137

Reihe Ökonomie

Economics Series

A Diffusion Approximation for the Riskless Profit under Selling of Discrete Time Call Options

Sergei A. Nagaev

137

Reihe Ökonomie

Economics Series

A Diffusion Approximation for the Riskless Profit under Selling of Discrete Time Call Options

Sergei A. Nagaev

September 2003

Institut für Höhere Studien (IHS), Wien
Institute for Advanced Studies, Vienna

Contact:

Sergei A. Nagaev
Department of Economics and Finance
Institute for Advanced Studies
Stumpergasse 56
1060 Vienna, Austria
☎: +43/1/599 91-232
fax: +43/1/599-91-555
email: sergej@ihs.ac.at

Abstract

A discrete time model of a financial market is considered. We focus on the study of a guaranteed profit of an investor which arises when the stock price jumps are bounded. The limit distribution of the profit as the model becomes closer to the classical model of the geometric Brownian motion is established. It is of interest that in contrast with the discrete approximation, no guaranteed profit occurs in the approximated continuous time model.

Keywords

Asymptotic uniformity, weak convergence in Skorokhod Space *D*[0, 1]

JEL Classification

G13, G24, C61

Comments
The author is grateful to Prof. Robert Kunst whose numerous comments and suggestions have helped to improve the presentation of this paper. The lemma is proved.

Contents

1 Introduction

Consider the simplest financial market in which securities of two types are circulating. The price evolution of the securites of the first type is given by the equations

$$b_k = b_0 \rho^k, \ k = 0, 1, 2, \ldots,$$

where $b_0 > 0, \ \rho > 1$. The prices are registered at the equidistant moments of time $t_k = a + kh$. With no loss of generality we put $a = 0, \ h = 1$, i.e. $t_k = k$.

The price of the security of the second type at the moment k is represented as

$$s_k = s_0 \xi_1 \cdots \xi_k, \ k = 0, 1, 2, \ldots,$$

where the relative jumps ξ_k are random.

The securities of the first type are *riskless* having the interest rate $(\rho - 1) \cdot 100\%$. Let us call them conventionally *bonds*. It is clear that possessing the securities of the second type is concerned with a risk of their devaluation. We call them conditionally *stocks*.

Taken together in certain amounts β and γ the securities of both types constitute a so-called *portfolio (writer's investment portfolio)* whose worth at the time moment k is $\beta b_k + \gamma s_k$. *Playing* in the considered financial market consists of successive changing of the portfolio content at the moments $k = 1, 2, \ldots, n-1$. The successive pairs $(\beta_0, \gamma_0), \ (\beta_1, \gamma_1), \ldots, (\beta_{n-1}, \gamma_{n-1})$ constitute a so-called *strategy* of the game. Obviously, as a basis for choosing (β_k, γ_k) serves the evolution of the stock price up to this moment i. e. $s_0, s_1, \ldots, s_k$. In other words

$$\beta_k = \beta_k(s_0, s_1, \ldots, s_k), \ \gamma_k = \gamma_k(s_0, s_1, \ldots, s_k).$$

The player is called a *writer (seller, investor)*.

A strategy is called *self-financing* if the changing of the portfolio content does not affect its value i.e.

$$\beta_k b_k + \gamma_k s_k = \beta_{k-1} b_k + \gamma_{k-1} s_k, \ k = 1, \ldots, n-1.$$

The final goal of the game is to meet the condition

$$x_n = \beta_{n-1} b_n + \gamma_{n-1} s_n \geq f(s_n) \tag{1.1}$$

where $f(s)$ is a so-called *pay-off function* of the simplest option of the *European* type having n as a *maturity date*. For more about the mathematical and substantial aspects of the option pricing theory see, e.g., Shiryaev (1999).

Basic problems of the mathematical theory of options are the evaluation of the so-called *rational option price* and, corresponding to it, a strategy leading to (1.1). Recall that the rational option price is the minimal initial capital x_0 which allows the investor to meet contract terms under proper behavior.

Both problems are easily solved within the framework of the so-called *binary* model, that is, in the case where ξ_k take only two values d and $u, \ d < \rho < u$. In this case (see, e.g., Ch. VI in Shiryaev (1999))

$$x_0 = \rho^{-n} \sum_{k=0}^{n} C_n^k p_*^k (1 - p_*)^{n-k} f(s_0 u^k d^{n-k}) \tag{1.2}$$

where

$$p_* = \frac{\rho - d}{u - d}.$$

It is worth emphasizing that (1.2) does not assume any restrictions imposed on the measure that governs the evolution of the stock price $(\xi_1, \ldots, \xi_n)$. Furthermore, there exists the unique self-financing strategy

$$(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \{(\beta_0, \gamma_0), \ (\beta_1, \gamma_1), \ldots, (\beta_{n-1}, \gamma_{n-1})\}$$

leading to the equality

$$x_n = \beta_{n-1} b_n + \gamma_{n-1} s_n = f(s_n). \tag{1.3}$$

The strategy is defined by the formulae

$$\beta_k = \frac{uf_{k+1}(s_k d) - df_{k+1}(s_k u)}{\rho b_k(u-d)} \tag{1.4}$$

and

$$\gamma_k = \frac{f_{k+1}(s_k u) - f_{k+1}(s_k d)}{s_k(u-d)} \tag{1.5}$$

where

$$f_k(s) = \rho^{-(n-k)} \sum_{j=0}^{n-k} C_{n-k}^j p_*^j (1-p_*)^{n-k-j} f(su^j d^{n-k-j}). \tag{1.6}$$

The successive values of the portfolio are

$$x_k = f_k(s_k), \ k = 0, 1, \ldots, n-1.$$

If $\xi_k, \ k = 1, 2, \ldots, n,$ take more than two values then it is impossible to guarantee the desired relation (1.3) with probability 1. However, sometimes it is possible to guarantee (1.1). For example, if $\xi_k \in [d, u]$ and $f(s)$ is convex then the minimal initial capital is evaluated by the same formula (1.2).

This fact was, first, proven in Tessitore and Zabczyk (1996) (see also Zabczyk (1996) and Motoczyński and Stettner (1998)). The proof follows the control theory lines. Later on in Shiryaev (1999) the rational price is derived as the solution of a extreme problem (see Theorem V.1c.1 ibidem). It seems that Shiryaev knew nothing about the works of his predecessors. At least in the rather rich list of references given in Shiryaev (1999) they are not presented.

