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A Model of B2B Exchanges

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Abstract

B2B exchanges are revolutionizing the way businesses will buy and sell a variety of intermediary products and services. It is estimated that most of the roughly $7 trillion worth of business transactions are likely to go through these new institutions within the next decade. This paper tries to understand the economics governing the transactions within B2B exchanges and analyze their likely evolution over time. In doing so, we start by providing rigorous definitions to a number of critical concepts broadly used in the context of B2B exchanges including "market fragmentation", "critical mass" and buyer-seller "connectivity". We describe equilibrium behavior in the exchange and analyze it as a function of these critical concepts. Next, we study the evolution of the exchange in a dynamic system where buyers and sellers enter (exit) the exchange based on the relative economic surplus (loss) they receive inside vs. outside the exchange. Our results have important implications for practice. For example, we show that equilibrium prices within the marketplace may not always decrease with lower search costs. However, buyer surplus rises with lower search costs even if prices are higher in the exchange. We also show that the general view that demand and supply (so-called "liquidity") either grows or shrinks in the marketplace may not always hold and it is quite possible to have a marketplace that is stable even though only a relatively small proportion of the market participants transact in it. Finally, we also provide conditions under which the exchange should subsidize buyers or sellers in order to achieve critical mass.

1 Introduction

Business-to-business (B2B) exchanges represent one of the key changes that mass-interactivity, fueled by the evolution of the Internet, brought to modern economic systems. While it is not clear what will be the ultimate form of B2B exchanges or even how they will generate revenue and profit, there is large scale agreement that trillions of dollars worth of products and services will change hand in these institutions in the coming years. Forrester Research for example, estimated that by 2001, 71% of business buyers and sellers will at least try online marketplaces. From $54 billion in 2000, the value of transactions over B2B exchanges is estimated to grow to $1.4 trillion by 2004.[1] B2B exchanges will be an important part of the economic infrastructure in modern economies.

The purpose of this paper is twofold. First, it tries to understand key concepts that are often used by academics as well as practitioners to describe the activity of B2B exchanges or to assess their potential success. For example, people often talk about *"market fragmentation"* in conjunction with e-marketplaces to describe industries or product markets where B2B exchanges are likely to make a difference or be successful. It is often argued that B2B exchanges create value primarily in markets that are highly fragmented. What exactly is meant by the term "market fragmentation" is far from being clear however. Similarly, there is a broad consensus that marketplaces can only succeed if they reach a certain "critical mass" or "liquidity", beyond which sellers and buyers have no other choice but to transact within the exchange. Again, it is not clear how exactly the concept of critical mass is defined. Our first goal therefore, is to put a rigorous structure on these concepts and provide precise definitions for them based on well understood economic fundamentals such as "search costs" or "consumer preferences".

The second objective of the paper is to understand economic behavior in B2B exchanges and the likely evolution of an exchange resulting from this behav-

[1]See *Business Week* (06/05/00, EB56). A comparable estimate by The Boston Consulting Group is reported in Cross (2000). For comparison, the total amount spent annually on components, supplies and services is estimated to be $7 trillion worldwide (*USA Today*, 2/7/00, B1). Lucking-Reiley and Spulber (2000) report larger absolute figures using commercial sources but their proportions are similar.

ior. In this context, we start by analyzing the effect of product and market characteristics on market equilibria in markets characterized by different levels of buyer search costs. Specifically, we study how the level of product commoditization, the pattern of buyer-seller connectivity and search costs are likely to influence equilibrium prices, buyers' surplus and sellers' profits. Subsequently, this allows us to examine the evolution of a dynamic system in which industry participants join (or quit) a newly established B2B exchange based on their switching costs and relative economic surplus generated respectively in the traditional and the new market medium.

Our results provide a number of interesting insights that may have important implications for practice. For example, we find that B2B exchanges may not exhibit the usual dynamics of increasing returns - the idea that buyers and sellers tend to either join or leave the exchange until its size either reduces to zero or includes all market participants. Rather, we find that exchanges may be stable even if they have a finite size where only part of the market's participants transact in the exchange (a so-called "stalled exchange"). We also find that, while sellers' prices may not necessarily decrease with lower search costs, buyers' surplus usually increases with dwindling search costs. The model also suggests that B2B exchanges are better off subsidizing buyers rather than sellers to achieve the "critical liquidity" after which the exchange is likely to grow without marketing subsidies. These results can provide normative insights to managers of B2B exchanges on launch strategies and optimal ways to solve the "chicken-and-egg problem" initially faced by all exchanges.

The paper is organized as follows. The next section reviews the relevant literature and motivates the model development. Section 3 describes the model of an "exchange" and Section 4 the rules governing the transition of buyers an sellers from the traditional medium to the new B2B exchange or vice versa. Throughout these sections, care is taken to relate our findings to some real-life phenomena. The paper ends with concluding remarks.

2 Related Literature

Given the variety of business models that can be observed today, it is hard to argue for a model based on any particular form of a B2B exchange. For example the above mentioned article in *Business Week* groups so-called e-marketplaces in 6 categories: Communities, Catalogs, Procurement Hubs, Auctions, Exchanges and Collaboration Hubs. While the article argues that many of these forms may not exist in the future they generally define "B2B marketplaces" as institutions that *"draw business buyers and sellers together in one virtual place, where participants can reduce transaction costs and reach new customers"*. Along the same lines, Wise and Morrison (2000) argue that B2B exchanges will evolve similarly to financial exchanges and current players will likely specialize to a variety of smaller activities that current marketplaces all encompass (see also Grover et al., 1999). McAfee (2000) even questions the need for e-marketplaces in a world that, according to him, will be dominated by peer-to-peer networks. In contrast, Sarkar et al. (1998) argue that independent entities (so-called "cybermediaries") have an important role for electronic markets based on the special (technological and coordination) skills required to operate such markets efficiently. They forecast that cybermediaries are likely to remain even after market participants have acquired experience with electronic transactions.

While there is consensus that many of the current formats may not necessarily prevail in the future there is also general agreement among academics and practitioners that e-marketplaces - including B2B exchanges - will be characterized by the following features (see e.g. Bakos 1991): (1) they reduce buyers' search costs, (2) participants' benefits depend on how many other organizations join the system and (3) there are significant switching costs in joining (leaving) the exchange including risk as well as system setup costs. Our goal therefore, is to analyze the behavior of economic systems with these general characteristics.

The above features have been individually studied by economists for a long time. As such our work is related to earlier work on commodity markets (see Stiglitz (1989) for a good review). In particular, our market model extends Perloff and Salop (1985) by adding buyer search to their model of product differentiation. A similar approach was initiated by Bakos (1997) in the context of electronic marketplaces. His model is based on Salop's (1979) classic model of circular city and considers sequential search (with replacement) by consumers. As such, his model better suits the context of consumer markets where sequential search is the norm and competition is "local", in the sense that brands typically compete with other brands in their neighborhood - whether neighborhood is defined in terms of physical distance or in terms of distance from consumers' ideal points. In contrast, our model fits the spirit of B2B exchanges in that consumers determine upfront the set of sellers that they intend to collect bids from (consistent with the idea of a formal Request For Proposal - RFP) and competition is not localized as all brands compete with one another for all consumers. This is consistent with fragmented commodity markets where consumer ideal points are not so clearly defined upfront as special services and terms and conditions are discovered by buyers during the evaluation of sellers' proposals.

In using the above approach we make several key assumptions. First, we refrain from studying marketplaces that are dominated by a few large players (either buyers or sellers). In contrast, the object of our analysis is exchanges where buyers and sellers are numerous and transaction sizes are roughly equal (Lucking-Reiley and Spulber, 2000, p. 16, Ramsdell, 2000, p. 176). This is the reason why we prefer to use the term "B2B exchange" as opposed to "e-marketplace"; the former terminology reflecting neutrality with respect to the size of players. Our choice is motivated by the simple argument that if the market is dominated by a few players then the whole idea of an *independent* exchange is questionable as such entity would depend on these players.[2] Second, as mentioned earlier, we assume away the exact market mechanism that governs the interactions between buyers and sellers (e.g. auctions, demand aggregation, etc.). We do this for a number of reasons. First, as argued before, it is not clear which market mechanism will ultimately dominate and it is also possible (in fact likely) that the particular market mechanisms used will depend on industry context. Second, most market mechanisms have similar broad implications in terms of equilibrium prices and buyer-seller surplus. For example, it is likely that decreasing buyer search costs will lead

[2] See, for example, Covisint (Ibid., p. EB62). Other examples dominated by large players include Oil and Chemical firms' plan to create an exchange (*Wall Street Journal*, April 12, 2000) or the major Specialty-Metals manufacturers' plan to create an electronic marketplace (*Wall Street Journal*, May 2, 2000).

to increased price competition between sellers. Finally, the third reason for our generalized approach is that our ultimate goal is to describe the dynamic evolution of a B2B exchange. "Evolution" is modeled with a dynamic system whose behavior is driven by the equilibrium surplus of participants interacting in markets which are characterized by different search costs. The modeling technology is similar to the one used in marketing to study new product diffusion (see e.g. Bass 1969) with the marked difference that here, system dynamics are driven by the economic agents' surpluses/losses generated under the equilibrium of a game. In particular, network externalities (feature 2 above) emerge indirectly. As market liquidity increases so does buyers' and sellers' surplus.

3 The marketplace model

Our goal is to understand how buyers and sellers will migrate to a newly formed B2B exchange. We recognize however, that market mechanisms have also been in place in the traditional world. Therefore, we assume that the general rules of buyer-seller interaction are the same in the traditional world (denoted TR) and within the new Exchange (denoted XC) and the two media only differ in that buyers' search cost is significantly lower in the latter medium. Thus, in what follows, we will start by building a general "marketplace" model and describe the participants' behavior as a function of the basic marketplace characteristic: buyer search cost.

Imagine an industry in which N_S identical sellers (indexed $\mu = 1, 2, ...N_S$) compete for the orders of N_B identical buyers (indexed $i = 1, 2, ...N_B$). Both buyers and sellers are risk neutral when they transact in the market.[3] Each buyer would like to purchase exactly volume $\mathcal{V}$ of a given product, which is on stock at sellers. In order to find the best offer, each buyer engages in a market search. During this search buyer i requests a proposal (RFP) from a random subset of sellers $\mathcal{K}_i$ who communicate their prices and product attributes. $\mathcal{K}_i$ is a random set because buyers have no a priori preference for potential sellers. Furthermore, $\mathcal{K}_i$ is the private knowledge of buyer i and its

[3]Risk neutrality will not be assumed when buyers and sellers choose to join a new market. Some of our model parameters will account for the risk associated with such a choice.

size will depend on the buyer's search cost.

The marketplace is characterized by the buyers' search cost, denoted a, which represents the cost of sending out and evaluating an RFP. Then, buyer i maximizes its expected payoff:

$$\beta_i = \max_{k_i}[\langle \max_{\mu \in \mathcal{K}_i}(Q + \xi_\mu - p_\mu)\rangle \mathcal{V} - ak_i], \tag{1}$$

where k_i is the total number of requested RFPs ($k_i \equiv |\mathcal{K}_i|$), $\langle . \rangle$ represents expectations over ξ and the randomness of $\mathcal{K}_i$, p_μ is seller μ's proposed price, Q is the fixed component of the utility derived from the product, and ξ is the utility's random component. One can think of ξ_μ as the random "fit" between the buyer and seller μ. In other words, ξ is some idiosyncratic aspect of the seller's product that the buyer can only assess through the careful evaluation of the seller's proposal (e.g. the subjective perception of product esthetics, special contract terms, etc.). We assume that ξ is independent across buyers and sellers, has density function $f(\xi)$ and cumulative density $F(\xi)$. All of these are common knowledge. In essence, the dispersion (variance) of ξ defines the extent of *"commoditization"* in the industry, i.e. the extent to which buyers are aware of sellers' offerings without evaluating them in detail. The higher the variance of ξ the *less* the product is a commodity. In contrast, when the variance of ξ is 0, the product is a perfect commodity.

