LEARNING, NETWORK FORMATION AND COORDINATION*

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A B S T R A C T

In many economic and social contexts, individual players choose their partners and also decide on a mode of behavior in interactions with these partners. This paper develops a simple model to examine the interaction between partner choice and individual behavior in games of coordination. An important ingredient of our approach is the way we model partner choice: we suppose that a player can establish ties with other players by investing in costly pair-wise links.

We show that individual efforts to balance the costs and benefits of links sharply restrict the range of stable interaction architectures; equilibrium networks are either complete or have the star architecture. Moreover, the process of network formation has powerful effects on individual behavior: if costs of forming links are low then players coordinate on the risk-dominant action, while if costs of forming links are high then they coordinate on the efficient action.

Keywords: Networks, social learning, equilibrium selection, coordination games, efficiency, risk dominance.

1 INTRODUCTION

In recent years, several authors have examined the role of interaction structure { di®erent terms like network structure, neighborhood in°uences, and peer group pressures, have been used { in explaining a wide range of social and economic phenomena. This includes work on social learning and adoption of new technologies, evolution of conventions, collective action, labor markets, and ¯nancial fragility.[1] This research suggests that the structure of interaction can be decisive in determining the nature of outcomes. This leads us to examine the reasonableness/robustness of di®erent structures and is the primary motivation for developing a model in which the evolution of the interaction structure is itself an object of study.

We propose a general approach to study this question. We suppose that individual entities can undertake a transaction only if they are `linked'. This link may refer to a social or a business relationship, or it may refer simply to awareness of the others. We take the view that links are costly, in the sense that it takes e®ort and resources to create and maintain them. This leads us to study the incentives of individuals to form links and the implications of this link formation for aggregate outcomes.

In the present paper, we apply this approach to a particular problem: the in°uence of link formation on individual behavior in games of coordination.[2] There is a group of players, who have the opportunity to play a 2 £ 2 coordination game with each other. We start with the case where two players

[1] See e.g., Allen and Gale (1998), Anderlini and Ianni (1997), Bala and Goyal (1998), Chwe (1996), Coleman (1966), Ellison and Fudenberg (1993), Ellison (1993), Ely (1996), Goyal and Janssen (1997), Granovetter (1974), Haag and Laguno® (1999), and Morris (2000), among others.

[2] Many games of interest have multiple equilibria. The study of equilibrium selection/problem of coordination therefore occupies a central place in game theory. We discuss the contribution of our paper to this research in greater detail below.

can only play with one another if they have a direct pair-wise link. Subsequently, we take up the case where players can play with one another if they are directly or indirectly connected. These links can be made on individual initiative but are costly to form. So each player prefers that others incur the cost and form links with him. For simplicity, the game is assumed to yield only positive payoffs in every bilateral interaction. Individuals care about aggregate payoffs and, therefore, they always accept any link supported (i.e. paid) by some other player. The link decisions of different players define a network of social interaction. In addition to the choice of links, each player also has to choose an action that she will use in all the games that she will engage in. We are interested in the nature of networks that emerge and the effects of link formation on social coordination.

In our setting, links as well as actions in the coordination game are chosen by individuals on an independent basis. This allows us to study the social process as a non-cooperative game. We start with a consideration of the situation in which two players can only play a game if they have a direct link between them. We find that a variety of networks – including the complete network, the empty network and partially connected networks – can be supported in equilibria of the static game. Moreover, the society can coordinate on different actions and conformism as well as diversity with regard to actions of individuals is possible in Nash equilibrium. This multiplicity motivates an examination of the dynamic stability of different outcomes.

We develop a dynamic model in which, at regular intervals, individuals choose links and actions to maximize their payoffs. Occasionally they make errors or experiment. Our interest is in the nature of long run outcomes, when the probability of these errors is small. We find that the dynamics generate clear-cut predictions both concerning the architecture of networks as well as regarding the nature of social coordination.

In particular, we show that the complete network is the unique stochastically stable architecture (except for the case where costs of link formation are very high and the empty network results).[3] Figure 1a gives an example of a complete network in a society with 4 players. This result shows that partially connected networks are not stable. We also find that, if players are at all connected, they always coordinate in the long run on the same action, i.e. social conformism obtains. However, the nature of coordination depends on the costs of link formation. We find that for low costs of link formation, players coordinate on the risk-dominant action, while for high costs of link formation they coordinate on the efficient action (Theorem 3.1). Thus our analysis reveals that, even though the eventual network is the same in all cases of interest, the process of network formation itself has serious implications for the nature of social coordination.

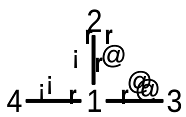

Figure 1a
Complete Network

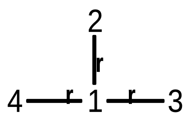

Figure 1b
Center-sponsored Star

The above result says that equilibrium networks are complete or empty. In practice, a variety of factors will lead to incomplete networks; these include capacity constraints on the time and budget of individual players, increasing costs to forming links, and it is also likely that indirect connections will facilitate transactions making complete networks unnecessary. In the present paper we explore the role of indirect linkages between players that facilitate transactions. As before, our focus is on the architecture of stable networks

[3] In a complete network, every pair of players is directly linked, while in a empty network there are no links.

and the in°uence of link formation on the behavior of players in the games with linked players.

We consider a model in which two players can play a game if they are directly or indirectly linked with each other.[4] In this setting, we ¯nd that the center-sponsored star is the unique stochastically stable architecture. This is a network in which one player forms a link with every other player and pays for all the links. Figure 1b provides an example of this architecture for a society with 4 players. We also ¯nd that there exists a critical cost of forming links, such that, for costs below this level, players coordinate on the risk-dominant action, while for costs above this level, they coordinate on the e±cient action (Theorem 4.1). We thus ¯nd that in the two settings the equilibrium network remains the same { the complete network in one case and the star in the other case { but the nature of individual behavior is very di®erent depending upon the cost of forming links. Moreover, the precise relationship between costs of forming links and the nature of coordination is robust across the di®erent settings. The general arguments we use in the proofs of Theorems 4 and 14 are quite similar. We now sketch them brie°y.

We consider the case where the cost of forming links is such that both types of coordination outcomes, the e±cient one as well as the ine±cient (risk-dominant) one can be sustained in a social equilibrium. We study the stochastic stability of the two outcomes: complete network with e±cient coordination and complete network with ine±cient (but risk-dominant) coordination. Roughly speaking, we need to assess the minimum number of `mutations' required to exit from each state. Suppose that we are in the following state: complete network with everyone choosing the ine±cient action. We assess the minimum number of mutations needed to exit from this

[4] More precisely, two players can play a game with each other if there is a path between them in the social network.

state as follows: given a particular network structure a certain minimum of players must be choosing the efficient action for a player to prefer to play this action. The first step is to find the minimum number of such players needed, across the set of all possible networks. This step also derives the network architecture that facilitates the transition. Clearly this number sets a lower bound on the number of mutations and is therefore necessary to exit from the efficient state. The second step then shows that this minimum computed in the first step is also sufficient for transition.

We now develop some intuition for the nature of the network that yields this minimum number of mutations. Here for the sake of concreteness, we focus on the case of direct links. In our model links are one-sided and this makes them a public good. An action α is particularly attractive for player i when every player choosing α forms a link with i, while no player choosing the other action β forms any links with i. In such a situation, if player i were to choose β then she would have to form a link with every player choosing the same action, while if she chooses action α then she can hope to `free-ride' on the links that the others have created.

Number the players from 1 to n. Suppose that a player i forms a a link with every other player with a higher index. This generates a complete network. Now suppose that the first k players have their strategies `mutated', and they all switch from the inefficient action (β) to the efficient action (α). Now consider the situation of player $k + 1$. This player is exactly in the situation described above. If the costs of forming links are very small then player $k + 1$ will choose to connect with everyone, irrespective of the choice of actions. Thus the free-riding aspect is relatively unimportant; the network is complete for all practical purposes and standard risk-dominance considerations prevail if cost of forming links is small. Next suppose that costs are relatively high in the sense that player $k + 1$ forms a link with players k + 2; :::; n only if she chooses action ¯, i.e., the costs of forming links are higher than the miscoordination payo®. In this case, the structure of network becomes important and we show that the existence of the k `passive' links makes it attractive to switch to the e±cient action, even with relatively few players actually playing this action. This in turn leads to the e±cient coordination outcome being stochastically stable when costs of forming links are relatively high.

We now place the paper and the results in context. Traditionally, sociologists have held the view that individual actions, and in turn aggregate outcomes, are in large part determined by interaction structure. By contrast, economists have tended to focus on markets, where social ties and the speci¯c features of the interaction structures between agents are typically not important. In recent years, economists have examined in greater detail the role of interaction structure and found that it plays an important role in shaping important economic phenomena (see the references given above, and also Granovetter, 1985). This has led to a study of the processes through which the structure emerges. The present paper is part of this general research program.

We relate the paper to the work in economics next. The paper contributes to two research areas: network formation games and equilibrium selection/coordination problems. We suppose that an individual players can form pair-wise links by incurring some costs, at their own initiative, i.e., link formation is one-sided. This allows us to model the network formation process as a non-cooperative game. This element of our model is similar to the work of Bala and Goyal (2000). Related work on network formation includes Dutta, van den Nouweland and Tijs (1995) and Jackson and Wolinsky (1996). Earlier work focuses on the architectural and the welfare properties of strategically stable networks. The primary contribution of the present paper is the presentation of a common framework within which the emergence of interaction networks and the behaviour of linked players can be studied.[5] In particular, the indirect links model in the present paper is closely related to the work in Bala and Goyal (2000); this allows us to use arguments from Bala and Goyal (2000) to prove results on equilibrium network structures in the current paper (cf. Proposition 11). The analysis of the indirect links model builds on this networks characterization result and its main contribution lies in establishing a clear relationship between the cost of forming links and behavior in coordination games between linked players. This is the content of Theorem 14.

