{\sigma_y(\lambda_j)\sigma_x(\lambda_j)} \qquad (7)$$

where $\sigma_x(\lambda_j)$ and $\sigma_y(\lambda_j)$ are the respective wavelet variance for y_t and x_t associated with scale λ_j, which can be easily derived from (5):

$$\widetilde{\sigma}_k^2(\lambda_j) = \frac{1}{\widetilde{N}_j}\sum_{t=L_j-1}^{N-1}\widetilde{w}_{k,t}^2 \quad for \quad k = x, y \qquad (8)$$

where $L_j = (2^j - 1)(L-1)+1$ is the length of the scale λ_j and $\widetilde{N}_j = N - L_j + 1$ is the number of coefficients unaffected by the boundary.

The wavelet correlation contains both the wavelet covariance for (y_t, x_t) and the wavelet variances of y_t and x_t, we can obtain an unbiased estimator of the wavelet correlation based on the MODWT:

$$\tilde{\rho}_{(y,x)}(\lambda_j) = \frac{\tilde{\gamma}_{(y,x)}(\lambda_j)}{\tilde{\sigma}_y(\lambda_j)\tilde{\sigma}_x(\lambda_j)} \tag{9}$$

where $\tilde{\gamma}_{(y,x)}(\lambda_j)$ is as estimated in (5), and $\tilde{\sigma}_y(\lambda_j)$ and $\tilde{\sigma}_x(\lambda_j)$ are as estimated in equation (8). The large sample properties for $\tilde{\rho}_{(y,x)}(\lambda_j)$ are established for both Gaussian and certain nonstationary processes (Whitcher, 1998).

An approximate $(1-\alpha)$ confidence interval for $\rho_{(y,x)}(\lambda_j)$ is given by

$$\tanh\left\{\tanh^{-1}\left[\tilde{\rho}_{(y,x)}(\lambda_j)\right] \pm \xi_{\frac{\alpha}{2}}\left[\frac{1}{\hat{N}_j - 3}\right]^{1/2}\right\} \tag{10}$$

.

where $\hat{N}_j$ is the number DWT coefficients associated with scale λ_j.

3. Empirical Investigations

3.1 Data and Estimation

To demonstrate the application of the wavelet approach to detect spurious regression, we conducted a Monte Carlo experiment where 200 observations for two independent random walks, y_t and x_t, were generated, with 100 replications. We performed a wavelet covariance analysis of the two series to measure how well these two series are associated with one another. Next, we determine the wavelet correlation of the two series, which is essentially the wavelet covariance normalized by the variability inherent in the observed wavelet coefficients. The properties of the wavelet covariance and correlation in the spurious regression are discussed in the results section.

Although these are sufficient to demonstrate how the wavelet approach can be used to detect spurious regression, we went one step further and demonstrate the properties of the wavelet covariance and correlation for a non-spurious bivariate time series. The bivariate time series used is the returns series $R_t = \log\ r_t - \log\ r_{t+1}$ for the Thai Baht- US Dollar (THB-USD) and Singapore Dollar-US Dollar (SINGD-USD).

3.2 Results

Figure 1 below presents the wavelet covariance between the two random walk series. The two lines in blue with characters " L" and " U" denote the lower and upper bounds for the approximate 95% confidence interval, respectively. The wavelet covariances are significantly close to zero (at the $\alpha = 0.05$ level of significance) except at higher scale. Hence, there appears to be little association between the two series. Notice that the confidence interval expands rapidly after scale 4. Any association at higher level is spurious.

Figure 2 shows the wavelet correlation for the two random walk series. Like the wavelet covariances, the wavelet correlations are significantly close to zero (at the $\alpha = 0.05$ level of significance) except at higher scale. Any association at higher level is spurious. Unlike those for the wavelet covariances, the confidence intervals are very wide, even at lower scale.