Denote

$$\bar{x}_k = f_k(s_k), \ k = 0, \ldots, n-1, \tag{1.7}$$

and let (β_k, γ_k) be defined as in (1.4) and (1.5).

Possessing after the $(k-1)-$th step the capital $\bar{x}_{k-1}$ distributed in portfolio in accordance with (1.4) and (1.5) at the next step k the investor gains the capital

$$x_k = \beta_{k-1} b_k + \gamma_{k-1} s_k = \frac{u - \xi_k}{u-d} f_k(s_{k-1}d) + \frac{\xi_k - d}{u-d} f_k(s_{k-1}u)$$

(for more detail see, e.g., A.Nagaev and S.Nagaev (2002a)).

If $\xi_k \in [d, u], \ k = 1, \ldots, n,$ then

$$\delta_k = x_k - \bar{x}_k = f_k(s_{k-1}d)\frac{u-\xi_k}{u-d} + f_k(s_{k-1}u)\frac{\xi_k - d}{u-d} - f_k(s_{k-1}\xi_k) \geq 0. \tag{1.8}$$

It is easily seen that $\delta_k = 0$ if and only if $\xi_k = d$ or $\xi_k = u$. Otherwise $\delta_k > 0$. Thus, if ξ_k takes at least one value lying in (d, u) then a profit arises. If the extreme values d and u belong to the support of the distribution of ξ_k then $\bar{x}_{k-1}$ is the minimal capital which allows such a profit. It implies that the policy determined by (1.4) and (1.5) forms the so-called *hedge* or, by the terminology adopted in [1], *upper hedge* while $(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{n-1})$ is the corresponding chain of hedging capitals.

The investor may dispose of the so arisen profit in various ways. The simplest one is to withdraw from the game the superfluous quota δ_k which to the maturity date acquires the value $\delta_k \rho^{n-k}$. So, the self-financing condition is fulfilled only in the part which bans any capital inflows.

Having withdrawn unnecessary quota one should follow the "binary" optimal strategy determined by (1.4) and (1.5). As a result to the maturity date the investor accumulates a riskless profit

$$\Delta_n = \delta_1 \rho^{n-1} + \delta_2 \rho^{n-2} + \cdots + \delta_n.$$

It is not easy to find the distribution of Δ_n even in the case of independent ξ_k. The question arises how to approximate it. It is one of such approximations that is a basic goal of the paper.

It is worth emphasizing that this problem was studied in A. Nagaev and S. Nagaev (2002b, 2003). In the first of these papers the authors consider the simplest case where the random variables $\xi_k, \ k = 1, 2, \ldots, n,$ are i.i.d. and the pay-off function is smooth. The second one is devoted to chaotic phenomena which arise when the pay-off function is not smooth. The typical example of such a function is provided by the call option. The basic goal of the present paper is to extend the main results of the latter work to a more general case. The generalization concerns the distribution of the stock price jumps (cf. the conditions on η in (2.10) below with the corresponding conditions in A. Nagaev and S. Nagaev (2002b, 2003)). Parallel to it the author considerably simplifies the proofs.

The paper is organized as follows. In Section 2 the basic results are formulated. The "local" profit in the case where the model converges to that of the geometrical Brownian motion is studied in Section 3. In Section 4 the limit value for the expected value of the total riskless profit is established. The limit distribution of the total riskless profit is given in Section 5. Concluding remarks are gathered together in Section 6. Auxiliary facts are given in the Appendix.

2 Basic results

$>$From now on we deal with the simplest case of the standard call option determined by the pay-off function

$$f(s) = (s-K)_+. \tag{2.9}$$

Put in (1.8)

$$\begin{cases} u = u_n & = \exp(hn^{-1} + xn^{-1/2}) \\ d = d_n & = \exp(hn^{-1} - yn^{-1/2}) \\ \rho = \rho_n & = \exp(\alpha n^{-1}) \\ \xi_k = \xi_{k,n} & = \exp(hn^{-1} + \eta_k n^{-1/2}) \\ s_{k,n} & = s_0 \xi_{1,n} \cdots \xi_{k,n} \end{cases} \tag{2.10}$$

where α, x and y are positive constants, a constant $h \in I\!R$ while random variables $\eta_k,\ k = 1, \ldots, n$, are independent copies of a random variable η taking values in $[-y, x]$ so that $\mathrm{E}\eta = 0,\ \mathrm{Var}\ \eta = \sigma^2 > 0$. We assume also that the extreme points $-y$ and x belong to the support of the distribution of η. Consider the stochastic process $z(t) = ht + \sigma w(t)$ where $w(t)$ is the standard Wiener process.

Define

$$\psi(t, z) = \frac{x+y}{\sqrt{xy(1-t)}} \varphi\left(\frac{\ln K - z + (1-t)(xy/2 - \alpha)}{\sqrt{xy(1-t)}}\right) \tag{2.11}$$

and

$$I(t) = \mathrm{E}\psi(t, z(t) + \ln s_0) = \frac{x+y}{\sqrt{t\sigma^2 + xy(1-t)}} \varphi\left(\frac{\ln(K/s_0) - ht + (1-t)(xy/2 - \alpha)}{\sqrt{t\sigma^2 + xy(1-t)}}\right). \tag{2.12}$$

Here $\varphi(v)$ is the density of the standard normal law.

The following two theorems contain the basic results of the present paper.

Theorem 2.1 *Let the distribution of η be non-lattice i.e.*

$$|\mathrm{E}e^{\imath t\eta}| \neq 1 \qquad \text{for all } t \neq 0.$$

Then as $n \to \infty$

$$\mathrm{E}\Delta_n = \frac{Kxy}{2(x+y)} \left(1 - \frac{\sigma^2}{xy}\right) \int_0^1 I(t)\mathrm{d}t + o(1)$$

where K is the strike price from (2.9).

Theorem 2.2 *Under the conditions of Theorem 2.1*

$$\Delta_n \xrightarrow{d} l(z(t)) = \frac{Kxy}{2(x+y)}\left(1-\frac{\sigma^2}{xy}\right)\int_0^1 \psi(t, z(t)+\ln s_0)\mathrm{d}t$$

where

$$z(t) = ht + \sigma w(t),$$

$w(t)$ is the standard Wiener process and $\psi(t,z)$ is defined as in (2.11).

It should be emphasized that the limit distribution of Δ_n depends on the underlying one only through σ.