In many games of interest, multiple equilibria arise naturally and so the problem of equilibrium selection occupies a central place in the theory of games. In recent years, there has been a considerable amount of research on equilibrium selection/coordination;[6] An important ¯nding of this work is that interaction structure and the mobility of players matters and that by varying the structure, the rate of change as well as the long run outcome can be signi¯cantly altered.[7] It is therefore worthwhile to examine the circumstances under which di®erent network structures emerge. From a theoretical point of view, a natural way to do this is by examining the strategic stability of di®erent interaction structures. This is the route taken in the present paper.

[5] In independent work, Droste, Gilles and Johnson (1999), and Jackson and Watts (1999) have developed a related model which addresses similar concerns. The primary di®erence between these papers and our paper is the model of link formation: the other papers consider two-sided link formation while we study one-sided link formation. Moreover, we allow for direct as well as indirect connections, while these papers consider only direct connections. These di®erences have a signi¯cant impact on the conclusions. We further discuss these papers in the conclusion.

[6] One strand of this work considers dynamic models. This work includes Blume (1993), Canning (1992), Ellison (1993), Kandori, Mailath and Rob (1993), and Young (1993), among others.
For a consideration of this same equilibrium selection problem from a di®erent (\eductive") perspective, the reader may refer to the work of Harsanyi and Selten (1988) or the more recent paper by Carlson and van Damme (1993).

[7] See, for example, Ellison (1993), Goyal (1996) and Morris (2000), among others.

A well known result on equilibrium selection in the learning and evolution literature is that risk-dominance considerations prevail over those of efficiency and, if there is a conflict between these two considerations then, an inefficient but risk-dominant equilibrium can be stable, in the long run. This finding have been re-examined by several authors and the result has been shown to be sensitive to different assumptions, such as the nature of the strategy revision rule, precise modeling of mutation, and mobility of players across locations.[8] The results in our paper are closely related to the work on mobility.

The basic insight of the work on player mobility is that if individuals can separate/insulate themselves easily from those who are playing an inefficient action (e.g., the risk-dominant action), then efficient "enclaves" will be readily formed and eventually attract the "migration" of others (who will therefore turn to playing efficiently). In a rough sense, one may be inclined to establish a parallel between easy mobility and low costs of forming links. However, the considerations involved in each case turn out to be very different, as is evident from the sharp contrast between our conclusions (recall the above summary) and those of the mobility literature.

There are two main reasons for this contrast. First, in our case, players do not indirectly choose their pattern of interaction with others by moving across a pre-specified network of locations (as in the case of player mobility). Rather, they construct directly their interaction network (with no exogenous restrictions) by choosing those agents with whom they want to play the game. Second, the cost of link formation (which are paid per link formed) act as a screening device that is truly effective only if it is high enough. In a heuristic sense, we may say that it is precisely the restricted "mobility" these costs induce which helps insulate (and thus protect) the individuals who are

[8] Bergin and Lipman (1996), Robson and Vega-Redondo (1996),, Ely (1996), Galesloot and Goyal (1997), Mailath, Samuelson and Shaked (1994), Oechssler (1997), Bhaskar and Vega-Redondo (1998) among others.

choosing the efficient action. If the link-formation costs are too low, the extensive interaction they facilitate may have the unfortunate consequence of rendering risk-dominance considerations decisive.

The rest of this paper is organized as follows. Section 2 describes the framework. Section 3 presents the results for the case of direct links, while Section 4 studies the case where players can play a game if they are either directly or indirectly connected to each other. Section 5 concludes.

2 THE MODEL

2.1 Networks

Let $N = \{1, 2, \ldots, n\}$ be a set of players, where $n \geq 3$. We are interested in modelling a situation where each of these players can choose the subset of other players with whom to play a fixed bilateral game. Formally, let $g_i = (g_{i,1}, \ldots g_{i,i-1}, g_{i,i+1}, \ldots g_{i,n})$ be the set of links formed by player i. We suppose that $g_{i,j} \in \{1, 0\}$, and say that player i forms a link with player j if $g_{i,j} = 1$. The set of link options is denoted by G_i. Any player profile of link decisions $g = (g_1, g_2 \ldots g_n)$ defines a directed graph, called a network. Abusing notation, the network will also be denoted by g.

Specifically, the network g has the set of players N as its set of vertices and its set of arrows, $\Gamma \subset N \times N$, is defined as follows, $\Gamma = \{(i, j) \in N \times N : g_{ij} = 1\}$. Graphically, the link (i, j) may be represented as an edge between i and j, a filled circle lying on the edge near agent i indicating that this agent has formed (or supports) that link. Every link profile $g \in G$ has a unique representation in this manner. Figure 1 below depicts an example. In it, player 1 has formed links with players 2 and 3, player 3 has formed a link with player 1, while player 2 has formed no links.[9]

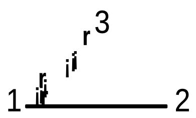

Figure 1

Given a network g; we say that a pair of players i and j are directly linked if at least one of them has established a linked with the other one, i.e. if $\max\{g_{i,j}, g_{j,i}\} = 1$. To describe the pattern of players' links, it is useful to define a modified version of g, denoted by $\bar{g}$, that is defined as follows: $\bar{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N. Note that $\bar{g}_{i,j} = \bar{g}_{j,i}$ so that the index order is irrelevant. We say there is a path in g between i and j if either $\bar{g}_{i,j} = 1$ or there exist agents $j_1, \ldots, j_m$ distinct from each other and i and j such that $\bar{g}_{i,j_1} = \cdots = \bar{g}_{j_k,j_{k+1}} = \cdots \bar{g}_{j_m,j} = 1$. We write $i \xrightarrow{\bar{g}} j$ to indicate a path between players i and j in network $\bar{g}$.

Let $N^d(i;g) \equiv \{j \in N : g_{i,j} = 1\}$ be the set of players in network g with whom player i has established links, while $\mu^d(i;g) \equiv |N^d(i;g)|$ is its cardinality. Similarly, let $N^d(i;\bar{g}) \equiv \{j \in N : \bar{g}_{i,j} = 1\}$ be the set of players in network g with whom player i is directly linked, while $\mu^d(i;\bar{g}) \equiv |N^d(i;\bar{g})|$ is the cardinality of this set. Let $N(i;g) \equiv \{j \in N : i \xrightarrow{\bar{g}} j\}$ be the players with whom player i has a path (is directly or indirectly linked) in a network g; we also define $\mu(i;\bar{g}) \equiv |N(i;\bar{g})|$ to be the cardinality of this set.

A subgraph $g' \subset g$ is called a component of g if for all $i, j \in g'$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in g'$ and $j \in g$, $g_{i,j} = 1$ implies $g'_{ij} = 1$. A network with only one component is called

[9] Since agents choose strategies independently of each other, two agents may simultaneously initiate a two-way link, as seen in the figure.

connected. Given any g; the notation $g + ij$ will denote the network obtained by replacing $g_{i,j}$ in network g by 1. Similarly, $g - g_{i,j}$ will refer to the network obtained by replacing $g_{i,j}$ in network g by 0. A connected network g is said to be minimally connected if the network obtained by deleting any single link, $g - g_{i,j}$, is not connected. A special example of minimally connected network is the center-sponsored star: a network g is called a center-sponsored star if there exists some $i \in N$ such that, for all $j \in N \setminus \{i\}$; $g_{ij} = 1$, and for all $j, k \in N \setminus \{i\}$; $j \neq k$;, $g_{jk} = 0$.

2.2 Social Game

Individuals located in a social network play a 2×2 symmetric game in strategic form with common action set. The set of partners of player i depends on her location in the network. We shall consider two different models: in the first model we will assume that two individuals can play a game if and only if they have a direct link between them. In this case, player i will play a game with all other players in the set $N^d(i; \bar{g})$. In the second model a player can play a game with all other players with whom she is directly or indirectly linked. In this case, player i will play a game with other players in the set $N(i; \bar{g})$.

We now describe the two-person game that is played between players. The set of actions is $A = \{\alpha, \beta\}$: For each pair of actions $a, a' \in A$; the payoff $\pi(a, a')$ earned by a player choosing a when the partner plays a' is given by the following table:

1 \ 2	α	β
α	d	e
β	f	b

Table I

We shall assume that this is a coordination game, with two pure strategy equilibria, ®;®) and ¯;¯). Without loss of generality we will assume that the (®;®) equilibrium is the e±cient one. Finally, in order to focus on the interesting case, we will assume that there is a con°ict between e±ciency and risk-dominance. These considerations are summarized in the following restrictions on the payo®s.[10]

$$d > f; \;\; b > e; \;\; d > b; \;\; d + e < b + f: \tag{1}$$

An important feature of our approach is that links are assumed costly. Specifically, every agent who establishes a link with some other player incurs a cost $c > 0$. Thus, we suppose that the cost of forming a link is independent of the number of links being established and is the same across all players. Another important feature of our model is that links are one-sided. This aspect of the model allows us to use standard solution concepts from non-cooperative game theory in addressing the issue of link formation. We shall assume that the payo®s in the bilateral game are all positive and, therefore, no player has any incentive to refuse links initiated by other players.[11]

[10] Our results extend in a natural way in case the risk-dominant equilibrium is also e±cient, i.e., if $d + e > b + f$. In particular, the network is either complete or a star (depending on the nature of links), while players coordinate on the (®;®) equilibrium, which is risk-dominant as well as e±cient, in the long run.

[11] There are di®erent ways in which the assumption of one-side links and positive payo®s in the coordination game can be relaxed. One route is to retain the one-sided links aspect but to incorporate negative payo®s. This motivates the following formulation: suppose that a link between players i and j formed by player i allows payo®s to °ow to player i only. In this case, both the payo® °ow (which may be negative) and costs are one-sided. This model may be interpreted as a model of peer groups and fashion, with asymmetric °ow of in°uence allowed. An analysis of this model suggests that the relationship between costs of forming links and equilibrium networks and actions is similar to the one obtained in Theorem 4.

Another way would be to allow for players to have the option of refusing links initiated by others (and possibly also negative payo®s.) This would lead to a model with two-sided links which requires di®erent methods of analysis and lies outside the scope of the present paper.

