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ABSTRACT

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Two firms choose locations (non-wage job characteristics) on the interval [0,1] prior to announcing wages at which they employ workers who are uniformly distributed; the (constant) marginal revenue products of workers may differ. Subgame perfect equilibria of the two-stage location-wage game are studied under laissez-faire and under a minimum wage regime. Up to a restriction for the existence of pure strategy equilibria, the imposition of a minimum wage is always welfare-improving because of its effect on non-wage job characteristics.

JEL Classification: D43, E24, J48

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1. INTRODUCTION

There is a growing literature on the theory of oligopsony, and the insights it can offer into the operation of labour markets (Bhaskar and To (1999, 2003), Kaas and Madden (2008), Manning (2003)). Analysis of the impact of minimum wage legislation is a major theme in this literature, and it is the focus of our paper.

It is well-known that the monopsony market structure (unlike perfect competition) can produce increases in aggregate employment and social welfare after a minimum wage imposition, a result that provides explanation of some empirical claims, albeit via the perhaps extreme monopsony assumption (see Manning (2003)). Bhaskar and To (1999) extend the argument to a wage-setting oligopsony, with exogenous, horizontal differentiation of firms' non-wage job characteristics (symmetric locations around a Salop circular city populated by a uniform distribution of workers) and firms of equal efficiency. They show that social welfare improvements from the imposition of minimum wages again emanate from the aggregate employment channel (see also Walsh (2003)). Secondly, in a similar differentiated oligopsony but now with firms of heterogeneous efficiency, Bhaskar and To (2003) provide explanations of certain empirical features of wage distributions and the impact of minimum wages. There is no welfare analysis in this model, and neither paper addresses the consequences of asymmetry between firms' non-wage job characteristics, or choice of these characteristics. We allow asymmetry in firms' efficiency, asymmetric non-wage job characteristics and firm choice of these characteristics. We demonstrate a new channel through which imposition of minimum wages is welfare improving[1], namely their impact on horizontally differentiated, non-wage job characteristics[2].

[1] Kaas and Madden (2008) provide yet another; minimum wages can improve social welfare (in a symmetric location Salop oligopsony) because they improve firms' investment, and reduce the "hold-up" problem.

To facilitate our analysis of job characteristics we switch to a Hotelling linear city duopsony model, analogous to the familiar Hotelling duopoly model (d'Apremont et al. (1979), Ziss (1993)). Workers are uniformly distributed along the line, and the non-wage job characteristic of each firm is a point on the line. One interpretation is that the line represents geographical location, so that the distance between a worker and a firm is the worker's commuting distance, and we usually adopt this interpretation since there is empirical evidence that commuting time is a significant non-wage job characteristic for workers (see Delfgaauw (2007))[3]. On the other hand, there are other interpretations where horizontal differentiation arises because multiple vertical characteristics are inherently linked. For example, longer opening hours (of a retail shop, say) entails more worker flexibility but, simultaneously, less convenient working hours.

We present a 2-stage game model where two firms choose locations at stage I and wages at stage II, and we study the impact of minimum wages on the laissez-faire equilibrium and welfare. We assume full employment throughout, so as to abstract completely from the known aggregate effect, and we find a new route whereby minimum wages can be a good thing; under laissez-faire jobs are too differentiated (as in d'Apremont et al. (1979), Ziss (1993)), but (up to a pure strategy existence limit) minimum wages always improve (aggregate) social welfare because they reduce the job differentiation. All individual workers benefit from increased minimum wages, but firm profits decline. Intuitively, with a binding minimum wage the firms' desire to soften wage competition by moving apart is limited; firms move closer to the centre to

[2] In an asymmetric information labour market model without our oligopsony features, de Fraja(1999) studies the effect of minimum wages on a vertically differentiated job characteristic (e.g. working conditions) and finds that minimum wage increases can make low-paid workers worse off because of a deterioration in working conditions.

[3] Using Dutch data, Delfgaauw (2007, p. 308) finds that from a mix of 15 horizontally and vertically differentiated job characteristics, "the main instigators of job search are dissatisfaction with (future) job duties, followed by dissatisfaction with the atmosphere at work, commuting time, and autonomy".

increase market share. By allowing for heterogeneity in firm efficiency, in equilibrium, the minimum wage binds either on the less efficient firm only, or on both firms. We demonstrate that the market share of the efficient firm is socially too low under laissez faire but that it increases when the minimum wage starts binding on its less efficient rival. Finally, we wish to emphasize that analogous policy conclusions can be obtained for price ceilings in the standard Hotelling duopoly model which is formally equivalent to the duopsony model of this paper.

Section 2 sets out our general framework, section 3 analyses the effect of minimum wages on the subgame perfect equilibria of the 2-stage game, and normative, welfare issues are addressed in section 4. Section 5 concludes.

2. THE FRAMEWORK

There are 2 firms (i =0,1) producing output from labour at constant marginal revenue product of ϕ_i where $\phi_0 \geq \phi_1$. It may be that the marginal physical product is higher at firm 0, or the difference may be caused by firm 0 selling in a more profitable output market than firm 1; for convenience we refer to firm 0 as the efficient firm when $\phi_0 > \phi_1$. The wage offered by firm i is w_i, i = 0,1 and is subject to minimum wage legislation whereby only $w_i \geq \overline{w}$ can be chosen; throughout we assume $\overline{w} \in [0, \phi_1)$ so that the minimum wage does not preclude positive profits for either firm.

Each firm also has a location (or more generally a non-wage job characteristic), $a \in$ [0,1] for firm 0 and $(1-b) \in [0,1]$ for firm 1, so firm 0 locates at distance a from the left-hand end of the linear city and 1 is b from the right-hand end. There is a continuum of workers of mass 1, whose ideal job locations are uniformly distributed over [0, 1]. Taking a job at firm i whose location is at a distance y from a worker's ideal provides the worker with job utility $w_i - ty^2$ where $t > 0$ is a parameter. Throughout we assume full employment, with each worker supplying one unit of labour to the firm that offers the higher job utility, so that the worker whose ideal job location is at $x \in [0,1]$ works for firm 0 if $w_0 - t(x-a)^2 > w_1 - t(1-b-x)^2$, at firm 1 if the inequality is reversed, with indifference if there is equality.[4] If $a \neq 1-b$ the solution to the equality is $\tilde{x}$ in (2.1) below, and the labour market shares or employment levels at firm i = 0, 1 are given in (2.2);

$$\tilde{x} = \frac{1}{2}(1-b+a) + (w_0 - w_1)/2t(1-a-b) \tag{2.1}$$

$$L_0 = \begin{cases} 0 & if\ \tilde{x} \leq 0 \\ \tilde{x} & if\ \tilde{x} \in [0,1] \\ 1 & if\ \tilde{x} \geq 1 \end{cases}, \quad L_1 = 1 - L_0 \quad if\ a \neq 1-b \tag{2.2}$$

We use $\delta = (\phi_0 - \phi_1)/t \geq 0$ to denote a measure of the between firm efficiency differential.

If $a = 1-b$, the firms co-locate and, when $\delta > 0$ we assume that the high wage firm gets the whole labour market if $w_0 \neq w_1$, but the efficient firm gets the whole market when $w_0 = w_1$, analogous to homogeneous product Bertrand duopoly models with asymmetric costs;

$$L_0 = \begin{cases} 0\ if\ w_0 < w_1 \\ 1\ if\ w_0 \geq w_1 \end{cases}, \quad L_1 = 1 - L_0 \quad if\ a = 1-b, \delta > 0 \tag{2.3}$$

If $a = 1-b$ and $\delta = 0$, we follow the standard, homogeneous product, symmetric cost Bertrand duopoly assumption;

[4] If the workers have a reservation utility of u, a sufficient (but not necessary) condition to ensure throughout our analysis that no worker would choose voluntary unemployment is $\phi_1 > 2t + u$, implicitly assumed from now on.

$$L_0 = \begin{cases} 0 & if\ w_0 < w_1 \\ 1 & if\ w_0 > w_1 \\ \frac{1}{2} & if\ w_0 = w_1 \end{cases} \quad , \ L_1 = 1 - L_0 \quad if\ a = 1-b, \delta = 0 \qquad (2.4)$$

Firm profits are;

$$\pi_0 = (\phi_0 - w_0)L_0, \quad \pi_1 = (\phi_1 - w_1)L_1. \qquad (2.5)$$

The model is a 2-stage game, where firms simultaneously choose locations first and then wages at the second stage. We study the subgame perfect equilibria (SPE), and compare with the social optimum. To avoid some implausible equilibria in the wage subgames when firms co-locate, we follow again a homogeneous product Bertrand duopoly lead (see Hurter and Lederer (1986) for a discussion) and assume throughout that firms cannot offer wages in excess of their marginal revenue product, so $w_0 \in [\overline{w}, \phi_0]$, $w_1 \in [\overline{w}, \phi_1]$.

Ziss (1993) has analysed a 3-stage entry-location-price game in a Hotelling duopoly with asymmetric firm efficiency and laissez-faire (i.e. without price controls). His main results have exact parallels for laissez-faire (i.e. $\overline{w} = 0$) in our duopsony model. First the SPE are as follows, analogous to Ziss (1993, Proposition 2, p. 536).

<u>Proposition 1</u> (a)If $\delta \in [0, \delta^*]$ where $\delta^* = 6 - 3\sqrt{3} \cong 0.81$ and if $\overline{w} = 0$, the unique (up to symmetry) pure strategy SPE outcome has maximum location differentiation of the firms (*a*=*b*=0 or 1) and the following wages, market shares and profits;

$$\begin{aligned} w_0^{**} &= \tfrac{2}{3}\phi_0 + \tfrac{1}{3}\phi_1 - t \quad \geq \quad w_1^{**} = \tfrac{1}{3}\phi_0 + \tfrac{2}{3}\phi_1 - t \\ L_0^{**} &= \tfrac{1}{2} + \tfrac{1}{6}\delta \quad \geq \quad L_1^{**} = \tfrac{1}{2} - \tfrac{1}{6}\delta \\ \Pi_0^{**} &= \tfrac{1}{18}t(3+\delta)^2 \quad \geq \quad \Pi_1^{**} = \tfrac{1}{18}t(3-\delta)^2 \end{aligned}$$

(b) If $\delta > \delta^*$ and if $\overline{w} = 0$, there is no SPE in pure strategies.

As in the earlier symmetric efficiency duopoly model of d'Apremont et al. (1979), the desire of both firms to move apart so as to soften the wage competition dominates the positive effects on market share that moving toward the rival would have at constant wages, producing maximum differentiation SPE for δ up to δ^*. For $\delta > \delta^*$, the inefficient firm wishes again to get as far away as possible from the rival, but the efficient firm wants to co-locate and force the rival out of the market with a wage of ϕ_1, leading to (b).

Turning to the social optimum and continuing to assume full employment, if $a \le 1-b$ it will be socially optimal that workers at locations $[0, L_0]$ work for firm 0 and those at $(L_0,1]$ work for 1, for some $L_0 \in [0,1]$. Market shares are then L_0 and $L_1 = 1 - L_0$, and social welfare is the aggregate surplus;

$$\begin{aligned} SW(a,b,L_0) &= \phi_0 L_0 + \phi_1(1-L_0) - t\int_0^{L_0}(x-a)^2\,dx - t\int_{L_0}^1(1-b-x)^2\,dx \\ &= \phi_0 L_0 + \phi_1(1-L_0) - \tfrac{1}{3}t\left[a^3 + b^3 + (L_0-a)^3 + (1-b-L_0)^3\right] \end{aligned} \quad (2.6)$$

Maximization of $SW(a,b,L_0)$ over $L_0 \in [0,1]$ and $(a,b) \in [0,1]^2$ with $a \le 1-b$ produces the social optimum market shares and locations (with 0 to the left of 1), and clearly there is also an optimum with the same market shares and 0 symmetrically to the right of 1. Proposition 2 is the result, analogous to Ziss (1993, Proposition 4(i), p.540);

<u>Proposition 2</u> The socially optimal locations and market shares are;

(a) $a^0 = \frac{1}{4} + \delta,\ b^0 = \frac{1}{4} - \delta$ or $a^0 = \frac{3}{4} - \delta,\ b^0 = \frac{3}{4} + \delta$, with $L_0^0 = \frac{1}{2} + 2\delta,\ L_1^0 = \frac{1}{2} - 2\delta$ if $\delta \in [0, \frac{1}{4})$,

(b) $a^0 = \frac{1}{2},\ b^0 \in [0,1]$, with $L_0^0 = 1,\ L_1^0 = 0$ if $\delta \ge \frac{1}{4}$.

With symmetric efficiency, the firms optimally locate at the quartiles, as in d'Apremont et al. (1979). As δ increases from 0, optimal locations remain a distance of $\frac{1}{2}$ apart, the efficient firm moving towards the centre of the line and employing more workers up to $\delta = \frac{1}{4}$, at which point the efficient firm is centrally located and employs the whole market.

Appendix B to this paper includes proofs of subsequent lemmas and of Propositions 1 and 2[5].