3 "Local" profit of investor

Let us convene to denote by c any positive constant whose concrete value is of no importance. Under such a convention we have e.g. $c+c=c,\ c^2=c$ etc. By $[\cdot,\cdot],((\cdot,\cdot])$ we denote a closed (closed from the right) interval and by θ any variable taking values in $[-1,1]$. By $[\cdot]$ and $\{\cdot\}$ we denote, respectively, the integer and fractional part of the embraced number.

Denote

$$p_n = \frac{\rho_n - d_n}{u_n - d_n},\ \ \lambda_{k,n} = \frac{\xi_{k,n}-d_n}{u_n-d_n}$$

and

$$a_{j,m} = u_n^j d_n^{m-j},\ \ b_{j,m} = C_m^j p_n^j (1-p_n)^{m-j}.$$

>From (1.6) it follows that the discounted "local"profit of the investor takes the form

$$\Delta_{k,n} = \delta_{k,n}\rho_n^{n-k} = \sum_{j=0}^{n-k} b_{j,n-k}(\lambda_{k,n} f(s_{k-1,n}u_n a_{j,n-k}) + (1-\lambda_{k,n})f(s_{k-1,n}d_n a_{j,n-k}) - \tag{3.13}$$

$$-f(s_{k-1,n}\xi_{k,n}a_{j,n-k})).$$

For time being we suppress the dependence of $\lambda_k,\ d,\ u,\ \xi_k$ and s_k on n. Let j be such that $s_{k-1}da_{j,n-k} > K$. Then

$$\lambda_k f(s_{k-1}ua_{j,n-k}) + (1-\lambda_k)f(s_{k-1}da_{j,n-k}) - f(s_{k-1}\xi_k a_{j,n-k}) = s_{k-1}(\lambda_k u + (1-\lambda_k)d - \xi_k)a_{j,n-k} = 0.$$

If $s_{k-1}ua_{j,n-k} \le K$ then

$$0 = f(s_{k-1}ua_{j,n-k}) \ge f(s_{k-1}\xi_k a_{j,n-k}) \ge f(s_{k-1}da_{j,n-k}).$$

It is worth reminding that $d \le \xi_{k-1} \le u$. Thus,

$$\Delta_{k,n} = \delta_{k,n}\rho_n^{n-k} = \sum_{r_{n-k}(u)<j\le r_{n-k}(d)} b_{j,n-k}(\lambda_k(s_{k-1}ua_{j,n-k}-K)_+ +$$

$$(1-\lambda_k)(s_{k-1}da_{j,n-k}-K)_+ - (s_{k-1}\xi_k a_{j,n-k}-K)_+)$$

where

$$r_m(z) = r_m(z, s_{k-1}) = \frac{\ln(K/(s_{k-1}zd^m))}{\ln(u/d)}.$$

The following lemma plays an important role.

Lemma 3.1 *If $0 < x' \le \min(x,y) \le \max(x,y) \le x'' < \infty$ then for $d \le z \le u,\ m \le n$*

$$r_m(z) = m\cdot\frac{y}{x+y} + n^{1/2}\left(\frac{\ln K}{x+y} - \frac{\ln s_{k-1}}{x+y} - \frac{m+1}{n}\cdot\frac{h}{x+y}\right) - \frac{w}{x+y}$$

where $\ln z = hn^{-1} + wn^{-1/2}$.

Proof. From (2.10) it follows that

$$\ln\frac{u}{d} = (x+y)n^{-1/2}$$

and, therefore,

$$\frac{\ln z}{\ln(u/d)} = \frac{w}{x+y} + \frac{h}{x+y}\cdot n^{-1/2}.$$

In particular,

$$\frac{\ln d}{\ln(u/d)} = -\frac{y}{x+y} + \frac{h}{x+y}\cdot n^{-1/2}$$

and the lemma follows.

It is easily seen that $r_m(d) - r_m(u) = 1$. Moreover,

$$\#\{j:\ r_m(u) < j \le r_m(d)\} = 1. \tag{3.14}$$

Taking into account (2.10) we obtain

$$u - d = (x+y)n^{-1/2} + \frac{x^2-y^2}{2}n^{-1} + O(n^{-3/2})$$

while

$$\rho - d = yn^{-1/2} + (\alpha - h - y^2/2)n^{-1} + O(n^{-3/2}).$$

Therefore,

$$p_n = \frac{y}{x+y} + \frac{\alpha - h - xy/2}{x+y}n^{-1/2} + O(n^{-1}).$$

By Lemma 3.1

$$r_m(d) - mp_n = n^{1/2}\left(\frac{\ln(K/s_{k-1})}{x+y} + \frac{m}{n}(\frac{xy}{2(x+y)} - \frac{\alpha}{x+y})\right) + O(1)$$

and, therefore,

$$\frac{r_m(d) - mp_n}{\sqrt{mp_n(1-p_n)}} = (m/n)^{-1/2}(xy)^{-1/2}\left(\ln(K/s_{k-1}) + (m/n)(\frac{xy}{2} - \alpha)\right) + O(m^{-1/2}). \tag{3.15}$$

Further,

$$\begin{aligned}\Delta_{k,n} = &\sum_{r_{n-k}(\xi_k)<j\le r_{n-k}(d)} b_{j,n-k}(\lambda_k(s_{k-1}ua_{j,n-k} - K) - (s_{k-1}\xi_k a_{j,n-k} - K)) + \\ \lambda_k &\sum_{r_{n-k}(u)<j\le r_{n-k}(\xi_k)} b_{j,n-k}(s_{k-1}ua_{j,n-k} - K) = \Delta'_{k,n} + \Delta''_{k,n}.\end{aligned} \tag{3.16}$$

By definition of $r_{n-k}(z)$ we have

$$s_{k-1}za_{j,n-k} = s_{k-1}zd^{n-k}(u/d)^j = K(u/d)^{j-r_{n-k}(z)}.$$

Hence

$$s_{k-1}ua_{j,n-k} = K(u/d)^{j-r_{n-k}(u)} = K(u/d)^{j+1-r_{n-k}(d)}$$

and

$$s_{k-1}da_{j,n-k} = K(u/d)^{j-r_{n-k}(d)}.$$

Since $\lambda_k u - \xi_k = -d(1-\lambda_k)$ we conclude that

$$\Delta'_{k,n} = (1-\lambda_k)K \sum_{r_{n-k}(\xi_k)<j\le r_{n-k}(d)} b_{j,n-k}\left(1 - (d/u)^{r_{n-k}(d)-j}\right) \tag{3.17}$$

while
$$\Delta''_{k,n} = \lambda_k K \sum_{r_{n-k}(u)<j\le r_{n-k}(\xi_k)} b_{j,n-k}\left((u/d)^{j+1-r_{n-k}(d)} - 1\right). \tag{3.18}$$

