Assessing the Credibility of Instrumental Variables Inference with Imperfect Instruments Via Sensitivity Analysis

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September 28, 2007

abstract

Consistent instrumental variables (IV) estimation requires instruments uncorrelated with the model errors, but this assumption is usually both suspect and untestable. Here the asymptotic sampling distribution of the IV parameter estimator is derived for any specified instrument-error covariance vector. This result makes it possible to quantify the sensitivity of any particular IV inference result to instrument-error correlations, allowing one to assess the robustness of such inferential conclusions to uncertainty in the validity of the instruments. An application illustrating the value of this sensitivity analysis is given to a study by Acemoglu et al. (2001).

JEL: C10, C12

Keywords: Instrumental variables, instrument validity, IV estimation, sensitivity analysis

[1]The author would like to thank an anonymous referee and the participants at the 2003 meeting of the Midwest Econometric Group for helpful comments and Simon Johnson for sharing his data set.

1. Introduction

Consistent instrumental variables (IV) estimation requires instruments which are valid, which is to say, at least asymptotically uncorrelated with the error term in the regression equation. In practice, however, this condition is hardly likely to be precisely satisfied. Moreover, this assumption is virtually impossible to check since the relevant error term is not directly observable. Consequently, the validity of IV-based parameter estimation and inference typically rests on an underlying statistical assumption which is generally both suspect and untested.

Note that this problem is distinct from that of “weak instruments,” as in Stock, Wright, and Yogo (2002) and Dufour (2003). IV estimates using instruments which are weak (i.e., only weakly correlated with the endogenous variables) are known to yield unreliable parameter inference even when these instruments are valid, in the sense of being asymptotically uncorrelated with the model error term. But there is a relationship between these two problems: unable to quantify the sensitivity of one’s IV inference results to modest amounts of correlation between the instruments and the error term, analysts have often turned to weak instruments in a search for instruments which are credibly uncorrelated with the model errors.

This paper approaches the problem of instrument validity from a new perspective by developing the sensitivity analysis tools necessary for quantifying the consequences of explicitly relaxing the assumption that the instruments are uncorrelated with the error term in the regression equation. Such tools are of practical value because, while one might not be comfortable assuming that an instrument is completely uncorrelated with the model error term, one might be credibly able to assume that the magnitude of whatever correlation does exist is less than, say,

.40. The large-sample distribution of the usual IV parameter estimates is derived below under the alternative assumption that the instruments are correlated with the model error term to a specified degree. Hahn and Hausman (2003) derive the analogous distribution for the special case where this correlation is vanishingly small (of order $1/\sqrt{N}$); here the size of this correlation is not assumed to be small. Using the distributional results obtained in Sections 2 and 3 below, one can readily assess which inferential conclusions are robust to a specified degree of uncertainty in the instrument validity and which conclusions are not.[2]

In particular, with this estimate of the IV parameter estimator sampling distribution in hand (as a function of an assumed value for the instrument-error correlation vector) an analyst can explicitly examine the sensitivity (to those correlations) of the p-value at which any particular null hypothesis depending on the model parameters can be rejected. This null hypothesis might be a simple restriction on a particular model coefficient. Or it might involve restrictions on a number of (possibly) nonlinear functions of the model parameters. Notably, some inferential conclusions one might want to draw from a particular model and data set might be substantially robust to reasonably likely departures from instrument validity, whereas other conclusions might not – the sensitivity analysis proposed here can settle that question.[3]

In contrast to the correlations between the instruments and the model errors – which are ordinarily assumed to be zero – the correlations between one or more of the explanatory variables

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[2]This suggestion is in the same spirit as Leamer (1985), which addresses the sensitivity of econometric modeling/inference to model specification search, and Ashley (1998), in which the credibility of bootstrap-based assessment of postsample model forecasting effectiveness is enhanced by a double bootstrap quantifying the uncertainty in the ordinary bootstrap inference.

[3]The sensitivity analysis proposed here is distinct from – and conceptually much simpler than – the alternative approach of attempting to construct wider confidence intervals for the IV-estimated parameters based on the perceived uncertainty in instrument-error correlations.

in a model and the model error are often assumed to be non-zero. Indeed, those correlations (and the resulting inconsistency of OLS parameter estimation) are the raisons d'être for IV estimation in the first place. Somewhat remarkably, results obtained in Section 3 below make it possible to use the sample discrepancy between the OLS and IV parameter estimates to consistently estimate these correlations between the endogenous variables and the model errors for any given degree of correlation between the instruments and the model errors.[4]

This result is very useful because – due to its role in producing inconsistency in the OLS estimates – there is a long and voluminous literature attempting to quantify such correlations between explanatory variables and the model errors and to thereby at least crudely assess the expected distortions in the OLS parameter estimates.[5] Whatever information is available in any particular setting as to the likely sign and/or size of these correlations between the endogenous variables and the model errors can thus be combined with the relationship alluded to above to provide likely bounds on the correlations between the instruments and the model errors and hence on the fragility or durability of the statistical inferences based on the IV parameter estimates.

For example, where the endogeneity is thought to arise from measurement error in an explanatory variable, it is well known that the correlation between the (uncorrupted) explanatory variable and the model error is opposite in sign to the coefficient with which the variable enters

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[4]In evaluating the plausibility of such a result existing, recall that the Hausman (1978) test for endogeneity exploits this sample parameter estimator discrepancy (and an assumption that the instrument-error correlation is zero) to obtain a test for correlations between the endogenous variables and the model errors.