Seller-μ's expected profit function is:

$$\pi_\mu = (p_\mu - c)\, V_\mu^{\mathrm{tot}} - b, \tag{2}$$

where c is the product's marginal cost, b is the seller's fixed cost and V_μ^{tot} is the *expected* total volume she sells in that period. Without loss of generality, we assume that $c = b = 0$.

The timing of the game is as follows:

1. Sellers set their prices and buyers choose their k_i values and their $\mathcal{K}_i$ sets simultaneously.

2. Buyers issue RFP-s according to $\mathcal{K}_i$ and learn their ξ values for each proposal.

3. Buyers compare proposals and announce the winner.

4. Fulfillment, settlement.

Notice that this structure introduces asymmetry between buyers and sellers with respect to their *connectivity*, which we define as the total number of potential partners they are connected to. Specifically, we assumed that buyers choose the optimal number of RFPs that they intend to evaluate. This means that, since buyers are identical, in equilibrium, each buyer will have the same connectivity $k_i^* = k_B$. Sellers, on the other hand will typically end-up with different connectivities, k_μ.

The above model is an extension to Perloff and Salop's (1985) model of product differentiation. Specifically, we added two elements to their model. First, we introduced the random connectivity between buyers and sellers. Second, and more importantly, we explicitly consider buyers' optimal search in determining the total number of sellers they are willing to evaluate. In this context, we are looking for symmetric equilibria of the game. The following proposition partially characterizes such an equilibrium by describing the equilibrium price *given* the optimal buyer connectivity.

Proposition 1 (Perloff and Salop, 1985) *Assume a symmetric equilibrium where the optimal buyer connectivity is k_B. Then, the equilibrium price and sales volume in the marketplace are*

$$p^* = \frac{1}{k_B(k_B - 1)\int_{-\infty}^{\infty} d\xi\, f^2(\xi)\, F^{k_B-2}(\xi)}. \tag{3}$$

and

$$V^* = \mathcal{V}\frac{N_B}{N_S}. \tag{4}$$

Proof: The key steps of the proof are similar to Perloff and Salop but the introduction of random connectivity requires careful consideration. First, to track buyer-seller connectivity we introduce the (random) connectivity matrix $\kappa_{\mu i}$, whose values are

$$\kappa_{\mu i} = \begin{cases} 1 & \text{if } \mu \in \mathcal{K}_i \\ 0 & \text{if } \mu \notin \mathcal{K}_i. \end{cases} \tag{5}$$

Next, let us introduce $W_{\mu i}$ as seller μ's conditional probability to win buyer i's order if they are "connected", i.e. if $\kappa_{\mu i} = 1$. With this,

$$V_\mu^{\mathrm{tot}} = \mathcal{V} \sum_i^{N_B} \langle \kappa_{\mu i} W_{\mu i} \rangle = \mathcal{V} \sum_i^{N_B} \frac{k_B}{N_S} W_\mu = \mathcal{V} \frac{N_B k_B}{N_S} W_\mu . \tag{6}$$

where in the second equation we used the fact that $\langle \kappa_{\mu i} \rangle = k_B / N_S$, i.e. buyer-seller connectivity is random, and that $W_{\mu i} = W_\mu$ is, in fact, i-independent because all buyers are identical. The total expected volume is the sum of the individual orders. However, winning an individual order is always a matter of chance, since the buyer's valuation is a random process. Even if seller μ sets a lower price than all other sellers do, there is a probability that he loses the order because ξ is low for some buyers. The contrary may occur as well.

Our main task is to express the probability of winning, W_μ. Clearly, this is a function of the number of sellers in $\mathcal{K}_i$, which we assumed to be an i-independent constant k_B, and the individual prices these sellers have set. Knowing the buyer's profit function in Eq. (1) we find

$$\begin{aligned} W_\mu \equiv W(p_\mu) = \mathrm{Prob}(\forall\, \nu \in \mathcal{K}_i, \nu \neq \mu : \quad & \xi_\mu - p_\mu > \xi_\nu - p_\nu , \\ \& \quad & Q + \xi_\mu - p_\mu > a k_B) \end{aligned} \tag{7}$$

where $\xi_{\mu,\nu}$ are the different realizations of fit between buyer i and sellers μ and ν. In the sequel we will omit the second condition assuming that Q is large enough compared to the competitive price. The probability W_μ, which depends on the prices of all sellers in $\mathcal{K}_i$ can trivially be reduced to quadrature:

$$W(p_\mu) = \int_{-\infty}^{\infty} d\xi \, f(\xi) \prod_{\substack{\nu = 1 \\ \nu \neq \mu}}^{k_B} F(\xi - p_\mu + p_\nu). \tag{8}$$

Using the above elements, substituting V_μ^{tot} into profit function (2) and differentiating each seller's profit with respect to their price, we obtain the following first order conditions:

$$W(p_\mu) + p_\mu \frac{\partial W(p_\mu)}{\partial p_\mu} = 0, \qquad \forall \mu. \tag{9}$$

Since sellers are identical, we assume a symmetric equilibrium, i.e. $p_\nu = p^*, \ \forall \nu$. Under this assumption the winning probability reduces to the intuitive form:

$$W(p^*) = \int_{-\infty}^{\infty} d\xi \, f(\xi) F^{k_B - 1}(\xi) = \frac{1}{k_B}, \tag{10}$$

while its first derivative is

$$\frac{\partial W(p^*)}{\partial p_\mu} = -(k_B - 1) \int_{-\infty}^{\infty} d\xi \, f^2(\xi) F^{k_B - 2}(\xi). \tag{11}$$

Substituting these expressions in the F.O.C. of Eq. (9) we obtain the equilibrium price of the proposition. The equilibrium volume follows from Eqs. (6) and (10) independent of the fit distribution. □

First, notice that p^* is independent from the seller's connectivity, k_μ. This is because sellers do not know their connectivity at the time when they set their prices. However, it is important to see that even when they discover their connectivity sellers have no incentive to change their prices. This is because a seller's total volume can be partitioned into a sum of k_μ independent terms. $V_\mu^{\text{tot}} = \mathcal{V} \sum_{i \in k_\mu} W_{\mu i} = \mathcal{V} k_\mu W_\mu$, which reduces to the optimization problem above. The basic idea is that for each order, the seller competes with the same number of other sellers because buyer connectivity is uniform.

Second, it is important to characterize the equilibrium price as a function of buyer connectivity in order to assess the impact of increasing connectivity (lower search cost) on sellers' profits. In our model the average sales volume is a constant, thus sellers' profit in equilibrium is directly proportional to the equilibrium sales price. In the following we show that p^* may increase, decrease, or tend to a finite limiting value as k_B increases depending on the distribution of ξ.[4]

We will limit our attention to fit functions $f(\xi)$ whose support is the finite or infinite interval (a, b), which are analytic and strictly positive in (a, b). We assume that finite limits $f(a) \equiv \lim_{\xi \to a} f(\xi) < \infty$ and $f(b) \equiv \lim_{\xi \to b} f(\xi) < \infty$

[4]Under special conditions more exotic behaviors, e.g. oscillations between two limiting values, are also possible.

exist at the boundaries. These conditions imply that $f(\xi)$ is bounded from above in $[a,b]$, but its derivatives may have singularities as $\xi \to a$ or $\xi \to b$. To exclude a couple of pathologies we will also assume that for any $n \geq 1$, the limits of the nth derivative $\lim_{\xi\to a} f^{(n)}(\xi)$ and $\lim_{\xi\to b} f^{(n)}(\xi)$ exist (but may be $\pm\infty$). Furthermore, we will suppose that $\int_a^b f(\xi)^2 d\xi < \infty$ and $\int_a^b \xi f(\xi) d\xi < \infty$, which are sufficient to ensure that the integrals we will cope with converge. In the following we will only consider this category of functions. Note, however, that some of our propositions may have a generalization (at the price of higher complexity in the proof) for a broader class of functions as well.

For technical reasons it is worth classifying the above functions into four classes according to the behavior of f and its logarithmic derivative $\partial_\xi \ln f(\xi) = f'(\xi)/f(\xi)$ near b:

Definition

Class-A: $f(b) > 0$.

Class-B1: $f(b) = 0$ and $\lim_{\xi\to b} \partial_\xi \ln f(\xi) = -\infty$.

Class-B2: $f(b) = 0$ and $\lim_{\xi\to b} \partial_\xi \ln f(\xi) = -C$, $0 < C < \infty$.

Class-B3: $f(b) = 0$ and $\lim_{\xi\to b} \partial_\xi \ln f(\xi) = 0$.

For functions in Class-A the upper boundary is necessarily finite $b < \infty$. From a practical perspective, it is easy to justify such distributions if one can safely assume that the random fit between the buyer and the seller has some upper limit. On the other hand, b is necessarily infinite $b = \infty$ for Class-B2/B3. Class-B1 collects functions with a faster than exponential decay, a typical such distribution being the Gaussian distribution. Similarly to distributions with a finite support (Class-A), the Gaussian distribution has special relevance for practice. If we assume that "fit" results from the sum of a large number of random factors, then the central limit theorem ensures that ξ is normally distributed.[5] It is easy to see that Class-B2 has a purely (negative) exponential asymptotic behavior as $\xi \to b = \infty$. Finally, Class-B3 collects distributions with a slower than exponential behavior for large

[5] Assuming that the second moment of the individual factors exists.

ξ. These are the so called "fat tail" distributions. The latter two categories describe situations where extreme values of "fit" between buyers and sellers are not too unlikely. Table 1 enlist some typical and conceptually important representatives of the different classes.

For later purposes we also introduce the second logarithmic derivative of the fit density:

$$\partial_\xi^2 \ln f(\xi) = \frac{f''(\xi) f(\xi) - f'(\xi)^2}{f(\xi)^2}. \tag{12}$$

Using this, within the above function classes we can prove the following:

Proposition 2 *Let k_B be the optimal buyer connectivity.*

Class-A/B1/B2: *If the distribution is such that the second logarithmic derivative $\partial_\xi^2 \ln f(\xi) < 0$ for all $\xi \in (a, b)$ then the equilibrium price $p^*(k_B)$ monotonically decreases for all $k_B \geq 2$.*

Class-B2/B3: *If the distribution is such that the second logarithmic derivative $\partial_\xi^2 \ln f(\xi) > 0$ for all $\xi \in (a, b)$ then the equilibrium price $p^*(k_B)$ monotonically increases for all $k_B \geq 2$.*

Proof: See Appendix.

As can be checked by an explicit calculation, the premise of the proposition holds for the Class-A "uniform", the Class-B1 "Gaussian" and the Class-B3 "Pareto" examples of Table 1. The Class-B2 "exponential" distribution in the Table is *marginal* in the sense that for all $\xi \in [0, \infty)$, $\partial_\xi^2 \ln f(\xi) = 0$, and $p^*(k_B)$ turns out to be a k_B-independent constant. Note, however, that Proposition 2 only gives a sufficient but not a necessary condition to ensure monotonic behavior of the equilibrium price. The proposition is especially weak in the Class-A case as it becomes evident in the proof.