In view of (2.10) and (3.14) we have uniformly in $k,\ \delta n \le k \le (1-\delta)n$,

$$1-(d/u)^{r_{n-k}(d)-j} = (x+y)n^{-1/2}(r_{n-k}(d) - j + O(n^{-1}))$$

and

$$(u/d)^{j+1-r_{n-k}(d)} - 1 = (x+y)n^{-1/2}(j+1-r_{n-k}(d) + O(n^{-1})).$$

Here $\delta > 0$ is arbitrarily small.

Taking into account (2.10) we conclude that

$$\Delta'_{k,n} = K(x-\eta_k + O(n^{-1/2}))n^{-1/2} \sum_{r_{n-k}(\xi_k)<j\le r_{n-k}(d)} b_{j,n-k}(r_{n-k}(d) - j + O(n^{-1}))$$

while

$$\Delta''_{k,n} = K(\eta_k + y + O(n^{-1/2}))n^{-1/2} \sum_{r_{n-k}(u)<j\le r_{n-k}(\xi_k)} b_{j,n-k}(j+1-r_{n-k}(d) + O(n^{-1})).$$

Both representations are valid uniformly in $k,\ \delta n \le k \le (1-\delta)n$.

By the uniform version of the Moivre-Laplace local limit theorem we obtain for $k,\ \delta n \le k \le (1-\delta)n$,

$$b_{j,n-k} = \frac{1}{\sqrt{(n-k)p_n(1-p_n)}}\varphi\left(\frac{j-(n-k)p_n}{\sqrt{(n-k)p_n(1-p_n)}}\right) + o(n^{-1/2}) = n^{-1/2}\psi(kn^{-1}, \ln s_{k-1}) + o(n^{-1/2})$$

or, taking into account (3.15) and (2.11)

$$b_{j,n-k} = n^{-1/2}\psi(kn^{-1}, \ln s_{k-1}) + o(n^{-1/2}). \tag{3.19}$$

It is worth emphasizing that (3.19) holds uniformly in s_{k-1}.

Thus,

$$\Delta'_{k,n} = K(x-\eta_k)n^{-1}\psi(kn^{-1}, \ln s_{k-1}) \sum_{r_{n-k}(\xi_k)<j\le r_{n-k}(d)} (r_{n-k}(d) - j) + O(n^{-3/2})$$

while

$$\Delta''_{k,n} = K(\eta_k + y)n^{-1}\psi(kn^{-1}, \ln s_{k-1}) \sum_{r_{n-k}(u)<j\le r_{n-k}(\xi_k)} (j+1-r_{n-k}(d)) + O(n^{-3/2}).$$

Both representations are valid uniformly in $k,\ \delta n \le k \le (1-\delta)n$. In view of (3.14) the interval $(r_{n-k}(u), r_{n-k}(d)]$ contains exactly one integer $j^* = [r_{n-k}(d)]$. So,

$$\sum_{r_{n-k}(\xi_k)<j\le r_{n-k}(d)} (r_{n-k}(d) - j) = \begin{cases} \{r_{n-k}(d)\} & \text{if } r_{n-k}(\xi_k) < [r_{n-k}(d)] \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$\sum_{r_{n-k}(u)<j\le r_{n-k}(\xi_k)} (j+1-r_{n-k}(d)) = \begin{cases} 0 & \text{if } r_{n-k}(\xi_k) < [r_{n-k}(d)] \\ 1-\{r_{n-k}(d)\} & \text{otherwise.} \end{cases}$$

It is worth reminding that $\{r_{n-k}(d)\}$ denotes the fractional part of $r_{n-k}(d)$.

Now we may combine (3.16) and the latest estimates in the following way

$$\Delta_{k,n} = Kn^{-1}\psi(kn^{-1}, \ln s_{k-1})\sigma_{k,n} + O(n^{-3/2}) \tag{3.20}$$

where

$$\sigma_{k,n} = \begin{cases} (x-\eta_k)\{r_{n-k}(d)\} & \text{if } r_{n-k}(\xi_k) < [r_{n-k}(d)] \\ (\eta_k + y)(1 - \{r_{n-k}(d)\}) & \text{otherwise.} \end{cases}$$

For the sake of brevity put

$$p = \frac{y}{x+y}, \;\; R = x + y.$$

Then the inequality $r_{n-k}(\xi_k) < [r_{n-k}(d)]$ can be rewritten as

$$\eta_k > R(\{r_{n-k}(d)\} - p).$$

Therefore,

$$\sigma_{k,n} = \begin{cases} (x-\eta_k)\{r_{n-k}(d)\} & \text{if } \eta_k > R(\{r_{n-k}(d)\} - p) \\ (\eta_k + y)(1 - \{r_{n-k}(d)\}) & \text{otherwise.} \end{cases} \tag{3.21}$$

So, we obtained the desired representation of the "local" profit.

4 Proof of Theorem 2.1

Represent the total profit Δ_n as

$$\Delta_n = \sum_{1\le k<\delta n} \Delta_{k,n} + \sum_{\delta n\le k\le(1-\delta)n} \Delta_{k,n} + \sum_{(1-\delta)n\le k\le n} \Delta_{k,n} = \Delta_n' + \Delta_n'' + \Delta_n''' \tag{4.22}$$

and estimate the expectations $\mathrm{E}\Delta_n'$, $\mathrm{E}\Delta_n''$ and $\mathrm{E}\Delta_n'''$ one after another.