[5]A minimal selection of relevant citations would go back to Gini (1921) and Frisch (1934) and include Klepper and Leamer (1984), Erickson (1993), Bound, et al. (1994), Card (1999), and Black, Berger and Scott (2000).

the model. Going beyond this, Card (1999, p. 1816), for example, cites a substantial body of research indicating that errors in measures of schooling induce correlations between actual and observed schooling of around .90, which imply specific bounds on the correlation of measured schooling and the errors in a model using measured schooling instead of actual schooling. There is also a fairly extensive literature on measurement error in labor market variables and its impact on estimated regression models in which these are used as explanatory variables – e.g., Bound, et al. (1994)'s study using the Panel Study of Income Dynamics Validation Study and numerous references cited therein. Alternatively, where an explanatory variable is a serially correlated time series corrupted by serially uncorrelated measurement errors, one can bound the correlation between this variable and the model errors using the spectral method given by Ashley and Vaughan (1985).[6]

The proposed sensitivity analysis is applied to Acemoglu, et al. (2001)'s study of the impact of institutions on post-colonial development, where the coefficient on an admittedly noisy index of the quality of a country's institutions is estimated using an instrument based on the mortality rates recorded for European settlers in the colonial period. In this instance the sensitivity analysis indicates that the relevant IV-based inferences are reasonably robust to possible correlations between the instrument used and the model errors, particularly when those instrument-error correlations are restricted to the interval of (in this case, negative) values consistent with the correlation between the institution-quality index and the model error term being primarily due to measurement error.

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[6] See also, Fuller (1987). Hausman and Watson (1985) and Hyslop and Imbens (2001) consider more sophisticated specifications of the measurement error process.

2. IV Estimation with Flawed Instruments

Consider the usual multiple regression model with N observations and k stochastic regressors:

$$Y = X\beta + \epsilon \tag{1}$$

where

$$\left[X_{i,1} \ldots X_{i,k}\, ,\, \epsilon_i\right] \sim \text{iid}\left(0\, ,\begin{bmatrix}\Sigma_{XX} & \Sigma_{X\epsilon}\\ \Sigma_{\epsilon X} & \sigma_\epsilon^2\end{bmatrix}\right). \tag{2}$$

Here it is assumed that OLS yields an inconsistent estimator for β because – due to simultaneity, omitted variables, measurement error, etc. – one or more of the first $p \le k$ explanatory variables are correlated with the model error, ϵ. That is, one or more of the first p components of the row vector $\Sigma_{X\epsilon}$ are non-zero. Consequently, β is estimated using the p column vectors $Z_1 \ldots Z_p$ as instruments for the first p columns of of X; the remaining k-p columns of Z are identical to the analogous columns of X; notationally,[7]

$$\left[Z_{i,1} \ldots Z_{i,k}\, ,\, X_{i,1} \ldots X_{i,k}\, ,\, \epsilon_i\right] \sim \text{iid}\left(0\, ,\begin{bmatrix}\Sigma_{ZZ} & \Sigma_{ZX} & \Sigma_{Z\epsilon}\\ \Sigma_{XZ} & \Sigma_{XX} & \Sigma_{X\epsilon}\\ \Sigma_{\epsilon Z} & \Sigma_{\epsilon X} & \sigma_\epsilon^2\end{bmatrix}\right). \tag{3}$$

It is assumed that $\lim_{N\to\infty} \frac{1}{N} E\left[Z^t X\right] = \Sigma_{ZX}$ is of full rank; thus, $\hat{\beta}^{IV}$ could potentially

[7] The assumption that $[Z_{i1} \ldots Z_{ik}\, , X_{i1} \ldots X_{ik}\, , \epsilon]$ is independently distributed is made for didactic clarity; as (briefly) indicated in the proofs below, this assumption can be weakened.

provide a consistent estimator for β, except that Z is known to be flawed as an instrument for X, in that one or more of the first p columns of Z are correlated with ϵ – that is, some or all of the first p components of the covariance vector $\Sigma_{Z\epsilon}$ are non-zero.

The object of this section is to obtain the asymptotic sampling distribution of $\hat{\beta}^{IV}$ under these circumstances, as a function of $\Sigma_{Z\epsilon}$ – which is taken as given – and of quantities (such as Σ_{XX}, Σ_{ZX}, and Σ_{ZZ}) which can be consistently estimated using the sample data.

Lemma 1 Inconsistency of $\hat{\beta}^{IV}$ (Definition of inconsistency vector δ and error term η)

Under the assumptions given above – i.e., Equation 1 and Equation 3 – and with $\hat{\beta}^{IV} = \left(Z^t X\right)^{-1} Z^t Y$,

$$\text{plim}(\hat{\beta}^{IV}) = \text{plim}\left[\left(\frac{1}{N} Z^t X\right)^{-1}\left(\frac{1}{N} Z^t Y\right)\right] = \beta + \Sigma_{ZX}^{-1}\,\Sigma_{Z\epsilon} = \beta + \delta, \qquad (4)$$

which defines the non-zero k-vector, δ.

Proof: This result follows directly from substitution of Equation 1 into the expression for $\hat{\beta}^{IV}$ and taking probability limits. The inverse of Σ_{ZX} exists because Σ_{ZX} is of full rank; thus, the probability limit in Equation 4 is well-defined. Because Σ_{ZX} is of full rank, δ – the inconsistency in $\hat{\beta}^{IV}$ – is zero if and only if $\Sigma_{Z\epsilon}$ is zero; the covariance vector $\Sigma_{Z\epsilon}$ is non-zero because the instrument Z is flawed.

A corollary to Lemma 1, however, provides a regression model for which Z **is** a valid instrument for X:

Corollary 1 A Regression Model in Which Z is a Valid Instrument for X

Under the assumptions given above, Z is a valid instrument for X in the model

$$Y = X(\beta + \delta) + (\epsilon - X\delta) = X(\beta + \delta) + \eta, \tag{5}$$

which defines the modified error term, η, for which $\Sigma_{Z\eta}$ is zero.