We shall assume that every player i is obliged to choose the same action in the (possibly) several bilateral games that she is engaged in. This assumption is natural in the present context; if players are allowed to choose a different action for every two-person game this will make the behaviour of players in any particular game insensitive to the network structure. The strategy space of a player can be identified with $S_i = G_i \times A$; where G_i is the set of possible link decisions by i and A is the common action space of the underlying bilateral game.[12]

We can now present the payoffs of the social game. First, we present the payoff for the case where only directly linked players can play with each other. Given the strategies of other players, $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, the payoff to a player i from playing some strategy $s_i = (g_i, a_i)$ is given by:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in N^d(i;g)} \pi(a_i, a_j) - \eta^d(i;g) \cdot c \tag{2}$$

We note that the individual payoffs are aggregated across the games played by him. In some of the earlier work, e.g., Ellison (1993), Kandori, Mailath and Rob (1993), authors have assumed that individuals care about average payoffs. In our framework, the number of games a player plays is endogenous, and we would like to explicitly account for the influence of the size of the neighborhood. This motivates the aggregate payoff formulation.

Second, we present the payoffs for the case where two players can play a game if they are either directly or indirectly linked with each other. Given

[12] In our formulation, players choose links and actions in the coordination game at the same time. An alternative formulation would be to have players choose links first and then choose actions, contingent on the nature of the network observed. We discuss the timing of moves in the concluding remarks.

the strategies of other players, $s_{-i} = (s_1, \ldots s_{i-1}, s_{i+1}, \ldots s_n)$, the payoff to a player i from playing some strategy $s_i = (g_i, a_i)$ is given by:

$$\hat{\Pi}_i(s_i, s_{-i}) = \sum_{j \in \hat{N}(i;g)} \pi(a_i, a_j) - \eta^d(i;g) \cdot c \tag{3}$$

These payoff expressions allow us to particularize the standard notion of Nash Equilibrium as follows. A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is said to be a Nash equilibrium for the game if, for all $i \in N$,

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i. \tag{4}$$

The set of Nash equilibria will be denoted by S^*. A Nash equilibrium is said to be strict if every player gets a strictly higher payoff with her current strategy than she would with any other strategy. The equilibrium notions for the indirect links model are obtained by substituting $\hat{\Pi}(.)$ in place of $\Pi(.)$.

2.3 Dynamics

Time is modeled as being discrete, $t = 1, 2, 3, \ldots$. At each t, the state of the system is given by the strategy profile $s(t) \equiv [(g_i(t), a_i(t))]_{i=1}^n$ specifying the action played, and links established, by each player $i \in N$. At every period t, there is a positive independent probability $p \in (0, 1)$ that any given individual gets a chance to revise her strategy.[13] If she receives this opportunity, we assume that she selects a new strategy

$$s_i(t) \in \arg\max_{s_i \in S_i} \Pi_i(s_i, s_{-i}(t-1)). \tag{5}$$

That is, she selects a best response to what other players chose in the preceding period. If there are several strategies that fulfill (5), then any one of them is taken to be selected with, say, equal probability. This strategy

[13] This formulation may be interpreted as saying that, with some positive probability, a player dies and is replaced by another player.

revision process defines a simple Markov chain on $S \equiv S_1 \times \cdots \times S_n$. In our setting, which will be seen to display multiple strict equilibria, there are several absorbing states of the Markov chain.[14] This motivates the examination of the relative robustness of each of them.

To do so, we rely on the approach proposed by Kandori, Mailath and Rob (1993), and Young (1993). We suppose that, occasionally, players make mistakes, experiment, or simply disregard payoff considerations in choosing their strategies. Specifically, we assume that, conditional on receiving a revision opportunity, a player chooses her strategy at random with some small "mutation" probability $\varepsilon > 0$. For any $\varepsilon > 0$, the process defines a Markov chain that is aperiodic and irreducible and, therefore, has a unique invariant probability distribution. Let us denote this distribution by μ_ε. We analyze the form of μ_ε as the probability of mistakes becomes very small, i.e. formally, as ε converges to zero. Define $\lim_{\varepsilon \to 0} \mu_\varepsilon = \mu^*$. When a state $s = (s_1, s_2, \ldots, s_n)$ has $\mu^*(s) > 0$, i.e. it is in the support of μ^*, we say that it is stochastically stable. Intuitively, this reflects the idea that, even for infinitesimal mutation probability (and independently of initial conditions), this state materializes a significant fraction of time in the long run.

3 DIRECT LINKS

This section provides an analysis of the model in which two players can undertake a transaction only if they have a direct link between them. We first characterize the Nash equilibrium of the social game. We then provide a complete characterization of the set of stochastically stable social outcomes.

[14] We note that the set of absorbing states of the Markov chain coincides with the set of strict Nash equilibria of the one-shot game.

3.1 Equilibrium outcomes

Let $s^* = [(g_i^*, a_i^*)]_{i=1}^n$ be a Nash equilibrium of the population game described and denote by $g^* \equiv (g_i^*)_{i=1}^n$ the corresponding equilibrium network.[15]

Our first result concerns the nature of networks that arise in equilibria. If costs of link formation are low ($c < e$), then a player has an incentive to link up with other players irrespective of the actions the other players are choosing. On the other hand, when costs are quite high (specifically, $b < c < d$) then everyone who is linked must be choosing the efficient action. This, however, implies that it is attractive to form a link with every other player and we get the complete network again. Thus, for relatively low and high costs, we should expect to see the complete network. In contrast, if costs are at an intermediate level ($f < c < b$) a richer set of configurations is possible. On the one hand, since $c > f (> e)$, the link formation is only worthwhile if other players are choosing the same action. On the other hand, since $c < b (< d)$, coordinating at either of the two equilibria (in the underlying coordination game) is better than not playing the game at all. This allows for networks with two disconnected components in equilibria. These considerations underlie the following result.

Proposition 1 Suppose (1 and (2) hold. (a) If $c < \min\{f, b\}$, then an equilibrium network is complete. (b) If $f < c < b$, then an equilibrium network is either complete or can be partitioned into two complete components.[16] (c) If $b < c < d$, then an equilibrium network is either empty or complete. (d) If $c > d$, then the unique equilibrium network is empty.

[15] The fact that links are costly immediately implies the absence of superfluous links, i.e. if $g_{i,j}^* = 1$ then $g_{j,i}^* = 0$.

[16] Our parameter conditions allow both $f < b$ and $b < f$. If the latter inequality holds, Part (b) of Proposition 1 (and also that of Proposition 2 below) applies trivially.

The proofs of Propositions 1 and 2 are given in Appendix A. We now characterize the Nash equilibria of the static game. First, we introduce some convenient notation. On the one hand, recall that g^e denotes the empty network characterized by $g^e_{ij} = 0$ for all $i, j \in N$ $(i \neq j)$. We shall say that a network g is essential if $g_{ij} g_{ji} = 0$, for every pair of players i and j. Also let $G^c \equiv \{g : \forall i, j \in N, \bar{g}_{ij} = 1, g_{ij} g_{ji} = 0\}$ stand for the set of complete and essential networks on the set N. Analogously, for any given subset $M \subset N$, denote by $G^c(M)$ the set of complete and essential subgraphs on M. Given any state $s \in S$, we shall say that $s = (g, a) \in S^h$ for some $h \in \{\alpha, \beta\}$ if $g \in G^c$ and $a_i = h$ for all $i \in N$. More generally, we shall write $s = (g, a) \in S^{\alpha\beta}$ if there exists a partition of the population into two subgroups, N^α and N^β (one of them possibly empty), and corresponding components of g, g^{α} and g^β, such that: (i) $g^{\alpha} \in G^c(N^\alpha)$, $g^\beta \in G^c(N^\beta)$; and (ii) $\forall i \in N^\alpha$, $a_i = \alpha$; $\forall i \in N^\beta$, $a_i = \beta$. With this notation in hand, we may state the following result.

Proposition 2 Suppose (1) and (2) hold. (a) If $c < \min\{f, b\}$, then the set of equilibrium states $S^* = S^\alpha \cup S^\beta$. (b) If $f < c < b$, then $S^\alpha \cup S^\beta \subset S^* \subset S^{\alpha\beta}$, the first inclusion being strict for large enough n. (c) If $b < c < d$, then $S^* = S^\alpha \cup \{(g^e, (\beta, \beta, ..., \beta))\}$. (d) If $c > d$, then $S^* = \{g^e\} \times A^n$.

Parts (a) and (c) are intuitive; we elaborate on the coexistence equilibrium identified in part (b). In these equilibria, there are two unconnected groups, with each group following a single action. The strategic stability of this configuration rests on the appeal of `passive' links. A link such as $g_{ij} = 1$ is paid for by player i, but both players i and j derive payoffs from it. We refer to g_{ij} as an active link for player i and a passive link for player j. In the mixed equilibrium configuration the links in each group are evenly distributed. This means that players enjoy the benefits of passive links. If a player were to switch actions, then to derive the full benefits of this switch, she would have to form (active) links with everyone in the new group. This lowers the incentives to switch. These incentives are decisive if the relative number of passive links is large enough (hence the requirement of large n.)

The above result indicates that, whenever the cost of links is not excessively high (i.e. not above the maximum payoff attainable in the game), Nash equilibrium conditions allow for a genuine outcome multiplicity. For example, under the parameter configurations allowed in Parts (a) and (c), this multiplicity permits alternative states where either of the two actions is homogeneously chosen by the whole population. Under the conditions of Part (b), the multiplicity allows for a wide range of possible states where neither action homogeneity nor full connectedness necessarily prevails. Therefore, the model raises a fundamental issue of equilibrium selection.[17]

3.2 Dynamics

This section resolves the problem of equilibrium selection using the techniques of stochastic stability. As a first step in this analysis, we establish convergence of the unperturbed dynamics. Let S^{∞} denote the set of absorbing states of this dynamics. Given the definition of the adjustment process, it follows that there is an one-to-one correspondence between S^{∞} and the class of strict Nash equilibria of the social game which are characterized in Proposition 2. If $c < b$; all Nash equilibria are strict, while if $b < c < d$; only the Nash equilibria in S^{α} are strict. Finally, if $c > d$; no strict Nash equilibria exist. In the next result, therefore, we focus on the case where $c < d$.