3. MINIMUM WAGES AND MARKET EQUILIBRIUM

We analyse first the Nash equilibria (NE) of stage II wage subgames at arbitrary stage I locations. For the laissez-faire case ($\overline{w} = 0$) we have the following lemma 3.1, where $H = \{(a,b) \in [0,1]^2 : a + b = 1\}$ defines the set of locations where the firms co-locate, so jobs are homogeneous, and $S = \{(a,b) \in [0,1]^2 : a + b < 1\}$ denotes locations where firms are separated with firm 0 to the left of firm 1; notice that the wage subgame at (a,b) $\in$ S has, from symmetry, the same outcome as that at $(1-a, 1-b) \in [0,1]^2$, so the following description of NE for (a,b) $\in$ $S \cup H$ suffices.

<u>Lemma 3.1</u> The unique NE wages, market shares and profits of the stage II subgame at locations (a,b) under laissez-faire ($\overline{w} = 0$) is as follows;

(a) if $(a,b) \in T = \{(a,b) \in S : \delta < (1-a-b)(3-a+b)\}$, then

$$w_0^*(a,b) = \tfrac{2}{3}\phi_0 + \tfrac{1}{3}\phi_1 - \tfrac{1}{3}t(1-a-b)(3+a-b), \qquad w_1^*(a,b) = \tfrac{1}{3}\phi_0 + \tfrac{2}{3}\phi_1 - \tfrac{1}{3}t(1-a-b)(3-a+b)$$

$$L_0^*(a,b) = \tfrac{1}{6}(3+a-b) + \tfrac{1}{6}\delta/(1-a-b), \qquad L_1^*(a,b) = \tfrac{1}{6}(3-a+b) - \tfrac{1}{6}\delta/(1-a-b)$$

$$\Pi_0^*(a,b) = \tfrac{1}{18}t(1-a-b)[3+a-b+\delta/(1-a-b)]^2, \quad \Pi_1^*(a,b) = \tfrac{1}{18}t(1-a-b)[3-a+b-\delta/(1-a-b)]^2$$

(b) if $\delta > 0$ and (a,b) $\in$ ($S \cup H$)\T, then

$$w_0^*(a,b) = \phi_1 + t(1-a-b)(1-a+b), \qquad w_1^*(a,b) = \phi_1$$

[5] The proofs of Propositions 1 and 2 use our duopsony setting and notation, and so are more immediately accessible for the reader of our paper. Also we correct a small error in the Ziss (1993) discussion of Proposition 1, and provide a proof of Proposition 2 that does not rely on the Ziss (1993) unproven Lagrangean concavity.

$L_0^*(a,b) = 1,$ $\qquad$ $L_1^*(a,b) = 0$

$\Pi_0^*(a,b) = \phi_0 - \phi_1 - t(1-a-b)(1-a+b)$, $\Pi_1^*(a,b) = 0$.

(c) if $\delta = 0$ and $(a,b) \in (S \cup H) \backslash T = H$, then

$w_0^*(a,b) = w_1^*(a,b) = \phi_0 = \phi_1, \quad L_0^*(a,b) = L_1^*(a,b) = \frac{1}{2}, \quad \Pi_0^*(a,b) = \Pi_1^*(a,b) = 0$

Proof See Appendix B.

Various remarks follow. First, T is the subset of locations where both firms receive positive market shares when $\delta > 0$. If $\delta \geq 3$, T is empty and firm 1's inefficiency is such that it never is active. If $\delta = 0$, T=S and both firms have positive market share (but zero profits) on H also[6]. Figure 3.1 illustrates for $\delta \in (0, \frac{1}{2})$, when U_1 (defined later with U_0, D, a_2, a_3) is non-empty[7].

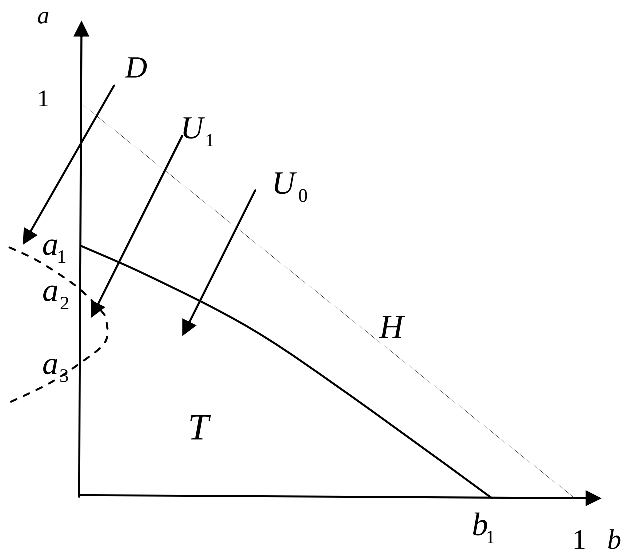

Figure 3.1; Various location subsets

Secondly, it follows from the formulae in Lemma 3.1 that the wages offered by both firms increase if either a or b increases. The intuition is standard; as the firms move closer together, wage competition becomes more severe as jobs are less differentiated.

[6] This is because (2.4) now replaces (2.3).

[7] The upper boundary intercepts of T are $a_1 = 2 - \sqrt{1+\delta}$, $b_1 = \sqrt{4-\delta} - 1$).

It follows that the lowest wages occur when jobs are maximally differentiated (a=b=0, as in the laissez-faire SPE), and that the lowest wage for either firm in any laissez-faire subgame is w_1^{**} (see Proposition 1). Thirdly, best response graphs behind Lemma 3.1 are upward-sloping linear functions (when market shares are positive), exactly as in Bhaskar and To (2003)[8], of the form $w_i = \alpha_i + \beta w_j, i \neq j = 0,1,$ where $\alpha_0 = \frac{1}{2}\{\phi_0 - t(1-a-b)(1+a-b)\}, \alpha_1 = \frac{1}{2}\{\phi_1 - t(1-a-b)(1-a+b)\}$ and $\beta = \frac{1}{2}$[9]. Thus all wage games entail strategic complementarity with equilibria that are stable in the usual best response dynamic, and the high wage firm will be the one with the higher α_i, so $w_1^*(a,b) > (<)\ w_0^*(a,b)$ if and only if $(a,b) \in U_1(U_0)$ where $U_0 = \{(a,b) \in T : \delta > 2(1-a-b)(a-b)\}, U_1 = \{(a,b) \in T : \delta < 2(1-a-b)(a-b)\}$ and U_1 is non-empty when $\delta < \frac{1}{2}$, as shown in figure 3.1[10] where $a_2 = \frac{1}{2}(1+\sqrt{1-2\delta}), a_3 = \frac{1}{2}(1-\sqrt{1-2\delta})$. If we restrict attention to symmetric locations (as do Bhaskar and To (2003)), then $a = b$ and the high wage firm is the efficient firm if $\delta > 0$, again as in Bhaskar and To (2003), and wages are equal if $\delta = 0$. At asymmetric locations when $\delta = 0$, the high wage firm is 1(0) if *a>(<)b.* When *a>b* (say) firm 1 gets the smaller market share at equal wages. If 1 increased its wage the marginal revenue would be the same as for 0 if it increased its wage, but the marginal cost for 1 is lower than for 0 because of its smaller market share. Thus, from equal wages when $\delta = 0$ and *a>b*, firm 1 has the greater incentive to raise wages, which leads to the NE with $w_1 > w_0$. At asymmetric locations when $\delta > 0$ and *a>b*, 1's marginal revenue (as well as its marginal cost) is lower, so the previous argument is no longer decisive. In fact if locations are nearly symmetric, marginal costs are

[8] With a change of notation these are the same as in Ziss (1993, p. 528, equation (5b)).
[9] These formulae are the same as in Bhaskar and To (2003) when the mass of their high reservation wage workers (characterised by unemployment, assumed away in our model) is zero.
[10] The dashed curve separating U_0 and U_1 is $D = \{(a,b) \in R^2 : \delta = 2(1-a-b)(a-b)\}$.

similar and 0's higher marginal revenue leads to $w_0 > w_1$. Also if locations are close together, then eventually 0 will want to take the whole market, again with $w_0 > w_1$. The residual set of locations that are not too symmetric and not too close is U_1 where $w_1 > w_0$.

Leaving the laissez-faire scenario, there are, in principle, 4 types of wage subgame equilibria that can occur for $\overline{w} \in (0, \phi_1)$: equilibria in which the minimum wage is binding on neither firm (type $\varnothing$ equilibrium in what follows, equivalent to laissez-faire), binding on both firms (type 01), binding only on firm 1 (type 1) and binding only on firm 0 (type 0). Clearly $\overline{w} \leq w_1^{**}$ will have no effect on any subgame equilibrium – the laissez-faire outcome will continue at all locations. As $\overline{w}$ increases from w_1^{**} it will impact first on the subgame equilibria at the maximum differentiation location (*a*=*b*=0) and those nearby. With *a*=*b*=0 and $\delta > 0$ the effect is as in Bhaskar and To (2003), forcing the inefficient firm to raise its wage in line with the minimum wage, and producing a smaller increase in the efficient firm's wage also, because of the strategic complementarity, compressing the wage distribution. Thus type 1 equilibrium emerges first at *a*=*b*=0. Eventually $\overline{w}$ will reach a level (denoted below by $\overline{w}_0(0,0)$) where it starts to bind on firm 0 also, and type 01 equilibria emerge. For other locations Lemma 3.2 provides a full description of the values of ($\overline{w}, a, b$) associated with each equilibrium type for all $\delta \geq 0$, illustrated in Figures 3.2 and 3.3 later. The notation $\overline{w}_0(a,b) = \phi_0 - t[(1-b)^2 - a^2]$, $\overline{w}_1(a,b) = \phi_1 - t[(1-a)^2 - b^2]$ is used, where $\overline{w}_i(a,b)$ is the lowest minimum wage which binds on both firms if firm *i* is the high-wage firm under laissez-faire, as with $\overline{w}_0(0,0)$ described earlier.

Lemma 3.2 The unique NE wages, market shares and profits of the stage II subgame at locations (a,b) for $\overline{w} \in (0,\phi_1)$ are as follows:

(a) if $(a,b) \in T$;

(i) type $\varnothing$, with the laissez-faire values described in lemma 3.1(a)

iff $\overline{w} \le \min[w_0^*(a,b), w_1^*(a,b)]$

(ii) type 01, with $w_0 = w_1 = \overline{w}$, $L_0 = \frac{1}{2}(1+a-b), L_1 = 1-L_0$,

$\Pi_0 = \frac{1}{2}(\phi_0 - \overline{w})(1+a-b), \Pi_1 = \frac{1}{2}(\phi_1 - \overline{w})(1-a+b)$

iff $\overline{w} \ge \max[\overline{w}_0(a,b), \overline{w}_1(a,b)]$

(iii) type 1, with $w_0 = \frac{1}{2}\{\phi_0 + \overline{w} - t[(1-b)^2 - a^2]\}$, $w_1 = \overline{w}$,

$L_0 = \{\phi_0 - \overline{w} + t[(1-b)^2 - a^2]\}/4t(1-a-b) = 1-L_1$,

$\Pi_0 = \{\phi_0 - \overline{w} + t[(1-b)^2 - a^2]\}^2/8t(1-a-b), \Pi_1 = (\phi_1 - \overline{w})L_1$,

iff $\overline{w}_0(a,b) \ge \overline{w} \ge w_1^*(a,b)$

(iv) type 0, with $w_0 = \overline{w}, w_1 = \frac{1}{2}\{\phi_1 + \overline{w} - t[(1-a)^2 - b^2]\}$

$L_0 = 1-L_1$, $L_1 = \{\phi_1 - \overline{w} + t[(1-a)^2 - b^2]\}/4t(1-a-b)$

$\Pi_0 = (\phi_0 - \overline{w})L_0$, $\Pi_1 = \{\phi_1 - \overline{w} + t[(1-a)^2 - b^2]\}^2/8t(1-a-b)$

iff $\overline{w}_1(a,b) \ge \overline{w} \ge w_0^*(a,b)$

(b) if $(a,b) \in (S \cup H) \setminus T$ the equilibrium is the laissez-faire equilibrium described in Lemma 3.1(b) if $\delta > 0$, and Lemma 3.1(c) if $\delta = 0$.

Proof See Appendix B.

To find the SPE, backward induction requires the NE of the “reduced form” stage I location game where firm 0 chooses $a \in [0,1]$, firm 1 chooses $b \in [0,1]$ and payoffs, now denoted $\overline{\Pi}_i(a,b,\overline{w}),\ i=0,1,$ are defined by Lemma 3.2. The resulting functions $\overline{\Pi}_i:(S\cup H)\times R_{++}\to R_+$ are continuous, and differentiable almost everywhere.

As a first step, we develop diagrams to illustrate Lemma 3.2 via the following 4 curves in the (*a*,*b*) plane for given $\overline{w}\in(w_1^{**},\phi_1)$ [11];

C1; $\overline{w}=w_0^*(a,b)=\frac{2}{3}\phi_0+\frac{1}{3}\phi_1-\frac{1}{3}t(1-a-b)(3+a-b)$

C2; $\overline{w}=w_1^*(a,b)=\frac{1}{3}\phi_0+\frac{2}{3}\phi_1-\frac{1}{3}t(1-a-b)(3-a+b)$

C3; $\overline{w}=\overline{w}_0(a,b)=\phi_0-t[(1-b)^2-a^2]$

C4; $\overline{w}=\overline{w}_1(a,b)=\phi_1-t[(1-a)^2-b^2]$

It is easily checked that: (a) these 4 curves intersect only on the curve *D* ($\delta=2(1-a-b)(a-b)$) in Figure 3.1, and each is a downward sloping, concave curve within the interior of *S*; (b) in U_0, *C4* is above *C2* which is above *C1* which is above *C3*, with the reverse ranking in U_1.