According to (3.20) we have

$$\mathrm{E}\Delta_n'' = Kn^{-1} \sum_{\delta n\le k\le(1-\delta)n} \mathrm{E}\psi(kn^{-1}, \ln s_{k-1,n})\sigma_{k,n} + c\theta n^{-1/2}.$$

Consider

$$A(u,v) = (x-v)u\chi(u,v) + (v+y)(1-u)(1-\chi(u,v)), \;\; (u,v) \in [0,1]\times[-y,x], \tag{4.23}$$

where

$$\chi(u,v) = \begin{cases} 1 & \text{if} \quad R(u-p) < v \le x, \;\; 0 \le u \le 1 \\ 0 & \text{if} \quad -y < v \le R(u-p), \;\; 0 \le u \le 1 \end{cases}$$

In view of (3.21) we have

$$\sigma_{k,n} = A(\{r_{n-k}(d)\}, \eta_k).$$

It is evident that $\chi(u,v)$ admits a monotone $\varepsilon-$approximation by means of $\chi_+(u,v)$ and $\chi_-(u,v)$ where

$$\chi_+(u,v) = \begin{cases} \frac{v-R(u-p)}{\varepsilon} + 1 & \text{if } R(u-p)-\varepsilon \le v \le R(u-p), \;\; 0 \le u \le 1 \\ 0 & \text{if } -y \le v \le R(u-p)-\varepsilon, \;\; 0 \le u \le 1 \\ 1 & \text{if } R(u-p) \le v \le x, \;\; 0 \le u \le 1 \end{cases}$$

and

$$\chi_-(u,v)=\begin{cases}\frac{v-R(u-p)}{\varepsilon} & \text{if } R(u-p)\le v\le R(u-p)+\varepsilon,\ 0\le u\le 1\\ 0 & \text{if } -y\le v\le R(u-p),\ 0\le u\le 1\\ 1 & \text{if } R(u-p)+\varepsilon\le v\le x,\ 0\le u\le 1.\end{cases}$$

Obviously, $\chi_\pm(u,v)$ are continuous in $[0,1]\times[-y,x]$ and

$$\chi_-(u,v)\le\chi(u,v)\le\chi_+(u,v).$$

Furthermore,

$$0\le\int\limits_{[0,1]\times[-y,x]}(\chi_+(u,v)-\chi_-(u,v))\mathrm{d}u\mathrm{d}F(v)\le\int\limits_{U_\varepsilon}\mathrm{d}u\mathrm{d}F(v)\le(2\varepsilon/R) \tag{4.24}$$

where

$$U_\varepsilon=((u,v):\ u\in(0,1),\ -y<v<x,\ |v-R(u-p)|\le\varepsilon).$$

Therefore

$$\mathrm{E}\psi(kn^{-1},\tfrac{\eta_1+\cdots+\eta_{k-1}}{\sqrt{n}}+h\tfrac{k-1}{n}+\ln s_0)A_-(\{r_{n-k}(d)\},\eta_k)\le$$

$$\mathrm{E}\psi(kn^{-1},\tfrac{\eta_1+\cdots+\eta_{k-1}}{\sqrt{n}}+h\tfrac{k-1}{n}+\ln s_0)\sigma_{k,n}=$$

$$\mathrm{E}\psi(kn^{-1},\tfrac{\eta_1+\cdots+\eta_{k-1}}{\sqrt{n}}+h\tfrac{k-1}{n}+\ln s_0)A(\{r_{n-k}(d)\},\eta_k)\le$$

$$\mathrm{E}\psi(kn^{-1},\tfrac{\eta_1+\cdots+\eta_{k-1}}{\sqrt{n}}+h\tfrac{k-1}{n}+\ln s_0)A_+(\{r_{n-k}(d)\},\eta_k)$$

where

$$A_\pm(u,v)=(x-y)u\chi_\pm(u,v)+(y+x)(1-u)(1-\chi_\mp(u,v)).$$

Obviously, the family $\psi(t,z)$, $\delta\le t\le1-\delta$, is contained in the class $\mathcal{G}$ defined in Appendix. So, we may apply Corollary 7.2.

By the corollary

$$\mathrm{E}\psi(kn^{-1},\tfrac{\eta_1+\cdots+\eta_{k-1}}{\sqrt{n}}+h\tfrac{k-1}{n}+\ln s_0)A_\pm(\{r_{n-k}(d)\},\eta_k)=$$

$$\mathrm{E}\psi(kn^{-1},\sigma\nu\sqrt{kn^{-1}}+hkn^{-1}+\ln s_0)\int\limits_{[0,1]\times[-y,x]}A_\pm(u,v)\mathrm{d}u\mathrm{d}F(v)+o(1)$$

uniformly in k, $\delta\le kn^{-1}\le1-\delta$. Here ν has the standard $(0,1)-$normal distribution and F is the distribution function of η.

In view of (4.24)

$$\int\limits_{[0,1]\times[-y,x]}A_\pm(u,v)\mathrm{d}u\mathrm{d}F(v)=\int\limits_{[0,1]\times[-y,x]}A(u,v)\mathrm{d}u\mathrm{d}F(v)+2\theta\varepsilon.$$

The straightforward calculations yield

$$a_F=\int\limits_{[0,1]\times[-y,x]}A(u,v)\mathrm{d}u\mathrm{d}F(v)=\frac{xy}{2(x+y)}\left(1-\frac{\sigma^2}{xy}\right).$$

Since ε is arbitrary we obtain

$$\mathrm{E}\psi(kn^{-1},\sigma\nu\sqrt{kn^{-1}}+hkn^{-1}+\ln s_0)\sigma_{k,n}=a_F\mathrm{E}\psi(kn^{-1},\sigma\nu\sqrt{kn^{-1}}+hkn^{-1}+\ln s_0)+o(1)$$

uniformly in k, $\delta \le kn^{-1} \le 1-\delta$.
Thus,

$$\mathrm{E}\Delta_n'' = Ka_F n^{-1} \sum_{\delta n \le k \le (1-\delta)n} \mathrm{E}\psi(kn^{-1}, \sigma\nu\sqrt{kn^{-1}} + hkn^{-1} + \ln s_0) + o(1).$$

Obviously,

$$I(t) = \mathrm{E}\psi(t, \sigma\nu\sqrt{t} + ht + \ln s_0) = \int \psi(t, \sigma v\sqrt{t} + ht + \ln s_0)\varphi(v)\mathrm{d}v$$

or after the straightforward calculations

$$I(t) = \frac{x+y}{\sqrt{t\sigma^2 + xy(1-t)}}\varphi\left(\frac{\ln(K/s_0) - ht + (1-t)(xy/2-\alpha)}{\sqrt{t\sigma^2 + xy(1-t)}}\right)$$

whence we deduce

$$\mathrm{E}\Delta_n'' = Ka_F \int_\delta^{1-\delta} I(t)\mathrm{d}t + o(1). \tag{4.25}$$

