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Abstract

In the past several decades, new-product diffusion models has been an active area of research in marketing (see, e.g., Mahajan, Muller, and Wind 2000, and Mahajan and Wind 1986). Such models are useful because they can provide important insights into the timing of initial purchase of new products by consumers. Much of the work in this area has been spawned by a seminal paper of Bass (1969), in which it was postulated that the trajectory of cumulative adoptions of a new product follows a *deterministic* function whose instantaneous growth rate depends on two parameters, one of which captures a consumer's intrinsic tendency to purchase, independent of the number of previous adopters, and the other captures a positive force of influence on a consumer by previous adopters. While Bass's model, or the *Bass Model* (BM), yields an S-shaped cumulative-adoptions curve that has proven to provide excellent empirical fit for a wide range of new-product-adoptions data sets (especially for consumer durables), there also has been a common belief (see, e.g., Eliashberg and Chatterjee 1986) that it would be of interest to have an appropriate *stochastic* version of his model. The purpose of this paper is to formulate and study a stochastic counterpart of the BM. Inspired by a very early paper of Taga and Isii (1959), we formulate the trajectory of cumulative number of adoptions as a pure birth process with a set of state-dependent birth rates that are judiciously chosen to closely parallel the roles played by the two parameters in the deterministic BM. We demonstrate that with our choice of birth rates, the resulting pure birth process exhibits characteristics that resemble those in the BM. In particular, we show that the fraction of individuals who have adopted the product by time t in our formulation agrees with (converges in probability to) the corresponding deterministic fraction in a BM with the same pair of parameters, when the total number of consumers in the target population approaches infinity. Our formulation, therefore, supports and expands the BM by having explicit micro-level stochastic interactions amongst individual adopters.

PURE BIRTH PROCESSES; DIFFUSION MODELS; NEW-PRODUCT ADOPTIONS; EPIDEMICS

1 Introduction

It is well known (see, e.g., Mahajan, Muller, and Wind 2000, Mahajan and Wind 1986, and Rogers 1995) that for a large variety of new products, the Bass model (Bass 1969) describes the empirical cumulative-adoptions curve extremely well. The *Bass Model* (BM) assumes that the instantaneous rate of adoption of a new product (or technology) at any time epoch depends on two forces, one is an intrinsic tendency for an individual (given that the individual has not yet adopted) to make a purchase, independent of the number of previous adopters in the target population, and the other is a positive influence by previous adopters on the remaining individuals in the population (via, e.g., word of mouth).

The mathematical formulation of the BM is as follows. Let p and q be two parameters that represent the extent of the above-mentioned two forces, let m be the size of a target population, and let $N(t)$ be the cumulative number of adopters of a new product by time t. Then, under the assumption that $N(t)$ is a continuous function with $N(0) = 0$, Bass postulates (Bass 1969, p. 217) that the following differential equation holds:

$$\frac{dN(t)}{dt} = [m - N(t)] \left[p + \frac{q}{m} N(t)\right], \qquad t \geq 0\,. \tag{1}$$

That is, the growth rate of $N(t)$ at time t is equal to the product of $m - N(t)$ and $p + (q/m)N(t)$, where $m - N(t)$ is the size of the remaining population and $p + (q/m)N(t)$ is the instantaneous adoption rate of every individual in the remaining population.

Notice that if we let $F(t)$ be the (continuous) fraction of individuals who have adopted the product by time t, i.e., let

$$F(t) \equiv \frac{N(t)}{m}\,, \tag{2}$$

then, equation (1) has the following equivalent form:

$$\frac{f(t)}{1 - F(t)} = p + qF(t), \qquad t \geq 0\,, \tag{3}$$

where $f(t)$ denotes the derivative of $F(t)$. When m is large, the fraction $F(t)$ as defined in (2) can intuitively be thought of as the "probability" for a randomly-selected individual in the target population to have adopted the product by time t. With this language, the left-hand side of (3) is the failure-rate (or hazard-rate) function associated with the "distribution" function F; and equation (3) says that the failure-rate function of F equals $p + qF(t)$ at time t.

In Bass (1969), it was shown that the solution of (3) is given by

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \dfrac{q}{p} e^{-(p+q)t}}\,, \qquad t \geq 0\,; \tag{4}$$

and that this S-shaped solution provides excellent empirical fit for the timing of initial purchase for a wide range of consumer durables.

Bass (1969) referred to the parameter p as the "coefficient of innovation" and the parameter q as the "coefficient of imitation." His terminology was motivated by the following behavioral rationale:

> "Initial purchases of the product are made by *both* "innovators" and "imitators," the important distinction between an innovator and an imitator being the buying influence. Innovators are not influenced in the timing of their initial purchase by the number of people who have already bought the product, while imitators are influenced by the number of previous buyers. Imitators "learn," in some sense, from those who have already bought."

Other researchers have referred to p as the coefficient of "external influence" and q as the coefficient of "internal influence." Thus, one can also interpret p and q as the respective intensities of the transmission of information from an external source (or via an external broadcast) and between any given pair of individuals within a target population.

Since the publication of Bass's paper more than three decades ago, the solution (4), or its derivative,

$$f(t) = \frac{\dfrac{(p+q)^2}{p} e^{-(p+q)t}}{\left(1 + \dfrac{q}{p} e^{-(p+q)t}\right)^2}\,,$$

has been used extensively to forecast the growth of sales volume of new products over time. In such applications, it is important to develop good estimates for the parameters p and q from historical data (see, e.g., Putsis and Srinivasan 2000, Section 11.2.1, for a review). A standard framework for this purpose is to conduct a regression analysis (ordinary least squares or nonlinear least squares) based on the assumption that the actual sales in successive time intervals can be modeled as the sum of two independent components: the (discretized) adoption-rate curve, $mf(t)$, and a sequence of independent and identically distributed (i.i.d.) error terms. Clearly, the adoption of such a framework can be attributed to the fact that the BM assumes that $N(t)$ is a deterministic function. In other words, while the BM is parsimonious, the assumption of a deterministic $N(t)$ effectively forces one to model deviations of the actual sales data from the adoption-rate curve as manifestations of the presence of independent random errors, as opposed to being a consequence of the underlying stochastic nature of the forces behind successive adoptions. This observation suggests that it would be of interest to have a stochastic version of the BM in which $\{N(t),\, t \geq 0\}$ is assumed to be a stochastic process.