Proposition 3 Suppose (1) and (2) hold and $c < d$. Then, starting from any initial strategy configuration, the best response dynamics converges, almost surely, to a strict Nash equilibrium of the social game.

[17] We note that the equilibria identified in parts (a)-(b) and those in S^{α} are also strict equilibria.

The proof of the above result is given in Appendix A. This result delimits the set of states that can potentially be stochastically stable since, obviously, every such state must be a limit point for the unperturbed dynamics. Let the set of stochastically stable states be given by $\hat{S} \equiv \{s \in S : \mu^*(s) > 0\}$. The following result summarizes our analysis of the dynamics in the direct-link model.

Theorem 4 Suppose (1) and (2) hold. There exists some $\hat{c} \in (e, b)$ such that if $c < \hat{c}$ then $\hat{S} = S^{\beta}$ while if $\hat{c} < c < d$ then $\hat{S} = S^{\alpha}$; provided n is large enough.[18] Finally, if $c > d$ then $\hat{S} = \{g^e\} \times A^n$.

In order to determine the support of the limit distribution μ^*, we use the well-known graph-theoretic techniques developed by Freidlin and Wentzell (1984) for the analysis of perturbed Markov chains, as applied by the aforementioned authors (Kandori et al. and Young) and later simplified by Kandori and Rob (1995). They can be summarized as follows. Fix some state $s \in S^{\infty}$. An s-tree is a directed graph on S^{∞} whose root is s and such that there is a unique (directed) path joining any other $s' \in S^{\infty}$ to s. For each arrow $s' \to s''$ in any given s-tree, a "cost" is defined as the minimum number of simultaneous mutations that are required for the transition from s' to s'' to be feasible through the ensuing operation of the unperturbed dynamics alone. The cost of the tree is obtained by adding up the costs associated with all the arrows of a particular s-tree. The stochastic potential of s is defined as the minimum cost across all s-trees. Then, a state $s \in S^{\infty}$ is found to be stochastically stable if it has the lowest stochastic potential across all $s \in S^{\infty}$.

In our framework, individual strategy involves both link formation and action choice in games with linked individuals. This richness in the strategy

[18] The proviso on n is simply required to deal with possible integer problems when studying the number of mutations required for the various transitions.

space leads to a corresponding variety in the nature of (strict) Nash equilibria of the social game. There are two facets of this variety: one, we obtain three different types of equilibria in terms of action configuration, S^{α}, S^{β} and $S^{\alpha\beta}$; and two, there are a large number of strategy profiles that support the equilibrium network configurations. For example, in a game with 10 players, there are 2^{45} different (links) strategy profiles that can support a complete network. This proliferation of equilibria necessitates the development of specific arguments to asses the stochastic potential of different equilibrium profiles.

The first step in the analysis is concerned with establishing a simple relationship between the strategy profiles within the sets S^h, for $h = \alpha, \beta$. Specifically, we show that states in each of these sets can be connected by a chain of single-mutation steps, each such step followed by a suitable operation of the best-response dynamics. To state this result precisely, it is convenient to introduce the metric $d(\cdot)$ on the space of networks that, for each pair of networks g and g', has their respective distance given by $d(g, g') = d(g', g) \equiv \sum_{i,j} |g_{i,j} - g'_{i,j}|/2$. In words, this distance is simply a measure of the number of links that are different across the two networks. With this metric in place, we have the following Lemma.

Lemma 5 For each $s \in S^h$, $h = \alpha, \beta$, there exists an s-tree restricted to S^h such that for all arrows $s' \to s''$ in it, $d(g', g'') = 1$, where g' and g'' are the networks respectively associated to s' and s''.

The proof of this lemma is given in Appendix A. This lemma implies that, provided $S^h \subset S^{\infty}$, the (restricted) tree established by Lemma 5 for any $s \in S^h$ involves the minimum possible cost $|S^h| - 1$. This Lemma also indicates that, in the language of Samuelson (1994), S^{α} (if $c < d$) and also S^{β} (if $c < b$) are recurrent sets. This allows each of them to be treated as a single "entity" in the following two complementary senses: (i) if any state in one of these recurrent sets is stochastically stable, so is every other state in this same set; (ii) in evaluating the minimum cost involved in a transition to, or away from, any given state in a recurrent set, the sole relevant issue concerns the minimum cost associated to a transition to, or away from, some state in that recurrent set. Using (i)-(ii), the analysis of the model can be greatly simplified. To organize matters, it is useful to consider the different range of c separately.

Let us start with the case where $0 < c < e$. In this range, Proposition 3 tells us that the set of absorbing states $S^{*} = S^{\alpha} \cup S^{\beta}$. Since, by Lemma 5, the sets S^{α} and S^{β} are each recurrent, the crucial point here is to assess what is the minimum mutation cost across all path joining some state in S^{h} to some state in $S^{h'}$ for each $h, h' = \alpha, \beta$; $h \neq h'$. Denote these mutation costs by $m^{hh'}$; (which are integer numbers).

Lemma 6 Suppose that $0 < c < e$. Then $m^{\beta\alpha} > m^{\alpha\beta}$; for large enough n.

The proof, given in Appendix A, reflects the standard considerations arising in much of the recent evolutionary theory when the fixed pattern of interaction involves every individual of the population playing with all others. Now, if costs are low ($c < e$); such full connectivity is not just assumed but it endogenously follows from players' own decisions, both at equilibrium (i.e. when the unperturbed best-response dynamics is at a rest-point) and away from it. In effect, this implies that the same basin-of-attraction considerations that privilege risk-dominance in the received approach also select for it in the present case.

We next examine the case where $e < c < \min\{f, b\}$. From Proposition 3, we have that $\hat{S}^{1} = S^{\alpha} \cup S^{\beta}$. If $e < c < \min\{f, b\}$ then players who choose action α no longer find it attractive to form links with other players who choose action β. This factor plays a crucial role in the analysis. The following result derives the relative magnitude of the minimum mutation costs.

Lemma 7 Suppose $e < c < \min\{f, b\}$. There is a $\tilde{e}$, $e < \tilde{e} \leq \min\{f, b\}$, such that if $c < \tilde{e}$, then $m^{\beta\alpha} - m^{\alpha\beta} > 0$, while if $c > \tilde{e}$, then $m^{\beta\alpha} - m^{\alpha\beta} < 0$, for large enough n.

The methods used to prove this lemma are quite general; we use them in the proofs of Theorem 4 as well as in Theorem 14. It is therefore useful to provide them in the text.

Proof of Lemma 7: Let s^α and s^β be generic states in S^α and S^β, respectively.

<u>Step 1:</u> Consider transitions from state s^β to state s^α and let k be the number of mutations triggering it. If this transition is to take place after those many mutations, there must be some player currently choosing β (i.e. who has not mutated) that may then voluntarily switch to α. Denote by q^h the number of active links this player chooses to support to players choosing h ($h = \alpha, \beta$) and let r^h stand for the number of passive links she receives from players choosing h ($h = \alpha, \beta$). If she chooses α, her payoff is given by:

$$\pi_\alpha = r^\alpha d + r^\beta e + q^\alpha (d - c); \tag{6}$$

where we implicitly use the fact that q^β must equal zero — since $c > e$, an agent who switches to α will not find it worthwhile to support any link to players choosing β. On the other hand, if the agent in question were to continue adopting β, her payoff would be equal to:

$$\pi_\beta = \hat{r}^\alpha f + \hat{r}^\beta b + \hat{q}^\alpha (f - c) + \hat{q}^\beta (b - c); \tag{7}$$

where $\hat{q}^h$ and $\hat{r}^h$ are interpreted as the active and passive links that would be chosen by the player if she decided to adopt β. Clearly, we must have $r^h = \hat{r}^h$ for each $h = \alpha, \beta$. Thus, if a switch to α is to take place, it must be that

$$\pi_\alpha - \pi_\beta = (r^\alpha + q^\alpha) d - (r^\alpha + \hat{q}^\alpha) f - r^\beta (b - e) - \hat{q}^\beta (b - c) \geq 0. \tag{8}$$

Note that $r^{\alpha} + q^{\alpha} = k$, since $c < d$ and therefore the player who switches to α will want to be linked (either passively or by supporting herself a link) to all other players choosing α; i.e. to the total number k of α-mutants. On the other hand, since $c < \min\{f, b\}$; we must also have that $\hat{r}^{\beta} + \hat{q}^{\beta} = n - k - 1$ and $r^{\alpha} + \hat{q}^{\alpha} = \hat{r}^{\alpha} + \hat{q}^{\alpha} = k$, i.e. the player who chooses β must become linked to all other players, both those choosing β and those choosing α:

We now ask the following question: What is the lowest value of k consistent with (8)? Since $c > e$; the desired payoff advantage of action α will occur for the lowest value of k when $r^{\beta} = \hat{r}^{\beta} = 0$ and therefore $\hat{q}^{\beta} = n - k - 1$. That is, if the desired transition is to take place, the necessary condition (8) holds for the minimum number of required mutations when the arbitrary agent that must start the transition has no passive links to individuals choosing action β: Recall that $m^{\beta\alpha}$ stands for the minimum number of mutations required for the transition. Now introducing the above observations in (8), we obtain the following lower bound

$$m^{\beta\alpha} \geq \frac{b - c}{(d - f) + (b - c)}(n - 1) \equiv H: \tag{9}$$

The above expression gives the minimum number of players choosing α that are needed to induce some player to switch to action α across all possible network structures. Next, we argue that this number of mutations is also sufficient to induce a transition from some s^{β} to some s^{α}. The proof is constructive. The main idea is to consider a particular state s^{β} where its corresponding (complete) network displays the maximal responsiveness to some suitably chosen mutations. Using the observations on the distribution of active and passive links, this occurs when there are some players who support links to all others – those are, of course, players with a "critical" role whose mutation would be most effective. Specifically, suppose that the network prevailing in s^{β} has every player $i = 1, 2, \ldots, n$ support active links to all $j > i$. (This means, for example, that player 1 supports links to every other player whereas player n only has passive links.) Then, denoting by $\lceil z \rceil$ the smallest integer no smaller than z, the most mutation-effective way of inducing the population to switch actions from β to α is precisely by having the players $\ell = 1, 2, \ldots, \lceil H \rceil$ simultaneously mutate to action α and maintain all their links. Thereafter, a transition to some state s^{α} will occur if subsequent strategy revision opportunities are appropriately sequenced so that every player with index $j = \lceil H \rceil + 1, \ldots, n$ is given a revision opportunity in order. This, in effect, shows that the lower bound in (9) is tight and $m^{\beta\alpha} = \lceil H \rceil$.