Additional insight into the behavior of the equilibrium price can be obtained by considering the large k_B limit of $p^*(k_B)$. The behavior for large k_B also has special importance since this is the limit which we believe will characterize the B2B exchange. It turns out that the large k behavior is determined by the tail of the fit distribution alone. Let us define

$$D = \lim_{\xi \to b} \frac{\partial_\xi^2 \ln f(\xi)}{f(\xi)}. \tag{13}$$

Using this we can show the following:[6]

Proposition 3 *Let k_B be the optimal buyer connectivity.*

Class-A/B1: *There always exists K such that for $k_B > K$ the equilibrium price $p^*(k_B)$ monotonically decreases to 0.*

Class-B2: *If $D < 0$ ($D > 0$) then there exists K such that for $k_B > K$ the equilibrium price $p^*(k_B)$ monotonically decreases (increases) to a positive constant.*

Class-B3: *There always exists K such that for $k_B > K$ the equilibrium price $p^*(k_B)$ monotonically tends to ∞.*

In all Classes, $\lim_{k_B \to \infty} p^{*\prime}(k_B) = 0$.

Proof: See Appendix.

The conventional Class-A/B1 case reflects conventional wisdom, according to which higher market transparency (larger connectivity) exerts downward pressure on prices. It is illustrative to see how the major input parameters like the buyer connectivity k_B and the standard deviation of the distribution influence the equilibrium price in the most simple, uniform distribution case. When ξ is uniformly distributed between $-\Delta/2$ and $\Delta/2$ the complicated expression in Eq. (3) reduces to the simple form

$$p^* = \frac{\Delta}{k_B}. \tag{14}$$

As mentioned before, Δ may be viewed as a measure of the "inverse" of commoditization of the product category. Thus the equilibrium price decreases as the product approaches perfect commodity (Δ decreases). Furthermore, p^* decreases with the buyers' optimal connectivity (k_B). Later, we will show

[6] A similar, although less detailed analysis can be found in Perloff and Salop (1985).

that higher equilibrium connectivity (k_B) is directly linked to lower search cost. This means that p^* also increases with the buyer's search cost - as expected.

In essence, Δ and k_B are related to the concept of *"market fragmentation"*, which is often used in the context of B2B exchanges. The notion is that a market is fragmented from the buyer's perspective if there are "many sellers to consider". Clearly, the number of sellers to consider is only meaningful in relation to the search cost and the benefit that the buyer gets by searching more sellers. In our definition, as k_B increases (the search cost decreases) fragmentation also decreases. Similarly, when the potential benefit from searching more sellers (Δ) increases fragmentation increases. Thus, one could naturally define market fragmentation as Δ/k_B. Then, for the uniform distribution the equilibrium price is simply equal to the fragmentation of the market. This is also consistent with the language used in the context of B2B exchanges. People consider that the central role of such exchanges is to "de-fragment" the market which will result in lower prices. An important insight here is that market fragmentation is driven by two basic fundamentals (commoditization and buyer search costs). Interestingly, we will see later that market fragmentation alone is not enough to characterize the evolution of the marketplace (in fact for more complicated distributions than the uniform, while the standard deviation can be factored out from the equilibrium price, k_B or a can not[7]). Rather, the impact of its different components need to be considered separately to understand equilibrium outcomes. While p^* takes a simple form for the uniform distribution case, it may not have a closed form solution for more general fit distributions.

Contrary to the conventional expectation, if the market is characterized by a broad enough (Class-B3, fat-tail) fit distribution the equilibrium price does not decrease but *increases* as a function of connectivity k_B. This is obviously profitable for sellers but rather unfavorable for buyers. In the following, however, we will demonstrate that even in this situation buyers increase their connectivity because their loss on prices is well compensated by the advantage of establishing a more efficient matching of their preferences with

[7] By a change of variable $x = \xi/\Delta$ in (3) one can easily obtain $p^* = \frac{\Delta}{\int_{-\infty}^{\infty} f_1(x)^2 F_1(x)^{k-2} dx}$, where $f_1(x) = \Delta f(x\Delta)$ has unit variance.

sellers' offerings.

To see this consider the buyers' problem of choosing their connectivity optimally. Our goal is to find the buyer's best response k_B given the sellers' price $p^*(k_B)$ in Eq. (3). According to Eq. (1), k_B maximizes the buyer's surplus

$$\beta(k) = \mathcal{V}[Q + M(k) - p^*(k)] - ak \tag{15}$$

where by elementary probability theory

$$M(k) \equiv \max_{\mu=1}^{k} \xi_\mu = k \int_a^b d\xi \; \xi \, f(\xi) F(\xi)^{k-1}. \tag{16}$$

First, we note that

Proposition 4 *For any distribution of fit for which $M(1) = \int_a^b \xi f(\xi)$ exists, the function $M(k)$ monotonically increases in k, it is concave, and it has the limits $\lim_{k\to\infty} M(k) = b$ and $\lim_{k\to\infty} M'(k) = 0$.*

Proof: See Appendix.

Given the search cost a, the optimal connectivity k_B is the solution of the first order condition $d\beta(k)/dk = 0$, i.e.,

$$\mathcal{V}[M'(k_B) - p^{*\prime}(k_B)] = a. \tag{17}$$

In general this equation may have several solutions for a given a. Nevertheless, in all the specific examples we have considered (the ones in Table 1 and many others not discussed explicitely) we found a one-to-one relationship between a and k_B. In particular we have found that k_B is a monotonically decreasing function of a. Although we cannot say anything firm for intermediate values of a and k_B, we have some controle over the small a, large k_B limit. Indeed, in virtue of Propositions 3 and 4, the formal value $k_B = \infty$ is a solution of Eq. (17) for $a = 0$. This is an indication to expect a solution (in most cases the unique solution) $k_B = k_B(a)$ becoming large as a going to zero. The existence and monotonicity of this "large-k_B" solution can be proven rigorously under the following conditions:

Proposition 5 *In the cases listed below there exists A small enough such that for any $a \in (0, A)$ Eq. (17) has a solution $k_B = k_B(a)$ which is a continuous, monotonically decreasing function of a, and has the limit $\lim_{a\to 0} k_B(a) = \infty$:*

Class-A/B1: *Always.*

Class-B2: *If* $D \equiv \lim_{\xi \to b} \partial_\xi^2 \ln f(\xi)/f(\xi) < 0$.

Class-B3: *In the subclass of Pareto distributions with* $\alpha > 2$.

Proof: See Appendix.

The conditions listed for Class-B2/B3 are sufficient conditions but by no means necessary. Infact, we conjecture that the large k_B solution probably exists for any fit distributions. Unfortunately, we were unable to extend the proof to the general case. Nevertheless, we were unable to find conterexamples either.

In the following we will disregard pathological cases with multiple equilibria or without a large-k_B solution (if such cases exist at all). Then Proposition 5 ensures that as search costs dwindle the equilibrium connectivity increases beyond any limits. In case of the B2B exchange where search costs are assumed to be negligibly small this will imply full connectivity (all buyers connected to all sellers within the exchange) as will be discussed in the next Section. Another important implication of Proposition 5 is that we should not necessarily use the search cost a as our "fundamental" marketplace parameter, but utilizing the one-to-one relationship between this and the connectivity itself, we can use k_B instead. This reparametrization renders the mathematical expressions simpler, and thus will be used throughout the paper.

Figure 1 shows the actual relationship between the search cost a and the equilibrium connectivity k_B for the fit distributions of Table 1. One can see that the relationship is indeed monotonic. Moreover, for small enough a we observe that the faster the decay of the fit density for large values of ξ, the less sensitive is the equilibrium connectivity to a decrease in search cost. In other words, one can ask: what is the critical value of search cost at which there is full connectivity in the exchange. Figure 1 shows that the faster the distribution decays, the lower is the critical value of search cost. The reason for this is quite intuitive. The more likely that buyers will discover large positive values of fit the more they have an incentive to collect bids from sellers, i.e. the more connected they want to be.

Irrespective whether k_B and a has a one-to-one relationship, or there are multiple equilibria, it is easy to show formally that dwindling search costs are always favorable for buyers in our model:

Proposition 6 *The equilibrium buyer surplus is a monotonically decreasing function of the search cost a.*

Proof: Consider k_B as a continuous parameter. We show that $d\beta^*/da$ is negative. Indeed, differentiating $\beta^*(k_B)$ in Eq. (15) with respect to a, recalling that $k_B = k_B(a)$, gives

$$\frac{d\beta^*(a)}{da} = \Big(\mathcal{V}[M'(k_B) - p'(k_B)] - a\Big)\frac{dk_B}{da} - k_B(a) = -k_B(a) < 0, \qquad (18)$$

where we have used the fact that the expression in the square bracket is zero by Eq. (17). Of course we should assume that a is small enough such that the solution of Eq. (17), k_B, is positive.[8] □

In the remaining part of the paper we will fix the fit distribution, and conduct all further analysis assuming ξ being distributed uniformly between $-\Delta/2$ and $\Delta/2$. We emphasize, however, that by the above propositions the qualitative behavior we are going to find is robust for a wider range of functions.

For the uniform distribution, explicit calculation gives that a is a monotonically decreasing single-valued function of k_B:

$$a = \Delta(\frac{1}{k_B^2} + \frac{1}{(1+k_B)^2}). \qquad (19)$$

In practice this relationship breaks down as a decreases below a certain level, where k_B simply becomes N_S, i.e. buyers are connected to all available sellers (full connectivity). Using the one-to-one correspondence between k_B and a, in what follows we will consider the buyers' connectivity k_B to be the relevant parameter of the marketplace instead of a. This will make computations as

[8] If Eq. (17) has multiple solutions, $d\beta^*/da$ may have jump discontinuities where $dk_B/da = \pm\infty$, but its sign cannot change.

well as the interpretation of the results much simpler.

With these elements, we are ready to define the traditional marketplace, TR and the new Exchange, XC. We assume that XC differs from TR in that the search cost is "significantly" lower in the former. By "significantly" we mean that in XC there is full connectivity on the buyer side, i.e. $k_B = N_S$. In contrast, $k_B << N_S$ in TR.

4 The dynamic evolution of the Exchange

In this section we analyze the evolution of the B2B Exchange (XC) using our model with identical buyers and sellers. We assume that buyers either join the Exchange or stay outside and use the traditional marketplace (TR). We refer to this assumption as "buyer dichotomy". On the seller side we do not have dichotomy. In other words, if a seller receives an RFP in the traditional marketplace, it will still respond to it even if it decided to join the new Exchange. These assumptions are consistent with practice. As mentioned before, the Exchange is defined by a negligibly low search cost, i.e., optimal buyer connectivity is N_S (full). On the other hand, the traditional marketplace is defined by an optimal connectivity $k_B < N_S$. Thus, while in TR the buyer's payoff is defined as in Eq. (1), it is slightly different in XC:

$$\beta_i^{\mathrm{XC}} = \langle \max_{\mu=1}^{N_S} [Q + \xi_\mu - p_\mu^{\mathrm{XC}}] \rangle \mathcal{V} - r, \qquad (20)$$

where p_μ^{XC} is seller μ's price in the XC channel (usually different from p_μ^{TR}).