[Insert Figure 1 Wavelet Covariance for Spurious Regression]

[Insert Figure 2 Wavelet Correlation for Spurious Regression]

One obvious advantage of the wavelet approach is that it offers a more direct way to detect spurious regression in bivariate time series. We reiterated our earlier argument that detection of spurious regression by looking out for high R^2 statistics and a low Durbin-Watson statistic is the logical fallacy of affirming the consequent. Moreover, the presence of R^2 statistics and a low Durbin-Watson statistic may be symptoms of other properties, such as misspecification and cannot be regarded as conclusive. In contrast, our approach looks at the bivariate time series directly and more objectively. We demonstrated that to detect a spurious regression, it is sufficient to ensure the wavelet covariance and correlation are significantly equal to zero. This is more intuitive and logically sound compared to trying to interpret high R^2 statistics and a low Durbin-Watson statistic.

An additional advantage of the wavelet approach is that we can graph both the wavelet covariance and correlation and their confidence intervals and detect any symptoms of spurious regression visually.

We take our analysis a step further by exploring how the wavelet covariance and correlation will be like for a non-spurious regression. Figure 3 and 4 shows respectively the wavelet covariance and correlation for the bivariate returns series for the Thai Baht-US Dollar (THB-USD) and Singapore Dollar-US Dollar (SINGD-USD). The monthly data series, dating from 1 January 1981 to 1 November 2005, are taken from Federal Reserve Bank of St. Louis.

From Figure 3, the wavelet covariances for the bivariate returns series are positive and appear to be significantly different from zero (at the $\alpha = 0.05$ level of significance) except at very high scale. Although there appears to be a positive association between the two returns series, we cannot compare wavelet scale easily due to the variability inherent in each scale. The wavelet correlation offers a simple way to resolve this difficulty by standardizing the variance of each series at each scale, thus making it possible to compare the magnitude of association across scales.

[Insert Figure 3: Wavelet Covariance for Non-spurious Regression]

[Insert Figure 4: Wavelet Correlation for Non-spurious Regression]

From Figure 4, we can observe that the wavelet correlation for the bivariate returns series are also positive and appear to be significantly different from zero (at the $\alpha = 0.05$ level of significance). The strength of the association increases with the scale though this is accompanied by a widening confidence interval at higher scales.

4. Conclusion

In this paper, we demonstrate an innovative application of a wavelet approach in bivariate time series, namely the detection of spurious regression using wavelet covariance and correlation. We highlighted some key advantages of the wavelet approach, which we reiterate briefly here. Unlike conventional approach that relies on the logically problematic detection of a high R^2 statistic and a residual test like the Durbin-Watson d statistic, the spurious relationship in a bivariate time series can detected more directly with the wavelet approach. There is no need to rely on assumptions which are critical to the Durbin Watson test but which may not be fulfilled in practice. Unlike the Durbin Watson residual test, the wavelet approach can deal with higher order correlation, non-stochastic explanatory variables, non-normally distributed disturbances and missing observations. The usefulness of graphical wavelet analysis of wavelet covariance and correlation to detect spurious regression cannot be overemphasized. A distinct advantage is the simplicity and efficiency of the decision rule compared to the complicated Durbin-Watson decision rules.

Although we have demonstrated our results using bivariate time series, extensions to multivariate time series and panel models are potential areas for future research. Doing so is important because it enables researchers to deal with spurious regression in complex panel data, an area that is currently underdeveloped (Entorf, 1997).

Above all, our application demonstrates that there is much potential that wavelet analysis can offer, beyond prediction of univariate time series and business cycle analysis. In particular, we have hopefully shown that wavelet analysis offers fresh and interesting perspectives on a conventional problem such as spurious regression.

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Appendix

Derivation of Confidence Intervals for Wavelet Covariance

Suppose we have a sequence of random variables $z_0, z_1, ..., z_{N-1}$ with mean $\mu = E(z_t)$ and variance $\sigma^2 = Var(z_t)$. The sample mean is given by $\bar{z} = N^{-1}\sum_0^{N-1} z_t$ and the sample variance $Var(\bar{z}) = N^{-1}\sigma^2$.