When $\delta=0$ the curve *D* degenerates, intersecting *S* in the line $a=b,\ 0\le a<\frac{1}{2}$. From (b) above and Lemma 3.2 the following diagram emerges for a typical $\overline{w}\in(w_1^{**},\phi_1)$ when $\delta=0$, indicating the subgame equilibrium type at the various locations. In this case the intersection of *C1*-*C4* at (*a*,*a*) increases monotonically from *a*=0 as $\overline{w}\to w_1^{**}$, converging to $a=\frac{1}{2}$ as $\overline{w}\to\phi_1$.

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[11] When $\overline{w}\in(0,w_1^{**})$ all locations produce type $\varnothing$ equilibrium.

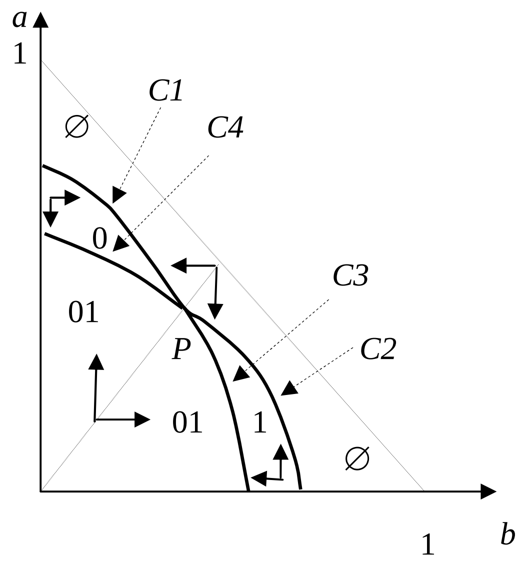


Figure 3.2: Wage subgame equilibrium type, $\delta = 0$

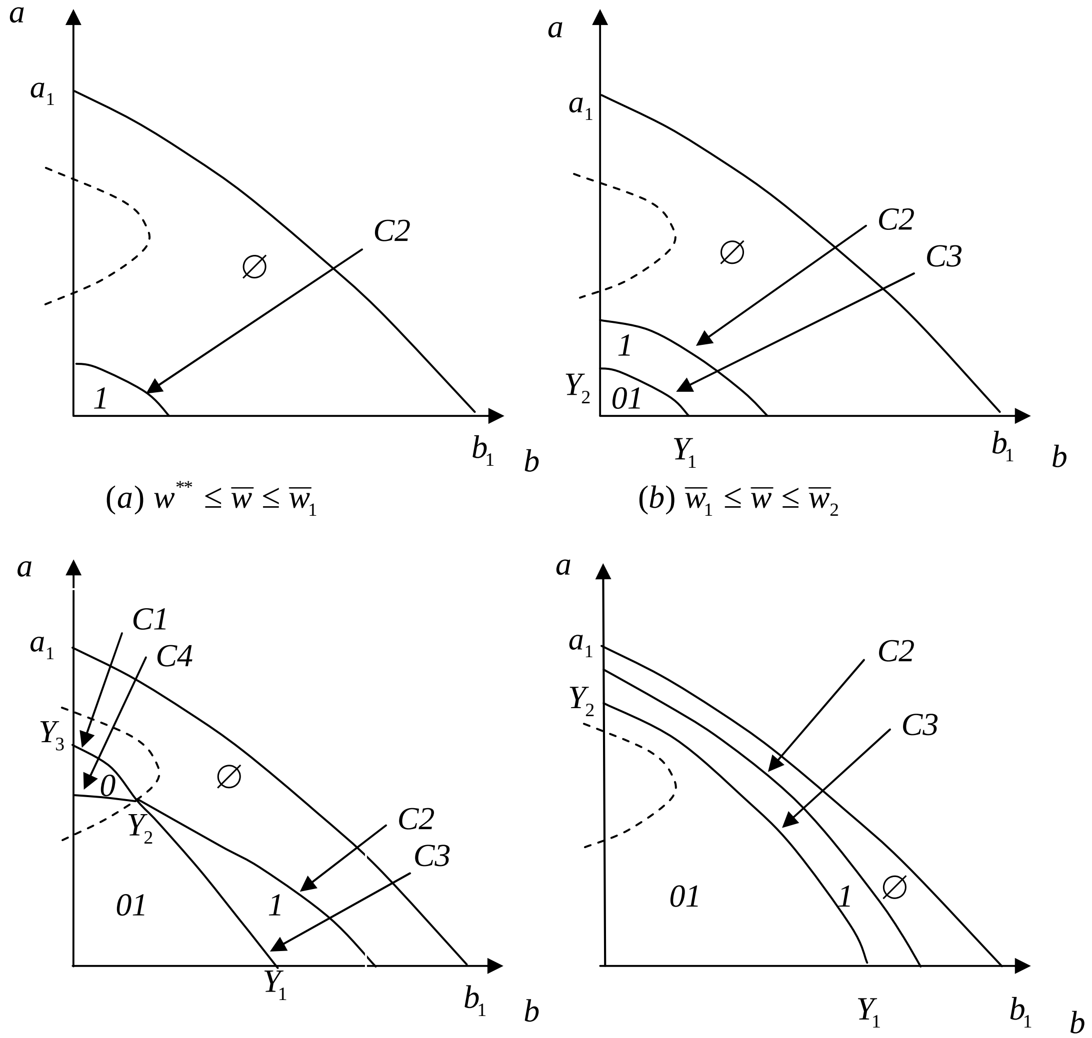
For the case $\delta = 0, \overline{w} \in (w_1^{**}, \phi_1)$, we now provide an intuitive explanation of the signs of the derivatives $\partial\overline{\Pi}_0 / \partial a$ and $\partial\overline{\Pi}_1 / \partial b$ on the interiors of regions $\varnothing$, 0, 1 and 01 in Figure 3.2, indicated by bold arrows. For $\varnothing$ (laissez-faire) the usual centrifugal force dominates, firms wanting to move away from the rival to avoid increased wage competition. But at any point in 01 the minimum wage binds on both firms and continues to do so at nearby locations, so the increased wage competition from moving closer to the rival is absent, and firms now want to move towards the rival to increase market share. In region 0(1) the minimum wage binds only on firm 0(1), so the effect of the last sentence means firm 1(0) wants to move towards the rival, but the centrifugal force remains dominant for firm 0(1). Hence firm 1's constrained location best response graph (where 1 is constrained to locate at or to the right of 0) follows the path *C3, C1* and the vertical axis up to $a = 1, b = 0$, as a increases from 0 to 1. Similarly the corresponding graph for firm 0 is *C4, C2* and then the horizontal axis as b increases from 0 to 1. It follows that the intersection of *C1*-*C4* (shown as *P* in figure 3.2) is the unique (up to symmetry) candidate for the reduced form game NE (the only intersection of the constrained best responses). And this is indeed the NE, since firm 1 (say) will not want to jump to a location in the relatively small region (length $< \frac{1}{2}$) to the left of 1 where its profits will be lower, and similarly for firm 0.

Theorem 1 If $\delta = 0$ and $\overline{w} \in \left(w_1^{**}, \phi_1\right)$ the unique (up to symmetry) pure strategy SPE outcome has $a = b = \frac{1}{2}\left(1 - \left(\phi_0 - \overline{w}\right)/t\right), L_0 = L_1 = \frac{1}{2}$ and $\pi_0 = \pi_1 = \frac{1}{2}\left(\phi_0 - \overline{w}\right)$.

Proof Appendix A contains a combined proof of Theorems 1 and 2.

The case $\delta > 0$ is more complicated, and to shorten and simplify exposition in the remainder of this paper we restrict attention to $\delta < \frac{1}{2}$ (where U_1 in Figure 3.1 is non-empty). Figure 3.2 gives way to Figure 3.3, demarcated by the following critical minimum wages where $w_1^{**} < \overline{w}_1 < \overline{w}_2 < \overline{w}_3 < \phi_1$;

$$\overline{w}_1 = \phi_0 - t \quad ; \quad \overline{w}_2 = \phi_0 - \tfrac{1}{2}t\left(1 + \delta + \sqrt{1 - 2\delta}\right) \quad ; \quad \overline{w}_3 = \phi_1 - \tfrac{1}{2}t\left(1 + \delta - \sqrt{1 - 2\delta}\right).$$



$(a)\ w^{**} \le \overline{w} \le \overline{w}_1$ $(b)\ \overline{w}_1 \le \overline{w} \le \overline{w}_2$

$(c)\ \overline{w}_2 \le \overline{w} \le \overline{w}_3$ $(d)\ \overline{w}_3 \le \overline{w} \le \phi_1$

Figure 3.3: Wage subgame equilibrium type, $0 < \delta < \frac{1}{2}$.

When $\overline{w} \in (w_1^{**}, \overline{w}_1)$, the intersection of *C1-C4* on *D* now lies outside *S*, with $a < \frac{1}{2}$ and *b*<0. As $\overline{w}$ increases in this range, the *C1-C4* intersection moves up along *D* but *C3* remains outside *S*, touching at (0,0) when $\overline{w} = \overline{w}_1$, whilst *C*2 intersects *S* as shown in figure 3.3(a), defining the boundary between type $\varnothing$ and 1 equilibria from Lemma 3.2. As $\overline{w}$ increases in $[\overline{w}_1, \overline{w}_2]$ the intersection of *C1-C4* on *D* remains outside *S* with *b*<0 and $a < \frac{1}{2}$, converging to the point on *D* where *b*=0 and $a < \frac{1}{2}$ as $\overline{w} \to \overline{w}_2$, but now *C*3 intersects *S* as in figure 3.3(b). For $\overline{w} \in [\overline{w}_2, \overline{w}_3]$ the *C*1-*C*4 intersection moves up round *D* but now in *S*, generating Figure 3.3(c), the intersection reaching the point on *D* where *b*=0 and $a > \frac{1}{2}$ when $\overline{w} = \overline{w}_3$. For $\overline{w} > \overline{w}_3$ the *C1-C4* intersection is again outside *S*, producing Figure 3.3(d).

For the inefficient firm the signs of $\partial\overline{\Pi}_1 / \partial b$ continue as in Figure 3.2 (positive in regions 01 and 0, negative in regions $\varnothing$ and 1), with the same intuition. Hence for firm 1's constrained best response problem in the reduced form location game ($\max_b \overline{\Pi}_1(a,b)$ subject to $a \le 1-b$), it follows that the constrained best response correspondence graph is, for $a \in [0, a_1]$;

(a) the vertical axis from the origin to a_1 in Figure 3.3(a),

(b) the locus Y_1, Y_2, a_1 in Figure 3.3(b),

(c) the locus Y_1, Y_2, Y_3, a_1 in Figure 3.3(c),

(d) the locus Y_1, Y_2, a_1 in Figure 3.3(d).

These constrained best responses are also unconstrained best responses if $a \in [0, \frac{1}{2}]$. When $a \in [\frac{1}{2}, a_1]$, firm 1 would jump to the location in the left half of the line which is the constrained best response by 1 to *a* when 1 is constrained to locate to the *left* of firm 0.

A new critical minimum wage appears between $\overline{w}_3$ and ϕ_1 when $a = \frac{1}{2}$ at Y_2 in Figure 3.3(c). This is $\overline{w} = \overline{\overline{w}} = \phi_0 - \frac{1}{2}t\left(\delta + \sqrt{2\delta}\right) \in (\overline{w}_2, \overline{w}_3)$, and Figure 3.4 shows in bold firm 1's unconstrained best response graphs[12] with $\overline{\overline{w}}$ now as the demarcation between parts (c) and (d) of the diagram; obviously these are the only candidates for SPE locations. It remains to identify which (if any) points on these bold segments are also best responses for firm 0 in the reduced form location game. This is point *P* (plus its

[12] Appendix B contains precise statements (with proofs) of firm 1's constrained and unconstrained best responses.

symmetric counterpart) in Figure 3.4 (a),(b), and (c). In Figure 3.4(a) and (b), a small upward (downward) deviation in *a* by firm 0 takes us into region $\varnothing(1)$, and, with the same intuition as in Figure 3.2, $\partial\overline{\Pi}_0 / \partial a < (>) 0$. Thus *P* is at least a local best location response for 0, and the next Theorem shows it is global. And similarly for *P* in Figure 3.4(c), except downward deviations now lead to type 01 equilibrium. But in Figure 3.4(d), the bold segments are on the 01/1 border and firm 0 now wants to move (locally) closer to firm 1 from all such points, precluding SPE with pure strategies.