<u>Step 2:</u> Consider next the transition from some state s^{α} to a state s^{β} and let again k be the number of mutations (now towards β) triggering it. Using arguments from part (1) above, it is easy to show that $m^{\alpha\beta}$, must satisfy:

$$m^{\alpha\beta} \geq \frac{d - f}{(d - f) + (b - e)}(n - 1) \equiv H'. \tag{10}$$

We can use arguments from part (1) to also show that $\lceil H' \rceil$ is sufficient.
<u>Step 3:</u> Finally, we wish to study the difference $m^{\beta\alpha} - m^{\alpha\beta}$ as a function of c. For low c (close to e), and large n, this difference is clearly positive in view of the hypothesis that $b - e > d - f$. Next, to verify that it switches strict sign at most once in the range $c \in (e, \min\{f, b\})$, note that $H - H'$ is strictly declining with respect to c in the interval $(e, \min\{f, b\})$. □

Lemma 7 applies both to the case where $b < f$ and that where $b > f$. Suppose first that $b < f$. Then, since $\Delta(b) < 0$, a direct combination of former considerations leads to the desired conclusion for the parameter range $c \in (e, b]$. We now take up the case $f < b$ and focus on the range $c \in (f, b)$. We first derive the relative magnitude of the minimum mutation costs for $s \in S^h$, where $h = \{\alpha, \beta\}$.

Lemma 8 Suppose $f < c < b$. There is a threshold $\tilde{c} \in [f, b)$ such that if $c < \tilde{c}$ then $m^{\beta\alpha} - m^{\alpha\beta} > 0$, while if $c > \tilde{c}$ then $m^{\beta\alpha} - m^{\alpha\beta} < 0$, for large enough n.

The arguments needed to establish this result are very similar to those used in the proof of Lemma 7; we provide the computations in Appendix A.

The principal complication in case $c \in [f, b)$ is that the set of absorbing states in not restricted to $S^{\alpha} \cup S^{\beta}$ but will generally include mixed states where the population is segmented into two different action components (cf. Propositions 2 and 3). Let $m^{h,\alpha\beta}$, for $h = \alpha, \beta$, denote the minimum number of mutations needed to ensure a transition from some $s \in S^h$ to some $s \in S^{\alpha\beta}$. The first point to note is that by the construction in Lemma 7, $m^{\alpha,\alpha\beta} \geq m^{\alpha\beta}$. Similarly, $m^{\beta,\alpha\beta} \geq m^{\beta\alpha}$. The following lemma characterizes the process of transiting from some $s \in S^{\alpha\beta}$ to a state in S^h for $h = \alpha, \beta$.

Lemma 9 Let $f < c < b$ and consider any equilibrium state $s \in S^{\alpha\beta}$ involving two non-degenerate (α and β) components, g^{α} and g^{β}, with cardinalities $|A(s)| > 0$ and $|B(s)| > 0$, respectively. Then, there is another equilibrium state s' with cardinality for the resulting α component $|A(s')| \geq |A(s)| + 1$ that can be reached from s by a suitable single mutation followed by the best-response dynamics. An identical conclusion applies to some equilibrium state s'' with $|A(s'')| \leq |A(s)| - 1$.

The proof of this Lemma is given in Appendix A. We briefly sketch the argument here. Fix some mixed state, and suppose the strategy of some player $i \in A(s)$ mutates as follows: she switches to action β, while everything else remains as before. Now, have all the players in the α group move; suppose that they wish to keep playing action α. Since $c < f$, their best response is to delete their links with player i. Next, have all the players in group β move; their best response is to form a link with player i. This is true since the original state was an equilibrium, and $c < b$. Finally, have player i choose a best response; since the original state was an equilibrium and $c > f$, her best response is to play action β and delete all links with players in the α group. We have thus increased the number of β players with a single mutation. This argument extends in a natural manner to prove the above result. We now have all the information to complete the proof of the Theorem.

Proof of Theorem 3.1: Take any such state $s \in S^h$ for some $h = \alpha, \beta$. With the help of Lemmas 5 and 9 we can infer that the minimum-cost s-tree for $s \in S^\alpha$ will have the following cost: $m^{\beta\alpha} + |S^\alpha| + |S^\beta| + |S^{\alpha\beta}| - 2$. For any $s' \in S^\beta$, the situation is symmetric, the minimum cost being equal to $m^{\alpha\beta} + |S^\alpha| + |S^\beta| + |S^{\alpha\beta}| - 2$. We also note that for any $s \in S^{\alpha\beta}$, the corresponding s-tree would have to display a path joining some state in S^α to s and some path joining some state in S^β to s. Proceeding as above, the minimum cost of such an s-tree will be at least $m^{\alpha\beta} + m^{\beta\alpha} + |S^\alpha| + |S^\beta| + |S^{\alpha\beta}| - 1$. This expression is greater than the mutation costs for $s \in S^h$, for $h = \alpha, \beta$, since each $m^{hh'} > 1$ if the population is large. We therefore conclude that a state $s \in S^{\alpha\beta}$ cannot be stochastically stable. Finally, in comparing the stochastic potential of any $s \in S^\alpha$ to that of $s \in S^\beta$, the key issue concerns the comparison of $m^{\alpha\beta}$ versus $m^{\beta\alpha}$. If $c < b$, then the proof now follows from this observation and the computations reported in Lemmas 6-8.

If $b < c < d$ the key point to observe is that the set of strict Nash equilibria and hence the set of absorbing states is simply $S^{\infty} = S^\alpha$. Similar considerations apply to the case where $c > d$, in which case Propositions 2 and 3 establish that $S^{\infty} = \{g^e\} \times A^n$. $\square$

4 INDIRECT LINKS

In this section, we turn to the context where each player interacts with all players to whom she is joined by a path in the network. As for the direct-link scenario, we ¯rst characterize the Nash equilibria of the social game and then provide a complete characterization of the set of stochastically stable states.

4.1 Equilibrium outcomes

Our ¯rst result, whose proof is provided in Appendix B, derives some basic properties of equilibrium networks and actions.

Proposition 10 Suppose (1 and (3) hold. Then; any equilibrium network is either minimally connected or empty. Furthermore, if the equilibrium network is connected, everyone chooses the same action and social conformism obtains.

Thus, when any pair of indirectly linked agents play the game, the social disconnectedness and heterogeneity allowed in the direct-link model is no longer possible: at equilibrium, any non-empty equilibrium network must now be connected (i.e. de¯ne a single component), every player then choosing the same action.

Equilibrium already impose some obvious, but nevertheless interesting, condition on associated networks, i.e. they must be minimal (or non-redundant). This minimal connectivity, however, is too permissive a requirement and allows for a wide range of network architectures. This motivates imposing the requirement of strictness on Nash equilibria, a condition which was obtained \for free" in the direct-link model when the social network is non-empty. Of course, a further justi¯cation for our interest in strict Nash equilibria is that, in view of the adjustment (best-response) mechanism postulated, all of the rest points of the unperturbed dynamics must correspond to Strict Nash equilibria of the social game.

The following proposition, proven in Appendix B, provides a characterization of strict Nash architectures.

Proposition 11 Suppose (1 and (3) hold. Then, a strict Nash network is either empty or a center-sponsored star.

To gain some intuition on the above result, consider any Nash equilibrium network (minimally connected, by virtue of Proposition 10) that is not a center-sponsored star. – recall that a center-sponsored star is a network in which a single agent supports a link with every other player. Then, there has to be a player i who forms a link with some player j; the latter in turn being linked to some third player $k \neq i$. To see that the underlying strategy configuration cannot define a Strict Nash equilibrium, simply note that player i can interchange her link with player j for a link with player k and still get the same payoffs. Of course, the only kind of network which is immune to this problem is one in which a single player supports all existing links. A center-sponsored star, in other words, is the only candidate for a strict Nash networks.

The above result on strict Nash networks helps us achieve a full characterization of strict Nash equilibria for different values of c, as established by the following result. Let G^{cs} stand for the collection of networks that define a center-sponsored star. Furthermore, denote $\mathcal{S}^{\alpha} \equiv G^{cs} \times \{\alpha, ..., \alpha\}$ and $\mathcal{S}^{\beta} \equiv G^{cs} \times \{\beta, ..., \beta\}$, whereas S^{**} represents the set of strict Nash equilibria. Then, we have the following result, whose proof is found in Appendix B.

Proposition 12 Suppose (1 and (3) hold: (a) if $0 < c < b$, $S^{**} = \mathcal{S}^{\alpha} \cup \mathcal{S}^{\beta}$; (b) if $b < c < d$, $S^{**} = \mathcal{S}^{\alpha}$; (c) if $c > d$, there is no strict Nash equilibrium.

The above result parallels for the present context the characterization provided by Proposition 2 for the direct-link scenario. The underlying intuition is somewhat akin to that discussed for that scenario, but also displays important differences. One of them is that, since all equilibrium networks must presently be connected (cf. Proposition 10), co-existence of the two actions, α and β, is ruled out at any equilibrium configuration. A second interesting difference is that the requirement of strictness amounts here to a genuine refinement criterion (i.e. many Nash equilibria are not strict), even when $c < b$.