Proof: That $\Sigma_{Z\eta} = \lim_{N\to\infty} \frac{1}{N} E[Z^t \eta]$ is zero follows directly from substitution (for η and δ) and taking the indicated limit.

In effect, then, when one uses Z as an instrument for X in estimating β in Equation 1, one is actually (consistently) estimating the parameter vector $\beta+\delta$ in the model of Equation 5. It follows that the sampling distribution of $\hat{\beta}^{IV}$ is fairly simple, but – due to the structure of the modified error term, η – a bit different from the usual result:

Theorem 1 Sampling Distribution of $\hat{\beta}^{IV}$

Under the assumptions given above by Equations 1 and 3,

$$\sqrt{N}\left(\hat{\beta}^{IV} - \beta - \delta\right) \underset{d}{\rightarrow} N(0,V), \tag{6}$$

where

$$V = \Sigma_{ZX}^{-1}\left(\sigma_\epsilon^2 \Sigma_{ZZ} + W\right)\Sigma_{XZ}^{-1} \tag{7}$$

and where the $(j, \ell)^{th}$ element of W is:

$$w_{j\ell} = \text{cov}\left(\epsilon_i^2, Z_{ij} Z_{i\ell}\right) + \sum_{u=1}^{k} \sum_{v=1}^{k} \delta_u \delta_v \, E\left(Z_{ij} Z_{i\ell} X_{iu} X_{iv}\right) - 2 \sum_{u=1}^{k} \delta_u \, E\left(Z_{ij} Z_{i\ell} X_{iu} \epsilon_i\right), \qquad (8)$$

$\delta = \Sigma_{ZX}^{-1} \Sigma_{Z\epsilon}$, and the indicated fourth moments in $w_{j\ell}$ are the same for all values of the index *i* and assumed to be finite. Note that the $\text{cov}\left(\epsilon_i^2, Z_{ij} Z_{i\ell}\right)$ term in the expression for $w_{j\ell}$ is zero under the usual homoscedasticity assumption.

Proof: Substituting Equation 5 into the expression for $\hat{\beta}^{IV}$ in Lemma 1 yields

$$\hat{\beta}^{IV} - \beta - \delta = \left(\frac{1}{N} Z^t X\right)^{-1} \left(\frac{1}{N} Z^t \eta\right)$$

From Corollary 1, $\Sigma_{Z\eta}$ equals zero, so that $\text{plim}\left(\hat{\beta}^{IV} - \beta - \delta\right)$ is zero also. Each row of [Z, X, ϵ] is assumed iid; the asymptotic normality of each of the k components of $\frac{1}{N} Z^t \eta$ follows from the definition of η and the usual Central Limit Theorem for iid data.

The asymptotic variance of $\sqrt{N}\left(\hat{\beta}^{IV} - \beta - \delta\right)$ is

$$\begin{aligned} V &= \text{plim}\left[N \left\{\hat{\beta}^{IV} - \beta - \delta\right\} \left\{\hat{\beta}^{IV} - \beta - \delta\right\}^t\right] \\ &= \text{plim}\left[N \left\{\left(\frac{1}{N} Z^t X\right)^{-1} \left(\frac{1}{N} Z^t \eta\right)\right\} \left\{\left(\frac{1}{N} \eta^t Z\right) \left(\frac{1}{N} X^t Z\right)^{-1}\right\}\right] \qquad (9) \\ &= \Sigma_{ZX}^{-1} \lim_{N \to \infty} \left(\frac{1}{N} E[Z^t \eta \, \eta^t Z]\right) \Sigma_{XZ}^{-1}. \end{aligned}$$

Because the modified error term in Equation 5 (η) depends on X when the instruments are flawed – so that δ is non-zero – the evaluation of the middle term in Equation 9 differs from the usual derivation. The $(j, \ell)^{th}$ element of $\left(\frac{1}{N} E[Z^t \eta \, \eta^t Z]\right)$ is:

$$\frac{1}{N} E\left[\sum_{i=1}^{N} Z_{ij}\,\eta_i \sum_{s=1}^{N} Z_{s\ell}\,\eta_s\right] = \frac{1}{N}\sum_{i=1}^{N}\sum_{\substack{s=1\\ s\neq i}}^{N} E\left[Z_{i,j}\eta_i\right] E\left[Z_{s\ell}\,\eta_s\right] + \frac{1}{N} E\left[\sum_{i=1}^{N} Z_{i,j}\,Z_{i\ell}\,\eta_i^2\right] \tag{10}$$

$$= \frac{1}{N} E\left[\sum_{i=1}^{N} Z_{i,j}\,Z_{i\ell}\left(\epsilon_i - \sum_{u=1}^{k} X_{iu}\delta_u\right)\left(\epsilon_i - \sum_{v=1}^{k} X_{iv}\delta_v\right)\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N} E\left(Z_{ij}\,Z_{i\ell}\left[\epsilon_i^2 + \sum_{u=1}^{k}\sum_{v=1}^{k} X_{iu}X_{iv}\,\delta_u\delta_v - 2\sum_{u=1}^{k} X_{iu}\,\epsilon_i\,\delta_u\right]\right)$$

where the sum over the index s is eliminated using the assumption that the rows of [Z, X, ϵ] are independent and the result (Corollary 1) that $\Sigma_{Z\eta}$ is zero.[8] Using the assumption that the rows of [Z, X, ϵ] are identically distributed, this expression for the $(j, \ell)^{th}$ element of $\left(\frac{1}{N} E[Z^t\,\eta\,\eta^t Z]\right)$ reduces to:

$$\left[\sum_{i=1}^{N} Z_{ij}\,\eta_i \sum_{s=1}^{N} Z_{s\ell}\,\eta_s\right] = \sigma_\epsilon^2\, E\left(Z_{ij}\,Z_{i\ell}\right) + \operatorname{cov}\left(\epsilon_i^2, Z_{ij}\,Z_{i\ell}\right) \tag{11}$$

$$+ \sum_{u=1}^{k}\sum_{v=1}^{k} \delta_u\delta_v\, E\left(Z_{ij}\,Z_{i\ell}X_{iu}\,X_{iv}\right) - 2\sum_{u=1}^{k} \delta_u\, E\left(Z_{ij}\,Z_{i\ell}X_{iu}\,\epsilon_i\right)$$

where the indicated expectations are the same for all values of the index i and are assumed to exist. The expression for $(j, \ell)^{th}$ element of W given in Equation 8 then follows immediately, completing the proof of Theorem 1.