Large sample properties allow us to derive the $(1-\alpha)$ confidence interval for μ :

$$\bar{z} \pm \xi_{\frac{\alpha}{2}}\left[\frac{\sigma^2}{N}\right]^{1/2} \tag{A1}$$

To obtain the estimated confidence interval for the wavelet covariance, we can substitute the point estimate for the wavelet covariance $\tilde{\gamma}_{(y,x)}(\lambda_j)$, an unbiased estimator of the spectral density function (SDF) $\hat{S}_{\bar{w},j}(0)$ and $\tilde{N}_j = N - L_j + 1$ into (A1):

$$\tilde{\gamma}_{(y,x)}(\lambda_j) \pm \xi_{\frac{\alpha}{2}}\left[\frac{\hat{S}_{\bar{w},j}(0)}{\tilde{N}_j}\right]^{1/2} \tag{A2}$$

Let $S_{(x,y)}(f)$ be the cross spectrum between y_t and x_t . Intuitively the wavelet covariance captures smaller and smaller of the cross spectrum as λ_j increases. Introducing an integer lag τ between $w_{y,j,t}$ and $w_{x,j,t}$ gives us a definition of the wavelet cross-covariance:

$$\gamma_{(x,y)}(\lambda_j) = \frac{1}{2\lambda_j} Cov(w_{y,j,t}, w_{x,j,t+\tau}) \tag{A3}$$

Using Parseval's relation and assuming an underlying Gaussian bivariate stochastic process, Whitcher (1998) has shown that an unbiased estimator of the spectral density function (SDF) to be

$$\hat{S}_{\overline{w},j}(0) = \frac{\hat{\gamma}_{y,j,0}\hat{\gamma}_{x,j,0}}{2} + \sum_{m=1}^{\tilde{N}_j - 1} \hat{\gamma}_{y,j,m}\hat{\gamma}_{x,j,m} + \frac{1}{2}\sum_{m=-(\tilde{N}_j - 1)}^{\tilde{N}_j - 1} \hat{\gamma}^2_{W,j,m} \qquad \text{(A4)}$$

where $\hat{\gamma}^2_{W,j,m}$ is the biased cross-covariance sequence (CCVS) between the λ_j wavelet coefficients $w_{y,j,t}$ and $w_{x,j,t}$.

If a non-Gaussian assumption is adopted, mutlitaper spectrum estimation can be employed to estimate $\hat{S}_{\overline{w},j}(0)$. Details can be found in Serrokh and Walden (2000)