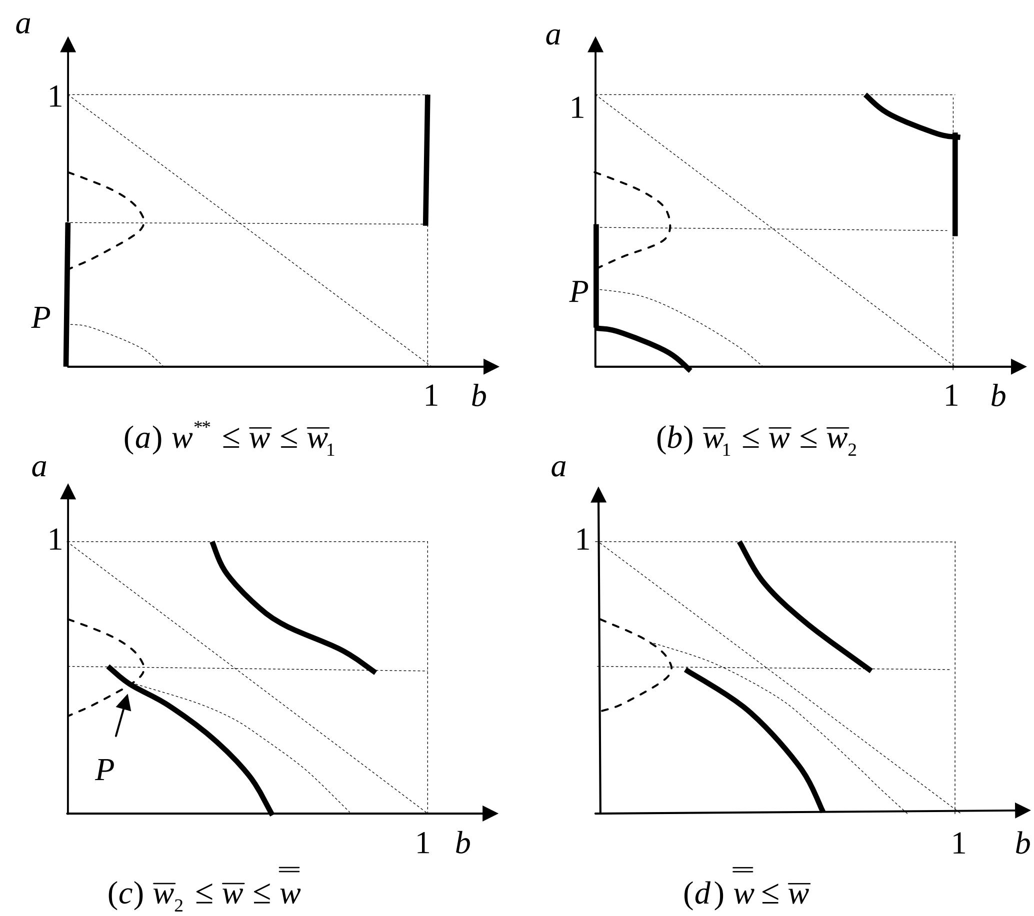


Figure 3.4 Firm 1’s location best responses with minimum wages

Theorem 2 If $\delta \in (0, \frac{1}{2})$ there are critical values for minimum wages $w_1^{**} < \overline{w}_2 < \overline{\overline{w}}$ such that:

(a) for $\overline{w} \in (w_1^{**}, \overline{\overline{w}})$ the unique (up to symmetry) pure strategy SPE outcome has the locations $P = (a,b)$ shown in Figure 3.4(a), (b), (c), with;

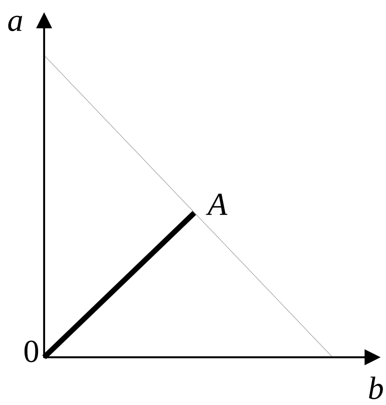
(i) (a,b) defined by $w_1^*(a,0) = \overline{w}$ and b=0, $w_0 = w_0^*(a,0) > w_1 = \overline{w}$, $L_i = L_i^*(a,0)$ and $\pi_i = \Pi_i^*(a,0), i = 0,1,$ if $\overline{w} \in (w_1^{**}, \overline{w}_2]$;

(ii) (a,b), (w_0,w_1) defined by $w_0^*(a,b) = w_1^*(a,b) = \overline{w} = w_0 = w_1$, $L_i = L_i^*(a,b)$, and $\pi_i = \Pi_i^*(a,b), i = 0,1,$ if $\overline{w} \in (\overline{w}_2, \overline{\overline{w}}]$;

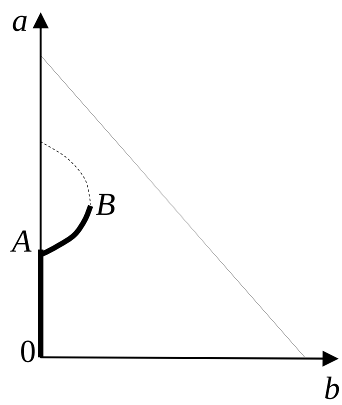
(b) for $\overline{w} \in (\overline{\overline{w}}, \phi_1)$, there is no pure strategy SPE.

Proof See Appendix A for a combined proof of Theorems 1 and 2.

Figure 3.5 illustrates the effects on SPE locations of increasing the minimum wage from w_1^{**} to $\overline{\overline{w}}$. When $\delta = 0$ locations are always symmetric, starting at maximum job differentiation when $\overline{w} = w_1^{**}$, the firms gradually moving closer together as $\overline{w}$ increases, converging to co-location at the centre of the line as $\overline{w} \to \overline{\overline{w}} = \phi_1$. Here minimum wages above w_1^{**} bind on both firms, so that moving towards the rival fails to produce the increased wage competition of laissez-faire, removing the centrifugal force that dominates under laissez-faire and bringing the firms closer together as $\overline{w}$ increases. When $\delta > 0$ the initial effect (for $\overline{w}$ increasing from w_1^{**} to $\overline{w}_2$, along 0A) is that the minimum wage binds only on the inefficient firm, removing the above centrifugal force for the efficient firm (only) which moves towards the rival, but leaving the inefficient firm at the extremity. Minimum wages above $\overline{w}_2$ bind on both firms, both centrifugal forces disappear and both firms move towards the rival, along *AB*.



(a) 0*A* for $\delta = 0$



(b) 0*AB* for $\delta \in (0, \frac{1}{2})$

Figure 3.5 SPE location paths as $\overline{w}$ increases from w_1^{**} to $\overline{\overline{w}}$

In all cases it is clear that the effect of increasing minimum wages is to reduce job differentiation (the distance $1-a-b$ between firms). The effects of the minimum wage on market shares also follow easily from Theorems 1 and 2;

Corollary to Theorems 1 and 2 For $\delta \in [0, \frac{1}{2})$, as $\overline{w}$ increases from w_1^{**} to $\overline{\overline{w}}$, the effects on the SPE are;

(a) a reduction in job differentiation,

(b) an increase (decrease) in the efficient (inefficient) firm market share when $\delta > 0$, and no change in market shares when $\delta = 0$,

Proof See Appendix A.

4. SOCIAL WELFARE AND MINIMUM WAGES

We now turn to the impact of minimum wages on welfare. With some abuse of notation, let $SW(\overline{w})$ denote the value of social welfare ($SW(a,b,L_0)$ defined in (2.6)) when a,b,L_0 are the SPE locations and market shares at minimum wage $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$, and let $SW(w_1^{**})$=SW^{**} denote its laissez-faire value and SW° its value at the social optimum described in Proposition 2.

The case $\delta = 0$ is straightforward. From the Corollary to Theorems 1 and 2, and from (2.6), for $\overline{w} \in [w_1^{**} = \phi_1 - t, \phi_1)$, $L_0 = \frac{1}{2}$ and locations follow 0A in figure 3.4(a), so;

$$SW(\overline{w}) = \phi_1 - \tfrac{2}{3}[a^3 + (1-a)^3], \text{ where } a(=b) = \tfrac{1}{2}\left(1 - \left(\phi_0 - \overline{w}\right)/t\right)$$

It is easy to check that $SW(\overline{w})$ is strictly concave with maximum at $\overline{w} = \phi_1 - \frac{1}{2}t$ where $a = \frac{1}{4}$, and as $\overline{w} \to \phi_1$, $SW(\overline{w}) \to \phi_1 - \frac{1}{6}t = SW^{**}$. This proves;

<u>Theorem 3</u> If $\delta = 0$, $SW(\overline{w})$ is strictly increasing for $\overline{w} \in \left(w_1^{**}, \phi_1 - \frac{1}{2}t\right)$ and strictly decreasing for $\overline{w} \in \left(\phi_1 - \frac{1}{2}t, \phi_1\right)$, attaining a unique maximum and the social optimum (SW°) when $\overline{w} = \phi_1 - \frac{1}{2}t$; $SW(\overline{w}) > SW^{**}$ for all $\overline{w} \in (w_1^{**}, \phi_1)$.

From the laissez-faire SPE, as the minimum wage increases, social welfare thus increases until $\overline{w} = \phi_1 - \frac{1}{2}t$ when the full social optimum is attained, and then declines but remains above its SPE level for all $\overline{w} \in (w_1^{**}, \phi_1)$ [13]. When $\delta > 0$ the picture is a bit more complicated. Note that all pure strategy SPE for $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$, occur in (on the border of) region $\varnothing$ (see Figure 3.2), so the efficient firm market share is

[13] Alternatively in the symmetric efficiency case, maximization of social welfare is equivalent to minimization of aggregate commuting time, which attains the same, largest value when firms are at the extremes or co-locating at the middle of the line, with lower values in between and the global minimum at the socially optimal quartile locations.

$L_0 = \frac{1}{6}(3+a-b)+\delta/6(1-a-b)$ from Lemma 3.1(a). Locations now follow 0*AB* in Figure 3.4(b), producing;

Theorem 4 Suppose $\delta \in \left(0,\frac{1}{2}\right)$ and $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$.

(a) $SW(\overline{w})$ is strictly increasing for $\overline{w} \in [w_1^{**}, \overline{w}_2]$.

(b) If $\delta \in \left[\frac{2}{5},\frac{1}{2}\right)$ then $SW(\overline{w})$ is strictly increasing for all $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$

(c) If $\delta \in \left(0,\frac{2}{5}\right)$ there is a unique minimum wage, $\overline{w}^*$ say, which maximizes $SW(\overline{w})$ over $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$; at $\overline{w}^*$, $a+b > \frac{1}{2}$.

(d) $SW(\overline{w}) > SW^{**}$ for all $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$.

(e) $SW^{\circ} > SW(\overline{w})$ for all $\overline{w} \in [w_1^{**}, \overline{\overline{w}}]$.

Proof See Appendix A.

Hence, as the minimum wage increases from w_1^{**}, the effect is to increase social welfare monotonically (up to the pure strategy existence limit $\overline{w} = \overline{\overline{w}}$) if $\delta \in \left[\frac{2}{5},\frac{1}{2}\right)$(part (b)). If $\delta \in \left(0,\frac{2}{5}\right)$ the minimum wage (again up to the limit $\overline{w} = \overline{\overline{w}}$) always improves on laissez-faire (part (d)), but now the improvement is not monotonic over the whole $\overline{w}$ range (parts (a) and (c)), social welfare reaching a maximum at some $\overline{w}^* \in \left(w_1^{**}, \overline{\overline{w}}\right)$, similar to Theorem 2. Although minimum wages (up to the pure strategy existence limit) always improve on laissez-faire, they never now allow attainment of the full social optimum (part (e)).

The above shows that in the "long-run", via its impact on the non-wage job characteristic, the imposition of a minimum wage is welfare-improving over laissez-faire.

It is instructive to consider also the “short-run” impact, where we assume that the locations remain fixed at the laissez-faire maximum differentiation. When $\delta = 0$, from Lemma 3.2 (a)(ii) and Figure 3.2, the short-run (subgame) equilibrium is type 01 with equal market shares for all $\overline{w} \in (w_1^{**}, \phi_1)$. Hence, as $\overline{w}$ increases, there is no effect on welfare in the short-run, and the above beneficial long-run effects are therefore due to the effects of the minimum wage on locations. Again when $\delta > 0$ the picture is different, and indeed the short-run effect is disadvantageous to welfare. Note first that with *a*=*b*=0, (2.6) becomes the following, strictly concave function of L_0 whose maximum is at $L_0 = \frac{1}{2}(1+\delta)$;

$$SW(0,0,L_0) = \phi_0 L_0 + \phi_1(1-L_0) - \tfrac{1}{3}t[L_0^3 + (1-L_0)^3]$$

The laissez-faire SPE market share for firm 0 is $L_0^{**} = \frac{1}{2} + \frac{1}{6}\delta < \frac{1}{2}(1+\delta)$, and so provides too low a share to the efficient firm in the short-run (i.e. given *a*=*b*=0). Because firms cannot perfectly wage discriminate among workers, the efficient firm does not internalise all social gains that are associated with a wage increase. As a result the wage premium offered by the efficient firm in the SPE is too low and its market share is too low. But minimum wages above w_1^{**} bind first (for $\overline{w} \in (w_1^{**}, \overline{w}_1)$, see Figure 3.3(a)) only on the inefficient firm, increasing its wage more than that of the efficient rival, compressing the wage differential and *decreasing* the efficient firm market share ($L_0 = \{\phi_0 - \overline{w} + t\}/4t$, see Lemma 3.2), thus reducing welfare in the short-run. Eventually the minimum wage binds on both firms (Figure 3.3(b), (c), (d)), market shares are equalised and welfare remains constant thereafter. Formally, define short-run social welfare as $SSW(\overline{w}) = SW(0,0,L_0)$ where L_0 is its subgame equilibrium value at locations *a*=*b*=0 and minimum wage $\overline{w}$. We have shown;

Theorem 5 For $\overline{w} \in (w_1^{**}, \phi_1)$ $SSW(\overline{w})$ is constant everywhere if $\delta = 0$, and if $\delta > 0$, $SSW(\overline{w})$ is strictly decreasing for $\overline{w} \in (w_1^{**}, \overline{w}_1)$, and constant for $\overline{w} \in [\overline{w}_1, \phi_1)$.