Interestingly, in 1959, a decade prior to Bass's work, Taga and Isii (1959) had introduced a stochastic model to study the pattern of communication between an information source and individuals within a social group. Specifically, Taga and Isii assume that transmissions of a given piece of information can take place either directly from the source to an individual or between individuals within the group; and that the growth of the number of individuals who have received the information follows a pure birth process with a set of state-dependent birth rates that are functions of two parameters that correspond to these two modes of information transmission. Observe that while the intended application context is different, the stochastic assumptions in Taga and Isii's model are remarkably similar in spirit to the deterministic ones in the BM.

In fact, in addition to Taga and Isii's work, there exists a large body of stochastic models of diffusion in the theory of epidemics (see Bailey 1957; Bartholomew 1982, Chapters 9 and 10; and Bartlett 1960). For detailed discussions of the stochastic diffusion literature, we refer the reader to Bailey (1975) and to Eliashberg and Chatterjee (1986).

The purpose of this paper is to formulate and study a stochastic counterpart of Bass's new-product diffusion model. Our formulation, which we refer to as the *Stochastic Bass Model* (SBM), is based on Taga and Isii (1959). We will, however, define a slightly different set of birth rates, one that closely parallels the manner in which the two forces are captured in the deterministic BM. Our primary aim is to prove that the fraction of individuals who have adopted the product by time t in a SBM agrees with (converges in probability to) the solution $F(t)$ of a corresponding BM with the same pair of parameters, when the size of the population approaches infinity. This asymptotic agreement shows that the family of SBMs supports and expands the BM in the sense of having explicit micro-level stochastic interactions amongst individual adopters (see, e.g., Roberts and Lattin 2000).

One potential application of our stochastic formulation of the BM is that it can serve as a starting basis for empirical studies of new-product diffusions. Work in this direction is currently in progress and will be reported in a subsequent paper.

The outline of the rest of this paper is as follows. In Section 2, we present the formulation of the SBM. In Section 3, we summarize our results; the basic theme is to describe properties of the SBM that constitute counterparts to those in the deterministic BM. Finally, in Section 4, we provide detailed proofs.

2 The Stochastic Bass Model

Consider a product that has a potential market size of m individuals. We assume that each individual in this potential market, which will be referred to as the target population, will eventually adopt (or purchase) exactly one unit of the product. The timing of this adoption is, however, uncertain. Let $A_m(t)$ be the cumulative number of adoptions by time t, with $A_m(0) \equiv 0$. Following Taga and Isii (1959), we assume that the cumulative-adoptions process $\{A_m(t),\, t \geq 0\}$ is a pure birth process. Our specific assumptions on the birth rates are described as follows.

If an individual has not yet adopted the product by time t, then we assume that the "intrinsic" probability for this individual to adopt the product during the time interval $(t,\, t+h)$ is (independently of everything else) given by

$$\alpha h + o(h)\,. \tag{5}$$

(A function $g(h)$ is said to be $o(h)$ if the ratio $g(h)/h$ converges to zero as h goes to zero.) If, on the other hand, an individual has already adopted the product by time t, then we assume that the probability for this individual to "induce" any other member of the remaining population at time t to adopt in $(t,\, t+h)$ is (independently of everything else) given by

$$\frac{\beta}{m-1}\, h + o(h)\,. \tag{6}$$

(If $m = 1$, we define $\beta/(m-1)$ as 0.) Thus, each individual in the target population has an *intrinsic adoption rate* and an *induction rate*, given by α and β respectively; moreover, the induction rate β associated with each individual is apportioned uniformly to all other members ($m-1$ in number) of the population. The parameters α and β correspond conceptually to the parameters p and q in the original BM.

Suppose $A_m(t) = j$, where $0 \leq j \leq m-1$. Then, according to (5) and (6), the probability for any individual in the remaining population at time t to adopt the product in $(t,\, t+h)$ is given by $[\alpha + j\beta/(m-1)]h + o(h)$. Since the size of this remaining population equals $m - j$, the probability for $A_m(t)$ to increase to $j+1$ (from j) in $(t,\, t+h)$ is given by $\lambda_{mj}h + o(h)$, where

$$\lambda_{mj} \equiv (m-j)\left(\alpha + \frac{\beta}{m-1}\, j\right), \qquad j = 0, 1, \ldots, m-1\,. \tag{7}$$

Since the growth of $A_m(t)$ stops upon reaching level m, it follows that the range for j in (7) can be extended to cover the case $j = m$ as well. We will refer to λ_{mj} as the birth (or diffusion) rate at state j, and the resulting pure birth process $\{A_m(t),\, t \geq 0\}$ with state-dependent birth rates $\{\lambda_{mj}\}_{j=0}^{m}$ as the *Stochastic Bass Model*.

We conclude this section by noting that the difference between the SBM and Taga and Isii's original formulation is that in the latter, the probability in (6) is defined as $\beta h + o(h)$ (Taga and Isii 1959, pp. 27–28). The apportionment, or scaling, of β in the SBM parallels the term q/m in (1) (apart from using $m-1$ in place of m); and it ensures that the total potential influence by any single individual on the rest of the population does not grow without bound as m increases to infinity.

3 Summary of Results

Clearly, the variable $A_m(t)$ is the stochastic counterpart to $N(t)$ in the BM. We say that the process $\{A_m(t),\, t \geq 0\}$ is in state j at time t if $A_m(t) = j$. The first question of interest is: What is the state distribution of $\{A_m(t),\, t \geq 0\}$ at time t? For pure birth processes in general, explicit formulas for the state distribution can be found in Bartlett (1955, Section 3.2), Taga and Isii (1959, p. 28), Bartholomew (1982, p. 252), or Ross (2000, p. 324). In terms of our notation, these formulas are:

$$P\{A_m(t) = j\} = \frac{1}{\lambda_{mj}} \sum_{i=0}^{j} c_{m;ij} \lambda_{mi} e^{-\lambda_{mi} t}, \qquad 0 \leq j \leq m-1\,, \tag{8}$$

where

$$c_{m;ij} \equiv \prod_{\substack{k=0 \\ k \neq i}}^{j} \frac{\lambda_{mk}}{\lambda_{mk} - \lambda_{mi}}, \qquad 0 \leq i \leq j \leq m-1\,. \tag{9}$$

We note that for the $c_{m;ij}$s in (9) to be well defined, it is necessary that $\lambda_{mi} \neq \lambda_{mj}$ whenever $i \neq j$. If $\lambda_{mi} = \lambda_{mj}$ for some $i \neq j$, then (8) requires a modification. Mechanically, the L'Hôpital's rule can be applied for this purpose. We will leave out this nonessential complication and assume similar qualifications without further comment for other related formulas below.