Figure 1 Wavelet Covariance for Spurious Regression

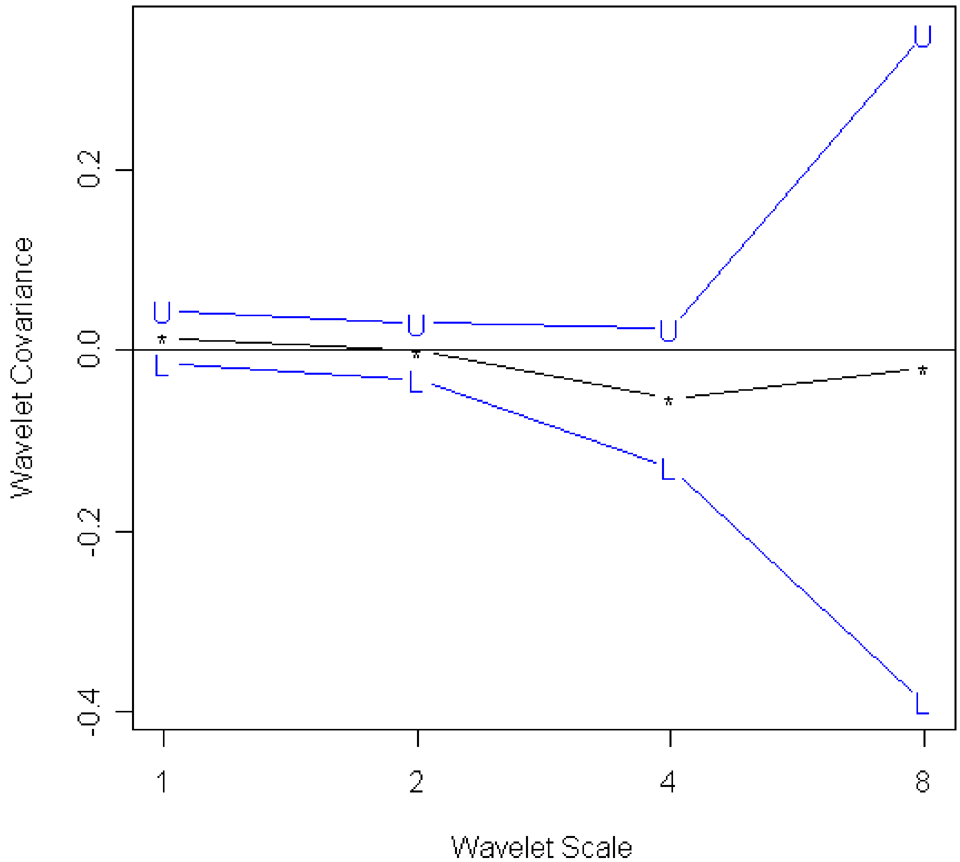

Figure 2 Wavelet Correlation for Spurious Regression

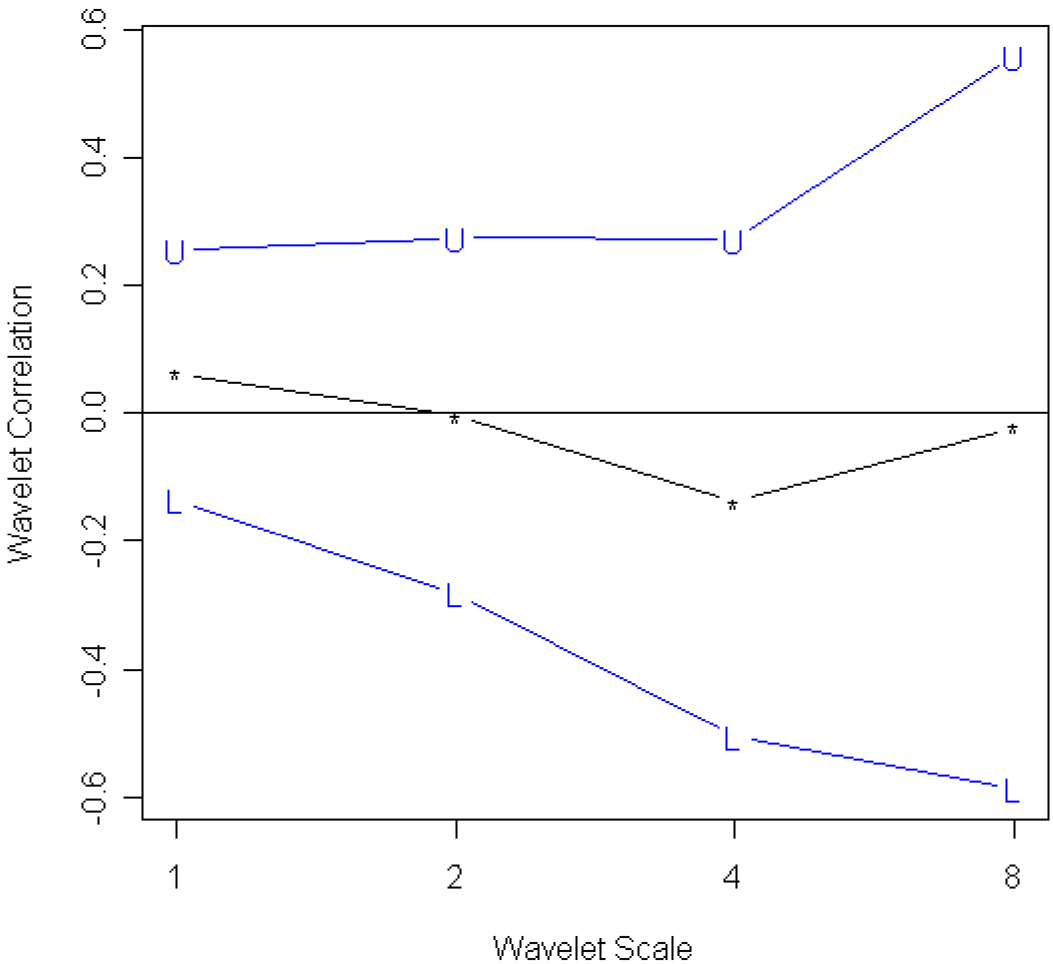