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[8]Weaker assumptions on the rows of [Z,X, ϵ] would suffice, at the cost of a more complicated expression here and an appeal to an appropriately modified version of the Central Limit Theorem.

The fourth moments needed in order to evaluate the elements of W can be consistently estimated using the corresponding sample moments.[9] However, for samples which are insufficiently large as to support such estimates – N < 500, say – it is preferable to make a distributional assumption on the rows of [Z, X, ϵ], so as to express these fourth moments in terms of more-easily estimated second moments. For example, if it is assumed that this vector is normally distributed, then it follows from the multivariate gaussian moment generating function that:

Corollary 2 Moments needed for evaluating $w_{j\ell}$ when the rows of [Z, X, ϵ] are normally distributed and an explicit formula for the sampling variance of $\hat{\beta}^{IV}$ in the single-instrument case.

Using the notation given by Equation 3 and additionally assuming that the rows of [Z, X, ϵ] are normally distributed, the expectations required for evaluating $w_{j\ell}$, given by Equation 11, are:

$$E\left(Z_{ij} Z_{i\ell} X_{iu} X_{iv}\right) = \Sigma_{ZZ}(j,\ell)\ \Sigma_{XX}(u,v) + \Sigma_{ZX}(j,u)\ \Sigma_{ZX}(\ell,v) + \Sigma_{ZX}(j,v)\ \Sigma_{ZX}(\ell,u) \quad (12)$$

$$E\left(Z_{ij} Z_{i\ell} \epsilon_i^2\right) = \Sigma_{ZZ}(j,\ell)\, \sigma_\epsilon^2 + 2\, \Sigma_{Z\epsilon}(j)\, \Sigma_{Z\epsilon}(\ell) \quad (13)$$

and

$$E\left(Z_{ij} Z_{i\ell} X_{iu} \epsilon_i\right) = \Sigma_{ZZ}(j,\ell)\ \Sigma_{X\epsilon}(u) + \Sigma_{ZX}(j,u)\ \Sigma_{Z\epsilon}(\ell) + \Sigma_{ZX}(\ell,u)\ \Sigma_{Z\epsilon}(j). \quad (14)$$

[9]The original model errors (ϵ) can be consistently estimated as $X\left(\hat{\beta}^{IV} - \delta\right) = X\left(\hat{\beta}^{IV} - \Sigma_{ZX}^{-1}\ \Sigma_{Z\epsilon}\right)$ for given $\Sigma_{Z\epsilon}$. Or one could simply use the usual robust standard errors based on the $\hat{\eta}$.

where, for example, $\Sigma_{ZZ}(j,\ell)$ is the $(j, \ell)^{th}$ element of Σ_{ZZ}. These follow directly from the appropriate fourth partial derivatives of the moment generating function of the multivariate gaussian distribution; similar results would obtain for alternative distributional assumptions.

For the single-instrument case, where j, ℓ, u, and v are all one (and Σ_{XX}, Σ_{XZ}, and Σ_{ZZ}, $\Sigma_{X\epsilon}$, and $\Sigma_{Z\epsilon}$ are all scalars), the asymptotic variance of $\hat{\beta}^{IV}$ in Theorem 1 reduces to:

$$\mathrm{var}(\hat{\beta}^{IV}) = \frac{1}{N\Sigma_{ZX}^2}\left[\sigma_\epsilon^2\,\Sigma_{ZZ} + 2\,\Sigma_{Z\epsilon}^2 + \delta^2\left(\Sigma_{ZZ}\,\Sigma_{XX} + 2\,\Sigma_{ZX}^2\right) - 2\,\delta\left(\Sigma_{ZZ}\,\Sigma_{X\epsilon} + 2\,\Sigma_{ZX}\,\Sigma_{Z\epsilon}\right)\right] \quad (15)$$

where δ is now just $\Sigma_{Z\epsilon}/\Sigma_{ZX}$. This completes the proof.

Thus, for given $\Sigma_{Z\epsilon}$, the quantities needed in order to estimate the sampling distribution of $\hat{\beta}^{IV}$ using Theorem 1 are estimates of Σ_{XX}, Σ_{XZ}, and Σ_{ZZ}, δ, W, $\Sigma_{X\epsilon}$, and σ_ϵ^2. The population moments Σ_{XX}, Σ_{XZ}, and Σ_{ZZ} can be consistently estimated via the analogous sample moments; δ can be consistently estimated from $\Sigma_{ZX}^{-1}\,\Sigma_{Z\epsilon}$ for any given value of $\Sigma_{Z\epsilon}$; and estimation of the fourth moments entering W was just discussed. Estimates of $\Sigma_{X\epsilon}$ and σ_ϵ^2 are necessary for implementing the results of Theorem 2 and Corollary 2, so as to examine the sensitivity with respect to $\Sigma_{Z\epsilon}$ of the p-values at which hypothesis tests regarding functions of β can be rejected. A second motivation for estimating $\Sigma_{X\epsilon}$ and σ_ϵ^2, however, is so that one can also compute the variation (with $\Sigma_{Z\epsilon}$) of the correlations, $R_{Z\epsilon} = [\mathrm{corr}(Z_{i1}, \epsilon_i), \ldots \mathrm{corr}(Z_{ik}, \epsilon_k)]$ and $R_{X\epsilon} = [\mathrm{corr}(X_{i1}, \epsilon_i), \ldots \mathrm{corr}(X_{ik}, \epsilon_k)]$, as these are more easily interpreted than the corresponding covariances. While $\Sigma_{X\epsilon}$, and σ_ϵ^2 can be estimated directly, using $X\left(\hat{\beta}^{IV} - \delta\right)$ as a large-sample estimator of ϵ, it is more graceful to estimate $\Sigma_{X\epsilon}$, and σ_ϵ^2 using the results obtained in the next section, which are based on $\Sigma_{Z\epsilon}$ and the observed discrepancy between $\hat{\beta}^{IV}$ and $\hat{\beta}^{OLS}$.