The state distribution (8) can, in principle, serve as the starting point for the calculation of many other characteristics of interest (e.g., moments) for the process $\{A_m(t),\, t \geq 0\}$. For its basic relevance and for self-containedness, we will sketch the standard proof of (8) in Section 4.1.

Denote by $\eta_m(t)$ the expected total number of adoptions by time t; that is, let $\eta_m(t) \equiv E[A_m(t)]$. The function $\eta_m(t)$ is the expected-value counterpart to the cumulative-adoptions curve $N(t)$ (or $mF(t)$) in the BM. In Section 4.2, we show that a formula for $\eta_m(t)$, stated next, can be derived easily from (8).

Theorem 1 *The expected total number of adoptions by time t in the SBM is given by:*

$$\eta_m(t) = m\left(1 - \sum_{i=0}^{m-1} a_{mi} e^{-\lambda_{mi} t}\right), \qquad t \geq 0\,, \tag{10}$$

where

$$a_{mi} \equiv \frac{1}{m}\sum_{j=i}^{m-1} c_{m;ij}\,, \qquad 0 \leq i \leq m-1\,. \tag{11}$$

Consider a randomly-selected individual in the target population, and refer to this individual as the *tagged individual.* Denote by $F_m(t)$ the distribution of the adoption time of the tagged individual; then, $F_m(t)$ can be taken as a counterpart (for another counterpart, see (17) below) to the fraction $F(t)$ in the BM. Observe that if $A_m(t) = k$, where $0 \leq k \leq m$, then the conditional probability for the tagged individual to have adopted the product by time t equals k/m. This observation immediately yields

$$F_m(t) = E\left[\frac{A_m(t)}{m}\right] = \frac{\eta_m(t)}{m}\,, \tag{12}$$

a basic relation that parallels (2).

One consequence of (12) is that the distribution $F_m(t)$ can be derived from the expectation $\eta_m(t)$. Thus, from (10) and (12), we immediately have the following result.

Theorem 2 *The distribution of time to adoption of a randomly-selected individual in the SBM is given by:*

$$F_m(t) = 1 - \sum_{i=0}^{m-1} a_{mi} e^{-\lambda_{mi} t}\,, \qquad t \geq 0\,. \tag{13}$$

Another observation regarding (12) is that it can be written in the form:

$$F_m(t) = \int_0^t \frac{1}{m}\, d\eta_m(y)\,,$$

which implies that we can interpret $d\eta_m(y)$ as the probability for having an adoption in the time interval $(y, y + dy)$ (this corresponds to the notion of "renewal" density in classical renewal theory; see Ross 1996, p. 114, Remark (2)) and $1/m$ as the probability for the tagged individual to be responsible for this adoption. It follows that

$$\eta'_m(t) \equiv \frac{d}{dt}\,\eta_m(t)$$

is the instantaneous *population* adoption rate at time t, and

$$f_m(t) \equiv \frac{\eta'_m(t)}{m} = \frac{d}{dt}\,F_m(t) \tag{14}$$

is the corresponding *individual* adoption rate. In other words, the functions $\eta'_m(t)$ and $f_m(t)$ constitute the counterparts to $mf(t)$ and $f(t)$, respectively, in the BM.

Theorem 2 (together with (11) and (9)) can be used to derive expressions for the F_ms that are explicitly in terms of the original parameters α and β. As examples, it can be shown (details omitted) that for $m = 1$ to 4, we have:

$$F_1(t) = 1 - e^{-\alpha t},$$

$$F_2(t) = 1 - \frac{-\beta}{\alpha - \beta} e^{-2\alpha t} - \frac{\alpha}{\alpha - \beta} e^{-(\alpha+\beta)t},$$

$$F_3(t) = 1 - \frac{\beta^2}{(2\alpha - \beta)(\alpha - \beta)} e^{-3\alpha t} - \frac{-\beta}{\alpha - \beta} e^{-2(\alpha+\beta/2)t} - \frac{2\alpha + \beta}{2\alpha - \beta} e^{-(\alpha+\beta)t},$$

and

$$F_4(t) = 1 - \frac{-2\beta^3}{(\alpha - \beta)(3\alpha - \beta)(3\alpha - 2\beta)} e^{-4\alpha t} - \frac{2\beta^2}{(\alpha - \beta)(3\alpha - \beta)} e^{-3(\alpha+\beta/3)t} - \frac{-9\beta}{(3\alpha - 2\beta)(3\alpha - \beta)} e^{-2(\alpha+2\beta/3)t} - \frac{3\alpha + 2\beta}{3\alpha - \beta} e^{-(\alpha+\beta)t}.$$

In principle, the above calculations can be executed up to any m. Observe, however, that the algebra quickly becomes extremely complicated as m increases. Despite this complexity, we prove in Section 4.3 that the sequence of F_ms converges; and this result is stated in the following theorem.

Theorem 3 *The distribution of time to adoption for a randomly-selected individual in the SMB agrees in the limit with the solution (4) in a BM with parameters $p = \alpha$ and $q = \beta$, when the size of the target population in the SBM approaches infinity. That is,*

$$\lim_{m\to\infty} F_m(t) = F_\infty(t)\,, \qquad t \geq 0\,,$$

where

$$F_\infty(t) \equiv \frac{1 - e^{-(\alpha+\beta)t}}{1 + \dfrac{\beta}{\alpha}e^{-(\alpha+\beta)t}}\,. \tag{15}$$

Denote by $\delta_m(t)$ the variance of the total number of adoptions by time t; that is, let $\delta_m(t) \equiv Var[A_m(t)]$. Clearly, this is one important characteristic (see, e.g., Cohen, Ho, and Matsuo 2000, p. 245) of the SMB that does not have a counterpart in the BM. From (8), it immediately follows that $\delta_1(t) = F_1(t)[1 - F_1(t)]$. For $m \geq 2$, we derive in Section 4.4 the following formula for $\delta_m(t)$.

Theorem 4 *For $m \geq 2$, the variance of the total number of adoptions by time t in the SBM is given by:*

$$\begin{aligned}\delta_m(t) = {} & \frac{m^2}{\beta}\left\{[1 - F_m(t)][\alpha + \beta F_m(t)] - f_m(t)\right\} \\ & + \frac{m}{\beta}\left\{f_m(t) - \alpha\,[1 - F_m(t)]\right\}, \qquad\qquad t \geq 0\,.\end{aligned} \tag{16}$$

Finally, define

$$B_m(t) \equiv \frac{A_m(t)}{m}\,; \tag{17}$$

that is, let $B_m(t)$ be the fraction of individuals who have adopted the product by time t in the SBM. Observe that in contrast with $F_m(t)$, which is an expectation (see (12)), the fraction $B_m(t)$ is the random-variable counterpart to $N(t)/m$ in the BM. In Section 4.5, we show that Theorem 3 can be strengthened to the following "weak law" for $B_m(t)$.