Figure 3: Wavelet Covariance for Non-spurious Regression

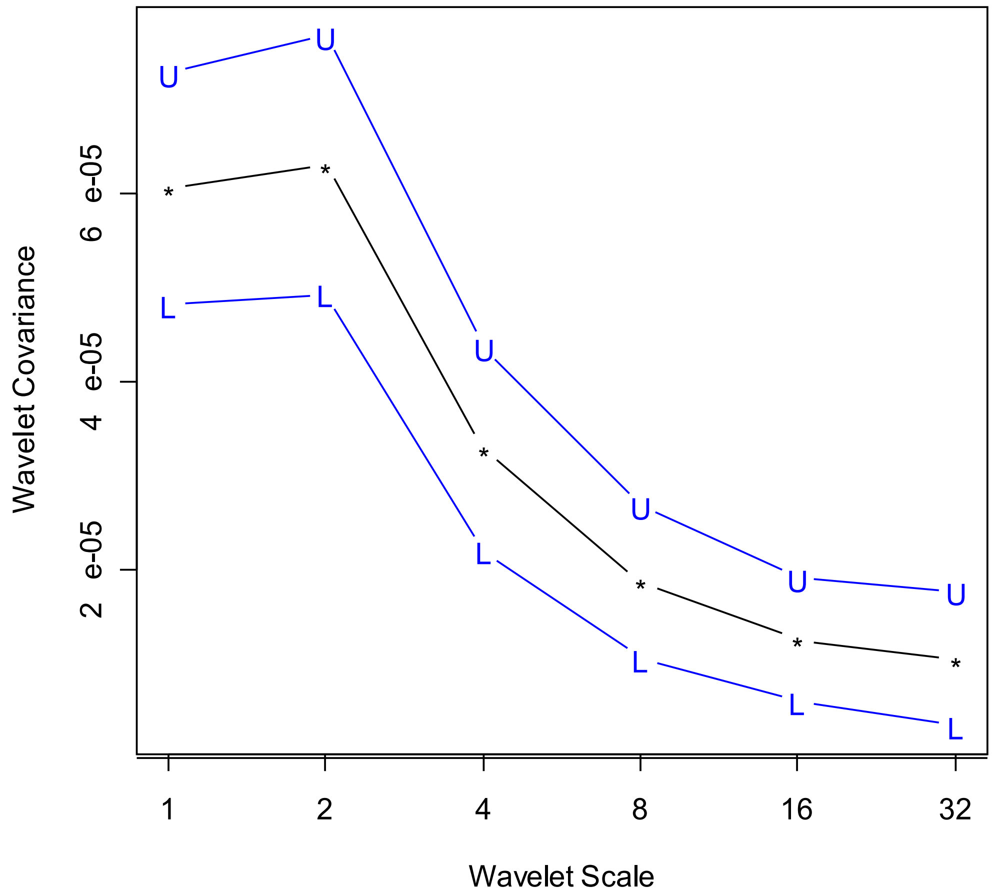

Figure 4: Wavelet Correlation for Non-spurious Regression