3. Estimation of σ_ϵ^2 and $\Sigma_{X\epsilon}$ for Given $\Sigma_{Z\epsilon}$

In the previous section Theorem 1 provides the sampling distribution of $\hat{\beta}^{IV}$ given $\Sigma_{Z\epsilon} = [\mathrm{cov}(Z_{i1}, \epsilon_i) \ldots \mathrm{cov}(Z_{i1}, \epsilon_k)]$, the covariance of the instruments with the model error. Using this result, one can readily examine the sensitivity of the p-value at which any particular null hypothesis regarding functions of β can be rejected to various assumptions about the covariance vector $\Sigma_{Z\epsilon}$.

Implementation of this result requires an estimate of $\Sigma_{X\epsilon}$ and of σ_ϵ^2. An estimate of $\Sigma_{X\epsilon}$ is additionally useful because it is often possible to sign $\Sigma_{X\epsilon}$, as in the empirical example of Section 4 below; thus a concomitant estimate of $\Sigma_{X\epsilon}$ allows one to restrict the sensitivity analysis to the most relevant values of $\Sigma_{Z\epsilon}$. An estimate of σ_ϵ^2 is additionally useful in allowing one to re-express $\Sigma_{Z\epsilon}$ and $\Sigma_{X\epsilon}$ in terms of the more-easily interpreted correlation vectors, $R_{Z\epsilon} = [\mathrm{corr}(Z_{i1}, \epsilon_i) \ldots \mathrm{corr}(Z_{i1}, \epsilon_k)]$ and $R_{X\epsilon} = [\mathrm{corr}(X_{i1}, \epsilon_i), \ldots \mathrm{corr}(X_{ik}, \epsilon_k)]$. Theorem 2 below provides consistent estimators of $\Sigma_{X\epsilon}$ and σ_ϵ^2, based on an assumed value for $\Sigma_{Z\epsilon}$ and the sample discrepancy between $\hat{\beta}^{IV}$ and $\hat{\beta}^{OLS}$:

Theorem 2 Expressions for $\Sigma_{X\epsilon}$ and σ_ϵ^2, given $\Sigma_{Z\epsilon}$ and $\mathrm{plim}(\hat{\beta}^{IV}) - \mathrm{plim}(\hat{\beta}^{OLS})$.

Under the assumptions given in Section 2 – i.e., Equations 1 and 3 – $\Sigma_{X\epsilon} = [\mathrm{cov}(X_{i1}, \epsilon_i) \ldots \mathrm{cov}(X_{i1}, \epsilon_k)]$ can be written:

$$\Sigma_{X\epsilon} = \Sigma_{XX}\left[\mathrm{plim}(\hat{\beta}^{OLS}) - \mathrm{plim}(\hat{\beta}^{IV})\right] + \Sigma_{XX}\Sigma_{ZX}^{-1}\Sigma_{Z\epsilon}, \tag{16}$$

providing a consistent estimator for $\Sigma_{X\epsilon}$, given $\Sigma_{Z\epsilon}$ and consistent estimates of Σ_{XX} and Σ_{ZX}.

Further,

$$\sigma_\epsilon^2 = \sigma_\eta^2 + 2\,\Sigma_{Z\epsilon}^t\,\Sigma_{XZ}^{-1}\,\Sigma_{XX}\left[\text{plim}(\hat\beta^{OLS}) - \text{plim}(\hat\beta^{IV})\right] + \Sigma_{Z\epsilon}^t\,\Sigma_{XZ}^{-1}\,\Sigma_{XX}\,\Sigma_{ZX}^{-1}\,\Sigma_{Z\epsilon}. \tag{17}$$

Since the IV regression which is actually estimated provides a consistent estimator of σ_η^2, Equation 17 can be used to construct a consistent estimator of σ_ϵ^2; this estimate of σ_ϵ^2 can then be used to convert estimates of the covariances – $\Sigma_{X\epsilon}$ and $\Sigma_{Z\epsilon}$ – into the corresponding correlation vectors.

Proof: From Lemma 1, $\text{plim}(\hat\beta^{IV}) - \Sigma_{ZX}^{-1}\,\Sigma_{Z\epsilon}$ equals β. For the special case of OLS, where $Z = X$, this implies that $\text{plim}(\hat\beta^{OLS}) - \Sigma_{XX}^{-1}\,\Sigma_{X\epsilon}$ also equals β. Setting these two expressions for β equal to one another and solving for $\Sigma_{X\epsilon}$ yields Equation 16.

It follows from $\eta = \epsilon - X\delta$ that $\eta^t\eta$ is equal to $\epsilon^t\epsilon - 2\delta^tX^t\epsilon + \delta^tX^tX\delta$. Dividing by N and taking limits yields

$$\sigma_\eta^2 = \sigma_\epsilon^2 - 2\delta^t\Sigma_{X\epsilon} + \delta^t\Sigma_{XX}\delta. \tag{18}$$

Equation 17 follows by solving Equation 18 for σ_ϵ^2 and substituting into the resulting expression $\Sigma_{ZX}^{-1}\,\Sigma_{Z\epsilon}$ for δ and Equation 16 for $\Sigma_{X\epsilon}$. This completes the proof.