4.2 Dynamics

We now examine the dynamic properties of the different equilibrium outcomes. We start by studying the (unperturbed) best response dynamics in the extended model. The following result summarizes our analysis.

Proposition 13 Suppose (1 and (3) hold. If $c < d$, the best response dynamics converges, almost surely, to one of the states identified in parts (a)-(b) of Proposition 12. If $c > d$, the best response dynamics converges to the set $\{g^e\} \times A^n$.

The proof of the above result follows from suitable adaptations of arguments used in Bala and Goyal (2000) for the present strategic context. Thus, since they are long and involved, we dispense with them here. In essence, it involves four steps, each of them establishing that the following corresponding transitions have positive probability:

(i) from any given state to one that is minimally connected;

(ii) if $c < d$, from a minimally connected state to one where every two players are "agglomerated" in the following sense: every two players have a path no longer than two links which joins them;

(iii) if $c < d$; from an agglomerated state, as described above, to a center-sponsored star where every agent plays the same action;

(iv) if $c > d$; from a minimally connected state to the empty network.

Proposition 13 establishes that the unperturbed dynamics converges a.s. to one of strict Nash equilibria identified in Proposition 12. This, of course, still leaves us with the need to tackle the essential multiplicity arising when $c < d$: Again, we shall approach the issue by perturbing the dynamics with small and independent mutation rates (recall Subsection 2.3), then focusing on the induced stochastically stable states. As before, the set of these states will be denoted by $\hat{S}$:

Theorem 14 Suppose (1 and (3) hold. There exists some $\hat{e} \in (e; b)$ such that if $c < \hat{e}$ then $\hat{S} = S^{\beta}$, while if $\hat{e} < c < d$ then $\hat{S} = S^{\alpha}$; provided n is large enough. Finally, if $c > d$ then $\hat{S} = \{g^e\} \times A^n$.

Concerning the long-run actions selected, this result is analogous to Theorem 4 for the direct-link scenario. That is, there is a certain threshold $\hat{e}$ (higher than the minimum payoff e but still lower than the minimum equilibrium payoff b) which separates the regions where the inefficient and efficient equilibrium actions are selected. However, pertaining to the associated networks, we find the interesting contrast between the two contexts that was already anticipated by our former preparatory analysis: if indirect links are allowed, a center-sponsored star is the only robust architecture that supports the full connectivity required when $c < d$:

Our proof of Theorem 14 closely mimics the steps undertaken in proving Theorem 4. First, we note that, in view of Proposition 13, only the states in S^{α} (if $c < d$) and also those in S^{β} (if $c < b$).are possible candidates for stochastic stability. Furthermore, it is easy to see that, as with their counterparts S^{α} and S^{β} in the direct-link context, each of these two sets are recurrent in each case (recall Lemma 5 and its ensuing discussion).

To verify the latter claim, assume for concreteness that $c < b$ and consider any two states $s, s' \in \tilde{S}^h$ $(h = \alpha, \beta)$. We argue that a transition from s to s' can be triggered by a single mutation followed by the operation of the unperturbed dynamics. Specifically, let i and j be the two central players in the center-sponsored stars defined by s and s', respectively. Then, if the central player i mutates at s and removes all her links (still keeping her former action), a transition to s' will materialize provided that, subsequently, player j alone is given a revision opportunity.

As explained, the fact that $\tilde{S}^\alpha$ and $\tilde{S}^\beta$ are recurrent states (provided $c < d$ or $c < b$, respectively) simplifies the analysis substantially. Consider first the case where $b < c < d$. Then, we know from Proposition 13 that $\hat{S} \subset \tilde{S}^\alpha$ and, consequently, since $\tilde{S}^\alpha$ is recurrent, $\hat{S} = \tilde{S}^\alpha$. On the other hand, if $c < b$ (and therefore $c < d$ as well), we have $\hat{S} \subset \tilde{S}^\alpha \cup \tilde{S}^\beta$ and the conclusion hinges upon the minimum number of mutations needed to implement the transitions from some state $s \in \tilde{S}^h$ $(h = \alpha, \beta)$ to some other state $s' \in \tilde{S}^{h'}$ $(h \neq h')$. In analogy with previous notation, denote by $\tilde{m}^{hh'}$ such a minimum number of mutations. Then, the recurrent set S^h selected (i.e. $\hat{S} = S^h$) is that one for which $\tilde{m}^{hh'} > \tilde{m}^{h'h}$.

Now, we compute $\tilde{m}^{hh'}$ $(h, h' = \alpha, \beta;\ h \neq h')$ for different subranges of c in the interval $(0, b)$. As direct counterparts of the corresponding results established for the direct-link scenario, we have the following Lemmata, whose proofs may be found in Appendix B (recall that $\lceil z \rceil$ denotes the smallest integer no smaller than z):

Lemma 15 Suppose that $0 < c < e$. Then,

$$\tilde{m}^{\beta\alpha} = \left\lceil \frac{b - e}{(d - f) + (b - e)}(n - 1) \right\rceil$$

$$\tilde{m}^{\alpha\beta} = \left\lceil \frac{d - f}{(d - f) + (b - e)}(n - 1) \right\rceil :$$

Lemma 16 *Suppose* $e \leq c \leq \min\{f, b\}$*. Then,*

$$\tilde{m}^{\beta\alpha} = \left\lceil \frac{b - c}{(d - f) + (b - c)}(n - 1) \right\rceil$$

$$\tilde{m}^{\alpha\beta} = \left\lceil \frac{d - f}{(d - f) + (b - e)}(n - 1) \right\rceil .$$

Lemma 17 *Suppose* $f \leq c \leq b$*. Then,*

$$\tilde{m}^{\beta\alpha} = \left\lceil \frac{b - c}{(d - f) + (b - c)}(n - 1) \right\rceil$$

$$\tilde{m}^{\alpha\beta} = \left\lceil \frac{d - c}{(d - c) + (b - e)}(n - 1) \right\rceil .$$

In view of Lemmas 15-17, the proof of Theorem 14 then follows from the observation that, for all $c \in (0, b)$, the minimum number of mutations required to make the transition from S^h to $S^{h'}$ $(h, h' = \alpha, \beta;\ h \neq h')$ in the present indirect-link model is exactly the same as that required to make the transition from S^h to $S^{h'}$ in the former direct-link model. Thus, $\tilde{m}^{hh'} = m^{hh'}$ for any c in each of the three considered subranges, which leads to the desired conclusions by relying on the same arguments as for Theorem 4.

5 CONCLUDING REMARKS

In many economic and social contexts, individuals can interact (e.g. undertake a transaction) only if they are 'linked' or related to each other. It is therefore natural to postulate that individual players invest effort and resources in forming links with others, their link decisions then defining the network of social interaction. In this paper, we have studied the nature of networks that form and the effects of link formation on social coordination.

We start with a basic model in which two players can interact, or play a game, only if they have a direct link between them. We then consider a variation of it where two players can play the game if they are directly or indirectly linked. Our results show that, in both settings, equilibrium networks have simple architectures. In the former model the unique equilibrium architecture is the complete network, while in the latter model, the unique equilibrium architecture is a star. We show that network formation is intimately related to equilibrium selection in both settings: at low costs of forming links, individuals coordinate on the risk-dominant action, while for high costs of forming links individuals coordinate on the e±cient action. Thus in each of these settings, while the network architecture remains the same, the nature of coordination varies with the cost of forming links. These results suggest that the process of network formation per se has powerful implications for the nature of social coordination.

An important aspect of our model is that link formation is one-sided. This formulation has the advantage that it allows us to study the social process of link formation and coordination as a non-cooperative game. In some settings, it is perhaps more natural to think of link formation as a two-sided process, i.e. any link being formed leads to both of the players involved incurring some costs. Naturally, this must have as well both players acquiesce in the formation of the link.

In independent work, Jackson and Watts (1999) study such a two-sided model with direct links, i.e. they focus on the case where two players can play a game only if they are directly connected. They ¯nd, just like us, that the equilibrium network is complete. However their results on social coordination are quite di®erent. For instance, they ¯nd that, if the costs of link formation are high, both classes of states where players choose a common action are stochastically stable. In contrast, we have found that, when the costs are high (but below the maximum achievable payo®), the only stable states involve players choosing the e±cient action. This di®erence arises out of di®erences in the way we model the link formation process as well as the accompanying assumptions on the timing of moves. Speci¯cally, Jackson and Watts postulate that individuals choose links and actions separately, i.e., players choose links taking actions as given while they choose actions taking the links as a given. By contrast, in our setting, any individual undertaking a revision is allowed to impinge on every dimension of her choice and change both her action and her supported links. This suggests that it should be interesting to study the e®ects of varying levels of °exibility in the two choice dimensions, links and actions{ for example, it seems natural to allow for the possibility that link revision be more rigid than action change, or even vice versa.

Another possible route of generalization of the present framework pertains to the nature and implications of links. Our analysis has examined the two polar cases with regard to the value of indirect links: either only directly linked players can interact (and indirect links are irrelevant), or indirect links are as good as direct links, irrespective of the length of the path between the players. In the future, we would like to examine a more general formulation which allows for `distant' links to be less valuable (e.g. depreciate at a given rate) as compared to `near by' links.

Finally, an additional interesting issue concerns the number of links allowed or, relatedly, the shape (e.g. concavity or convexity) of the underlying cost function. In our model, we have imposed no limit on the number of links a player can support and the marginal cost of any additional link has been assumed constant. However, it seems interesting to contemplate the possibility that players might be constrained in the number of links they can support. This could be imposed directly on the model (by establishing a ¯xed upper bound) or indirectly derived from a su±ciently convex cost function. In either case, one may conjecture that it might have important e®ects on some aspects of the model. For example, it could destroy the full connectivity of stochastically stable states by creating disjoint components ("islands") of players. Building upon insights gleaned from existing evolutionary literature (recall the Introduction), this should in turn have signi¯cant implications in at least two respects. On the one hand, it should favor the long-run selection of e±ciency by allowing the creation of isolated havens or beachheads, from which e±cient behavior can spread throughout the whole population. On the other hand, it should also speed up convergence to the long-run (stochastically stable) states by dispensing with the need of resorting to a large number of simultaneous mutations to trigger the required transitions.