The parameter r represents the buyer's net "cost" per time period of doing business on XC once it has joined in. A number of things may contribute to r, some positively, others negatively. For example, the Exchange may charge a subscription fee, and software license and maintenance fees, which impact r positively. Similarly, accepting offers from previously unknown suppliers in the XC may enhance the buyer's risk which can also be incorporated into r. On the other hand, by moving to the Exchange, the buyer may be able to dissolve (part of) the purchase department or reduce its acquisition costs through better logistics etc. As a result of all these factors r can be either positive or negative. We define

$$\delta\beta = \beta^{\mathrm{XC}*} - \beta^{\mathrm{TR}*}. \tag{21}$$

as the difference in the equilibrium payoffs of the buyers in the two marketplaces. When $\delta\beta$ is positive (negative) the buyer has an incentive to join (quit) the Exchange.

As mentioned earlier, sellers transact either in TR or in both media (TR and XC). We call seller μ to belong to class-I if she only transacts in TR, while to class-II if she transacts both in TR and XC. Due to the buyers' dichotomy the pool of buyers is different in the two channels. Obviously class-II sellers have a larger pool. Sellers in the two classes have the following profit functions:

$$\pi_\mu^I = p_\mu^{\mathrm{TR}} V_\mu^{\mathrm{TR}} \tag{22}$$

$$\pi_\mu^{II} = p_\mu^{\mathrm{TR}} V_\mu^{\mathrm{TR}} + \varepsilon p_\mu^{\mathrm{XC}} V_\mu^{\mathrm{XC}} - z \tag{23}$$

where marginal costs are omitted as before, $0 < \varepsilon < 1$ encompasses the overall effects of *volume-dependent* factors such as transaction commissions and savings in process costs, while $z > 0$ represents the *volume-independent* cost arising from subscription and software fees.[9] V_μ^{TR} and V_μ^{XC} denote the total volume seller μ expects to sell in the two channels. We define

$$\delta\pi = \pi^{II*} - \pi^{I*} = \varepsilon p^{\mathrm{XC}*} V^{\mathrm{XC}*} - z \tag{24}$$

as the difference in the equilibrium profit of a seller depending on whether she has joined the Exchange or not. When $\delta\pi$ is positive (negative) the seller has an incentive to join (quit) the Exchange.

Our goal is to formulate dynamic equations which model the evolution of the participation in the Exchange. Initially, when a new Exchange is created participation is small. Buyers and sellers join in if their discounted excess surplus and profit, $\delta\beta$ and $\delta\pi$, within the Exchange is higher then the required one-time investment (switching cost). We introduce the notation s_B

[9] Instead of the two constants ε and z, we have only introduced one, r, in Eq. (20) to represent the buyer's net cost. In that case a separation between volume-dependent and volume-independent factors is unnecessary, because the volume a buyer purchases is assume to be constant $\mathcal{V}$.

and s_S to denote the buyers' and sellers' respective switching costs associated with the *act* of joining the Exchange. These switching costs may arise from upfront investment in software, reorganization in the purchasing department, cutting ongoing contracts, etc. Notice that s_B and s_S are different from r and z and are always positive. For simplicity we assume that there are no switching costs when players quit the Exchange.

We assume that the dynamics of the Exchange is described by the following first order ordinary differential equations ("master equations"):

$$\begin{aligned} \frac{dN_B}{dt} &= w_B^{\mathrm{TR}\to\mathrm{XC}}(N_B^{\mathrm{tot}} - N_B) + w_B^{\mathrm{XC}\to\mathrm{TR}} N_B \\ \frac{dN_S}{dt} &= w_S^{\mathrm{TR}\to\mathrm{XC}}(N_S^{\mathrm{tot}} - N_S) + w_S^{\mathrm{XC}\to\mathrm{TR}} N_S \end{aligned} \tag{25}$$

where N_B and N_S are redefined to denote the number of buyers and sellers who are members of the Exchange at time t (we omit the t arguments), N_B^{tot} and N_S^{tot} denote the total number of buyers and sellers in the universe, and $w_{B,S}$ are transition rates. The fundamental concept behind these equations is the assumption that although players are basically identical in the marketplace, and thus have the very same assessment of the value the Exchange provides at a given time, they are on *different* individual timescales in realizing the new opportunity and in making their individual decisions. Therefore, when considering the transition rates we simply assume that i) flows into and out of the Exchange are mutually exclusive, and ii) the rates can be expanded in powers of the discounted utility difference and it is legitimate to only keep the lowest order (linear) term. Thus we postulate

$$\begin{aligned} w_B^{\mathrm{TR}\to\mathrm{XC}} &= (\delta\beta - s_B)\Theta(\delta\beta - s_B), \qquad w_B^{\mathrm{XC}\to\mathrm{TR}} = \delta\beta\,\Theta(-\delta\beta), \\ w_S^{\mathrm{TR}\to\mathrm{XC}} &= (\delta\pi - s_S)\,\Theta(\delta\pi - s_S), \qquad w_S^{\mathrm{XC}\to\mathrm{TR}} = \delta\pi\,\Theta(-\delta\pi), \end{aligned} \tag{26}$$

where $\Theta(x)$ is Heaviside's step function: $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ otherwise. Since $\delta\beta = \delta\beta(N_B, N_S)$ and $\delta\pi = \delta\pi(N_B, N_S)$ depend on N_B and N_S, the equations in Eq. (25) are, in fact, coupled nonlinear equations.[10]

[10]In Eq. (26) we implicitly assumed that the base rates of entry and exit are the same for buyers and sellers and are normalized to 1. This may not be the case in practice (for example buyers may be less sensitive to not joining the Exchange then sellers) but the qualitative insights do not change as a result of this simplification.

The dynamic system in (25)-(26) reflects the idea that buyers and sellers have a positive switching cost joining the Exchange. Doing business in the Exchange needs to bring sufficient value to buyers and sellers to overcome this switching cost. Once they have joined however, their ongoing excess surplus/profit is strictly positive and they are a finite "distance" away from the verge of quitting the Exchange. We will see that this feature of the model may lead to an Exchange that is stable (does not grow or decline) even though it only contains a relatively small proportion of market participants. Nevertheless, before indulging into the complexity of this situation, it is worth analyzing a special case of the model where switching costs are absent.

4.1 Evolution without switching costs

In the case of zero switching costs, $s_B = s_S = 0$, Eq. (26) suggests that there is an inflow of buyers (sellers) into XC when $\delta\beta$ ($\delta\pi$) is positive, and an outflow when it is negative. The flow rate of buyers (sellers) is only zero when $\delta\beta = 0$ ($\delta\pi = 0$). These conditions define two curves in the (N_S, N_B) plane along which buyers and sellers, respectively, are indifferent with respect to joining or quitting the Exchange. The crossing point of these indifference curves defines a fixed point of the dynamics.[11] Other fixed points arise at the border of the "phase space" rectangle whose corners are defined by: $N_B = 0$, $N_B = N_B^{\text{tot}}$, $N_S = 0$, $N_S = N_S^{\text{tot}}$. These latter fixed points are generated either by the zeros of the $w_{B,S}$'s or by the depletion of the pool of available buyers/sellers on the RHS of Eq. (25).

To make the analysis more specific, for the uniform fit distribution case we find:

Proposition 7 *Assume that $s_B = s_S = 0$, ξ is distributed uniformly in $[-\Delta/2, \Delta/2]$, and that the TR connectivity is k_B. Then the Exchange is in equilibrium (i.e no sellers or buyers move in or out of the Exchange) under the following three conditions:*

1. *"Low participation" (LP) fixed point: $N_S^{LP} = 0$ and $N_B^{LP} = 0$.*

[11] In a general setup the topology of the indifference curves may be more complex, and there may be several fixed points.

2. *"Critical mass" (CM) fixed point:*

$$N_S^{CM} = \frac{5 + (25 + 8(1 - 2C_1))^{1/2}}{2(1 - 2C_1)}, \qquad N_B^{CM} = \frac{z}{\varepsilon\Delta}(N_S^{CM})^2 \qquad (27)$$

where

$$C_1 = \frac{-4 - 9k_B - 6k_B^2 + k_B^3}{2k_B(k_B + 1)^2} + \frac{r}{\Delta}. \qquad (28)$$

3. *"High participation" (HP) fixed point:* $N_S^{HP} = \min(N_S^{\text{tot}}, zN_S^{\text{tot}}/\varepsilon\Delta)$ *and* $N_B^{HP} = N_B^{\text{tot}}$.

The proposition tacitly assumes that the parameters are such that $0 < N_S^{CM} < N_S^{\text{tot}}$ and $0 < N_B^{CM} < N_B^{\text{tot}}$. Incompatible cases are economically less relevant and will not be discussed here.

Proof:
We have to show that at the special points defined above $dN_B/dt = dN_S/dt = 0$. When the fit is uniformly distributed the equilibrium prices and volumes are

$$p^{\text{TR}*} = \frac{\Delta}{k_B}, \qquad p^{\text{XC}*} = \frac{\Delta}{N_S} \qquad (29)$$

and

$$V^{\text{TR}*} = \frac{N_B^{\text{tot}} - N_B}{N_S^{\text{tot}}}, \qquad V^{\text{XC}*} = \frac{N_B}{N_S} \qquad (30)$$

where we now suppose that each buyer buys exactly volume $\mathcal{V} = 1$ of the product. With these

$$\begin{aligned} \pi^{\text{I}} &= \frac{(N_B^{\text{tot}} - N_B)\Delta}{k_B N_S^{\text{tot}}}, \\ \pi^{\text{II}} &= \frac{(N_B^{\text{tot}} - N_B)\Delta}{k_B N_S^{\text{tot}}} + \varepsilon\frac{N_B\Delta}{N_S^2} - z, \\ \delta\pi &= \varepsilon\frac{N_B\Delta}{N_S^2} - z \end{aligned} \qquad (31)$$

and using Eq. (19) and first column of Table 1:

$$\beta^{\mathrm{TR}} = Q + \Delta \frac{-4 - 9k_B - 6k_B^2 + k_B^3}{2k_B(k_B+1)^2} \quad (32)$$

$$\beta^{\mathrm{XC}} = Q + \Delta \frac{N_S^2 - 3N_S - 2}{2N_S(N_S+1)} - r. \quad (33)$$

Both β^{TR} and β^{XC} are monotonically increasing functions of k_B and N_S. Recall, however, that k_B is the optimal buyer connectivity in TR, which is a fixed parameter now. Using these expressions we find

$$\delta\beta = \Delta \left[\frac{N_S^2 - 3N_S - 2}{2N_S(N_S+1)} - C_1 \right], \quad (34)$$

where C_1 is as defined in the proposition. The solution of the second order equation $\delta\beta = 0$ yields N_S^{CM} and through this the CM fixed point.

On the other hand, $\delta\beta(N_S = 0) < 0$ thus $N_B = 0$ is sufficient to ensure $dN_B/dt = 0$. This, together with the fact that the LP point is on the indifference curve for sellers is enough to guarantee that $N_B = N_S = 0$ is indeed a fixed point. Similar arguments are required to see that the HP point of the Proposition is also a fixed point. □

In order to understand the flow diagram on the (N_S, N_B) plane, we have to investigate the stability of the fixed points. We find

Proposition 8 *The LP and HP fixed points are stable, while the CM fixed point is a saddle point.*

Proof: See Appendix.