These results (and Theorem 1) are applied in the next section to a single-instrument regression model due to Acemoglu, Johnson and Robinson (2001), providing explicit estimates as to how the p-value for their key hypothesis test varies with $\text{corr}(Z_{i1}, \epsilon_i)$, the correlation of their instrument with the model error term.

4. An Illustrative Example: Using IV Estimation to Account for Measurement Error in a Development Equation

Acemoglu, Johnson and Robinson (2001) examine the relationship between variation in per capita income across countries and an index of the protection against government appropriation of assets. In particular, over a sample of 64 countries which were at one time European colonies, they find that OLS estimation of the regression model

$$\log\ y_i = \mu + \alpha R_i + x_i^t \gamma + \epsilon_i$$

yields what appear to be positive values for the parameter α, where $\log\ y_i$ is the logarithm of 1995 per capita GDP (on a PPP basis) for country i and R_i is the "protection from risk of expropriation" index from Political Risk Services, averaged over the period 1985-95.[10] They also consider the effects of a number of additional explanatory variables $\left(x_i^t\right)$ but these are not included in their "base" model and will not be further considered here.

They obtain the OLS model estimates,

$$\log\ y_i = \underset{(0.41)}{4.66} + \underset{(0.06)}{0.52}\, R_i + e_i \qquad \bar{R}^2 = .533$$

where the figures in parentheses are estimated standard errors. But they observe that this estimate of α is of somewhat dubious value since it is likely that R_i is correlated with ϵ_i, due to reverse causality ("rich economies may be able to afford, or perhaps prefer, better institutions'),

[10] Acemoglu, et al. also investigate a number of other measures of expropriation risk and other institutional measures – see their footnotes 3 and 11. R_i is measured on a scale from zero to ten, with a higher value indicating lower risk. For example, R_i is 10.00 for the U.S., 9.32 for Singapore, 8.27 for India, 6.27 for Ghana, and 4.00 for Mali.

omitted variables which are correlated with institutional quality,[11] and due to measurement error in R_i. Arguably, in this case, measurement error ("broadly construed," as they put it) is the most important problem with R_i; they note:

> In reality the set of institutions that matter for economic performance is very complex, and any single measure is bound to capture only part of the "true institutions," creating a typical measurement error problem. Moreover, what matters for current income is presumably not only institutions today, but also institutions in the past. Our measure of institutions which refers to 1985-1995 will not be perfectly correlated with these.
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> [Acemoglu, et al. (2001, p. 1385-6)]

Consequently, they estimate α via instrumental variables using as an instrument for R_i the logarithm of an estimate of the mortality rate experienced by European settlers during the time period in which the country was colonized. This instrument is denoted "$mort_i$" below.

Acemoglu, et al. (2001, p. 1370) argue persuasively that this mortality rate is correlated with the current institutions in a country:

> 1. There were different types of colonization policies which created different sets of institutions. At one extreme, European powers set up "extractive states," exemplified by the Belgian colonization of the Congo. These institutions did not introduce much protection for private property, nor did they provide checks and balances against government appropriation. In fact, the main purpose of the extractive state was to transfer as much of the resources of the colony to the colonizer.
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> At the other extreme, many Europeans migrated and settled in a number of colonies, creating what the historian Alfred Crosby (1986) calls "Neo-Europes." The settlers tried to replicate European institutions, with strong emphasis on private property and checks against government power. Primary examples of this

[11] Their α estimates are not terribly sensitive to including covariates such as dummy variables for the dominant religion, being a former British colony, being a former French colony, or a French origin for the legal system. In the interest of expositional simplicity these variations on the base model are not included in the analysis reported here, but this insensitivity suggests to them that omitted-variables bias is less important here than reverse causality or measurement error.

include Australia, New Zealand, Canada, and the United States.
2. The colonization strategy was influenced by the feasibility of settlements. In places where the disease environment was not favorable to European settlement, the cards were stacked against the creation of Neo-Europes, and the formation of the extractive state was more likely.
3. The colonial state and institutions persisted even after independence.
[Acemoglu, et al. (2001, p. 1370)]

And, indeed, the sample correlation of $mort_i$ with R_i is -.520; evidently, $mort_i$ is not a "weak" instrument.

This instrument choice yields the (IV) estimated model:

$$\log\ y_i = \underset{(1.03)}{1.91} + \underset{(0.16)}{0.94}\, R_i + n_i \qquad \bar{R}^2 = .174$$

in which the null hypothesis that α is zero can clearly be rejected.

The authors' argument that $mort_i$ is uncorrelated with the model error term (pp. 1371-72) is less persuasive, however. This, no doubt, is why much of the remainder of their paper is devoted to various alternative formulations of the model, each of which presumably yields a different set of model errors, in the hope of demonstrating that their inference result is robust.

Table 1 below displays the results obtained with this model by instead considering the sensitivity of inference results on α to varying degrees of assumed correlation between the instrument ($mort_i$) and the original model error term.

Clearly, "α," "$\log y_i$," "R_i," and "$mort_i$" here play the roles denoted "β_1," "Y_i," "X_{i1}," and "Z_{i1}" in the theoretical results obtained in Sections 2 and 3 above. And it is assumed here that the sample length is sufficiently large that it is reasonable to replace [Σ_{YX}, Σ_{YZ}, Σ_{XX}, Σ_{XZ}, Σ_{ZZ}, σ_η^2] by the corresponding sample moments [1.126150, -.906735, 2.156920, -.960240, 1.582520,

.899330].[12] Table 1 then displays – for a grid of different posited values for $\Sigma_{Z\epsilon}$ – the implied values for $\text{plim}(\hat{\beta}_1^{IV}) - \delta$, corr($Z_{i1}$, ϵ_i), corr(X_{i1}, ϵ_i) and the p-values at which H_o: $\beta_1 = 0.0$ and H_o: $\beta_1 = 0.5$ can be rejected.[13] For convenience of interpretation, β_1, Z_{i1}, and $X_{i,1}$ are expressed as α, $mort_i$, and R_i in Table 1 and below to correspond with the notation in the Acemoglu, et al. (2001) study.