Thus in the short-run minimum wages cannot be beneficial to social welfare, and the positive long run welfare impact of minimum wages is therefore totally driven by and dependent on the effect of minimum wages on the non-wage job characteristics. Moreover, the change in job characteristics turns the policy's impact on market shares upside down; since the efficient firm moves closer to the centre when the minimum wage binds on its rival, its market share *increases* relative to laissez-faire.

Theorems 3 and 4 relate to social welfare in aggregate; finally we note the consequences for individual firms and workers. Not surprisingly firm profits fall as the minimum wage increases. The worker located at 0 sees the wage increase from laissez-faire, but the commuting time also increases as we move up the SPE paths shown in figure 3.5. However the net effect is advantageous[14], and this worker is the most likely to suffer a utility loss; hence all workers are better off.

Theorem 6 Suppose $\delta \in [0, \frac{1}{2})$ and $\overline{w} \in (w_1^{**}, \phi_1)$. As $\overline{w}$ increases, SPE profits of both firms decline, but the utility of all workers increases.

Proof See Appendix A.

6. CONCLUSIONS

We have shown how the imposition of minimum wages can be welfare improving on laissez-faire, because of their impact on firms' choice of non-wage job characteristics. In the context of a Hotelling duopsony, jobs are horizontally differentiated (e.g. by

[14] In the vertical job differentiation model of de Fraja (1999) this is not so – minimum wage workers may end up worse off because of the deterioration in working conditions.

location) and the effect of the minimum wage is to narrow the gap between locations chosen by firms compared to the maximum differentiation chosen under laissez-faire , in a welfare improving way. The paper thus provides a new route through which minimum wages can be "a good thing". Following the quite different labour market model of de Fraja (1999), the paper also generates a natural question for further research in the differentiated oligopsony framework, namely the effect of minimum wages on vertically differentiated job characteristics, as opposed to, or in addition to, our horizontal differentiation.

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Appendix A: Proofs of the main theorems

Proof of Theorems 1 and 2 The proof involves a number of steps.

Step 1 We have the following derivatives $\partial\overline{\Pi}_0 / \partial a$ when $(a,b) \in T$.

(A) In region $\varnothing$, $\overline{\Pi}_0(a,b) = \Pi_0^*(a,b)$, firm 0's laissez-faire profit, and so, $\partial\overline{\Pi}_0 / \partial a = \partial\Pi_0^* / \partial a = \frac{1}{18}t\left(\frac{\delta}{1-a-b}+3+a-b\right)\left(\frac{\delta}{1-a-b}-1-3a-b\right)$ whose sign is that of $F(a,b) = \delta-(1-a-b)(1+3a+b)$. When $\delta > 0$ the curve $F(a,b) = 0$ intersects the boundary of T where $\delta = (1-a-b)(3-a+b)$ uniquely at $a = \frac{1}{2}$, $b = \frac{1}{2}\sqrt{9-4\delta}-1$, the boundary of T where $a = 0$ uniquely at $b = \sqrt{1-\delta}$, and is downward sloping in T between these intercepts; $\partial\Pi_0^* / \partial a > 0$ to the right of the curve and $\partial\Pi_0^* / \partial a < 0$ to the left. When $\delta = 0$, $\partial\Pi_0^* / \partial a < 0$ everywhere on S.

(B) In region 01, $\partial\overline{\Pi}_0 / \partial a = \frac{1}{2}(\phi_0 - \overline{w}) > 0$.

(C) In region 1, $\partial\overline{\Pi}_0 / \partial a$ has the sign of $\phi_0 - \overline{w} + t(1-a-b)(1-b-3a)$. But in region 1, $\overline{w} < \phi_0 - t[(1-b)^2 - a^2]$, so $\phi_0 - \overline{w} > t(1-a-b)(1-b+a)$. It follows that $\partial\overline{\Pi}_0 / \partial a > 0$ since $1-b+a > 3a+b-1$ (i.e. $1 > a+b$).

(D) In region 0, $\partial\overline{\Pi}_0 / \partial a$ has the sign of $t(1-a-b)^2 - (\phi_1 - \overline{w})$. But in region 0, $\phi_1 - \overline{w} > t(1-a-b)(1-a+b) > t(1-a-b)^2$, so $\partial\overline{\Pi}_0 / \partial a < 0$.

Step 2 Suppose $\overline{w} \in (w_1^{**}, \overline{w}_2]$ ($= \varnothing$ if $\delta = 0$). From Figure 3.3(a), (b), the positive derivatives in (B) and (C) above imply that b=0 (or, symmetrically, 1) in any SPE. Given b=0, (A), (B) and (C) above imply that the point P in Figure 3.4 (a), (b) is the best response of firm 0 amongst $a \in [0, a_1]$. Amongst $a \in [a_1, 1]$, firm 0's best response to b=0 is to co-locate (a=1), from Lemma 3.1(b), so P will be the unique SPE location (up to symmetry) firm 0's profits at P exceed those at a=1, b=0. The required inequality becomes $H(a,\delta) = (1-a)(3+a+\frac{\delta}{1-a})^2 - 18\delta \geq 0$, where a is its value at P, certainly in $[0, \frac{1}{2}]$ and so it is sufficient for the result if $H(a,\delta) \geq 0$ everywhere on $(a,\delta) \in [0,\frac{1}{2}]^2$. It is easy to check H is decreasing in both a and δ on this domain, and $H(\frac{1}{2},\frac{1}{2}) > 0$, proving (a)(i).

Step 3 Consider now $\overline{w} \in [\overline{w}_2, \overline{\overline{w}}]$, and for convenience let (a^*, b^*) denote P, shown in Figure 3.2 for $\delta = 0$ and Figure 3.4(c) for $\delta > 0$; note $b^* < \frac{1}{2}$. From (C) and (D), P is the only remaining SPE candidate (up to symmetry). The maximum possible value of $b^* = \frac{1}{2} - \sqrt{\frac{\delta}{2}} < \frac{1}{2}\sqrt{9-4\delta} - 1$, so a^* is the best response by 0 to b^* over $(a, b^*) \in T$.

We show next that firm 0 does not want to deviate from P to co-locate; this requires,

$$\overline{\Pi}_0(a,b) = \tfrac{1}{18}t(1-a-b)[3+a-b+\frac{\delta}{1-a-b}]^2 \geq \phi_0 - \phi_1$$

where $\delta = 2(1-a-b)(a-b)$. Equivalently,

$$(1-a-b)[3+a-b+\frac{\delta}{1-a-b}]^2 \geq 18\delta = 36(1-a-b)(a-b)$$

which becomes $[1-(a-b)^2] \geq 0$, and clearly is satisfied. It remains to show that firm 0 does not want to deviate from P to any location strictly to the right of firm 1. From symmetry the profits attainable by firm 0 from such right deviations are the same as when firm 1 is at $1-b^*$ and firm 0 chooses strictly to the left of firm 1. From step 1 the only candidates for local maxima of $\overline{\Pi}_0(a,b)$ on this latter set are;

(a) along the $\varnothing/1$ border ($C2$) to the right of P

(b) in $\varnothing$ with a=0 if $(b^* <) 1-b^* < \sqrt{1-\delta}$.

The proof of (a) (ii) is completed by showing;

(i) along the $\varnothing/1$ border firm 0's profits decrease with b.

(ii) in $\varnothing$ where $a = 0$ and $b < \sqrt{1-\delta}$, $\partial\overline{\Pi}_0 / \partial b < 0$.

For (i): here $\overline{\Pi}_0(a,b) = \frac{1}{18}t(1-a-b)\,[3+a-b+\dfrac{\delta}{1-a-b}]^2$,

$\overline{w} = w_1^*(a,b) = \frac{1}{3}\phi_0 + \frac{2}{3}\phi_1 - \frac{1}{3}t(1-a-b)(3-a+b)$ so that

$da/db = -(1+b)/(2-a)$, and $\delta < (1-a-b)(3-a+b)$.

Differentiating $\overline{\Pi}_0(a,b)$ totally with respect to b and using the *da/db* expression shows that $d\overline{\Pi}_0/db < 0$ if $\delta < (1-a-b)(9-a+b)$ which follows since $\delta < (1-a-b)(3-a+b)$.

For (ii): it follows straightforwardly that, with $a = 0$, $\partial\overline{\Pi}_0/\partial b < 0$ iff $\delta/(1-b) < 5-3b$. Using the restriction $b < \sqrt{1-\delta}$, $\delta/(1-b) < 1+b$ and the required inequality follows as $b < 1$.

Finally, for $\overline{w} \in (\overline{\overline{w}}, \phi_1)$ ($= \emptyset$ if $\delta = 0$), (B) and (C) ensure that *P* in Figure 3.4(d) (the only candidate) is not an equilibrium completing (b). ■

<u>Proof of Corollary to Theorems 1 and 2</u> (a) is established in the text. For (b), from Theorem 1, the efficient firm market share when $\delta > 0$ and $\overline{w} \in \left[w_1^{**}, \overline{w}_2\right]$ (along *0A* in Figure 3.4) is $L_0 = \frac{1}{6}(3+a+\delta/(1-a))$, which is increasing in *a*, in turn increasing in $\overline{w}$. When $\delta > 0$ and $\overline{w} \in \left[\overline{w}_2, \overline{\overline{w}}\right]$ (along *AB* in Figure 3.5), the SPE value of L_0 is defined by (from Theorem 1);

$L_0 = \frac{1}{6}(3+a-b+\delta/(1-a-b))$ (1)

$\delta = 2(1-a-b)(a-b)$ (2)

Now (2) implies $da/db = (1-2b)/(1-2a)$ and hence (1) implies that

$$\frac{dL_0}{db} = \frac{1}{6}\left[\frac{da}{db} - 1 - \frac{\delta}{(1-a-b)^2}\left(-\frac{da}{db} - 1\right)\right] = \frac{1}{3(1-2a)}\left[a - b + \frac{\delta}{1-a-b}\right] > 0$$

So along *AB* as *b* increases, *a* increases, $\overline{w}$ increases and L_0 increases. ■

Proof of Theorem 4

First note that substitution of the laissez-faire SPE ($a = b = 0,\ L_0 = \frac{1}{2} + \frac{1}{6}\delta$) into $SW(a, b, L_0)$ produces, after some manipulation:

$$t^{-1}\ SW^{**} = \beta / t - \tfrac{1}{12} + \tfrac{1}{2}\delta + \tfrac{5}{36}\delta^2$$

(a) For $\overline{w} \in [w_1^{**}, \overline{w}_2]$, SPE locations are $b = 0$ and $a = a(\overline{w})$, where $a(\overline{w})$ is defined by $\overline{w} = w_1^*(a,0) = \frac{1}{3}\phi_0 + \frac{2}{3}\phi_1 - \frac{1}{3}t(1-a)(3-a)$. It follows that $a'(\overline{w}) = 3t^{-1}(4-2a)^{-1} > 0$. Social welfare can then be written as the following function of a, on the domain $a \in [0, \frac{1}{2}), \delta \geq 2a(1-a)$ with $L_0 = \frac{1}{6}(3+a) + \dfrac{\delta}{6(1-a)}$;

$$SW(a) = \phi_0\ L_0 + \phi_1(1-L_0) - \tfrac{1}{3}t[a^3 + (L_0 - a)^3 + (1-L_0)^3]$$

Since $a'(\overline{w}) > 0$ it suffices to show that $SW'(a) > 0$ on its domain. Differentiation and manipulation produces

$$36t^{-1}SW'(a) = 5\frac{\delta^2}{(1-a)^2} + 10\delta + 9 - 26a - 15a^2$$

Since $\delta/(1-a) \geq 2a$ on the domain, $5\delta^2/(1-a)^2 \geq 20a, 10\delta \geq 20a - 20a^2$, and so;

$36t^{-1}SW'(a) \geq 9 - 6a - 15a^2 > 0$ since $a \in [0, \frac{1}{2})$, completing the proof of (a).