6 APPENDIX A

Proof of Proposition 1: The proof of part (a) follows directly from the fact that $c < f$ and is omitted. We provide a proof of part (b). In this case $f < c < b$. We ¯rst show that $a_i = a_j = a$, if i, j belong to the same component. Suppose not. If $g_{ij} = 1$, then it follows that the player forming a link can pro¯tably deviate by deleting the link, since $c > f$. Similar arguments apply if i and j are indirectly connected. We next show that if $i \in g'$ and $j \in g''$, where g' and g'' are two components in an equilibrium network g, then $a_i \neq a_j$. If $a_i = a_j$ then the minimum payo® to i from playing the coordination game with j is b. Since $c < b$, player i gains by forming a link $g_{ij} = 1$. Thus g is not an equilibrium network. The ¯nal step is to note that since there are only two actions in the coordination game, there can be at most two distinct components. We note that the completeness of each component follows from the assumption that $c < b$.

We next prove part (c). There are two subcases to consider: $c > \max\{b, f\}$ or $b < c < f$. (Note, of course, that the former subcase is the only one possible if

$b > f$:) Suppose first that $c > \max\{b, f\}$, and let g be an equilibrium network which is non-empty but also incomplete. from above arguments in (b) it follows that if $g_{i,j} = 1$, then $a_i = a_j = \alpha$. Moreover, if $a_j = \beta$, then player j can have no links in the network. (These observations follow directly from the hypothesis that $c > \max\{b, f\}$.) However, since g is assumed incomplete, there must exist a pair of agents i and j such that $\bar{g}_{ij} = 0$. First, suppose that $a_i = a_j = \alpha$. Then, since $c < d$, it is clearly profitable for either of the two players to deviate and form a link with the other player. Suppose next that $a_i = a_j = \beta$. Then, players i and j can have no links and, furthermore, since g is non-empty, there must be at least two other players $k, l \in N$ such that $a_k = a_l = \alpha$. But then player i can increase her payoff by choosing action α and linking to player k. Finally, consider the case where $a_i \neq a_j$ and let player i choose β. Then, if this player deviates to action α and forms a link with player j she increases her payoff strictly. We have thus shown that $\bar{g}_{ij} = 0$ cannot be part of an equilibrium network. This proves that a non-empty but incomplete network cannot be an equilibrium network in the first subcase considered.

Consider now the case $b < c < f$ and suppose, for the sake of contradiction, that g is an equilibrium network which is non-empty but incomplete. Since $b < c < d$, it follows directly that not every player chooses action α or β. Moreover, in the mixed configuration, all the players who choose α are directly linked (since $c < d$), there is a link between every pair of players who choose dissimilar actions (since $c < f$), but there are no links between players choosing β (since $b < c$). But then it follows that every player choosing β can increase her payoff by switching to action α. This contradicts the hypothesis that the mixed configuration is an equilibrium. This completes the argument for part (c).

Part (d) is immediate from the hypothesis that $c > d$. $\square$

Proof of Proposition 2: We start proving Part (a). In view of Part (a) of Proposition 1 and the fact that the underlying game is of a coordination type, the inclusion $S^{\alpha} \cup S^{\beta} \subset S^{*}$ is obvious. To show the converse inclusion, take any profile s such that the sets $A(s) \equiv \{i \in N : a_i = \alpha\}$ and $B(s) \equiv \{j \in N : a_j = \beta\}$ are both non-empty. We claim that such an s cannot be an equilibrium.

Assume, for the sake of contradiction, that such a state s is a Nash equilibrium of the game and denote $u \equiv |A(s)|$; $0 < u < n$. Recall from Proposition 1 that every Nash network in this parameter range is complete. This implies that for any player $i \in A(s)$; we must have:

$$(u-1)d + (n-u)e - \nu^d(i;g) \cdot c \geq (u-1)f + (n-u)b - \nu^d(i;g) \cdot c \quad (11)$$

and for players $j \in B(s)$:

$$(n-u-1)b + uf - \nu^d(j;g) \cdot c \geq (n-u-1)e + ud - \nu^d(j;g) \cdot c. \quad (12)$$

It is easily verified that (11) and (12) are incompatible.

Now, we turn to Part (b). The inclusion $S^{\alpha} \cup S^{\beta} \subset S^{*}$ is trivial, in view of Part (b) of Proposition 1. To show that the inclusion $S^{*} \subset S^{\alpha\beta}$ holds strictly for large enough n; consider a state s where both $A(s)$ and $B(s)$; defined as above, are both non-empty and complete components. Specifically, focus attention on those configurations that are symmetric within each component, so that every player in $A(s)$ supports $\frac{u-1}{2}$ links and every player in $B(s)$ supports $\frac{n-u-1}{2}$ links. (As before, u stands for the cardinality of $A(s)$ and we implicitly assume, for simplicity, that u and $n-u$ are odd numbers.) For this configuration to be a Nash equilibrium, we must have that the players in $A(s)$ satisfy:

$$d(u-1) - \frac{u-1}{2}c \geq f\frac{u-1}{2} + b(n-u) - c(n-u) \quad (13)$$

where we use the fact that, in switching to action β; any player formerly in $A(s)$ will have to support herself all links to players in $B(s)$ and will no longer support any links to other players in $A(s)$ – of course, she still anticipate playing with those players from $A(s)$ who support links with her. On the other hand, the counterpart condition for players in $B(s)$ is:

$$(n-u-1)b - \frac{n-u-1}{2}c \geq du + e\frac{n-u-1}{2} - cu \tag{14}$$

where, in this case, we rely on considerations for players in $B(s)$ that are analogous to those explained before for players in $A(s)$: Straightforward algebraic manipulations show that (13) is equivalent to:

$$\frac{u}{n} \geq \frac{1}{n}\frac{2d-c-f}{2b+2d-3c-f} + \frac{2(b-c)}{2b+2d-3c-f} \tag{15}$$

and (14) is equivalent to:

$$\frac{u}{n} \quad \frac{1}{n}\frac{c+e-2b}{2b+2d-3c-e} + \frac{2b-c-e}{2b+2d-3c-e}. \tag{16}$$

We now check that, under the present parameter conditions:

$$\frac{2b-c-e}{2b+2d-3c-e} > \frac{2(b-c)}{2b+2d-3c-f}. \tag{17}$$

Denote $Y \equiv 2b-c$, $Z \equiv 2b+2d-3c$, and rewrite the above inequality as follows:

$$\frac{Y-e}{Y-c} > \frac{Z-e}{Z-f} \tag{18}$$

which is weaker than:

$$\frac{Y-e}{Y-f} > \frac{Z-e}{Z-f} \tag{19}$$

since $c > f$. The function $\zeta(z) \equiv \frac{z-e}{z-f}$ is uniformly decreasing in z since $b > f > e$. Therefore, since $Y < Z$, (19) obtains, which implies (18). Hence it follows that, if n is large enough, one can ¯nd suitable values of u such that (15) and (16) jointly apply. This completes the proof of Part (b).

We now present the proof for part (c). We know from Proposition 1 that the complete and the empty network are the only two possible equilibrium networks. Since c > b > f > e, it is immediate that, in the complete network, every player must choose ® and this is a Nash equilibrium. First note that for the empty network to be an equilibrium, it should be the case that no player has an incentive to form a link. This implies that every player chooses ¯. On the other hand, it is easy to see that the empty network with everyone choosing ¯ is a Nash equilibrium.

The proof of part (d) follows directly from the hypothesis c > maxfd; b; f ; eg.

2

Proof of Proposition 3: It is enough to show that, from any given state !0, there is a ¯nite chain of positive-probability events (bounded above zero, since the number of states is ¯nite) that lead to a rest point of the best response dynamics.

Choose one of the two strategies, say ¯; and denote by B(0) the set of individuals adopting action ¯ at !0. Order these individuals in some pre-speci¯ed manner and starting with the ¯rst one suppose that they are given in turn the option to revise their choices (both concerning strategy and links). If at any given stage ¿, the player i in question does not want to change strategies, we set B(¿ + 1) = B(¿) and proceed to the next player if some are still left. If none is left, the ¯rst phase of the procedure stops. On the other hand, if the player i considered at stage ¿ switches from ¯ to ®; then we make B(¿ + 1) = B(¿)nfig and, at stage ¿ + 1; re-start the process with the ¯rst-ranked individual in B(¿ + 1); i.e. not with the player following i: Clearly, this ¯rst phase of the procedure must eventually stop at some ¯nite ¿1.