For reasonable parameter values a numerically calculated flow diagram is depicted in Figure 2. Since CM is a saddle point, there exists two eigendirections which direct the flow around CM: the stable eigendirection defines the separatrix which divide the phase space into two domains. When the initial condition of the dynamic equations is given in the lower left side (LP side) of the separatrix the system flows into the LP fixed point. On the other hand, initial condition on the upper right side (HP side) of the separatrix evolve towards the HP fixed point. Only if we start exactly on the separatrix, can we end up at the unstable CM fixed point.

What is the meaning of this structure? It is reasonable to identify the separatrix with the "critical mass" or "critical liquidity" of the Exchange. Clearly, if the Exchange manages to sign up enough buyers and/or sellers so that the Exchange starts to evolve on its own from above the separatrix, then this evolution will lead to high participation. In other words, under this scenario, eventually all sellers and buyers will transact in the Exchange. In contrast, if initial marketing efforts only manage to guarantee a starting point that is below the separatrix then the Exchange will gradually shrink as buyers and sellers will quit for the traditional medium. It is important to realize that the critical mass of an exchange is a curve (a relationship between the critical number of buyers and sellers) rather than just two independent values.

If we consider the Exchange as an active economic agent whose marketing problem is to steer the Exchange to an initial point after which it can evolve to the HP fixed point, then this agent has several choices. In particular, it can choose to entice (subsidize) buyers or sellers. Given the nature of the critical mass, the Exchange has to decide on how to tradeoff the allocation of marketing resources between buyers or sellers. This choice will clearly depend on two things: (i) the response to marketing effort by buyers and sellers and (ii) the dynamics of the Exchange once it evolves without subsidies. While the model cannot say anything about the first factor, inspecting Figure 2 helps us understand the second one. The model suggests that the dynamics are asymmetric across buyers and sellers (the arrows on the left side of the diagram are longer than the ones on the right side).[12] In particular, for a given number of buyers the sellers tend to join the exchange much faster than buyers would for the same number of sellers. This suggests that for comparable marketing response, the exchange is better off marketing initially to buyers. Sellers are likely to follow quickly and the Exchange can expect a faster evolution to the HP fixed point.

The qualitative behavior we have found above for the uniform distribution is

[12]This feature is not specific to the parameters used in generating Figure 2 but it is robust across the parameter space.

rather robust. We expect roughly identical flow diagrams for other reasonable distributions. An explicit check for the Gaussian fit distribution has been carried out (but not shown here).

4.2 Evolution with switching costs

The situation gets complicated in the presence of switching costs. In this case, we can define two pairs of indifference curves: one pair for joining the Exchange when the buyer/seller is not a member yet, and another pair for quitting when they are members already. Since we neglect switching costs associated with quitting the Exchange, the indifference curves for quitting are the same as in the former subsection. The critical lines for joining, however, are shifted towards the HP fixed point. Formally, their location is determined by the same formulae as above with the replacement $r \to r + s_B$ and $z \to z + s_S$. The situation is demonstrated in Figure 3.

The new features of the flow diagram can be summarized by the following proposition:

Proposition 9 *In the case of nonnegligible switching costs*

1. *the CM fixed point develops into a finite region of fixed points, and*
2. *the critical mass line broadens into a critical stripe of finite width.*

Proof: The proposition is rather intuitive. When N_B and N_S happens to be in the region limited by the four indifference curves the buyers and sellers already in the Exchange have no incentive to leave the Exchange, since their excess profit $\delta\beta$ and $\delta\pi$ is positive. However, players outside the Exchange have no incentive to join either, because $\delta\beta - s_B$ and $\delta\pi - s_S$ are still negative. Thus the number of participants does not change in time.

On the other hand, the critical stripe is defined by the points which flow into the above region of fixed points. Since this is an extended object on the two-dimensional (N_B, N_S) plane, we necessarily obtain an extended stripe of such points. □

The fact that the CM fixed point becomes a finite region in the presence of switching costs is quite interesting and has important practical implications.

It basically means that the exchange can remain in a state where it does not evolve in any direction (a so-called "stalled exchange"). Recognizing this situation is critical because under this scenario high participation can only be assured if further marketing effort is used to increase participation beyond the limits of the CM region.

The critical mass of the Exchange in this case can be naturally defined as the upper/right boundary of the stripe that leads into the CM region. How can we interpret the critical stripe? It can be naturally associated with the notion of the "chasm" introduced by Moore (1999). Indeed, the presence of the stripe creates a situation where the exchange can experience a certain degree of success without really being able to corner the market. If marketing effort pushes the exchange beyond the critical mass frontier however, the exchange rapidly evolves into a situation where all market participants transact in it.

5 Conclusion

In this paper we have developed a model of B2B exchanges. We have described the interaction of buyers and sellers within the exchange with a differentiated product market that reduces buyers' search cost to the degree that buyers can afford to connect to all sellers. Next, we have built a dynamic system where market participants join (quit) the exchange based on their surplus/profit.

The goal of building such a model was twofold. First, we tried to put rigorous structure to some of the terminology (used by managers as well as academics) to describe the scope of B2B exchanges. Within our framework we could provide precise definitions to the notions of market fragmentation and critical mass or liquidity of the exchange.

Second, we hoped to gain managerial insights on how to promote and bring to succes a newly born B2B exchange. With respect to the second goal, we found that prices within an exchange may not necessarily decrease with lower sarch costs (or increased buyer-seller connectivity). In particular, when the fit between buyers' needs and sellers' offerings is uncertain (i.e. extreme values of fit or mis-fit are not rare) then prices may increase. In contrast, we found that irrespective of the nature of fit between buyers and sellers, buyer surplus always increases as search costs decrease.

With respect to dynamics, we identified the key driving forces which can motivate players to join the exchange. In case of buyers there are two components: one is, of course, dwindling prices (however note the fat-tail distribution case!), the other being equally important is the substantial improvement in the matching process the exchange brings about. For sellers there are quite different forces in action: they join the exchange (beside keeping their presence in the traditional medium too) mainly for maintaining relationship with and not losing potential buyers who have migrated to the new medium. We found that the rate at which buyers and sellers join the exchange is asymmetric. Sellers tend to react fast to an increase in the number of buyers while buyers are slower in reacting to increasing seller participation. This suggests that for identical marketing response the exchange is better off marketing to (subsidizing) buyers than sellers. We have also found that in the presence of switching costs the dynamic system exhibits hystereses. This means that it is possible to have an exchange that is stable in size in which only part of the market interacts. Market participants in the exchange do not have an incentive to leave, but buyers and sellers outside the exchange do not find it beneficial to join either. Such a situation may resonate to the notion of "chasm" advanced by Moore (1999).

The analysis we presented in this paper had been built on several simplifying assumptions. Indeed, our general marketplace model is a so-called "mean field" model, which treats the players and their relationships almost completely homogeneously. We assumed that all buyers and sellers have identical utility functions and that the connectivity structure is sufficiently randomized in agent space (and in time) to allow a simple mathematical formalism.

In this respect our model could be extended in a number of directions. First, it may be possible to study models with heterogeneous agents (e.g., buyers purchasing different quantities, having different levels of risk aversion; having different valuation for quality, etc; or sellers being differentiated in the quality of their products/services, etc.) Second, and partly in connection with the first, the topology of buyer-seller connectivity may get some additional structure like predefined buyer preferences for a set of "trusted" sellers, or for sellers with lower/higher quality, etc. Third, realizing that in the present paper our model of buyer-seller interaction is in fact a reduced form model only, it could be interesting to explicitly model the actual market mechanism that the exchange proposes to participants (auction, demand aggregation, etc.). Finally, the paper only considered the competition between one Exchange and the traditional marketplace. It would be interesting to investigate what would occur when several exchanges compete for critical liquidity. Does such competition always lead to winner-take-all situations or is it conceivable that competing exchanges can co-exist? We left these questions for future research.

Appendix

Before the proofs let us define

$$I(k) := k(k-1)\int_a^b f(\xi)^2 F(\xi)^{k-2} d\xi, \tag{35}$$

where $\int_a^b f(\xi)^2 d\xi < \infty$ is assumed to ensure that $I(k)$ exists for any $k \geq 2$. With this $p^*(k_B) = 1/I(k_B)$. In the following we will treat k as a continuous variable. After the change of variable $u = F(\xi)$ we obtain

$$\begin{aligned} I(k) &= k(k-1)\int_0^1 du\, \phi(u) u^{k-2}, && (36)\\ I'(k) &= \int_0^1 du\, \phi(u) u^{k-2}[2k-1+k(k-1)\ln u], && (37)\\ I''(k) &= \int_0^1 du\, \phi(u) u^{k-2}[2+2(2k-1)\ln u + k(k-1)\ln^2 u], && (38)\end{aligned}$$

where

$$\phi(u) \equiv f(F^{-1}(u)), \qquad u \in [0,1]. \tag{39}$$

Analyticity of $f(\xi)$ in (a,b) implies analyticity of $\phi(u)$ in $(0,1)$. Whereas ϕ remains bounded at the boundaries, $\phi(0) = f(a)$ and $\phi(1) = f(b)$, its derivatives may (and in some of the cases discussed here actually do) diverge there. Note that

$$\begin{aligned} \phi'(u) &= \left.\frac{f'(\xi)}{f(\xi)}\right|_{\xi=F^{-1}(u)} = \partial_\xi \ln f(\xi)\big|_{\xi=F^{-1}(u)}, && (40)\\ \phi''(u) &= \left.\frac{f''(\xi)f(\xi) - f'(\xi)^2}{f(\xi)^3}\right|_{\xi=F^{-1}(u)} = \left.\frac{\partial_\xi^2 \ln f(\xi)}{f(\xi)}\right|_{\xi=F^{-1}(u)}. && (41)\end{aligned}$$

and

$$F^{-1}(u) = \int_{1/2}^u \frac{1}{\phi(u)} + \text{const}, \tag{42}$$

the latter providing us with a way to reconstruct the original distribution (up to an irrelevant constant shift) if $\phi(u)$ is given.

Proof of Proposition 2:

In order to prove monotonicity of $p^*(k)$ we need to analyze

$$\frac{dp^*(k)}{dk} = -\frac{I'(k)}{I(k)^2} \tag{43}$$

where $I'(k)$ is given in Eq. (37). Since the denominator is always positive, monotonic decrease (increase) of p^* is ensured if $I'(k) > 0$ ($I'(k) < 0$). After integrating by parts two times we obtain

$$I'(k) = \Big[\phi(u)u^{k-1}(1 + k\ln u)\Big]_0^1 - \Big[\phi'(u)u^k \ln u\Big]_0^1 + \int_0^1 du\, \phi''(u)u^k \ln u \tag{44}$$

Using our assumption that ϕ is bounded the first term reduces to $\phi(1) = f(b)$. The second term is less evident, since $\phi'(u)$ may diverge at the boundary. The following Lemma, however, implies that it is zero.