(b)/(c) For $\overline{w} \in (\overline{w}_2, \overline{\overline{w}})$ SPE locations are $a(\overline{w})$ and $b = b(a)$ where $b(a)$ is defined by $\delta = 2(1-a-b)(a-b)$ for $a \in (\frac{1}{2}(1-\sqrt{1-2\delta}), \frac{1}{2}]$ and (e.g.) $a(\overline{w})$ is defined by $\overline{w} = w_1^*(a, b(a)) = \frac{1}{3}\phi_0 + \frac{2}{3}\phi_1 - \frac{1}{3}t(1-a-b(a))(3-a+b(a))$. From the $b(a)$ definition $b'(a) = (1-2a)/(1-2b)$, and it follows from $\overline{w} = w_1^*(a, b(a))$ that $a'(\overline{w}) = \frac{1}{2}t^{-1}(1-2b)(1-a-b)^{-1} > 0$. Now social welfare is the following function of a, where $a \in (\frac{1}{2}(1-\sqrt{1-2\delta}), \frac{1}{2}]$, $b = b(a)$ and $L_0 = \frac{1}{2}(1-b+a)$, and where b increases from *0* to $\frac{1}{2} - \sqrt{\delta/2}$ as a increases over its domain;

$$SW(a) = \phi_0 L_0 + \phi_1(1-L_0) - \tfrac{1}{3}t[a^3 + b^3 + (L_0 - a)^3 + (1-b-L_0)^3]$$

Since $a'(\overline{w}) > 0$ it suffices again to show that $SW'(a) > 0$ on its domain. Differentiation and manipulation produces;

$$t^{-1}(1-2b)\,SW'(a) = \delta(a-b) - a^2(1-2b) - b^2(1-2a) + \tfrac{1}{2}(1-a-b)^3$$

Substituting $a-b = \tfrac{1}{2}\delta/(1-a-b)$, s=a+b (which increases from $\tfrac{1}{2}\left(1-\sqrt{1-2\delta}\right)$ to $1-\sqrt{\delta/2}$ as a varies over its domain) and manipulating:

$$8t^{-1}(1-s)^2(1-2b).\,SW'(a) = \delta^2(3-5s) - 4(2s-1)(1-s)^3 = m(s), \text{say}$$

It is easy to check that $m(s) > 0$ for $s \in (\tfrac{1}{2}\left(1-\sqrt{1-2\delta}\right), \tfrac{1}{2}]$, that $m(s)$<0 for $s > \tfrac{3}{5}$,that $m'(s) < 0$ for $s \in \left[\tfrac{1}{2}, \tfrac{3}{5}\right]$, and that $m(1-\sqrt{\delta/2})$ has the sign of $5\delta - 2$. Hence, if $\delta \geq \tfrac{2}{5}$, *m(s)* is positive over its domain, proving (b). If $\delta \in (0, \tfrac{2}{5})$ *m(s)* has a unique maximum at some $s \in \left(\tfrac{1}{2}, 1-\sqrt{\delta/2}\right)$, so $a + b > \tfrac{1}{2}$, which corresponds to a unique maximum of *SW(a)* over its domain; (c) follows with $\overline{w}^*$ as the associated minimum wage.

(d) From (a), (b) and (c) it suffices to show that $SW\left(\overline{\overline{w}}\right) > SW^{**}$. At $\overline{\overline{w}}$, $a = \tfrac{1}{2}, b = \tfrac{1}{2} - \sqrt{\delta/2}$ and the required inequality becomes after manipulation;

$$n(y) = 20y^3 - 45y^2 + 18y - 9 < 0$$

where $y = \sqrt{\delta/2} \in (0,1)$. Now $n(0) < 0$, $n(1) < 0$, n is concave on $\left[0, \tfrac{3}{4}\right]$, convex on $\left[\tfrac{3}{4}, 1\right]$. A straightforward calculation shows that at the unique stationary point on $\left[0, \tfrac{3}{4}\right]$, $n(y) < 0$ which ensures the result.

(e) This follows since at the unique social optimum $a^\circ + b^\circ = \tfrac{1}{2}$ whereas at the SPE location which maximizes social welfare $a + b > \tfrac{1}{2}$. ■

Proof of Theorem 6 The fall in profit is immediate from Theorem 1 when $\delta = 0$. From Theorem 2(a)(i), when $\delta > 0$ and $\overline{w} \in (w_1^{**}, \overline{w}_2]$, $\pi_i = \Pi_i^*(a,0), i = 0,1,$ and a increases along *OA* in Figure 3.4(b) as $\overline{w}$ increases in this range. It is easy to check that $\Pi_i^*(a,0)$ is decreasing in a $(< \frac{1}{2})$, so profits fall as $\overline{w}$ increases. Similarly, from Theorem 2(a)(ii) when $\delta > 0$ and $\overline{w} \in (\overline{w}_2, \overline{\overline{w}}]$, $\pi_i = \Pi_i^*(a,b), i = 0,1,$ with (a,b) increasing along *AB* in Figure 3.4(b) as $\overline{w}$ increases in this range. Along *AB* $\delta = 2(1-a-b)(a-b)$, so $db/da = (1-2a)/(1-2b)$, and it is easy to check that $\Pi_i^*(a,b)$ is decreasing as a $(< \frac{1}{2})$ increases along *AB*, so again profits fall as $\overline{w}$ increases.

If (a,b,w_0,w_1) are their SPE values when $\overline{w} \in (w_1^{**}, \phi_1)$, the equilibrium utility of the workers at the extreme locations is, with obvious notation, $u(0) = w_0 - ta^2$ and $u(1) = w_1 - tb^2$. Since (a,b,w_0,w_1) never decrease as $\overline{w}$ increases in this range, it is clear that the extreme workers are the most likely to suffer a utility loss from an increase in the minimum wage since their commuting times increase the most. But when $\delta = 0$, and using Theorem 1, $\partial u(0)/\partial \overline{w} = \partial u(1)/\partial \overline{w} = 1 - a > 0$, so the utility of all workers increases monotonically with $\overline{w}$. When $\delta > 0$ the worker at 0 is unambiguously most likely to suffer a utility fall, because an increase in the minimum wage produces a smaller wage increase but larger increase in commuting time than that for the worker at 1. Using Theorem 2 it is easy to check that u(0) is again monotonically increasing in $\overline{w}$, both for $\overline{w} \in (w_1^{**}, \overline{w}_2]$ (Theorem 2(a)(i), along *OA* in Figure 3.4(b)) and for $\overline{w} \in (\overline{w}_2, \overline{\overline{w}}]$ (Theorem 2(a)(ii), along *AB* in Figure 3.4(b)). Again the utility of all workers increases monotonically with $\overline{w}$. ■

Appendix B: Proofs of lemmas and propositions

Proof of Lemma 3.1

For a wage subgame with location $(a,b) \in S$, we first show that the following (i) – (iii) describe firm 0's best responses and (iv) – (vi) those of firm 1, where $\gamma_0 = \phi_1 - t(1-a-b)(3+a-b)$,

$\Lambda_0 = \phi_1 + t(1-a-b)(1-a+b)$, $\gamma_1 = \phi_0 - t(1-a-b)(3-a+b)$,
$\Lambda_1 = \phi_0 + t(1-a-b)(1-b+a)$;

(i) $w_0 = w_1 + t(1-a-b)(1-a+b)$ if $w_1 < \gamma_1$

(ii) $w_0 = \frac{1}{2}\{\phi_0 + w_1 - t(1-a-b)(1+a-b)\}$ if $w_1 \in [\gamma_1, \Lambda_1)$

(iii) $w_0 = [0, \phi_0]$ if $\Lambda_1 \le w_1$

(iv) $w_1 = w_0 + t(1-a-b)(1+a-b)$ if $w_0 < \gamma_0$

(v) $w_1 = \frac{1}{2}\{\phi_1 + w_0 - t(1-a-b)(1-a+b)\}$ if $w_0 \in [\gamma_0, \Lambda_0)$

(vi) $w_1 = [0, \phi_1]$ if $\Lambda_0 \le w_0$

From the definitions of π_0 and $\tilde{x}$;

(1) $\pi_0 = 0$ if $\tilde{x} \le 0$, i.e. $w_0 \le w_1 - t[(1-b)^2 - a^2]$

(2) $\pi_0 = (\phi_0 - w_0)\tilde{x}$

if $\tilde{x} \in (0,1]$, i.e. $w_1 - t[(1-b^2 - a^2] < w_0 \le w_1 + t[(1-a)^2 - b^2]$

(3) $\pi_0 = \phi_0 - w_0$ if $\tilde{x} \ge 1$, i.e. $w_1 + t[(1-a)^2 - b^2] \le w_0$

It is easy to check that (1), (2) and (3) define π_0 as a continuous, quasi-concave function of w_0 over the whole range $[0, \phi_0]$ (constant at 0 over the range of (1), strictly concave over (2) and linear, decreasing over (3)).

If $w_1 \ge \Lambda_1$ then $\tilde{x} \le 0$, and so $L_0 = \pi_0 = 0$, for all $w_0 \in [0, \phi_0]$. Thus any $w_0 \in [0, \phi_0]$ is a best response for firm 0 to $w_1 \ge \Lambda_1$. If $w_1 < \Lambda_1$ then strictly positive profits are attainable by firm 0 (by choosing $w_0 = \phi_0 - \varepsilon$, ε small enough), and a best response must lie in the range of (2) above. In this range, π_0 is a strictly concave function of w_0 with stationary point $w_0 = \frac{1}{2}\{\phi_0 + w_1 - t[(1-b)^2 - a^2]\}$ which lies in the range of (2), and so is the best response, iff $w_1 \in [\gamma_1, \Lambda_1)$. If $w_1 < \gamma_1$, π_0 is increasing over the range of (2) so the maximum of π_0 occurs at $w_0 = w_1 + t[(1-a)^2 - b^2]$, which is therefore the best response. Interchanging 0/1 subscripts, *a*/*b* and ϕ_0 / ϕ_1, and replacing $\tilde{x}\ by\ (1-\tilde{x})$ produces the firm 1 result. Thus the set of subgame NE for $(a,b) \in S$ correspond to simultaneous solutions of one of (i) – (iii) with one of (iv) – (vi), where $w_0 \in [0, \phi_0]$ and $w_1 \in [0, \phi_1]$.

(a) Assume $(a,b) \in T$ and consider the (ii)/(v) pairing. The equations intersect at $w_i = w_i^*(a,b), i = 0,1$ and the resulting w_0, w_1 satisfy the inequalities in (ii)/(v) iff $\delta < (1-a-b)(3-a+b)$ (or $(a,b) \in T$). It is straightforward to check that no pairings produce any other NE for $(a,b) \in T$, which completes the proof of (a), using the NE wages to derive the corresponding market shares and profits.

(b) Assume $(a,b) \in S/T$. Consider the (i)/(vi) pairing where $w_1 = \phi_1$ in (vi). The resulting wages $(w_0 = \phi_1 + t[(1-a)^2 - b^2], w_1 = \phi_1)$ satisfy the required inequalities iff $\delta > (1-a-b)(3-a+b)$. The (ii)/(vi) pairing with $w_1 = \phi_1$ produces $w_0 = \frac{1}{2}\{\phi_0 + \phi_1 - t[(1-b)^2 - a^2]\}, w_1 = \phi_1$, which satisfies the inequalities iff $\delta = (1-a-b)(3-a+b)$, in which case $w_0 = \phi_1 + t[(1-a)^2 - b^2]$. Again one can check that no pairings produce other NE for $(a,b) \in S/T$ and that the NE wages produce the market shares and profits in (b). Consider now the case $(a,b) \in H$. From the definition of L_0 for this case, if $w_1 = \phi_1$, firm 0 attains $\pi_0 = \phi_0 - \phi_1 > 0$ with $w_0 = \phi_1$, which cannot be improved upon $(w_0 > \phi_1 \Rightarrow \pi_0 = \phi_0 - w_0 < \phi_0 - \phi_1, w_0 < \phi_1 \Rightarrow \pi_0 = 0)$. If $w_0 = \phi_1$, firm 1 can do no better than choose $w_1 = \phi_1$, giving $\pi_1 = 0$. Thus $w_0 = w_1 = \phi_1$ is a NE. This is the unique NE: if $w_1 < \phi_1$ then $w_0 = w_1$ is again 0's best response giving $\pi_1 = 0$, but $w_1 + \varepsilon\ (< \phi_1, \varepsilon > 0)$ strictly improves for 1; market shares and profits are as claimed, completing the proof. ■

Discussion and Proof of Proposition 1

To find the SPE of the laissez-faire game we need the NE of the "reduced form" stage I location game where firm 0 chooses $a \in$ [0, 1], firm 1 chooses $b \in$ [0, 1] and payoffs are given by $\Pi_i^*(a,b)$ in lemma 3.1. It turns out that the inefficient firm always wants to locate as far as possible from the rival, because of the usual centrifugal force; it moves away to soften wage competition and avoid the zero profits near co-location. In contrast, the efficient firm gets positive profits when it co-locates, and these can overcome the centrifugal tendency. If δ is small ($\delta < 1/4$) the centrifugal force dominates. But if $\delta \in [1/4, \delta^*]$ where $\delta^* = 6 - 3\sqrt{3} \cong 0.81$, the efficient firm co-locates if the rival is near the mid-point (b=1/2) and the centrifugal force is therefore small since th efficient firm cannot get "very far" from the rival; otherwise it moves to the extremity. Lemmas B.1 and B.2 provide the formal statements.

Lemma B.1 The best response of firm 1 in the reduced form stage I game when

$\delta \in (0, 5/4)$, is $b = 0$ if $a < \frac{1}{2}$, $b = 1$ if $a > \frac{1}{2}$ and $b = \{0,1\}$ *if* $a = \frac{1}{2}$.

Proof We look first at the "constrained best response" of the inefficient firm in this game, which solves:

$\max_b \Pi_1^*(a,b)$ s.t. $0 \le b \le 1-a$. We denote this solution $\psi_1(a)$, and $\tilde{\Pi}_1(a)$ are the resulting profits.