Now we are going to estimate $\mathrm{E}\Delta_n'''$. For the extreme "local" profit $\Delta_{n,n}$ we obtain

$$\Delta_{n,n} = \delta_{n,n} = (s_{n-1,n}d_n - K)_+ \frac{u_n - \xi_n}{u_n - d_n} + (s_{n-1,n}u_n - K)_+ \frac{\xi_n - d_n}{u_n - d_n} - (s_{n-1,n}\xi_{n.n} - K)_+$$

whence

$$\Delta_{n,n} = \begin{cases} 0 & \text{if} \quad s_{n-1,n}u_n \le K \text{ or } s_{n-1,n}d_n > K \\ \theta(s_{n-1,n}u_n - K) & \text{if} \quad K/u_n < s_{n-1,n} \le K/d_n. \end{cases}$$

Therefore,

$$\mathrm{E}\Delta_{n,n} \le K(u_n/d_n - 1) \le cn^{-1/2}.$$

For $m = n-k \ge 1$ in view of (3.16) – (3.18)

$$\Delta_{n-m,n} \le c\max_j b_{j,m}\left((u_n/d_n)^2 - 1\right)$$

or taking into account (2.10) and (7.36)

$$\Delta_{n-m,n} \le cm^{-1/2}n^{-1/2}.$$

Thus, for all sufficiently large n

$$\mathrm{E}\Delta_n''' \le c\delta^{1/2}. \tag{4.26}$$

Similarly,

$$\mathrm{E}\Delta_n' \le c\delta. \tag{4.27}$$

Since δ is arbitrary in view of (4.22), (4.25), (4.26) and (4.27) the theorem follows.

5 The limit distribution of the riskless profit

Consider the representation (4.22). From (3.20) it follows that

$$\Delta_n'' = Kn^{-1} \sum_{\delta n \le k \le (1-\delta)n} \psi(kn^{-1}, \ln s_{k-1,n})\sigma_{k,n} + O(n^{-1/2}).$$

Put

$$\Delta_n^* = n^{-1} \sum_{\delta n \le k \le (1-\delta)n} \psi(kn^{-1}, \ln s_{k-1,n})\sigma_{k,n}.$$

Then

$$\Delta''_n = K\Delta^*_n + O(n^{-1/2}).$$

Represent Δ^*_n as follows

$$\Delta^*_n = a_F n^{-1} \sum_{\delta n \le k \le (1-\delta)n} \psi(kn^{-1}, \ln s_{k-1,n}) + n^{-1} \sum_{\delta n \le k \le (1-\delta)n} \psi(kn^{-1}, \ln s_{k-1,n})(\sigma_{k,n} - a_F).$$

In view of (4.26) and (4.27)

$$n^{-1} \sum_{\delta n \le k \le (1-\delta)n} \psi(kn^{-1}, \ln s_{k-1,n}) = n^{-1} \sum_{1 \le k \le n} \psi(kn^{-1}, \ln s_{k-1,n}) + \theta c \delta^{1/2}.$$

Denote

$$l_n = n^{-1} \sum_{1 \le k \le n} \psi(kn^{-1}, \ln s_{k-1,n})$$

and

$$m_n = n^{-1} \sum_{\delta n \le k \le (1-\delta)n} \psi(kn^{-1}, \ln s_{k-1,n})(\sigma_{k,n} - a_F).$$

Then

$$\Delta''_n = K a_F l_n + m_n + O(n^{-1/2}). \tag{5.28}$$

We are going to prove that

$$\mathrm{E} l_n^2 \le c\delta. \tag{5.29}$$

In view of (4.26), (4.27), (5.28) and (5.29) it follows that for all sufficiently large n

$$\Delta_n = K a_F l_n + \omega(n, \delta)$$

where

$$\mathrm{E}|\omega(n,\delta)| \le c\delta^{1/2}.$$

Therefore, for any $\varepsilon > 0$ and $\delta' > 0$

$$\mathrm{P}(|\Delta_n - K a_F l_n| \ge \varepsilon) \le c\delta^{1/2}/\varepsilon \le \delta'$$

provided δ is sufficiently small. It implies that the limit distributions of Δ_n and $K a_F l_n$ coincide. So, it remains to establish the limit distribution for l_n and to prove (5.29). First, we tackle l_n. Consider the stochastic process

$$z_n(t) = \ln(s_{k,n}/s_0) = \frac{\eta_1 + \cdots + \eta_k}{\sqrt{n}} + hkn^{-1},\ (k-1)n^{-1} \le t < kn^{-1},\ k = 1, 2, \ldots, n.$$

It is well known that $z_n(t)$ weakly converges to $z(t) = ht + \sigma w(t)$ where $w(t)$ is the standard Wiener process (see, e.g., Billingsley (1968) Ch.3). It is easily seen that l_n is a continuous functional on $D[0,1]$. Note also that from (2.11) we obtain

$$\sup_z \psi(t,z) \le (2\pi(1-t))^{-1/2}.$$

Since $z_n(t)$ weakly converges to $z(t) = ht + \sigma w(t)$ we conclude that

$$l_n \xrightarrow{d} \int_0^1 \psi(t, z(t) + \ln s_0)\mathrm{d}t. \tag{5.30}$$

It remains to establish (5.29). It is easily seen that

$$\mathrm{E} m_n^2 = n^{-2} \sum_{\delta \le kn^{-1} \le 1-\delta} \mathrm{E}\psi^2(kn^{-1},\ \ln s_{k-1,n})(\sigma_{k,n} - a_F)^2 +$$

$$2n^{-2} \sum_{\delta n \le l < k \le n(1-\delta)} \mathrm{E}\psi(ln^{-1},\ \ln s_{l-1,n})(\sigma_{l,n} - a_F)\psi(kn^{-1},\ \ln s_{k-1,n})(\sigma_{k,n} - a_F) + \tag{5.31}$$

$$\Sigma_1 + 2\Sigma_2.$$

Obviously,

$$\Sigma_1 = O(n^{-1}). \tag{5.32}$$

Split Σ_2 in the following way

$$\Sigma_2 = n^{-2} \sum_{\delta n \le l < k \le n(1-\delta),\ k-l<\delta n} + n^{-2} \sum_{\delta n \le l < k \le n(1-\delta),\ k-l\ge\delta n} = \Sigma_{21} + \Sigma_{22}. \tag{5.33}$$