Theorem 5 *For any $\epsilon > 0$, we have*

$$\lim_{m\to\infty} P\{|B_m(t) - F_\infty(t)| > \epsilon\} = 0\,, \qquad t \geq 0\,. \tag{18}$$

That is, for every $t \geq 0$, the sequence of random variables $B_m(t)$ converges in probability to the constant $F_\infty(t)$ as $m \to \infty$.

If we interpret (3) as

$$\frac{f_\infty(t)}{1 - F_\infty(t)} = \alpha + \beta F_\infty(t), \qquad t \geq 0\,, \tag{19}$$

then Theorem 5 says that (3) can be viewed as an attempt at a "direct formulation" of the limiting trajectory of $B_m(t)$ in a family of SBMs indexed by m. It is in this sense that the family of SBMs supports and expands the BM.

4 Proofs

The pure birth process $\{A_m(t),\, t \geq 0\}$ can also be defined by specifying a sequence of inter-adoption times as follows. For $j = 1, 2, \ldots, m$, denote by X_{mj} the jth inter-adoption time. Then, it is well known (see, e.g., Ross 2000, pp. 323–324 and pp. 330–331) that the model specification in Section 2 is tantamount to the assumption that $\{X_{mj},\, j = 1, 2, \cdots, m\}$ is a sequence of independent exponential random variables with parameters $\{\lambda_{mj}\}_{j=0}^{m-1}$. In other words, for $1 \leq j \leq m$, the inter-adoption time X_{mj} has density $\lambda_{m,j-1}e^{-\lambda_{m,j-1}t}$, $t \geq 0$; and the X_{mj}s are independent.

For $j = 1, 2, \ldots, m$, denote by A_{mj} the jth adoption epoch. We will next derive the density function of A_{mj}, which we denote by $f_{mj}(t)$. Clearly, we have

$$A_{mj} = \sum_{i=1}^{j} X_{mi}\,; \tag{20}$$

therefore,

$$f_{mj}(t) = \frac{d}{dt} P\{X_{m1} + \cdots + X_{mj} \leq t\}, \qquad j = 1, 2, \ldots, m\,.$$

By conditioning on $X_{m1} + \cdots + X_{m,j-1}$, we obtain

$$\begin{aligned} f_{mj}(t) &= \int_0^t \lambda_{m,j-1}e^{-\lambda_{m,j-1}(t-y)} dP\{X_{m1} + \cdots + X_{m,j-1} \leq y\} \\ &= \int_0^t \lambda_{m,j-1}e^{-\lambda_{m,j-1}(t-y)} f_{m,j-1}(y)\, dy\,; \end{aligned} \tag{21}$$

and repeated applications of this recursion, starting with the initial condition $f_{m1}(t) = \lambda_{m0}e^{-\lambda_{m0}t}$, lead to (see, e.g., Ross 2000, pp. 253–255)

$$f_{mj}(t) = \sum_{i=0}^{j-1} c_{m;i,j-1}\lambda_{mi}e^{-\lambda_{mi}t}, \tag{22}$$

where the $c_{m;i,j-1}$s are defined by (9).

We are now ready for the proofs of (8) and Theorem 1.

4.1 Proof of (8) The distribution of $A_m(t)$ can be linked to the $f_{mj}(t)$s via the following simple relation:

$$f_{m,j+1}(t) = P\{A_m(t) = j\}\,\lambda_{mj}, \qquad j = 0, 1, , \ldots, m-1\,. \tag{23}$$

To see this, note that

$$P\{A_m(t) = j\} = \int_0^t e^{-\lambda_{mj}(t-y)} P\{A_m(y) = j-1\}\lambda_{m,j-1}\,dy\,, \tag{24}$$

which follows by observing that $P\{A_m(y) = j-1\}\lambda_{m,j-1}dy$ is the probability for the jth adoption to occur in the (infinitesimal) time interval $(y,\, y+dy)$ (i.e., for $A_m(y) = j-1$ and $A_m(y+dy) = j$) and $e^{-\lambda_{mj}(t-y)}$ is the (conditional) probability for the $(j+1)$th adoption not to occur during $(y,\, t]$. Comparison of (21) and (24) then establishes (23).

To complete the proof, we rewrite (23) as $P\{A_m(t) = j\} = f_{m,j+1}(t)/\lambda_{mj}$, which, upon substitution of (22), yields (8). □

4.2 Proof of Theorem 1 Denote by $I_{mj}(t)$ the indicator function of the event that the jth adoption occurs no later than time t (i.e., of the event $\{A_{mj} \leq t\}$); then,

$$A_m(t) = \sum_{j=1}^{m} I_{mj}(t)\,.$$

It follows that

$$\eta_m(t) = E\left[\sum_{j=1}^{m} I_{mj}(t)\right] = \sum_{j=1}^{m} P\{A_{mj} \leq t\}\,. \tag{25}$$

From (22), we have

$$\begin{aligned}P\{A_{mj} \le t\} &= \int_0^t f_{mj}(y)\,dy \\ &= \sum_{i=0}^{j-1} c_{m;i,j-1}\left(1 - e^{-\lambda_{mi}t}\right) \\ &= 1 - \sum_{i=0}^{j-1} c_{m;i,j-1}e^{-\lambda_{mi}t}, \end{aligned} \tag{26}$$

where the last equality is due to the fact that $\lim_{t\to\infty} P\{A_{mj} \le t\} = 1$. Now, upon substitution of (26), (25) evaluates to

$$\begin{aligned}\eta_m(t) &= \sum_{j=1}^{m}\left(1 - \sum_{i=0}^{j-1} c_{m;i,j-1}e^{-\lambda_{mi}t}\right) \\ &= m - \sum_{j=1}^{m}\sum_{i=0}^{j-1} c_{m;i,j-1}e^{-\lambda_{mi}t} \\ &= m\left[1 - \sum_{i=0}^{m-1}\left(\frac{1}{m}\sum_{j=i}^{m-1} c_{m;ij}\right)e^{-\lambda_{mi}t}\right];\end{aligned}$$

and this completes the proof. □

We will next establish several preliminary lemmas that are needed for the proofs of Theorems 3–5.

Let T_{mj} be the adoption time of the jth individual. Note that T_{mj} is not the same as A_{mj}, the time of the jth adoption. In the next lemma, we relate the T_{mj}s to the A_{mj}s.