For $a \in [\bar{a},1]$, $\Pi_1^*(a,b) = 0$ for all $b \in [0,1-a]$, so $\psi_1(a) = [0,1-a]$ *and* $\tilde{\Pi}_1(a) = 0$. For $a \in [0,\bar{a})$, firm 1 can attain positive profit only by choosing *b* so that (*a*,*b*) $\in$ *T*, but then, from lemma 3.1(b);

$$\partial \Pi_1^* / \partial b = \tfrac{1}{18} t\,(3-a+b-\delta/(1-a-b))(-1-a-3b-\delta/(1-a-b)) < 0$$

Thus $\psi_1(a) = 0$ and $\tilde{\Pi}_1(a) = \Pi_1^*(a,0)$ for $a \in [0,\bar{a})$, completing the description of firm 1's constrained best responses. The function $\tilde{\Pi}_1(a)$ thus defined is easily seen to be continuous, strictly decreasing on $[0,\bar{a})$ and constant at 0 on $[\bar{a},1]$.

In firm 1's unconstrained best response problem, it can also choose $b \in [1-a\,,1]$. From symmetry the maximum attainable profit over this *b* interval is $\tilde{\Pi}_1(1-a)$, and the unconstrained best response profit for firm 1 is $\max[\tilde{\Pi}_1(a),\ \tilde{\Pi}_1(1-a)]$ attained at the best responses $\psi_1(a)$ *if* $\tilde{\Pi}_1(a) > \tilde{\Pi}_1(1-a)$, $1-\psi_1(1-a)$ if $\tilde{\Pi}_1(1-a) > \tilde{\Pi}_1(a)$ and at $\{\psi_1(a), 1-\psi_1(1-a)\}$ if $\tilde{\Pi}_1(a) = \tilde{\Pi}_1(1-a)$. When $\delta < 5/4,\ \bar{a} > \tfrac{1}{2}$, which completes the proof of the Lemma. ■

<u>Lemma B.2</u> The best response of firm 0 in the reduced form stage I game is;

(a) for $\delta \in (0, 1/4)$, $a = 0$ if $b < \tfrac{1}{2}$, $a = 1$ if $b > \tfrac{1}{2}$, $a = \{0,1\}$ if $b = \tfrac{1}{2}$

(b) for $\delta \in [1/4, \delta^*]$, there is a strictly decreasing function $b(\delta)$ with $b(\tfrac{1}{4}) = \tfrac{1}{2}$, $b(\delta^*) = 0$ such that $a = 0$ if $b < b(\delta)$, $a = \{0,1-b\}$ if $b = b(\delta)$, $a = 1-b$ if $b \in (b(\delta), 1-b(\delta))$, $a = \{1-b, 1\}$ if $b = 1-b(\delta)$ and $a = 1$ if $b > 1-b(\delta)$

<u>Proof</u> Suppose $0 < \delta \le \delta^*$. From lemma 3.1 we have;

(i) $\partial\Pi_0^*/\partial a = 2t(1-a) > 0$ when $(a, b) \in (S \cup H)\backslash T$ and $a < 1$.

(ii) When $(a, b) \in T$, $\partial\Pi_0^*/\partial a = \frac{1}{18}t\left(\frac{\delta}{1-a-b}+3+a-b\right)\left(\frac{\delta}{1-a-b}-1-3a-b\right)$

whose sign coincides with that of $F(a,b) = \delta-(1-a-b)(1+3a+b)$. The curve $F(a,b) = 0$ intersects the boundary of T where $\delta=(1-a-b)(3-a+b)$ uniquely at $a=\frac{1}{2}$, $b=\frac{1}{2}\sqrt{9-4\delta}-1$, the boundary of T where $a = 0$ uniquely at $b=\sqrt{1-\delta}$, and is downward sloping in T between these intercepts when $\delta \le \frac{3}{4}$. For $\delta \in (\frac{3}{4}, \delta^*]$, the curve slopes down when $3a+2b > 1$, but is upward sloping when $3a+2b < 1$ (with a turning point at $b = 2 - \sqrt{3\delta}$, $a = \frac{1}{3}(1-2b)$). In each case, $\partial\Pi_0^*/\partial a > 0$ to the right of the curve and $\partial\Pi_0^*/\partial a < 0$ to the left.

Consider 0's constrained best response problem: $\max_a \Pi_0^*(a,b)$ s.t. $a \in [0, 1-b]$. Define $G(b,\delta) = \Pi_0^*(0,b) - \Pi_0^*(1-b,b)$ on the domain $b \in [0, \sqrt{1-\delta}], \delta \in (0, \delta^*]$. Then;

$$G(b,\delta) = \frac{1}{18}t\left[(1-b)\left(\frac{\delta}{1-b}+3-b\right)^2 - 18\delta\right],$$

$$\partial G/\partial b = \frac{1}{18}t\left(\frac{\delta}{1-b}+3-b\right)\left(\frac{\delta}{1-b}+3b-5\right) \le \frac{1}{18}t\left(\frac{\delta}{1-b}+3-b\right)(4b-4) < 0$$

and $\partial G/\partial\delta = \frac{1}{9}t\left(\frac{\delta}{1-b}-b-6\right) < 0 \left(\text{using } \frac{\delta}{1-b} \le 1+b \text{ on the domain}\right)$.

Thus there is a decreasing function $b(\delta)$ on the domain $\delta \in (0, \delta^*]$ such that $b=b(\delta)$ iff $G(b,\delta)=0$, $b < b(\delta)$ iff $G(b,\delta) > 0$ and $b > b(\delta)$ iff $G(b,\delta) < 0$. Moreover $\lim_{\delta\to 0} b(\delta) = 1$ (since $G(1,\delta) \to 0$ as $\delta\to 0$), $b\left(\frac{1}{4}\right) = \frac{1}{2}$ (since $G\left(\frac{1}{2},\frac{1}{4}\right)=0$) and $b(\delta^*) = 0$ (since $G(0,\delta) = \frac{1}{18}t[(\delta+3)^2 - 18\delta] = 0$ when $\delta = \delta^*$).

In the case where $\delta \in \left(0, \tfrac{3}{4}\right]$, the derivative signs in (i) and (ii), and the downward slope of the curve $F(a,b) = 0$ imply that $a = 0$ and $a = 1-b$ are the only 2 candidates for 0's constrained best response when $b \in [0, \sqrt{1-\delta}\,]$, and it follows from the previous paragraph that $a = 0$ if $b < b(\delta)$, $a = \{0, 1-b\}$ if $b = b(\delta)$ and $a = 1-b$ if $b > b(\delta)$. Moreover $\Pi_0^*(0,b) = \tfrac{1}{18} t(1-b)\left(\dfrac{\delta}{1-b} + 3 - b\right)^2$ is continuous and strictly decreasing in b, and $\Pi_0^*(1-b,b) = \alpha - \beta$ independent of b. When $\delta \in \left(0, \tfrac{1}{4}\right)$, $b(\delta) > \tfrac{1}{2}$ and using the symmetry of the (a,b) and $(1-a, 1-b)$ subgames, 0's unconstrained best response is as described in (a). When $\delta \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right]$, $b(\delta) \le \tfrac{1}{2}$ and the symmetry ensures the unconstrained best response of (b).

When $\delta \in (\tfrac{3}{4}, \delta^*]$, the above arguments ensure the unconstrained best responses in (b) if $b \in [0, \sqrt{1-\delta}\,]$ or if $b \in [2 - \sqrt{3\delta}, 1]$. When $b \in \left(\sqrt{1-\delta}, 2 - \sqrt{3\delta}\right)$ the candidates for 0's constrained best response are $a = 1-b$ and the value of a where $(a,b) \in T$ is on the upward sloping part of the $F(a,b) = 0$ curve; let $a = a(b)$ denote this curve, defined by $F(a,b) = 0$ and $3a+2b < 1$ for $b \in \left(\sqrt{1-\delta}, 2 - \sqrt{3\delta}\right)$. Along this curve 0's profit is $\Pi_0^*(a(b), b)$ whose derivative with respect to b is $-4(1-a-b) < 0$. Also $G\left(\sqrt{1-\delta}, \delta\right) = \tfrac{1}{9} t\left(8 - 9\delta - 8\sqrt{1-\delta}\right) < 0$ so $b(\delta) < \sqrt{1-\delta}$, and $a = 1-b$ is 0's constrained best response to any $b > b(\delta)$, as in the last paragraph, producing again the (b) statement. ■

Best response graphs are shown in Figures B.1 and B.2.

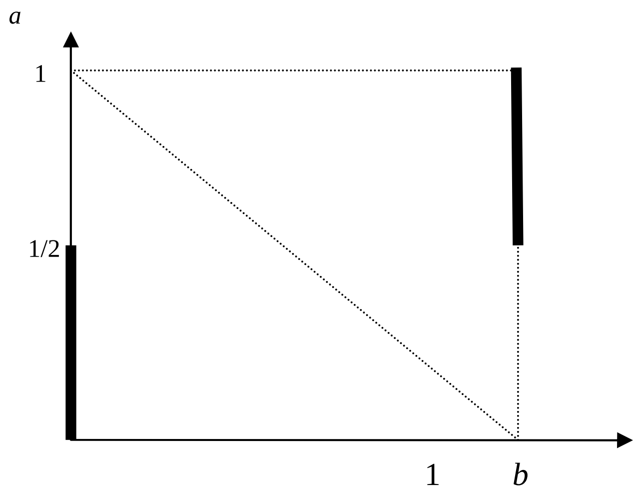
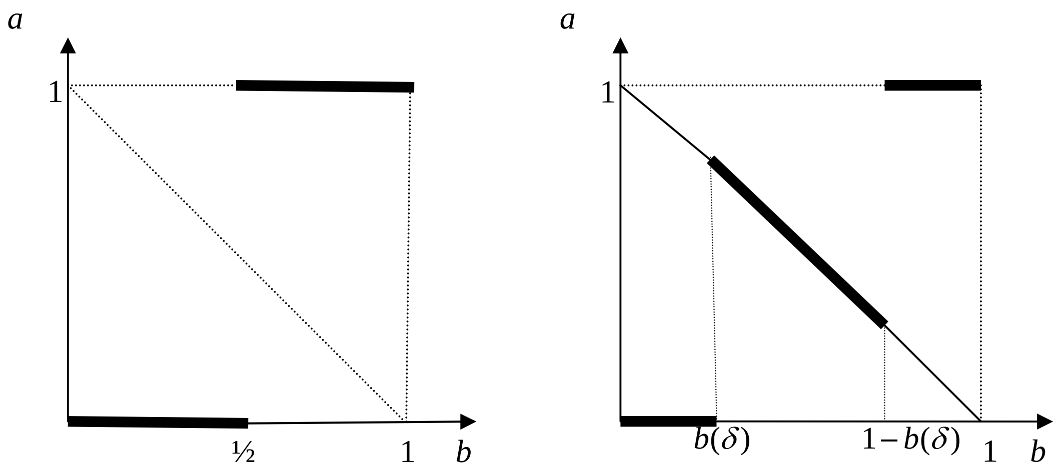


Figure B.1; 1’s laissez-faire best location response graph



(a) $0 < \delta < \frac{1}{4}$ (b) $\frac{1}{4} \leq \delta \leq \delta^*$

Figure B.2; 0’s laissez-faire best location response graph

Superimposing figure B.1 on figure B.2 (a) or (b) establishes Proposition 1(a) when $\delta > 0$, and it is straightforward to extend arguments for $\delta = 0$. On the other hand, when $\delta > \delta^*$, the simultaneous-move location game has no equilibrium in pure strategies: firm 1 wants to locate as far as possible from firm 0, and firm 0 wants to co-locate with firm 1.

REMARK; Ziss (1993) suggests that figure B.2 (a) is 0’s best response graph for all $\delta \in [0, \delta^*]$, overlooking figure B.2 (b).

Proof of Lemma 3.2

For $(a, b) \in S$, Lemma 3.1(a) and the quasi-concavity of π_i as a function of w_i noted in its proof ensure that the best responses of firm 0 are described by (i) – (iii) below, and those of firm 1 by (iv)-(vi):

(i) $w_0 = \max[\overline{w}, w_1 + t[(1-a)^2 - b^2]]$ if $w_1 < \gamma_1$

(ii) $w_0 = \max\left[\overline{w}, \frac{1}{2}\left\{\phi_0 + w_1 - t\left[(1-b)^2 - a^2\right]\right\}\right]$ if $w_1 \in \left[\gamma_1, \Lambda_1\right)$

(iii) $w_0 = \left[\overline{w}, \phi_0\right]$ if $\Lambda_1 \le w_1$

(iv) $w_1 = \max[\overline{w}, w_0 + t[(1-b)^2 - a^2]]$ if $w_0 < \gamma_0$

(v) $w_1 = \max\left[\overline{w}, \frac{1}{2}\left\{\phi_1 + w_0 - t\left[(1-a)^2 - b^2\right]\right\}\right]$ if $w_0 \in \left[\gamma_0, \Lambda_0\right)$

(vi) $w_1 = \left[\overline{w}, \phi_1\right]$ if $\Lambda_0 \le w_0$

Thus NE for subgames with $(a, b) \in S$ and $\overline{w} \in (0, \phi_1]$ correspond to solutions for $w_0 \in \left[\overline{w}, \phi_0\right], w_1 \in \left[\overline{w}, \phi_1\right]$ of one of (i)-(iii) coupled with one of (iv)-(vi).