Obviously,

$$\Sigma_{21} \le c\delta. \tag{5.34}$$

So, it remains to estimate Σ_{22}. Utilizing (3.20) and (4.23) we obtain

$$M_{l,k,n} = \mathrm{E}\psi(ln^{-1},\ \ln s_{l-1,n})(\sigma_{l,n} - a_F)\psi(kn^{-1},\ \ln s_{k-1,n})(\sigma_{k,n} - a_F) =$$

$$\mathrm{E}\psi(ln^{-1},\ \ln s_{l-1,n})(A(\{r_{n-l}\},\eta_l) - a_F)\psi(kn^{-1},\ \ln s_{k-1,n})(A(\{r_{n-k}(d)\},\eta_k) - a_F).$$

Denote by

$$F_{m,n}(z) = \mathrm{P}(n^{-1/2}(\eta_1 + \cdots + \eta_m) + \ln s_0 < z).$$

Then we may represent $M_{l,k,n}$ as

$$M_{l,k,n} = \int_{I\!R^1\times[-x,x]} \psi(ln^{-1},\ z' + (l-1)n^{-1}\mu)(A(\{r_{n-l}(d)\},v') - a_F)\mathrm{d}F_{l-1,n}(z')\mathrm{d}F(v')\cdot$$

$$\int_{I\!R^1\times[-x,x]} \psi(kn^{-1},\ z' + z + n^{-1/2}v + (k-1)n^{-1}\mu)(A(\{r_{n-k}(d)\},v) - a_F)\mathrm{d}F_{k-l-1,n}(z)\mathrm{d}F(v).$$

From (3.1) it follows that given $n^{-1/2}(\eta_1 + \cdots + \eta_{l-1}) = z',\ \eta_l = v'$ we have

$$r_{n-k}(d) = a_{k,n} - \frac{\eta_1 + \cdots + \eta_{k-1}}{x+y} = a'_{k,n} - \frac{\eta_{l+1} + \cdots + \eta_{k-1}}{x+y} \overset{d}{=} a'_{k,n} - \frac{\eta_1 + \cdots + \eta_{k-l-1}}{x+y}.$$

Further, note that for $\delta \le kn^{-1} \le 1-\delta$ the functions $\psi(kn^{-1},\ z'+z),\ |z'| < Z$, belong to the class $\mathcal{G}$ for any $Z > 0$. Since $k - l \ge \delta n$ and $A(u,v)$ admits the monotone $\varepsilon-$approximation we may apply Corollary 7.2. Thus

$$\lim_{n\to\infty} \sup_{\delta n \le l < k \le n(1-\delta),\ k-l\ge\delta n} \sup_{|z'|<Z} \Big| \int_{I\!R^1\times[-y,x]} \psi(kn^{-1},\ z' + z + n^{-1/2}v +$$

$$(k-1)n^{-1}\mu)(A(\{r_{n-k}(d)\},v) - a_F)\mathrm{d}F_{k-l-1,n}(z)\mathrm{d}F(v) \Big| = 0.$$

It implies that

$$\Sigma_{22} \le \sup_{\delta n \le l < k \le n(1-\delta),\ k-l\ge\delta n} |M_{l,k,n}| = o(1). \tag{5.35}$$

Combining (5.31) – (5.35) yields (5.29) that completes the proof. Theorem is proved.

6 Concluding remarks

The incompleteness of a discrete time financial market leads to such a phenomenon that optimal strategy is not self-financing and the riskless profit arises. Mathematically, the riskless profit is a functional defined on the sample path of the risk price evolution. This is a quite general fact that takes place when the relative stock price jumps are bounded, i.e., $d \le \xi_k \le u,\ k = 1,\ldots,n$, while the pay-off function is convex whatever be the measure that governs the risk price evolution. The diffusion approximation studied here is based on the assumption that $\ln\xi_k,\ k = 1,\ldots,n$ are i.i.d. random variables. This assumption makes the scheme rather far from reality. However, even this simplest model highlights main features of the riskless profit. It turns out that such infinitesimal characteristic of the riskless profit as the the "local"profit admits a representation that contains a chaotic multiplier

$\sigma_{k,n}$. This multiplier arises because the derivative of the pay-off function that determines call option has a jump at $s = K$. The asymptotic analysis of such random variables requires special tools. The given in the Appendix Lemma 7.1 and its Corollary 7.2 give the impression on how to analyze the random variables of a chaotic nature.

The results presented in this paper should be regarded as the first step on the way to exhaustive analysis of much more realistic schemes. To the moment it is clear that the methods utilized here can be applied to much more general schemes. In particular, the case where relative stock price jumps are independent but non-identically distributed. Such a scheme enables us to take into account such a typical property of financial markets as the *volatility.* Another way of possible extension of our results provides the case where the stock price evolution is mixing or, in other words, possesses short range memory.

However, all such generalizations are still rather far for the conditions of the real financial practice. They do not take into account the transaction costs and the various restrictions that regulates the investor activity. Nevertheless, they highlight certain rather delicate sides of the financial practice. In this connection it should be of great interest to implement the retrospective analysis of the historical data from the view point of riskless profit. The authors is going to take part in such studies in the nearest future.

7 Appendix. Local limit theorems

Let $\eta, \eta_1, \eta_2, \dots$ be i.i.d. random variables such that

$$\mathrm{E}\eta = 0, \ \mathrm{Var}\ \eta = \sigma^2 < \infty.$$

Consider

$$\zeta_n = \eta_1 + \dots + \eta_n.$$

If the distribution of η is non-lattice then for any fixed y', y'', $0 < y' < y'' < \infty$

$$\sup_{x,\ y' \le y \le y''} |\mathrm{P}(x \le \zeta_n < x + y) - \frac{y}{\sigma\sqrt{n}}\varphi(\frac{x}{\sigma\sqrt{n}})| = o(n^{-1/2}). \tag{7.36}$$

This is a slightly generalized version of the Shepp local limit theorem (see, e.g., A. Nagaev (1973)).