Lemma 1 *Let $\pi_1, \pi_2, \ldots, \pi_m$ be a random permutation of $1, 2, \ldots, m$; then,*

$$(T_{m1}, T_{m2}, \ldots, T_{mm}) =^d (A_{m\pi_1}, A_{m\pi_2}, \ldots, A_{m\pi_m}), \tag{27}$$

where $=^d$ denotes equality in distribution.

Proof This is clearly a consequence of symmetry. Formally, observe that the rates α, $\beta/(m-1)$, and λ_{mj} in (5), (6), and (7) are, at any time epoch t, symmetric with respect to the pool of remaining individuals in the target population. Therefore, from a standard property of the exponential distribution (see, e.g., Ross 2000, p. 249, equation (5.6), and/or p. 296, Exercise 10), we can reinterpret the inter-adoption time X_{mj}, for any j, as the minimum of $m-j+1$ competing i.i.d. exponential random variables. This implies that the identity of the individual who adopts at time A_{m1} is equally likely to be any one of the m individuals in the target population. Similarly, at time A_{m2}, regardless of who was responsible for the adoption at time A_{m1}, each of the remaining $m-1$ individuals has an equal probability of generating the second adoption. Continuation of this argument now shows that the vector $(T_{m1}, T_{m2}, \ldots, T_{mm})$ is stochastically identical to one that is obtained from $(A_{m1}, A_{m2}, \ldots, A_{mm})$ by a random permutation of its components; and this establishes (27). □

Recall that a random vector is said to be *exchangeable* if all vectors obtained from it by permuting its components have the same joint distribution. It follows from Lemma 1 that the vector $(T_{m1}, T_{m2}, \ldots, T_{mm})$ is exchangeable, and hence that the T_{mj}s (while being dependent) are identically distributed. Moreover, the adoption time of the tagged individual is distributed as A_{mJ_m}, where J_m is a random index distributed uniformly over 1, 2, …, m (independently of the A_{mj}s). In other words, we have

$$\begin{aligned} F_m(t) &= \sum_{j=1}^{m} P\{A_{mJ_m} \le t \mid J_m = j\} P\{J_m = j\} \\ &= \frac{1}{m} \sum_{j=1}^{m} P\{A_{mj} \le t\}\,. \end{aligned} \tag{28}$$

While the relation (28) is exact, its evaluation is extremely complicated. The key idea in our proof of the convergence of the F_ms is to develop a family of tight upper bounds for the F_ms. The starting point is the following lemma.

Lemma 2 *For $m = 1$, we have*

$$\frac{f_1(t)}{1 - F_1(t)} = \alpha\,, \qquad t \ge 0\,;$$

and for $m \ge 2$, we have

$$\frac{f_m(t)}{1 - F_m(t)} \le \alpha + \beta_m F_m(t)\,, \qquad t \ge 0\,, \tag{29}$$

where

$$\beta_m \equiv \frac{m\beta}{m-1}\,. \tag{30}$$

Proof The statement for $m = 1$ is an immediate consequence of the standard fact that an exponential random variable with parameter α has constant failure rate α.

We now assume $m \geq 2$. Observe that with $A_m(t)$ replacing j in (7), we have

$$\lambda_{mA_m(t)} = [m - A_m(t)] \left[\alpha + \frac{\beta}{m-1} A_m(t)\right]; \tag{31}$$

and upon taking expectations and dividing by m, this becomes

$$\frac{E\left[\lambda_{mA_m(t)}\right]}{m} = \alpha\, \frac{E\left[m - A_m(t)\right]}{m} + \beta\, \frac{E\{[m - A_m(t)]\, A_m(t)\}}{m(m-1)}\,. \tag{32}$$

By conditioning on $A_m(t)$, we have

$$\begin{aligned} \frac{E\left[\lambda_{mA_m(t)}\right]}{m} &= \frac{1}{m} \sum_{k=0}^{m-1} \lambda_{mk}\, P\left\{A_m(t) = k\right\} \\ &= \frac{1}{m} \sum_{j=1}^{m} f_{mj}(t) \\ &= f_m(t)\,, \end{aligned} \tag{33}$$

where the second equality is due to (23) and the third equality is due to (28) and (14). Next, from (12), we have

$$\frac{E\left[m - A_m(t)\right]}{m} = 1 - \frac{E\left[A_m(t)\right]}{m} = 1 - F_m(t)\,. \tag{34}$$

Finally, since $E\{[A_m(t)]^2\} \geq \{E[A_m(t)]\}^2$, we have

$$E\{[m - A_m(t)]\, A_m(t)\} \leq E\left[m - A_m(t)\right] E\left[A_m(t)\right]\,;$$

and therefore,

$$\begin{aligned} \frac{E\{[m - A_m(t)]\, A_m(t)\}}{m(m-1)} &\leq \frac{m}{m-1}\, \frac{E\left[m - A_m(t)\right]}{m}\, \frac{E\left[A_m(t)\right]}{m} \\ &= \frac{m}{m-1}\, \left[1 - F_m(t)\right] F_m(t)\,. \end{aligned} \tag{35}$$

Substitution of (33), (34), and (35) into (32) now leads to (29), and the proof is complete. □

Observe that if we replace the inequality in (29) with an equality, then we will have a corresponding differential equation that is of the same form as (3). This observation naturally suggests that we compare F_m against the solution of this corresponding differential equation.

Formally, we define, for every $m \geq 2$, a BM with parameters $p = \alpha$ and $q = \beta_m$; and denote by $G_m(t)$ the fraction of individuals who have adopted by time t in this model. Then, according to (3), the G_ms satisfy

$$\frac{g_m(t)}{1 - G_m(t)} = \alpha + \beta_m G_m(t)\,, \qquad t \geq 0\,,$$

where $g_m(t)$ denotes the derivative of $G_m(t)$; moreover, in light of (4), we have

$$G_m(t) = \frac{1 - e^{-(\alpha+\beta_m)t}}{1 + \dfrac{\beta_m}{\alpha} e^{-(\alpha+\beta_m)t}}\,. \tag{36}$$

To have full correspondence between the G_ms and the F_ms, we further define

$$G_1(t) \equiv \begin{cases} 1\,, & \text{for } t \geq 0, \\ 0\,, & \text{otherwise.} \end{cases}$$

The function G_1 can be viewed as the limiting solution of the BM when $q \to \infty$ (for any fixed p).