The issue for these authors is clearly whether the parameter α is positive or whether it is zero: they are not going to be rejecting H_o: $\alpha = 0.0$ based on negative estimates. Therefore, one-tailed p-values are reported in Table 1 for this hypothesis test. Note also that $\alpha \geq 0$ implies that it is mainly negative values for $\Sigma_{R\epsilon}$ and corr(R_i, ϵ_i) that are relevant, since Acemoglu, et al. (2001) identify measurement error in R_i as the primary reason that $\Sigma_{R\epsilon} \neq 0$.

Reference to the column of Table 1 for H_o: $\alpha = 0.0$ shows that this null hypothesis can be rejected at the 5% level for all values of corr(R_i, ϵ_i) less than .53 – including all of the possible negative values, which (given $\alpha \geq 0$) are the only ones consistent with measurement error being the primary problem in the OLS regression model. Moreover, in view of the observed discrepancy between the OLS and IV estimates of α, the results in Table 1 show that values of corr(R_i, ϵ_i) greater than .53 would correspond to values of |corr($mort_i$, ϵ_i)| exceeding .65, which is to say an instrument which is *quite* highly correlated with the model error term. Thus, the

[12] It is additionally assumed, as is necessary for Theorem 1and Corollary 2, that $mort_i$, R_i, and ϵ_i are normally, identically and independently distributed to a reasonable approximation. Per the discussion preceding Corollary 2, normality is assumed so as to obtain estimates with this modest sample of the fourth moments required for Theorem 1. Based on a visual examination of histograms and on formal tests – such as the skewness-kurtosis and Shapiro-Wilk tests – there is no evidence that $mort_i$, R_i, or the IV regression fitting errors are non-gaussian,

[13] The p-values reported are obtained using tail areas from the normal distribution given by Theorem 1; using tail areas from the Student's t distribution do not noticeably affect the conclusions; indeed, if it did, then the assumption that the sample size was sufficiently large for IV estimation would hardly be reasonable. With a two-instrument model, one might want to display the results as a two-dimensional array in which the entry for a particular [cov(Z_{i1},ϵ_i), cov(Z_{i2},ϵ_i)] pair is one if the null hypothesis can be rejected at the 5% level and zero otherwise.

sensitivity analysis shows that the authors' rejection of the null hypothesis H_o: $\alpha = 0.0$ is very robust to likely flaws in the colonial mortality rate instrument, even if measurement error is not the only problem.

Note that the results of the sensitivity analysis are appropriately sensitive to the null hypothesis considered. Here the null hypothesis H_o: $\alpha = 0.0$ – which is the hypothesis which is relevant to whether or not the quality of a country's institutions matter for growth – is highly robust to likely flaws in the colonial mortality rate instrument. But inferential results on other null hypotheses one might consider – such as H_o: $\alpha = 0.50$ – are clearly sensitive to modest values of $|corr(mort_i, \epsilon_i)|$ and hence what one might call "fragile."

Table 1

Sensitivity Analysis Results Regarding α,

the Coefficient on the Expropriation Protection Measure, R_i[a]

$\Sigma_{mort,\epsilon}$	$\text{plim}(\hat{\alpha}^{IV}) - \delta$	corr($mort_i$, ϵ_i)	corr(R_i, ϵ_i)	p-value for rejecting H_o: $\alpha = 0.0$	p-value for rejecting H_o: $\alpha = 0.5$
-1.00	-0.097	-0.686	0.785	0.7325	0.0001
-0.95	-0.045	-0.687	0.758	0.6133	0.0005
-0.90	0.007	-0.686	0.726	0.4821	0.0016
-0.85	0.059	-0.683	0.688	0.3529	0.0048
-0.80	0.111	-0.678	0.644	0.2388	0.0130
-0.75	0.163	-0.670	0.592	0.1485	0.0314
-0.70	0.215	-0.657	0.532	0.0845	0.0689
-0.65	0.267	-0.638	0.462	0.0438	0.1372
-0.60	0.319	-0.614	0.383	0.0206	0.2487
-0.55	0.372	-0.582	0.295	0.0088	0.4117
-0.50	0.424	-0.543	0.198	0.0034	0.6254
-0.45	0.476	-0.496	0.095	0.0012	0.8764
-0.40	0.528	-0.443	-0.011	0.0004	0.8594
-0.35	0.580	-0.385	-0.117	0.0001	0.6102
-0.30	0.632	-0.324	-0.219	0.0000	0.3996
-0.25	0.684	-0.263	-0.314	0.0000	0.2400
-0.20	0.736	-0.203	-0.401	0.0000	0.1316
-0.15	0.788	-0.146	-0.478	0.0000	0.0657
-0.10	0.840	-0.093	-0.546	0.0000	0.0298
-0.05	0.892	-0.044	-0.604	0.0000	0.0122
0.00	0.944	0.000	-0.654	0.0000	0.0045
0.05	0.996	0.040	-0.696	0.0000	0.0015
0.10	1.048	0.075	-0.733	0.0000	0.0005
0.15	1.100	0.107	-0.764	0.0000	0.0001
0.20	1.153	0.136	-0.790	0.0000	0.0000

[a]$Mort_i$ is the mortality rate instrument used by Acemoglu, et al. (2001); the sensitivity analysis is conducted over a grid of posited values for $\Sigma_{mort,\epsilon}$ = cov($mort_i$, ϵ_i); the figures in all of the other columns are thus estimates implied by the value of this entry and the sample estimates of Σ_{YX}, Σ_{YZ}, Σ_{XX}, Σ_{XZ}, Σ_{ZZ}, and σ_η^2. In particular, $\text{plim}(\hat{\alpha}^{IV}) - \delta$ is the implied (consistent) estimate of α; similarly, corr($mort_i$, ϵ_i) and corr(R_i, ϵ_i) are the implied correlations of the mortality rate instrument and of the explanatory variable with the model error term. As noted in the text, the p-value for H_o: $\alpha = 0.0$ is for a one-tailed test and that for H_o: $\alpha = 0.5$ is for a two-tailed test.