Lemma 1 *Suppose that* $\psi(u)$ *is analytic and bounded in* $(0,1)$, $\lim_{u\to 0^+}\psi(u) = \psi(0) < \infty$ *finite, and* $\lim_{u\to 0^+}\psi'(u) = \infty$. *Then* $\lim_{u\to 0^+}\psi'(u)u = 0$, *i.e.,* $\psi'(u) = o(1/u)$ *as* $u \to 0^+$.[13]

Proof of Lemma: It is obvious that $\lim_{u\to 0^+}\psi'(u)u \geq 0$. Suppose that $\lim_{u\to 0^+}\psi'(u)u > 0$. This means that there exists $\epsilon > 0$ and $\delta > 0$ such that $\psi'(u) > \epsilon/u$ for $u \in (0,\delta)$. Since $\psi(u)$ is bounded and analytic we can write

$$\infty > \psi(\delta) = \psi(0) + \int_0^\delta \psi'(u)du > \psi(0) + \int_0^\delta \frac{\epsilon}{u}du \tag{45}$$

However, the last term diverges, which is a contradiction. □

Now consider first the lower boundary. Since $\lim_{u\to 0^+}\phi'(u)u^k \ln u = -\lim_{u\to 0^+}\phi'(u)u^k$ the Lemma implies that it is zero for $k \geq 1$. For the upper boundary $\lim_{u\to 1^-}\phi'(u)u^k\ln u = -\lim_{v\to 0^+}\phi'(1-v)v$ which is again zero by the Lemma recalling that $\lim_{v\to 0^+}\phi(1-v) = \phi(1)$ is finite.

[13] In the following we will frequently use the standard mathematical notations: $g(x) = \mathcal{O}[h(x)]$ as $x \to c$, if there exists a region around $x = c$ in which $|g(x)/h(x)|$ is bounded, and $g(x) = o[h(x)]$ as $x \to c$, if $\lim_{x\to c} g(x)/h(x) = 0$.

After the above analysis $I'(k)$ reduces to

$$I'(k) \;=\; \phi(1) + \int_0^1 du\, \phi''(u)\, u^k \ln u. \tag{46}$$

In virtue of Eq. (41) and the fact that $f(\xi) > 0$ the premise of the proposition implies that $\phi''(u) < 0$ ($\phi''(u) > 0$) for all $u \in (0,1)$. Since $u^k \ln k < 0$ for all $u \in (0,1)$ this assures that the integral in Eq. (46) is positive (negative). Notice that this is only a sufficient but not a necessary condition. In Case-B1/B2/B3 it is the integral alone which determines the sign of I', but for Class-A there is another positive term $\phi(1) > 0$ present. Thus in the latter case the proposition is weaker. $\Box$

Proof of Proposition 3:

For large k the major contribution to $I(k)$ and its derivatives in Eqs. (36-38) is coming from values near $u \approx 1$. Contributions near $u \approx 0$ are exponentially suppressed due to the factor u^{k-2} in the integrals and do not influence the $k \to \infty$ limit. In order to see the asymptotic behavior of $I(k)$ we make another transformation, introducing $u = 1 - z/k$. With this

$$I(k) \;=\; (k-1) \int_0^k dz\, \phi(1 - \frac{z}{k}) \left(1 - \frac{z}{k}\right)^{k-2}. \tag{47}$$

It is legitimate to take the large k limit of the integrand and neglect an exponentially small error by moving the upper integration limit to infinity. This yields

$$I(k) = k \int_0^\infty dz\, \phi(1 - \frac{z}{k})\, e^{-z} \left[1 + \mathcal{O}(\frac{1}{k})\right], \tag{48}$$

which is valid asymptotically for large k. After integrating by parts and omitting the subleading correction term

$$I(k) \;\approx\; k\phi(1) - \int_0^\infty dz\, \phi'(1 - \frac{z}{k})\, e^{-z}. \tag{49}$$

Differentiating Eq. (49) with respect to k or alternatively taking the large k limit of Eq. (46) directly gives

$$I'(k) \;\approx\; \phi(1) - \frac{1}{k^2} \int_0^\infty dz\, \phi''(1 - \frac{z}{k})\, z\, e^{-z}. \tag{50}$$

Using these formulae, in the following we evaluate the large k limit of $p^*(k)$ and $p^{*\prime}(k)$ for the different function classes separately.

Class-A: The integrals on the RHSs of Eqs. (49) and (50) exist since $\phi(1)$ is finite by assumption. This fact limits the possible type of divergence of $\phi'(u)$ and $\phi''(u)$ as $u \to 1$. Indeed Lemma 1 ensures that $\phi'(u) = o[(1-u)^{-1}]$ for $u \to 1$, i.e., if it diverges slower than $(1-u)^{-1}$. Similarly, $\phi''(u) = o[(1-u)^{-2}]$. However, since k appears in the combination z/k in the integrals, this also means that the integrals on the RHSs of Eqs. (49) and (50) are at most $o(k)$ and $o(1)$ as $k \to \infty$,[14] respectively. Thus $I(k) = kf(b) + o(k)$ and $I'(k) = f(b) + o(1)$, which allows us to write

$$\text{Class} - \text{A}: \quad p^*(k) = \frac{1}{kf(b)} + o(k^{-1}), \tag{51}$$

$$p^{*\prime}(k) = -\frac{1}{k^2 f(b)} + o(k^{-2}) \tag{52}$$

The correction terms decay faster then the leading ones and this ensures that there exists K large enough such that for all $k > K$ they can be safely neglected. Clearly, in the large k limit p^* and $p^{*\prime}$ go to zero monotonically, as stated in the proposition.

Class-B1: In this case we have $\phi(1) = 0$, and the leading order behavior is determined by the integrals. In virtue of Eq. (40), the divergence of ϕ', $\lim_{u\to 1} \phi'(u) = -\infty$, is in fact the defining condition of the class. This condition also implies that $D = \lim_{u\to 1} \phi''(u) = -\infty$. Besides the $o(k)$ and $o(1)$ upper bounds on $I(k)$ and $I'(k)$ shown above, we can also prove that necessarily $I(k) \to \infty$ [but, of course, slower than $\mathcal{O}(k)$]. To show this we write

$$I(k) \approx -\lim_{k\to\infty} \int_0^\infty dz\, \phi'(1-\frac{z}{k})e^{-z} > -\lim_{k\to\infty} \int_0^\infty dz\, \phi_c'(1-\frac{z}{k})e^{-z} = c > 0 \tag{53}$$

where by definition $\phi_c'(u) := \max\{\phi'(u), -c\}$ is the function ϕ' chopped at $-c < 0$ near the singularity. Since $\lim_{u\to 1} \phi'(u) = -\infty$, for any $c > 0$ arbitrarily large there exists a finite interval $u \in [1-\delta, 1]$ such that $\phi_c'(u) \equiv -c$. This interval expands into the whole $z > 0$ half-line for $\phi_c'(1-z/k)$ as $k \to \infty$

[14] $o(1)$ is a shorthand for a term which goes to zero as $k \to \infty$.

justifying the last equality of Eq. (53). Since c is arbitrarily large, the integral should diverge in k as was claimed. In short we have proven that $I(k)$ goes to infinity, but sublinearly. For $I'(k)$ Eq. (50) and the fact that $D < 0$ assures that $I'(k) > 0$ for large enough k. Using the $o(1)$ upper bound found above we see that $I'(k)$ must go to zero eventually. In terms of p^* these mean that for large enough k values $p^*(k) \to 0$ monotonically as $k \to \infty$ but slower than $1/k$, and $\lim_{k\to\infty} p^{*\prime}(k) = 0$.

Example: It is illustrative for Class-B1 to consider in some detail the Gaussian case in Table 1. First we have to determine $\phi(u) \equiv f(F^{-1}(u))$ for $u \approx 1$. For large ξ we have (Abramowitz and Stegun, 1965):

$$u = F(\xi) = 1 - \frac{e^{-\xi^2/2}}{\sqrt{2\pi}\,\xi}\left[1 + \mathcal{O}(\xi^{-2})\right]. \qquad (54)$$

Taking the logarithm of both sides leads to the relation

$$\xi^2 = -\ln[2\pi(1-u)^2] - \ln\xi^2 + \mathcal{O}(\xi^{-2}) \qquad (55)$$

which can be solved for ξ iteratively: $\xi_n = -\ln[2\pi(1-u)^2] - \ln\xi_{n-1}^2$ with $\xi_0 = 1$. For our purposes it is enough to calculate the first two steps, which yields:

$$\xi^2 = -\ln[2\pi(1-u)^2] - \ln\ln[2\pi(1-u)^2] + \mathcal{O}\left(\frac{\ln\ln[2\pi(1-u)^2]}{\ln[2\pi(1-u)^2]}\right). \qquad (56)$$

From this (omitting the correction term)

$$\phi(u) \approx (1-u)\ln^{1/2}\frac{1}{2\pi(1-u)^2} \qquad (57)$$

valid for $u \approx 1$, and thus

$$\phi\left(1-\frac{z}{k}\right) \approx \frac{z}{k}\ln^{1/2}\frac{k^2}{2\pi z^2} \approx \frac{z}{k}\ln^{1/2}k^2 + \mathcal{O}\left(\frac{1}{\ln^{1/2}k^2}\right) \qquad (58)$$

valid for large k. Using this in Eq. (48) we finally obtain that for large k

$$p^*(k) \approx \frac{1}{\ln^{1/2}k^2} + \mathcal{O}\left(\frac{1}{\ln^{3/2}k^2}\right) \qquad (59)$$

in accordance with Table 1. Obviously, $\lim_{k\to\infty} p^*(k) = \lim_{k\to\infty} p^{*\prime}(k) = 0$ and these limits are taken monotonically, but logarithmically slowly, for large enough k.

Class-B2: In case of Class-B2 we can integrate Eq. (49) by parts once again

$$I(k) \quad \approx \quad -\phi'(1) + \frac{1}{k}\int_0^\infty dz\, \phi''(1-\frac{z}{k})\, e^{-z}. \tag{60}$$

The RHS exists, since $\phi'(1) = -C = \text{const}$ for Class-B2. As before, Lemma 1 implies that $\phi''(1-z/k)$ is $o(1/z)$ as $z \to 0$. Since k appears in the combination z/k this implies that the correction term is $o(1)$ and decays to zero as $k \to \infty$. Thus $\lim_{k\to\infty} p^*(k) = 1/C$. This limit is approached monotonically from above or from below for large enough k depending on the sign of D. As can be read of from Eq. (60), $I(k) \approx C + D/k$, i.e., $p^*(k) \approx 1/C - D/(C^2k)$, $p^{*\prime}(k) \approx D/(C^2k^2) \to 0$ as $k \to 0$.[15] The sign of $p^{*\prime}$ depends on the sign of D.

Class-B3: For Class-B3 $\phi(1) = \phi'(1) = 0$ in Eq. (60). This is only compatible with $D \geq 0$. If $D = 0$ we can integrate Eq. (60) by parts until we reach the first nonvanishing derivative $\phi^{(n)}(1) > 0$ for some $n > 2$ giving $I(k) \approx (-)^n k^{1-n}\int_0^\infty dz\, \phi^{(n)}(1-z/k)\, e^{-z} \approx (-)^n k^{1-n}\phi^{(n)}(1)$, which converges to zero for large k. If the first nonvanishing derivative $\phi^{(n)}(1) = \infty$, this expression does not hold. Nevertheless, Lemma 1 implies that $\phi^{(n)}(1-z/k) = o(1/z)$ as $z \to 0$, hence $I(k) = o(k^{2-n})$. If $\phi''(1) = D$ is already finite the result is $I(k) \approx D/k$. In any case $I(k)$ tends to zero and thus $p^*(k) \to \infty$ as $k \to \infty$.