Let D_1 and D_2 be two distribution functions and denote by $\bar{D}_1$ and $\bar{D}_2$, respectively, their corresponding tail distributions (i.e., let $\bar{D}_i(t) \equiv 1 - D_i(t)$ for $i = 1, 2$). Recall that D_1 is said to be stochastically less than D_2 whenever the inequality $\bar{D}_1(t) \leq \bar{D}_2(t)$ holds for all $t \geq 0$ (see, e.g., Ross 1996, pp. 404–405). In the next lemma, we show that G_m is stochastically less than F_m, for all $m \geq 1$.

Lemma 3 *For all $m \geq 1$, we have*

$$\bar{G}_m(t) \leq \bar{F}_m(t)\,, \qquad t \geq 0\,. \tag{37}$$

Proof Since $\bar{G}_1(t) = 0$ for all $t \geq 0$, the lemma is clearly true for $m = 1$.

We now assume $m \geq 2$. Observe that (29) is equivalent to:

$$\frac{f_m(t)}{[1 - F_m(t)][\alpha + \beta_m F_m(t)]} \leq 1\,. \tag{38}$$

Next, it is easily seen that the left-hand side of (38) can be expanded as:

$$\frac{1}{\alpha+\beta_m}\,\frac{f_m(t)}{1-F_m(t)}+\frac{\beta_m}{\alpha+\beta_m}\,\frac{f_m(t)}{\alpha+\beta_m F_m(t)}\,. \tag{39}$$

Now, substituting (39) into (38) and integrating both sides of the resulting inequality from 0 to t yields (after a little bit of algebra)

$$-\frac{1}{\alpha+\beta_m}\ln\left(\frac{1-F_m(t)}{1+\dfrac{\beta_m}{\alpha}F_m(t)}\right)\le t\,.$$

It follows that

$$\frac{1-F_m(t)}{1+\dfrac{\beta_m}{\alpha}F_m(t)}\ge e^{-(\alpha+\beta_m)t},$$

which, after a rearrangement, becomes

$$F_m(t)\le\frac{1-e^{-(\alpha+\beta_m)t}}{1+\dfrac{\beta_m}{\alpha}e^{-(\alpha+\beta_m)t}}\,. \tag{40}$$

Finally, since the right-hand side of (40) is precisely $G_m(t)$ (see (36)), we see that (40) is equivalent to (37), and this completes the proof. □

Lemma 3 can also be rephrased as that the function G_m lies entirely above the function F_m for every m. In the next lemma, we consider the region bounded between G_m and F_m; and we prove that as a function of m, the areas of these regions converge to 0 when m increases to infinity.

Lemma 4 *As $m\to\infty$, the sequence of integrals (or areas)*

$$\int_0^\infty [G_m(t)-F_m(t)]\,dt$$

converges to 0*. Moreover, the convergence is monotone.*

Proof It is easily seen that

$$\int_0^\infty [G_m(t)-F_m(t)]\,dt=\int_0^\infty [\bar{F}_m(t)-\bar{G}_m(t)]\,dt\,. \tag{41}$$

Denote by μ_D the mean of a given distribution function D, and recall the standard formula that $\mu_D = \int_0^\infty \bar{D}(t)\,dt$. Then, the right-hand side of (41) can be evaluated as $\mu_{F_m} - \mu_{G_m}$, provided that both μ_{F_m} and μ_{G_m} are finite. We will, therefore, examine μ_{F_m} and μ_{G_m} separately.

We begin with μ_{G_m}. Since $\mu_{G_1} = 0$, we will consider μ_{G_m} for $m \geq 2$. From (4), it easily follows that

$$\bar{F}(t) = \frac{(p+q)e^{-(p+q)t}}{p + qe^{-(p+q)t}}\,. \tag{42}$$

By differentiating (42) with respect to q, it is straightforward to show that $\bar{F}(t)$ is strictly decreasing in q (for fixed p and t). Since β_m (see (30)) is strictly decreasing in m with $\lim_{m\to\infty} \beta_m = \beta$ and since $\bar{F}(t)$ is continuous in q, it follows from (36) that for all $t \geq 0$, the $\bar{G}_m(t)$s converge monotonically from below to $\bar{F}_\infty(t)$, where $F_\infty(t)$ is given by (15). (In other words, the G_ms increase stochastically to F_∞.) With $p = \alpha$ and $q = \beta$ in (42), it is easily shown that

$$\mu_{F_\infty} = \int_0^\infty \bar{F}_\infty(t)\,dt = \frac{1}{\beta}\ln\left(\frac{\alpha+\beta}{\alpha}\right) \tag{43}$$

(a result noted in Bass 1969, p. 219); and with β_m replacing β in (43), we also have

$$\mu_{G_m} = \int_0^\infty \bar{G}_m(t)\,dt = \frac{1}{\beta_m}\ln\left(\frac{\alpha+\beta_m}{\alpha}\right).$$

It follows that μ_{G_m} is strictly increasing in m with

$$\lim_{m\to\infty} \mu_{G_m} = \mu_{F_\infty} < \infty\,. \tag{44}$$

We now turn our attention to μ_{F_m}. From (28), we have

$$\mu_{F_m} = \frac{1}{m}\sum_{j=1}^{m} E[A_{mj}]\,. \tag{45}$$

Since $E[X_{mi}] = 1/\lambda_{m,i-1}$, it follows from (20) that

$$E[A_{mj}] = \sum_{i=0}^{j-1} \frac{1}{\lambda_{mi}}\,, \tag{46}$$

where the λ_{mi}s are given by (7). Finally, substitution of (46) into (45), followed by an interchange of the order of summation, yields

$$\mu_{F_m} = \frac{1}{m} \sum_{i=0}^{m-1} \frac{1}{\alpha + \dfrac{\beta}{m-1} i} . \tag{47}$$

It follows easily from (47) that $\mu_{F_1} = 1/\alpha$ and

$$\mu_{F_2} = \frac{1}{2} \left(\frac{1}{\alpha} + \frac{1}{\alpha + \beta} \right) ;$$

therefore, we have $\mu_{F_1} > \mu_{F_2}$. We will next consider μ_{F_m} for $m \geq 2$, and prove that the sequence of μ_{F_m}s decreases monotonically to μ_{F_∞}. (Actually, we conjecture that the F_ms decrease stochastically to F_∞, but have been unable to prove this stronger result.)