Consider the sequence of the measures

$$Q_n(A) = \sigma\sqrt{2\pi n}\mathrm{P}(\zeta_n \in A).$$

The statement (7.36) implies that Q_n weakly converge to the Lebesgue measure that is for any continuous compactly supported function $g(u)$

$$\int g(u)Q_n(\mathrm{d}u) \to \int g(u)\mathrm{d}u. \tag{7.37}$$

Let $\mathcal{G}$ be the class of equicontinuous functions defined on $(-\infty, \infty)$ such that

$$\lim_{t\to\infty} \sup_{g\in\mathcal{G}} \int\limits_{|u|>t} |g(u)|\mathrm{d}u = 0.$$

It is easily seen that (7.37) holds uniformly in $G \in \mathcal{G}$. More precisely,

$$\lim_{n\to\infty} \sup_{g\in\mathcal{G}} |\int g(u)Q_n(\mathrm{d}u) - \int g(u)\mathrm{d}u| = 0. \tag{7.38}$$

Consider the family of the random variables $\tau_n(a) = \{\lambda\zeta_n + a\}$, $a \in I\!R^1$ where $\lambda \neq 1$ is constant. It is worth comparing the following statement with the basic result in S. V. Nagaev and Mukhin (1966).

Lemma 7.1 *If the distribution of* η *is non-lattice then for any fixed* u', u'', $0 < u' < u'' < 1$ *and* z', z'', $-\infty < z' < z'' < \infty$ *as* $n \to \infty$

$$\sup_a |\mathrm{P}(u' \le \tau_n(a) < u'',\ z' \le n^{-1/2}\zeta_n < z'') - (u'' - u')\left(\Phi(z''/\sigma) - \Phi(z'/\sigma)\right)| = o(1).$$

Proof. Let $k = k(a) = [a]$, $\theta = \theta(a) = \{a\}$. Suppose that $\lambda > 0$. It is easily seen that

$$P_n = \mathrm{P}(u' \le \tau_n(a) < u'',\ z' \le n^{-1/2}\zeta_n < z'') =$$

$$\sum_k \mathrm{P}(k + u' \le \lambda\zeta_n + a < k + u'',\ z'n^{1/2} \le \zeta_n < z''n^{1/2}) =$$

$$\sum_{k' \le k \le k''} \mathrm{P}(\tfrac{k+u'-\theta}{\lambda} \le \zeta_n < \tfrac{k+u''-\theta}{\lambda}) + \mathrm{P}(\tfrac{k''+u''-\theta}{\lambda} \le \zeta_n < z''n^{1/2}) +$$

$$\mathrm{P}(z'n^{1/2} \le \zeta_n < \tfrac{k'+u'-\theta}{\lambda})$$

where

$$k' = \min(k:\ \frac{k + u' - \theta}{\lambda} \ge z'n^{1/2}),\ k'' = \max(k:\ \frac{k + u'' - \theta}{\lambda} \le z''n^{1/2}).$$

According to (7.36)

$$P_n = \frac{u'' - u'}{\lambda\sigma\sqrt{n}} \sum_{k' \le k \le k''} \varphi(\frac{k}{\lambda\sigma\sqrt{n}}) + O(n^{-1/2}).$$

It remains to recall that

$$k' = z'\lambda\sqrt{n}(1 + o(1)),\ k'' = z''\lambda\sqrt{n}(1 + o(1)).$$

Lemma 7.1 has the following evident corollary (cf. (7.38)).

Corollary 7.2 *Let* $\chi(u, v)$ *be a bounded continuous function defined on* $[0, 1] \times I\!R^1$*. Under the conditions of Lemma 7.1*

$$\lim_{n\to\infty} \sup_{g\in\mathcal{G}} \sup_a |\mathrm{E}g(n^{-1/2}\zeta_n)\chi(\{\lambda\zeta_n + a\}, \eta_n) - \int g(\sigma z)\varphi(z)\mathrm{d}z \int_{[0,1]\times I\!R^1} \chi(u, v)\mathrm{d}u\mathrm{d}F(v)| = 0$$

where F *is the distribution function of* η.

Acknowledgments

The author is grateful to prof. Robert Kunst whose numerous comments and suggestions have helped to improve the presentation of this paper. The lemma is proved.

References

BILLINGSLEY P. (1968) *Convergence of Probability Measures.* Wiley, N-Y etc.

NAGAEV, A. V. (1973) Remarks on multidimensional local theorems. *Math. Notes* **14**, 4, 559–563 (in Russian).

NAGAEV, A. V. AND NAGAEV, S. A. (2002a) Two remarks on a discrete model of financial market. *Economics and Math. Methods* **38**, 3, 129–133 (in Russian).

NAGAEV, A. V. AND NAGAEV, S. A. (2002b) On the riskless profit of investor under smooth pay-off function. *Journal of Mathematical Sciences* (to appear)

NAGAEV, A. V. AND NAGAEV, S. A. (2003) Asymptotics of riskless profit under selling of discrete time call options. *Applicationes Mathematicae* **30**, 2, 173–191.

NAGAEV S. V. AND MUKHIN A. B (1966) A case of convergence to the uniform distribution on a segment. In: Limit Theorems and Statistical Inference. Fan, Tashkent, 113–117 (in Russian).

PETROV, V. V. (1975) *Sums of Independent Variables,* Springer, Berlin etc.

SHIRYAEV, A. N. (1999) *Essentials of Stochastic Finance.* World Scientific Publ., Singapore.

SHIRYAEV, A. N., KABANOV, YU. M., KRAMKOV, O.D. AND MEL'NIKOV, A. V. (1994) Towards the theory of of pricing of options of both European and American types. II. Continuous time. *Theory of Probability and its Applications (Special issue)* **39**, 1, 61–102.

TESSITORE, G. AND ZABCZYK, J. (1996) Pricing options for multinomial models. Bull. Polish Acad. Sci. **44**, 3, 363–380.

ZABCZYK, J. (1996) Pricing options by dynamic programming. *In: Stochastic Processes and Related Topics, Ed. H.J. Engelbert, H. Föllmer, J. Zabczyk, Stochastic Monographs Vol. 10, Gordon and Breach Publ.*, 55–72.

MOTOCZYŃSKI, M. AND STETTNER, L. (1998) On option pricing in the multinomial Cox-Ross-Rubinstein model. *Applications Math.* **25**, 1, 55–72.

Author: Sergei A. Nagaev

Title: A Diffusion Approximation for the Riskless Profit under Selling of Discrete Time Call Options

Reihe Ökonomie / Economics Series 137

ISSN: 1605-7996

Stumpergasse 56, A-1060 Vienna • ☎ +43 1 59991-0 • Fax +43 1 59991-555 • http://www.ihs.ac.at

ISSN: 1605-7996