Example: In the conceptually important case of Pareto distributions $\phi(1 - z/k) \sim (z/k)^{\alpha/(\alpha-1)}$. Indeed, inserting this expression into Eq. (42) leads to $1 - F(\xi) \sim \xi^{1-\alpha}$, and thus $f(\xi) \sim \xi^{-\alpha}$ for large ξ.[16] While this gives $\phi(1) = \phi'(1) = 0$, the second derivative diverges in z as $\phi(1-z/k) \sim (z/k)^{(2-\alpha)/(\alpha-1)}$. Using Eq. (60) this yields $I(k) \sim k^{1/(1-\alpha)}$, i.e., $p^*(k) \sim k^{1/(\alpha-1)}$ for large k as given in Table 1. □

[15] If $D = \pm\infty$ we can only claim that $p^*(k) \approx 1/C + \text{sign}(D)\, o(1)$

[16] $M(1) = \int_a^b d\xi\, \xi f(\xi)$ in Eq. (16) only exists if $\alpha > 2$ which we can assume.

Proof of Proposition 4:

After the change of variable $u = F(\xi)$ the function $M(k)$ and its derivatives can be written as

$$M(k) = k \int_0^1 du\, F^{-1}(u) u^{k-1}, \tag{61}$$

$$M'(k) = \int_0^1 du\, F^{-1}(u) u^{k-1}(1 + k \ln u), \tag{62}$$

$$M''(k) = \int_0^1 du\, F^{-1}(u) u^{k-1} \ln u\, (2 + k \ln u). \tag{63}$$

$M(1)$ exists iff $\int_0^1 du\, F^{-1}(u)$ exists, which then implies that for all k the integrals in Eqs. (61)-(63) exist. Notice that the function u^{k-1} is positive for $u \in (0, 1)$, but $u^{k-1}(1 + k \ln u)$ and $u^{k-1} \ln u(2 + k \ln u)$ changes sign within the interval. Moreover, $F^{-1}(u)$ is analytic in $(0, 1)$, it increases monotonically, and $\lim_{u \to 1} F^{-1}(u) = b$.

First we show that $M(k)$ increases monotonically, i.e., $M'(k) > 0$ for all k. Rewrite $M'(k)$ as

$$M'(k) = F^{-1}(u_0) \int_0^1 du\, u^{k-1}(1 + k \ln u) \tag{64}$$

$$+ \int_0^1 du\, [F^{-1}(u) - F^{-1}(u_0)] u^{k-1}(1 + k \ln u) \tag{65}$$

where $0 < u_0(k) < 1$ is the point where $u^{k-1}(1 + k \ln u)$ changes sign for a given k. For $u > u_0$ ($u < u_0$) the expression $u^{k-1}(1 + k \ln u)$ is positive (negative). While the first integral is zero for any k as can be checked explicitly, the second is positive by the very definition of u_0 and the monotonic increase of $F^{-1}(u)$. Thus $M'(k) > 0$ for all k.

The claim that $M(k)$ is concave, i.e., $M''(k) < 0$ for all k can be proven similarly. In this case we write the integral as

$$M''(k) = F^{-1}(u_1) \int_0^1 du\, u^{k-1} \ln u(2 + k \ln u) \tag{66}$$

$$+ \int_0^1 du\, [F^{-1}(u) - F^{-1}(u_1)] u^{k-1} \ln u(2 + k \ln u), \tag{67}$$

where $0 < u_1(k) < 1$ is the point where $u^{k-1} \ln u(2 + k \ln u)$ changes sign for a given k. Again the first integral is identically zero, and the second is negative in virtue of the definition of u_1 and the monotonicity of $F^{-1}(u)$.

In order to see the $k \to \infty$ limit we calculate the large k asymptotic expressions. After introducing $u = 1 - z/k$ and evaluating the asymptotic form of the integrands we obtain

$$M(k) \approx \int_0^\infty dz\, F^{-1}(1 - \frac{z}{k})\, e^{-z}, \quad (68)$$

$$M'(k) \approx \frac{1}{k^2} \int_0^\infty dz\, \frac{1}{\phi(1 - \frac{z}{k})}\, ze^{-z}, \quad (69)$$

which are valid asymptotically for large k. In Eq. (69) we have used Eq. (42).

In case of Class-A, $\lim_{u \to 1} F^{-1}(u) = b < \infty$ which yields $\lim_{k \to \infty} M(k) = b$ through Eq. (68). Using the fact that $\phi(1) = f(b) > 0$ the asymptotic behavior of $M'(k)$ can be read of from Eq. (69) $M'(k) \approx 1/[f(b)k^2]$, which tends to zero monotonically.

For Class-B1/B2/B3 $\phi(1) = 0$. If $b < \infty$ then $\lim_{u \to 1} F^{-1}(u) = b < \infty$ which gives $\lim_{k \to \infty} M(k) = b$ through Eq. (68). If $b = \infty$ this mean that for any $c > 0$ there exists a region $u \in [1 - \delta, 1]$ such that $F^{-1}(u) > c$. Introducing $F_c^{-1}(u) = \min\{F^{-1}(u), c\}$ we can write

$$M(k) \approx \int_0^\infty dz\, F^{-1}(1 - \frac{z}{k})\, e^{-z} > \int_0^\infty dz\, F_c^{-1}(1 - \frac{z}{k})\, e^{-z} \to c > 0. \quad (70)$$

Since c is arbitrarily large we get $\lim_{k \to \infty} M(k) = b = \infty$. Although $M(k)$ increases without bounds, its derivative tends to zero for large k. The argument is that $\phi(1 - z/k)$ must decay to zero at $z = 0$ slower than z^2, otherwise the integral in Eq. (69) does not exist. However this also implies that $1/\phi(1 - z/k) = o(k^2)$ in Eq. (69), and thus $M'(k) = o(1)$ as $k \to \infty$. □

Proof of Proposition 5:

Consider Eq. (17). If we can prove that $\lim_{k \to \infty} M'(k) - p^{*\prime}(k) = 0$, and that there exists K such that for all $k > K$ the LHS is positie and decreases monotonically, i.e., $M''(k) - p^{*\prime\prime}(k) < 0$ then the proposition holds. The first statement is true by Propositions 3 and 4. Proposition 4 also claims that $M''(k) < 0$ always. Thus it is a sufficient condition (but by no means necessary) to have $p^{*\prime\prime}(k) > 0$. We can write:

$$p^{*\prime\prime}(k) = \frac{2I'(k)^2 - I(k)I''(k)}{I(k)^3}. \tag{71}$$

Since $I(k)$ and $I'(k)^2$ are positive for all k the condition $p^{*\prime\prime}(k) > 0$ is satisfied if either $I''(k) < 0$ or II'' becomes negligible with respect to I'^2 for large enough k. Differentiating Eq. (46) with respect to k we find

$$I''(k) \;=\; \int_0^1 du\, \phi''(u) u^k \ln^2 k \approx \frac{1}{k^3}\int_0^\infty dz\, \phi''(1-\frac{z}{k}) z^2 e^{-z}. \tag{72}$$

In virtue of Lemma 1, $\phi''(1-\frac{z}{k}) = o(z^2)$ as $z \to 0$ and thus it is $o(k^2)$ as $k \to \infty$. This implies that $I''(k) = o(1/k)$.

In case of Class-A we have seen that $I'(k) \approx f(b) > 0$ goes to a positive constant, whereas $I(k) \sim k$ and thus $I(k)I''(k) = o(1)$ goes to zero. This implies that there exists K large enough such that for all $k > K$ the II'' term can safely be neglected and $p^{*\prime\prime}(k) > 0$. For Class-B1 the proof is even easier: in this case $D = -\infty$ thus necessarily $I''(k) < 0$ for large enough k. In case of Class-B2, postulating $D < 0$ is a sufficient condition by the same reasoning.

Finally, for Class-B3 we necessarily have $D > 0$, consequently the above proof does not work. We find $p^{*\prime\prime}(k) < 0$ and $M''(k) - p^{*\prime\prime}(k)$ is a difference of two negative numbers. Although we cannot rigorously prove that the difference is always negative for large enough k, we see that this claim is true in the case of the Pareto distribution with $\alpha > 2$ in Table 1. Indeed, by explicitly calculating the integrals we find

$$M(k) - p(k) = k\, B\left(k, \frac{\alpha-2}{\alpha-1}\right) - \frac{1}{\alpha k B\left(k, \frac{\alpha}{\alpha-1}\right)} \approx c(\alpha) k^{\frac{1}{\alpha-1}}, \tag{73}$$

where

$$c(\alpha) = \Gamma\left(\frac{\alpha-2}{\alpha-1}\right) - \frac{1}{\alpha\Gamma\left(\frac{\alpha}{\alpha-1}\right)} > 0 \qquad \text{for all } \alpha > 2. \tag{74}$$

The last expression in Eq. (73) is valid for large k. From this we see that $M''(k) - p''(k) \approx -c(\alpha)\alpha/(\alpha-1)^2 k^{-(2\alpha-1)/(\alpha-1)}$ is negative thus the proposition holds. □

Proof of Proposition 8:

We start by showing that the CM fixed point is a saddle point. The standard method is to linearize the flow equations (25) around the fixed point and analyze the eigenproblem of the resulting linear equations. An extra complication in the present case is that the RHS of Eq. (25) is not analytic along the indifference curves. The two indifference curves cut the phase plain into four quadrants (which we denote by Q-I, Q-II, Q-III, and Q-IV, respectively, starting from the upper right quadrant, and proceeding counter-clockwise), and the RHS depends on the quadrant in question. After introducing the relative variables $n_B = N_B - N_B^{\mathrm{CM}}$ and $n_S = N_S - N_S^{\mathrm{CM}}$ the structure of the equations for n_S, n_B small is

$$\begin{aligned} \frac{dn_B}{dt} &\approx \delta\beta(n_B, n_S)\; c_B, \\ \frac{dn_S}{dt} &\approx \delta\pi(n_B, n_S)\; c_S, \end{aligned} \tag{75}$$

where up to linear order

$$\begin{pmatrix} \delta\beta \\ \delta\pi \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} n_B \\ n_S \end{pmatrix} \tag{76}$$

with

$$\begin{aligned} A_{11} &= 0, \qquad A_{12} = \frac{1 + 2N_S^{\mathrm{CM}} + (N_S^{\mathrm{CM}})^2}{(N_S^{\mathrm{CM}})^2\,[1 + (N_S^{\mathrm{CM}})^2]}, \\ A_{21} &= \frac{\Delta\,\epsilon}{(N_S^{\mathrm{CM}})^2}, \quad A_{22} = -\frac{2\Delta\,\epsilon\,N_B^{\mathrm{CM}}}{(N_S^{\mathrm{CM}})^3}. \end{aligned} \tag{77}$$

The constants are $c_B = N_B^{\mathrm{CM}}$ or $N_B^{\mathrm{tot}} - N_B^{\mathrm{CM}}$, and $c_S = N_S^{\mathrm{CM}}$ or $N_S^{\mathrm{tot}} - N_S^{\mathrm{CM}}$, depending on the quadrant in question. Although the positive constants c_B and c_S change the quantitative behavior near the fixed point, they only mean a sign-conserving rescaling of the flow velocity. Thus the general structure of the fixed point can be understood by formally setting $c_B = c_S = 1$ in Eq. (75). In this case stability is determined by the matrix A alone. Since $A_{12} >$, $A_{21} > 0$ and $A_{22} < 0$ we get $\det(A) < 0$. This immediately implies that the CM fixed point is a saddle point (see, e.g., Jordan and Smith 1987).

Recalling that the flow is exactly "horizontal" along the indifference curve of buyers ($dn_B/dt \sim \delta\beta = 0$), and "vertical" along that of sellers ($dn_S/dt \sim \delta\pi = 0$), we can directly see that the unstable eigendirection is situated in Q-I and Q-III, while the stable eigendirection (separatrix) in Q-II and Q-IV.