(a) Suppose $(a, b) \in T$. Comparing the above best responses (i)-(vi) with those of Lemma 3.1(a) it is immediate that the laissez-faire outcomes continue as NE iff $\overline{w} \le \min\left[w_0^*(a,b), w_1^*(a,b)\right]$, completing the proof of (i).

Solutions with $w_0 = w_1 = \overline{w}$ can be generated by the (ii)/(v) pairing iff:

(1) $\overline{w} \ge \overline{w}_0(a,b) = \phi_0 - t[(1-b)^2 - a^2]$ (2) $\overline{w} \ge \overline{w}_1(a,b) = \phi_1 - t[(1-a)^2 - b^2]$

(3) $\overline{w} \ge \gamma_1 = \phi_0 - t(1-a-b)(3-a+b)$ (4) $\overline{w} \ge \gamma_0 = \phi_1 - t(1-a-b)(3+a-b)$

But (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4). Thus (ii)/(v) produce NE with $w_0 = w_1 = \overline{w}$ (and the corresponding market shares and profits in (ii)) iff (1) and (2) hold. It is straight forward to check that no pairings produce other NE with $w_0 = w_1 = \overline{w}$, completing the proof of (ii).

The (ii)/(v) pairing produces solutions with $w_0 \geq \overline{w} = w_1$ iff

$w_0 = \frac{1}{2}\left\{\phi_0 + \overline{w} - t\left[(1-b)^2 - a^2\right]\right\}$, $w_1 = \overline{w}$ and

(5) $\overline{w} \leq \overline{w}_0(a,b) = \phi_0 - t[(1-b)^2 - a^2]$ (6) $\overline{w} \geq \gamma_1 = \phi_0 - t\,(1-a-b)(3-a+b)$

(7) $\overline{w} \geq \frac{1}{2}\left\{\phi_1 + w_0 - t\left[(1-a)^2 - b^2\right]\right\}$ (8) $\Lambda_0 > w_0 \geq \gamma_0 = \phi_1 - t(1-a-b)(3+a-b)$

Substitution of w_0 shows (7) is equivalent to $\overline{w} \geq w_1^*(a,b)$. For $(a, b) \in T$ the inequalities in (5) and (7) imply those of (6) and (8), so (ii)/(v) produce NE with $w_0 \geq w_1 = \overline{w}$ (and market shares and profits of (c)) iff (5) and (7) hold. Again no pairings produce other NE with $w_0 \geq w_1 = \overline{w}$, completing (iii). The proof of (iv) is symmetric to that for (iii).

(b) For $(a, b) \in (S \cup H)/T$, the laissez-faire outcomes in Lemma 3.1(b) and 3.1(c) always continue as NE since for i=0,1 $w_i^*(a,b) \geq \phi_1 \geq \overline{w}$. It is straightforward to check that no pairings (of (i)-(iii) with (iv)-(vi)) produce any other NE.

■

Proof of Lemma 3.3

Consider first firm 1's constrained best response. The proofs of (b) and (a)(i) follow immediately from the text arguments.

(a)(ii) and (iv). In figure 3.3(b) and (d), Y_1Y_2 is the curve *C3*, defined by $\overline{w} = \overline{w}_0(a,b) = \phi_0 - t[(1-b)^2 - a^2]$. When b=0 this implies $(\phi_0 - \overline{w})/t = 1 - a^2$, so $a = \sqrt{1 - \frac{(\phi_0 - \overline{w})}{t}}$ which is then the a-value at Y_2. Hence, for $a \in [0, \sqrt{1 - \frac{(\phi_0 - \overline{w})}{t}}]$, the constrained best response along Y_1Y_2 is given by the solution in *b* to $\overline{w} = \overline{w}_0(a,b)$, so $b = 1 - \sqrt{a^2 + \frac{(\phi_0 - \overline{w})}{t}}$. When $a \in [\sqrt{1 - \frac{(\phi_0 - \overline{w})}{t}}, a_1]$, *b*=0 is the constrained best response.

<u>(a)(iii)</u>. In figure 3.3, Y_1Y_2 is again the curve *C3*, and Y_2 is now defined by the intersection of *C3* and the dashed curve (*D*), so $\overline{w} = \overline{w}_0(a,b)$ and $\delta = 2(1-a-b)(a-b)$. Writing $r = 2(\phi_0 - \overline{w})/t$, these 2 conditions are; (1) $r = 2(1-a-b)(1+a-b)$ and (2) $\delta = 2(1-a-b)(a-b)$ which imply $r(a-b) = \delta(1+a-b)$, so $r(1-b) = \delta a + r(1-a)$. Substituting back into (1) gives $\left[a\delta + r(1-a)\right]^2 - a^2(r-\delta)^2 - \frac{1}{2}r(r-\delta)^2 = 0$, which simplifies to produce $a = \frac{r}{2(r-\delta)} - \frac{1}{4}(r-\delta) = \frac{1}{2} - \frac{1}{4}(r-\delta) + \frac{\delta}{2(r-\delta)} = \underline{a}$. For $a \in \left[0, \underline{a}\right]$, the solution of (1) for *b* in terms of *a* produces the required constrained best response.

In figure 3.3(c), Y_2Y_3 is the curve *C1*, defined by (3) $\overline{w} = w_0^*(a,b)$ $= \frac{2}{3}\phi_0 + \frac{1}{3}\phi_1 - \frac{1}{3}t(1-a-b)(3+a-b)$. Y_3 is defined by the additional condition *b*=0, which produces $a = \overline{a}$. The general solution of (3) for *b* in terms of *a* then produces the required best response for $a \in [\underline{a}, \overline{a}]$, with b=0 for $a \in [\overline{a}, a_1]$.

Turning to the unconstrained best responses, let $\tilde{\pi}_1(a)$ denote firm 1's constrained best response profits when $a \in \left[0,1\right]$. $\tilde{\pi}_1(a) = 0$ for $a \in \left[a_1, 1\right]$; also $\tilde{\pi}_1(a)$ is a continuous function, from Figure 4.1 since $\hat{\pi}_1(a,b)$ is a continuous function. We now show that $\tilde{\pi}_1(a)$ is a strictly decreasing function on $\left[0, a_1\right)$; the symmetry arguments used earlier to establish lemmas B.1 and B.2 then complete the proof.

Along vertical segments of firm 1's constrained best response graph ((0, 0) to $(a_1, 0)$) in Figure 3.3(a), Y_2 to $(a_1, 0)$ in Figures 4.1(b) and (d), Y_3 to $(a_1, 0)$ in Figure 4.1(c)), $\tilde{\pi}_1(a) = \Pi_1^*(a,0) = \frac{1}{18}t\left(3 - a - \frac{\delta}{1-a}\right)^2$ and $\partial\tilde{\pi}_1/\partial a < 0$. Along segments of 1's constrained best response graph that coincide with the *C1* $(Y_2$ to Y_3 in figure $3.3(c))$

$\tilde{\pi}_1(a) = \Pi_1^*(a, b(a))$ where $b(a)$ is defined by $\overline{w} = \overline{w}_0^*(a, b(a))$ so that $b'(a) = -(1+a)(2-b)^{-1} \in (-1,0)$ which ensures $\partial\tilde{\pi}_1 / \partial a < 0$ here also. Finally, along segments of 1's constrained best response graph that coincide with *C3* (Y_1 to Y_2 in Figures 3.3(b), (c) and (d)), $\tilde{\pi}_1(a) = \frac{1}{2}(\phi_1 - \overline{w})(1 + b(a) - a)$ where *b(a)* is now defined by $\overline{w} = \overline{w}_0(a, b(a))$ so $b'(a) = -a(1-b)^{-1} < 0$ and again $\partial\tilde{\pi}_1 / \partial a < 0$. ■

Discussion and Proof of Proposition 2

REMARK; The Ziss(1993) proof of Proposition 2 uses Lagrangeans without proof of a supporting concavity statement that would ensure sufficiency of the resulting conditions. We found the required concavity elusive, and offer instead an alternative proof.

Suppose without loss of generality that $a + b \leq 1$. From (2.6) in the text, social welfare is

$$SW(a, b, L_0) = \phi_0 L_0 + \phi_1(1 - L_0) - \tfrac{1}{3}t[a^3 + b^3 + (L_0 - a)^3 + (1 - b - L_0)^3]$$

Given (a,b) the socially optimal L_0, $L_0(a,b)$, equates $\phi_0 - t(L_0 - a)^2$ *to* $\phi_1 - t(1 - b - L_0)^2$ if the resulting $L_0 \in [0,1]$, otherwise $L_0 = 1$. Hence;

$$L_0(a,b) = \begin{cases} \delta / 2(1 - b - a) + \frac{1}{2}(1 - b + a) & \text{if} \quad \delta \leq (1-a)^2 - b^2 \\ 1 & \text{if} \quad \delta \geq (1-a)^2 - b^2 \end{cases}$$

Substituting the top branch here into the SW formula and writing $\ell = 1 - a - b$ produces the function;

$$f(a,b) = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}\delta^2 t \ell^{-1} + \tfrac{1}{2}\delta t(a - b) - \tfrac{1}{3}t\left\{a^3 + b^3 + \tfrac{1}{8}(\delta\ell^{-1} + \ell)^3 + \tfrac{1}{8}(\ell - \delta\ell^{-1})^3\right\}$$

Similar substitution of the bottom branch produces;

$g(a,b) = \alpha - \frac{1}{3}t[a^3 + (1-a)^3]$

Hence the maximum social welfare attainable at locations $(a,b) \in S \cup H$ is;

$$SW(a,b,L_0(a,b)) = \begin{cases} f(a,b) & \text{if} \quad \delta \le (1-a)^2 - b^2 \\ g(a,b) & \text{if} \quad \delta \ge (1-a)^2 - b^2 \end{cases}$$

Note the following features of $f(a,b)$;

(i) $$\frac{1}{t}\frac{\partial f}{\partial a} = \frac{1}{2}\delta^2\ell^{-2} + \frac{1}{2}\delta - a^2 - \frac{1}{8}\left(\delta\,\ell^{-1} + \ell\right)^2\left(\delta\,\ell^{-2} - 1\right) + \frac{1}{8}\left(\ell - \delta\,\ell^{-1}\right)^2\left(\delta\,\ell^{-2} + 1\right)$$

$$= \frac{1}{2}\delta - a^2 + \frac{1}{4}\ell^2 + \frac{1}{4}\delta^2\ell^{-2}$$

(ii) $$\frac{1}{t}\frac{\partial f}{\partial b} = -\frac{1}{2}\delta - b^2 + \frac{1}{4}\ell^2 + \frac{1}{4}\delta^2\ell^{-2}$$

(iii) Equating (i) and (ii) to 0, f has a unique stationary point $a = \frac{1}{4} + \delta, b = \frac{1}{4} - \delta$ with $f(a,b) = f^* = \alpha - \frac{1}{2}\delta t + \delta^2 t - \frac{1}{48}t.$ Now consider problem 1: $\max_{(a,b)} g(a,b)$ s.t. $\delta \ge (1-a)^2 - b^2, (a,b) \in S \cup H.$ The solutions are $a = \frac{1}{2}$, $b \in [\sqrt{\frac{1}{4} - \delta}, \frac{1}{2}]$ if $\delta < \frac{1}{4}$, and $a = \frac{1}{2}$, $b \in \left[0, \frac{1}{2}\right]$ if $\delta \ge \frac{1}{4}$; in both cases the optimal value is $g^* = \alpha - \frac{1}{12}t.$ If $\delta \ge 1,$ the feasible set for problem 1 is $S \cup H$ and the solution to problem 1 is then necessarily the social optimum.

Suppose $\delta < 1$ from now on.

Next consider problem 2: $\max_{(a,b)} f(a,b)$ s.t. $\delta \le (1-a)^2 - b^2, (a,b) \in S \cup H.$ The feasible set is nonempty (with a non-empty interior) and compact, so there is a solution. But solutions cannot occur,

(1) on the feasible set boundary where $a = 0, b \in [0, \sqrt{1-\delta})$ since $t^{-1}\partial f / \partial a = \frac{1}{4}\left(\frac{\delta}{1-b} + 1 - b\right)^2 > 0;$

(2) on the feasible set boundary where $b = 0, a \in [0,1-\sqrt{\delta})$ since $t^{-1}\partial f/\partial b = \frac{1}{4}\left(1-a-\frac{\delta}{1-a}\right)^2 > 0$ there.

In addition, when $\delta \geq \frac{1}{4}$ the (unique) stationary point is not interior to the feasible set, so any solution to problem 2 belongs to the boundary where $\delta = (1-a)^2 - b^2$ and $a,b \geq 0$. But f and g coincide on this boundary which was also feasible, but not optimal, in problem 1. It follows that the solution to problem 1 provides the social optimum for all $\delta \geq \frac{1}{4}$, completing (b). Finally, when $\delta < \frac{1}{4}$, the stationary point of f is interior to the feasible set of problem 2 with value f^*; moreover $f^* > g^*$ then. Thus the (unique) stationary point is the only solution candidate interior to the feasible set for problem 2, and there cannot be a boundary solution. So the stationary point solves problem 2 and, since $f^* > g^*$, provides the social optimum; hence (a).

■