5. Conclusions

Instrumental variables estimation is a powerful and useful technique, but it involves assuming that the instruments are at least asymptotically uncorrelated with the model error term. Since this assumption is often open to question and is (in practice) almost impossible to check, the credibility of the resulting IV parameter estimates and confidence intervals suffers. By providing the means to easily check the sensitivity of IV estimation/inference to failures of this assumption, the results obtained above allow us to increase the credibility of the conclusions we draw from models estimated using IV methods.

The empirical example, drawn from the Acemoglu, et al. (2001) study of the impact of institution quality on a country's growth rate, illustrates how this sensitivity analysis can either confirm or disconfirm the proposition that the IV results are in fact robust with respect to the inference of interest. The results in this case provide strong support for the credibility of the IV results with respect to the inference which is arguably at issue, but they also illustrate the fragility of other possible inferences.

The basic idea used here is both simple and broadly applicable; it can be extended to 2SLS (where the model is over-identified), to GMM estimation, and in a variety of other contexts.[14] Estimation and inference in econometric modeling often requires us to make assumptions which are difficult or impossible to check: that an instrument is uncorrelated with an error term, or that a particular moment condition equals zero, etc. The approach here is to

[14]The analogous sensitivity analysis for models estimated using 2SLS and GMM methods is being explicitly considered in other work. In those particular instances where the crucial inferential question can be boiled down to the size of a single parameter, the kind of sensitivity analysis proposed here could alternatively be subsumed in a confidence interval estimate which has been suitably augmented to account for the uncertainty in $\Sigma_{Z\epsilon}$. Such augmentation poses some analytical challenges, but Imbens and Manski (2004) provide a good starting point.

suggest that it is generically useful to parametrically relax this assumption/condition and critically examine the sensitivity of one's inference results to this relaxation. If the most crucial inferential conclusions are insensitive to relaxing these assumptions or conditions, then it can be concluded that one's inference results are robust and credible. If the results are, in contrast, quite sensitive to such departures, then it must be concluded that these inference results are fragile and not very credible.

References

Acemoglu, D., S. Johnson and J.A. Robinson (2001) "The Colonial Origins of Comparative Development: An Empirical Investigation" *American Economic Review 91(5)*, 1369-1401.

Ashley, R. (1998) "A New Technique for Postsample Model Selection and Validation" *Journal of Economic Dynamics and Control 22*, 647-665.

Ashley, R. and D. Vaughan (1985) "Measuring Measurement Error in Economic Time Series" *Journal of Business & Economic Statistics 4(1)*, 95-103.

Black, D. A., M.C. Berger and F. A. Scott (2000) "Bounding Parameter Estimates with Nonclassical Measurement Error," *Journal of the American Statistical Association 95*, 739-748.

Bound, J. C. Brown, G. J. Duncan, and W. L. Rodgers (1994) "Evidence on the Validity of Cross-Sectional and Longitudinal Labor Market Data," *Journal of Labor Economics 12(3)*, 345-368.

Card, D. (1999) "The Causal Effect of Education on Earnings" *Handbook of Labor Economics 3A*, (Ashenfelter, O. and D. Card, eds.) Elsevier: Amsterdam, 1801-1863.

Dufour, J.-M. (2003) "Identification, Weak Instruments and Statistical Inference in Econometrics" *Canadian Journal of Economics 36(4)*, 767-808.

Erickson, T. (1993) "Restricting Regression Slopes in the Errors-in-Variables Model by Bounding the Error Correlation," *Econometrica 61*, 959-969.

Frisch, R. (1934) *Statistical Confluence Analysis by Means of Complete Regression Systems*. University Institute of Economics: Oslo.

Fuller, W. A. (1987) *Measurement Error Models*. Wiley: New York.

Gini. C. (1921) "Sull'interpolazione di una Retta Quando i Valori della Variabile Indipendente Sono Affetti da Errori Accidentali," *Metroeconomica 1*, 63-82.

Hausman, J. (1978) "Specification Tests in Econometrics," *Econometrica 46*, 125-1271.

Hausman, J. and M. W. Watson (1985) "Errors in Variables and Seasonal Adjustment Procedures," *Journal of the American Statistical Association 80(391)*, 531- 540.

Hahn, J and J. Hausman (2003) "IV Estimation With Valid and Invalid Instruments" http://econ-www.mit.edu/faculty/?prof_id=hausman

Hyslop, D. R. and G. M. Imbens (2001) "Bias From Classical and Other Forms of Measurement Error," *Journal of Business and Economic Statistics 19(4)*, 475-481.

Klepper, S. and E. E. Leamer (1984) 'Consistent Sets of Estimates for Regressions with Errors in All Variables," *Econometrica 52*, 163-184.

Leamer, E. E. (1985) "Sensitivity Analysis Would Help" *American Economic Review 75*, 308-313.

Stock, J. H. , J. H. Wright, and M. Yogo (2002) "A Survey of Weak Instruments and Weak Identification in Generalized Method of Moments," *Journal of Business and Economic Statistics 20(4)*, 518-529.