For $0 \leq y \leq 1$, define

$$\phi(y) \equiv \frac{1}{\alpha + \beta y} ;$$

and observe that in terms of the function ϕ, (47) can be written as $\mu_{F_m} = E[\phi(U_m)]$, where U_m is distributed uniformly over $i/(m-1)$ for $i = 0, 1, \ldots, m-1$. Now, consider U_m and U_{m+1}, for any $m \geq 2$; and observe further that the probability-mass function of U_m is, intuitively, more "spread out" than that of U_{m+1}. Since the function ϕ is strictly convex in y, these observations naturally suggest that the following inequality should hold:

$$E[\phi(U_m)] > E[\phi(U_{m+1})] . \tag{48}$$

We will prove that (48) is valid via a coupling argument.

The first step is to construct U_m from U_{m+1} as follows. Let $U_m = U_{m+1}$ if U_{m+1} equals either 0 or 1; and let

$$U_m = \begin{cases} \dfrac{i-1}{m-1} , & \text{with probability } \dfrac{i}{m} \\[2ex] \dfrac{i}{m-1} , & \text{with probability } 1 - \dfrac{i}{m} \end{cases}$$

if $U_{m+1} = i/m$, for $i = 1, \ldots, m-1$. Note that for this construction to be valid, the resulting U_m must satisfy $P\{U_m = i/(m-1)\} = 1/m$ for all $i = 0, 1, \ldots, m-1$; this can be easily verified, and we omit the details.

Next, observe that the coupling above can be restated as

$$U_m = U_{m+1} + Z_{m+1}\,, \tag{49}$$

where, by definition, $Z_{m+1} = 0$ if U_{m+1} equals either 0 or 1, and

$$Z_{m+1} = \begin{cases} -\dfrac{m-i}{m(m-1)}\,, & \text{with probability } \dfrac{i}{m} \\[2ex] \dfrac{i}{m(m-1)}\,, & \text{with probability } 1 - \dfrac{i}{m} \end{cases}$$

if $U_{m+1} = i/m$, for $i = 1, \ldots, m-1$. Moreover, it is easily shown that we have

$$E[Z_{m+1} \mid U_{m+1}] = 0 \tag{50}$$

with probability 1. (Relations (49) and (50) show that U_m is greater than U_{m+1} in the sense of what is known as convex order.) It now follows in a standard manner from (49), Jensen's inequality, and (50) that

$$\begin{aligned} E[\phi(U_m)] &= E[\phi(U_{m+1} + Z_{m+1})] \\ &= E[E[\phi(U_{m+1} + Z_{m+1}) \mid U_{m+1}]] \\ &> E[\phi(U_{m+1} + E[Z_{m+1} \mid U_{m+1}]] \\ &= E[\phi(U_{m+1})]\,; \end{aligned}$$

and this proves that for $m \geq 2$, μ_{F_m} is strictly decreasing. In addition, recall that $\mu_{F_1} = 1/\alpha > \mu_{F_2}$; therefore, it also follows that μ_{F_m} is finite for all m.

To determine the limit of the μ_{F_m}s, observe that (47) can be written as

$$\mu_{F_m} = \frac{m-1}{m}\, s_{m-1} + \frac{1}{m}\,\frac{1}{\alpha+\beta}\,, \tag{51}$$

where

$$s_{m-1} \equiv \frac{1}{m-1} \sum_{i=0}^{m-2} \frac{1}{\alpha + \beta\dfrac{i}{m-1}}\,.$$

Now, the fact that the function ϕ is decreasing implies that s_{m-1} is an upper Riemann sum of ϕ in the interval [0, 1]. Since ϕ is integrable, it follows that s_{m-1} converges to

$$\int_0^1 \phi(y)\,dy = \int_0^1 \frac{1}{\alpha+\beta y}\,dy = \frac{1}{\beta}\ln\left(\frac{\alpha+\beta}{\alpha}\right),$$

which is μ_{F_∞} (see (43)); and this, together with the fact that the second term in (51) converges to 0, proves that

$$\lim_{m\to\infty} \mu_{F_m} = \mu_{F_\infty} \,. \tag{52}$$

Finally, we return to (41) and rewrite its right-hand side as

$$\int_0^\infty [\bar{F}_m(t) - \bar{G}_m(t)]\, dt = (\mu_{F_m} - \mu_{F_\infty}) + (\mu_{F_\infty} - \mu_{G_m}) \,.$$

It now follows from (52) and (44) that both $\mu_{F_m} - \mu_{F_\infty}$ and $\mu_{F_\infty} - \mu_{G_m}$ converge to 0 as $m \to \infty$. Since we have also shown that the convergence is, for both cases, monotone, this completes the proof of the lemma. □

We are finally in position to prove Theorems 3, 4, and 5.

4.3 Proof of Theorem 3 The strategy is to establish, for all $t \geq 0$, the following two inequalities:

$$F_\infty(t) \geq \limsup_{m\to\infty} F_m(t) \tag{53}$$

and

$$F_\infty(t) \leq \liminf_{m\to\infty} F_m(t) \,. \tag{54}$$

Observe that if both (53) and (54) hold, then

$$F_\infty(t) \leq \liminf_{m\to\infty} F_m(t) \leq \limsup_{m\to\infty} F_m(t) \leq F_\infty(t) \,;$$

and since this implies (see, e.g., Rudin 1976, pp. 56-57) that $\lim_{m\to\infty} F_m(t)$ exists and the limit is equal to $F_\infty(t)$, the theorem follows.

Consider (53) first. Since $G_m(t) \geq F_m(t)$ (Lemma 3), we have

$$\limsup_{m\to\infty} G_m(t) \geq \limsup_{m\to\infty} F_m(t) \,, \qquad t \geq 0 \,. \tag{55}$$

In the proof of Lemma 4, we showed that for all $t \geq 0$, $\bar{G}_m(t)$ converges monotonically from below to $\bar{F}_\infty(t)$. The convergence of $\bar{G}_m$, and hence of G_m, implies that

$$\limsup_{m\to\infty} G_m(t) = \lim_{m\to\infty} G_m(t) = F_\infty(t) \,;$$

and this, together with (55), proves (53).

We now turn our attention to (54), which we prove by contradiction. Consider an arbitrary fixed t, say t^*; and suppose (54) does not hold at t^*. Then, there exists a positive ϵ and a subsequence $\{n_k\}_{k \geq 1}$ of positive integers such that

$$F_{n_k}(t^*) \leq F_\infty(t^*) - \epsilon \tag{56}$$

for all $k \geq 1$. Now, consider the function G_{n_k} and recall from Lemma 3 that $G_{n_k}(t) \geq F_{n_k}(t)$ for all $t \geq 0$. Moreover, as a consequence of Lemma 4, we have that the sequence of areas bounded between G_{n_k} and F_{n_k} converges to 0 as $k \to \infty$, that is,

$$\lim_{k \to \infty} \int_0^\infty [G_{n_k}(t) - F_{n_k}(t)]\, dt = 0\,. \tag{57}$$

We will show that (56) is in contradiction with (57).