The claims that the LP and HP fixed points are stable can be proven similarly.
□

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	Uniform	Gaussian	Exponential	Pareto
Class	A	B1	B2	B3
Support	$[-\Delta/2, \Delta/2]$	$(-\infty, \infty)$	$[0, \infty)$	$[1, \infty)$
$f(\xi)$	$1/\Delta$	$\dfrac{e^{-\xi^2/2\sigma}}{\sqrt{2\pi\sigma}}$	$Ce^{-C\xi}$	$(\alpha - 1)\xi^{-\alpha}$
$p^*(k)$	Δ/k	no closed form	$1/C$	$\dfrac{1}{\alpha k B\left(k, \frac{\alpha}{\alpha-1}\right)}$
	$\sim 1/k$	$\sim 1/\ln^{1/2} k$	$\sim$ const	$\sim k^{1/(\alpha-1)}$
$\max_{\mu=1}^{k} \xi_\mu$	$\dfrac{\Delta}{2}\dfrac{k-1}{k+1}$	no closed form	$\gamma + \psi(k+1)$	$k\, B\left(k, \frac{\alpha-2}{\alpha-1}\right)$
	$\sim$ const	$\sim \ln^{1/2} k$	$\sim \ln k$	$\sim k^{1/(\alpha-1)}$

Table 1: Some important "fit" distributions. B denotes the beta function, γ is Eulers's constant, ψ is the digamma function. Leading asymptotic terms for large k are also given.

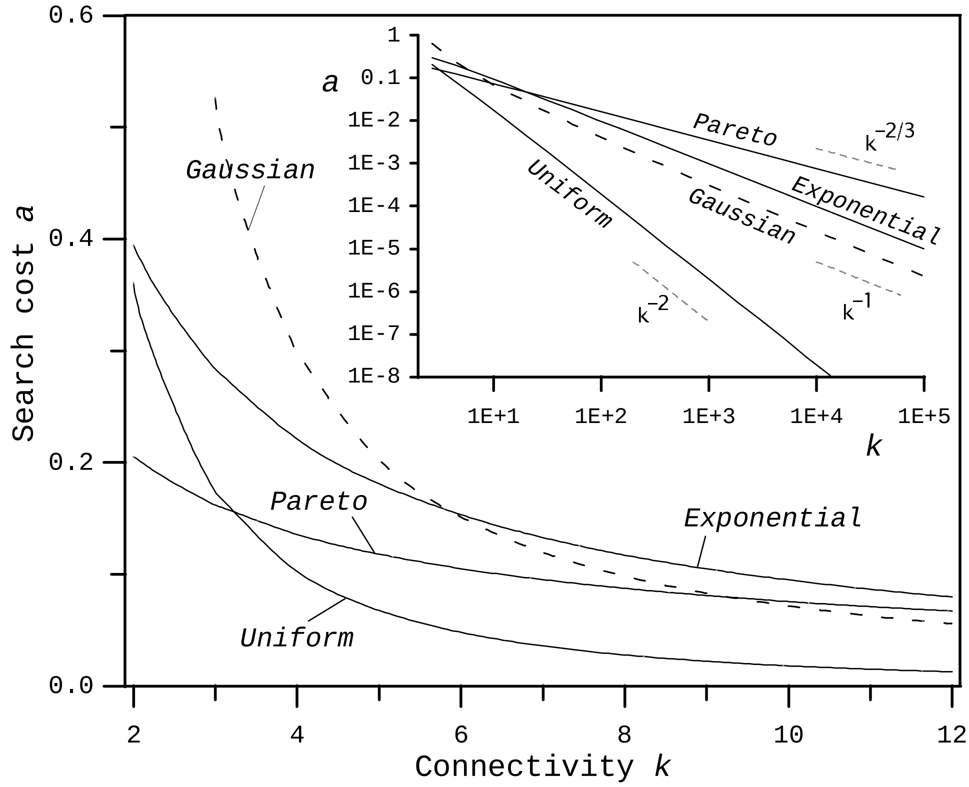

Figure 1: Search cost a vs optimal connectivity k calculated numerically for the four probability distributions of Table 1. The parameters are $\Delta = 1$, $\sigma = 1$, $C = 1$, and $\alpha = 4$. Inset shows the large k behavior on a log-log plot, together with the expected asymptotic forms.

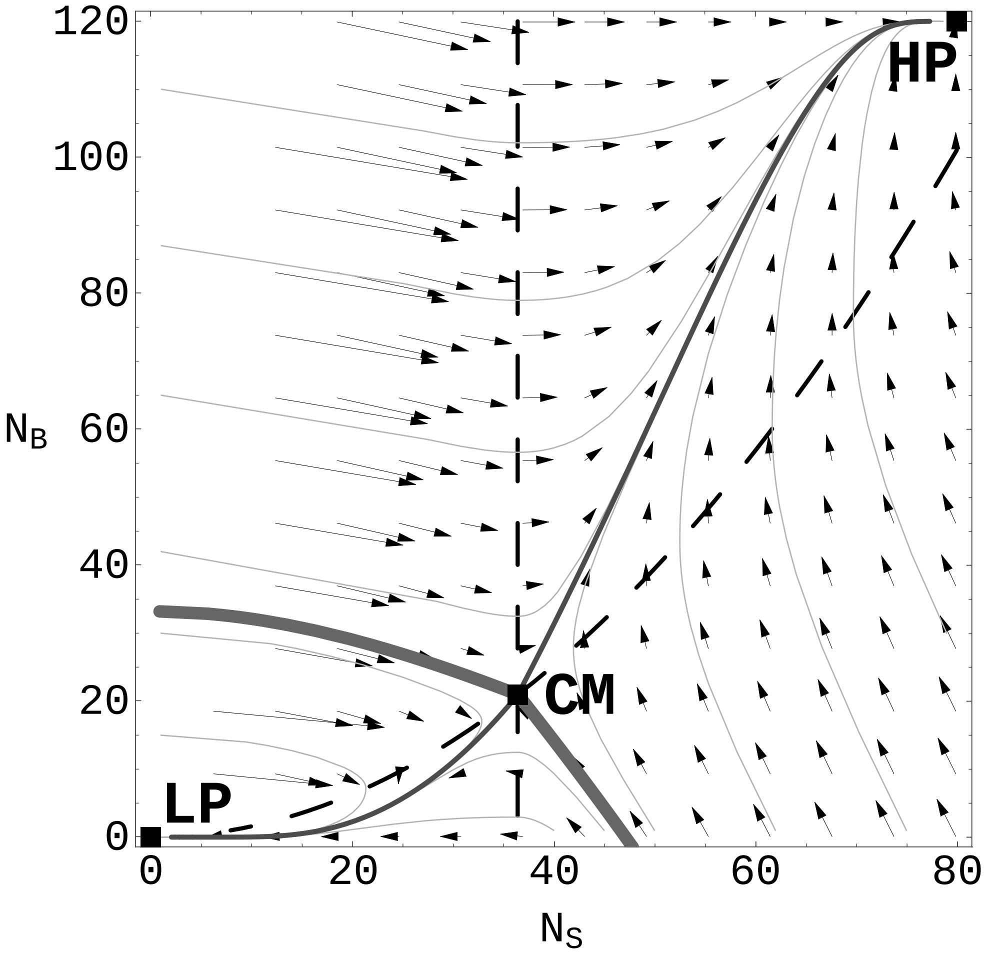

Figure 2: Flow diagram without switching costs for parameters $N_B^{\text{tot}} = 120, N_S^{\text{tot}} = 80; k_B = 3, r = 1.05, \varepsilon = .95$ and $z = .015$. Arrow length represents flow velocity. Black rectangles denote the fixed points: Low Participation (LP), Critical Mass (CM), and High Participation (HP). The dashed curves are the indifference lines. The thick grey curve is the "critical mass" curve.

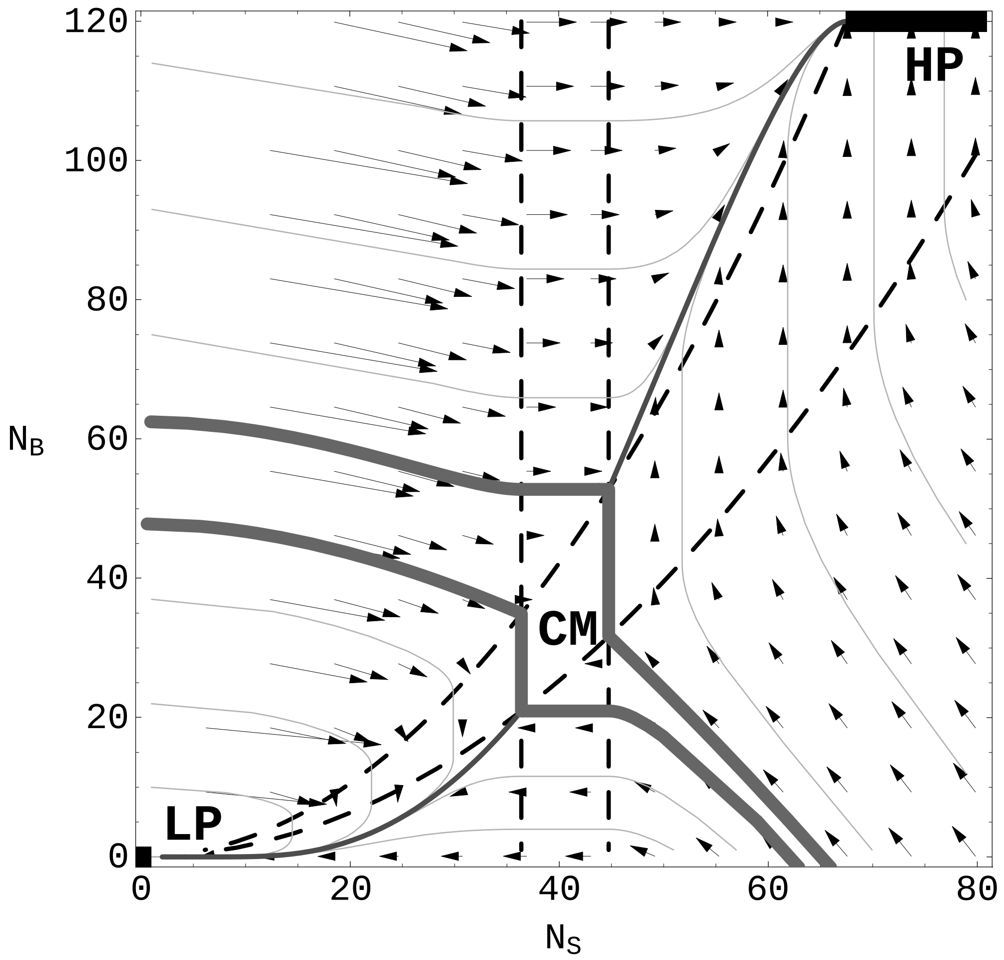

Figure 3: Flow diagram in the presence of switching costs for parameters $N_B^{\rm tot} = 120, N_S^{\rm tot} = 80; k_B = 3, r = 1.05, \varepsilon = .95, z = .015$, and $s_S = s_B = 0.01$. Arrow length represents flow velocity. Black rectangles denote fixed points/intervals: Low Participation (LP) and High Participation (HP). The dashed curves are the indifference lines for joining and quitting. The region between the two thick grey curves is the "critical stripe".