Suppose (56) holds. For any given k, define a distribution function H_{n_k} as follows:

$$H_{n_k}(t) \equiv \begin{cases} G_{n_k}(t)\,, & \text{for } 0 \leq t < G_{n_k}^{-1}(F_{n_k}(t^*))\,, \\ F_{n_k}(t^*)\,, & \text{for } G_{n_k}^{-1}(F_{n_k}(t^*)) \leq t < t^*, \\ G_{n_k}(t)\,, & \text{for } t^* \leq t < \infty\,, \end{cases} \tag{58}$$

where the superscript "-1" in $G_{n_k}^{-1}$ denotes functional inverse. Since $G_{n_k}(t)$ and $H_{n_k}(t)$ agree at $t = G_{n_k}^{-1}(F_{n_k}(t^*))$ and at $t = t^*$ and since the function G_{n_k} is strictly increasing in t, we have $G_{n_k}(t) \geq F_{n_k}(t^*)$ for $G_{n_k}^{-1}(F_{n_k}(t^*)) \leq t < t^*$; and therefore, $G_{n_k}(t) \geq H_{n_k}(t)$ for all $t \geq 0$. Moreover, since the function F_{n_k} is strictly increasing in t, so that $F_{n_k}(t) < F_{n_k}(t^*)$ for $G_{n_k}^{-1}(F_{n_k}(t^*)) \leq t < t^*$, and since $G_{n_k}(t) \geq F_{n_k}(t)$ for all t, we also have that $H_{n_k}(t) \geq F_{n_k}(t)$ for all $t \geq 0$. Thus, the function H_{n_k} is, by construction, sandwiched between G_{n_k} and F_{n_k}. It follows that

$$\int_0^\infty [G_{n_k}(t) - F_{n_k}(t)]\, dt \geq \int_0^\infty [G_{n_k}(t) - H_{n_k}(t)]\, dt\,. \tag{59}$$

Now, observe that

$$G_{n_k}^{-1}(F_{n_k}(t^*)) \leq G_{n_k}^{-1}(F_\infty(t^*) - \epsilon) < F_\infty^{-1}(F_\infty(t^*) - \epsilon) < F_\infty^{-1}(F_\infty(t^*)) = t^*;$$

and that these inequalities, together with (58), (56), and $G_{n_k}(t) > F_\infty(t)$, imply that the right-hand side of (59) can be further bounded as follows:

$$\begin{aligned}\int_0^\infty [G_{n_k}(t) - H_{n_k}(t)]\,dt &= \int_{G_{n_k}^{-1}(F_{n_k}(t^*))}^{t^*} [G_{n_k}(t) - F_{n_k}(t^*)]\,dt \\ &\geq \int_{F_\infty^{-1}(F_\infty(t^*)-\epsilon)}^{t^*} [G_{n_k}(t) - F_{n_k}(t^*)]\,dt \\ &\geq \int_{F_\infty^{-1}(F_\infty(t^*)-\epsilon)}^{t^*} [F_\infty(t) - (F_\infty(t^*) - \epsilon)]\,dt\,. \qquad (60)\end{aligned}$$

Finally, since F_∞ is strictly increasing and since $F_\infty^{-1}(F_\infty(t^*) - \epsilon) < t^*$, the last bound in (60) is positive; moreover, notice that this lower bound is independent of k. We have, therefore, arrived at a contradiction to (57). This establishes (54), and the proof of the theorem is complete. □

4.4 Proof of Theorem 4 Upon taking expectations, (31) becomes

$$E[\lambda_{mA_m(t)}] = \alpha\{m - E[A_m(t)]\} + \frac{\beta}{m-1}\{mE[A_m(t)] - E\{[A_m(t)]^2\}\},$$

which, after a rearrangement, yields

$$E\{[A_m(t)]^2\} = \frac{(m-1)\alpha}{\beta}\{m - E[A_m(t)]\} + mE[A_m(t)] - \frac{m-1}{\beta}E[\lambda_{mA_m(t)}]\,.$$

Hence,

$$\begin{aligned}\delta_m(t) &= E\{[A_m(t)]^2\} - \{E[A_m(t)]\}^2 \\ &= \left\{\frac{(m-1)\alpha}{\beta} + E[A_m(t)]\right\}\{m - E[A_m(t)]\} - \frac{m-1}{\beta}E[\lambda_{mA_m(t)}]\,.\end{aligned}$$

Upon substitution of $E[A_m(t)] = mF_m(t)$ and $E[\lambda_{mA_m(t)}] = mf_m(t)$ (see (33)), the last expression rearranges straightforwardly to (16), and this completes the proof. □

4.5 Proof of Theorem 5 From (17) and (16), we have

$$\begin{aligned}Var[B_m(t)] &= \frac{1}{m^2}\,\delta_m(t) \\ &= \frac{1}{\beta}\,\{[1 - F_m(t)][\alpha + \beta F_m(t)] - f_m(t)\} \\ &\quad + \frac{1}{m\beta}\,\{f_m(t) - \alpha\,[1 - F_m(t)]\}\,.\end{aligned}$$

It is easily seen from (29) that $f_m(t) - \alpha\,[1 - F_m(t)]$ is uniformly bounded; hence,

$$\lim_{m\to\infty} \frac{1}{m\beta}\,\{f_m(t) - \alpha\,[1 - F_m(t)]\} = 0\,.$$

Next, since $F_\infty(t)$ satisfies (3) with $p = \alpha$ and $q = \beta$, Theorem 3 implies that

$$\lim_{m\to\infty} \{[1 - F_m(t)][\alpha + \beta F_m(t)] - f_m(t)\} = 0\,.$$

It follows that

$$\lim_{m\to\infty} Var[B_m(t)] = 0\,. \tag{61}$$

Finally, from Markov's inequality, we have, for any positive ϵ,

$$\begin{aligned} P\{|B_m(t) - F_\infty(t)| > \epsilon\} &= P\{[B_m(t) - F_\infty(t)]^2 > \epsilon^2\} \\ &\le \frac{E\{[B_m(t) - F_\infty(t)]^2\}}{\epsilon^2} \\ &= \frac{Var[B_m(t)] + \{E[B_m(t)] - F_\infty(t)\}^2}{\epsilon^2}\,, \end{aligned}$$

which, together with (61) and Theorem 3, yields (18) upon taking limits. This completes the proof. □

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