Then, consider the players choosing strategy α at τ_1 and denote this set by $A(\tau_1) \equiv N \backslash B(\tau_1)$: Proceed as above with a chain of unilateral revision opportunities given to players adopting α in some pre-specified sequence, restarting the process when anyone switches from α to β: Again, the second phase of the procedure ends at some finite τ_2:

By construction, in this second phase, all strategy changes involve an increase in the number of players adopting β, i.e. $B(\tau_2) \supseteq B(\tau_1)$: Thus, if the network links affecting players in $B(\tau_1)$ remain unchanged throughout, it is clear that no player in this set would like to switch to α if given the opportunity at τ_2+1. However, in general, their network links will also evolve in this second phase, because individual players in $A(\tau_1)$ may form or delete links with players in $B(\tau_1)$. In principle, this could alter the situation of individual members of $B(\tau_1)$ and provide them with incentives to switch from β to α. It can be shown, however, that this is not the case. To show it formally, consider any given typical individual in $B(\tau_1)$ and denote by $\hat\nu^h$; $h = \alpha;\beta$; the number of links received (but not supported) by this player from players choosing action h. On the other hand, denote $\hat r \equiv |A(\tau_1)|$. Then, since the first phase of the procedure stops at τ_1; one must have:

$$\begin{aligned} &\max_{q^\alpha;q^\beta} b(q^\beta + \hat\nu^\beta) + f(q^\alpha + \hat\nu^\alpha) - c(q^\alpha + q^\beta) \\ &\geq \max_{q^\alpha;q^\beta} e(q^\beta + \hat\nu^\beta) + d(q^\alpha + \hat\nu^\alpha) - c(q^\alpha + q^\beta) \end{aligned} \quad (20)$$

for all $q^\alpha; q^\beta$ such that $0 \leq q^\alpha \leq \hat r - \hat\nu^\alpha;\ 0 \leq q^\beta \leq n - \hat r - 1 - \hat\nu^\beta$: Now denote by $\tilde\nu_h$ and $\tilde r$ the counterpart of the previous magnitudes ($\hat\nu^h$ and $\hat r$) prevailing at τ_2: By construction, we have $\tilde r \leq \hat r$; $\tilde\nu^\alpha \leq \hat\nu^\alpha$, and $\tilde\nu^\beta \geq \hat\nu^\beta$: We note that $\tilde r \leq \hat r$ by construction of the process. Next note that if $\tilde\nu^\alpha > \hat\nu^\alpha$ then this implies that some player who chooses action α has formed an additional link with player i in the interval between τ_1 and τ_2. This is only possible if $c < e$. It also implies that player i did not have a link with this player at τ_1. This is only possible if $c > f$, a contradiction. Thus $\tilde\nu^\alpha \leq \hat\nu^\alpha$. Finally note that

$\tilde{r}^{\beta} \geq \hat{r}^{\beta}$ follows from the fact that the all the players choosing β at τ_1 do not revise their decisions in the interval between τ_1 and τ_2.

Therefore, (20) implies:

$$\max_{q^{\alpha},q^{\beta}} b(q^{\beta} + \tilde{r}^{\beta}) + f(q^{\alpha} + \tilde{r}^{\alpha}) - c(q^{\alpha} + q^{\beta})$$
$$\geq \max_{q^{\alpha},q^{\beta}} e(q^{\beta} + \tilde{r}^{\beta}) + d(q^{\alpha} + \tilde{r}^{\alpha}) - c(q^{\alpha} + q^{\beta})$$

for all q^{α}, q^{β} such that $0 \leq q^{\alpha} \leq \tilde{u} - \tilde{r}^{\alpha}$; $0 \leq q^{\beta} \leq n - \tilde{u} - 1 - \tilde{r}^{\beta}$. This allows us to conclude that the concatenation of the two phases will lead the process to a rest point of the best response dynamics, as desired. $\square$

Proof of Lemma 5: The proof is constructive. Let $s \in S^h$, $h = \alpha, \beta$; and order in some arbitrary fashion all other states in $S^h \setminus \{s\}$. Also order in some discretionary manner all pairs $(i, j) \in P \times P$ with $i \neq j$. For the first state in $S^h \setminus \{s\}$, say s_1, proceed in the pre-specified sequence across pairs (i, j) reversing the links of those of them whose links are different from what they are in s. This produces a well-defined path joining s_1 to s, whose constituent states define a set denoted by Q_1. Next, consider the highest ranked state in $S^h \setminus Q_1$, say s_2. Proceed as before, until state s_2 is joined to either state s or a state already included in Q_1. Denote the states included in the corresponding path by Q_2. Clearly, when a stage n is reached such that $S^h \setminus (\cup_{\ell=1}^{n} Q_{\ell}) = \emptyset$, the procedure described has fully constructed the desired s-tree restricted to S^h. $\square$

Proof of Lemma 6: Let s^{α} and s^{β} be generic states in S^{α} and S^{β}, respectively. We want to determine the minimum number of mutations needed to transit across a pair of them in either direction.

(1). First, consider a transition from s^{β} to s^{α} and let k be the number of mutations triggering it. If this transition is to take place via the best-response dynamics after those many mutations, there must be some player currently choosing β (i.e. who has not mutated) that may then voluntarily switch to α. As before, denote by q^h the number of active links this player supports to players choosing h ($h = \alpha, \beta$) and let r^h stand for the number of passive links she receives from players choosing h ($h = \alpha, \beta$). The payoff from choosing α for that player is given by:

$$\pi_\alpha = r^\alpha d + r^\beta e + q^\alpha (d - c) + q^\beta (e - c). \qquad (21)$$

On the other hand, the payoff to choosing β is given by:

$$\pi_\beta = \hat{r}^\alpha f + \hat{r}^\beta b + \hat{q}^\alpha (f - c) + \hat{q}^\beta (b - c), \qquad (22)$$

where $\hat{q}^h$ and $\hat{r}^h$ have the same interpretation of active and passive links as before, now associated to the possibility that the player chooses β. Clearly, we have $q^h = \hat{q}^h$ and $r^h = \hat{r}^h$ for each $h = \alpha, \beta$. Concerning the passive links, this is immediate; for active links, it follows from the fact that, since $c < e$, a player will want to create links to all unconnected players, independently of what they do. Analogous considerations also ensure that (i). $r^\alpha + q^\alpha = k$ and (ii). $r^\beta + q^\beta = n - k - 1$. Thus, in sum, for a transition from some state in S^β to a state in S^α to be triggered, one must have:

$$\begin{aligned} \pi_\alpha - \pi_\beta &= (r^\alpha + q^\alpha)(d - f) - (r^\beta + q^\beta)(b - e) \\ &= k(d - f) - (n - k - 1)(b - e) \geq 0 \end{aligned}$$

Let $m^{\beta,\alpha}$ stand for the minimum number of mutations which lead to such a transition. The above considerations imply that

$$m^{\beta,\alpha} \geq \frac{b - e}{(d - f) + (b - e)}(n - 1), \qquad (23)$$

which gives us the minimum number of mutations that are necessary for a transition from any state s^β to some s^α. However, denote by $\lceil z \rceil$ the smallest integer no smaller than z and suppose that the strategies of $\lceil \frac{b-e}{(d-f)+(b-e)}(n -$

$1)\rceil$ players undergo a simultaneous mutation from any particular state s^{β} (i.e. these players maintain their links but switch from β to α). Thereafter, the repeated operation of the best-response dynamics is sufficient to induce a transition to a state s^{α}. Thus the necessary number of mutations computed above is also sufficient to induce a transition from any s^{β} to some s^{α}: That is, the inequality in (23) holds with equality.

(2). Consider on the other hand, the transition s^{α} to s^{β}: Using the expressions (21) and (22), we can deduce that the minimum number of mutations $m^{\alpha,\beta}$ needed to transit from some state in S^{α} to a state in S^{β} satisfies:

$$m^{\alpha,\beta} \geq \frac{d-f}{(d-f)+(b-e)}(n-1). \tag{24}$$

As in the first case, this gives us the minimum number of mutations needed for a transition. However, consider any state s^{α} and suppose that the strategies of $\lceil \frac{d-f}{(d-f)+(b-e)}(n-1)\rceil$ players undergo a simultaneous mutation (i.e. they maintain their links but switch from α to β). It again follows that the operation of the best-response dynamics suffices to induce a transition to a state s^{β}. That is, (24) holds with equality.

To conclude, simply note that, if n is large enough,

$$\left\lceil \frac{b-e}{(d-f)+(b-e)}(n-1) \right\rceil < \left\lceil \frac{d-f}{(d-f)+(b-e)}(n-1) \right\rceil$$

since $d-f < b-e$. $\square$

Proof of Lemma 8 (Sketch): The proof proceeds in the same way as the proof of Lemma 7. We therefore only spell out the main computations.

(1). First, consider transitions from state s^{β} to state s^{α} and let k be the number of mutations triggering it. We focus on a player currently choosing β and aim at finding the most favorable (i.e. least mutation-costly) conditions that would induce her to switch to α. Along the lines explained in the proof of Lemma 7, this leads to the following lower bound:

$$m^{\beta,\alpha} \geq \frac{b-c}{(d-f)+(b-c)}(n-1) \equiv H; \qquad (25)$$

which again can be seen to be tight in the sense that, in fact, $m^{\beta,\alpha} = \langle H \rangle$ – recall that $\lceil z \rceil$ stands for the smallest integer no smaller than z.

(2). Analogous considerations for a transition from state s^{α} to state s^{β} leads to the lower bound

$$m^{\alpha,\beta} \geq \frac{d-c}{(b-e)+(d-c)}(n-1) \equiv H'; \qquad (26)$$

which is also tight, i.e. $m^{\beta,\alpha} = \lceil H' \rceil$.

(3). Finally, to study how the sign of $m^{\beta,\alpha} - m^{\alpha,\beta}$ changes for large n as a function of c, note that

$$H - H' \equiv \Delta(c) = \frac{(b-c)(b-e)-(d-f)(d-c)}{[(d-f)+(b-c)][(b-e)+(d-c)]}(n-1). \qquad (27)$$

Observe that the denominator of $\Delta(c)$ is always positive, the numerator is decreasing in c, and is moreover negative at $c = b$. This completes the proof. $\square$

Proof of Lemma 9: Fix some $s \in S^{\alpha\beta}$, with the players $A(s)$ and $B(s)$ of the α and β components displaying respective cardinalities $|A(s)| \equiv u > 0$ and $|B(s)| \equiv n - u > 0$, respectively. To address the first part of the Lemma, suppose that a player $i \in B(s)$ experiences a mutation, which has the effect of switching her action from β to α and the deletion of all her links with players in $B(s)$. Now consider the players in the set $B(s)\backslash\{i\}$. There are two possibilities: either all of them wish to retain action β, or there is a player who wishes to switch actions.

In the former case, let all of them move and they will retain their earlier strategy except for one change: they will each delete their link with player i, since $f < c < b$. We now get players in $A(s)$ to move and they all form a link with player i, since $f < c < b < d$. It may be checked that we have reached an equilibrium state s', with $A(s') \geq A(s) + 1$.

Consider now the second possibility. Pick a player $j \in B(s) \setminus \{i\}$, who wishes to switch actions from β to α. It follows that this player will delete all her links with players in $B(s)$ and form links with all players in $A(s)$ (since $e < f < c < b < d$). We then examine the incentives of the players still choosing action β, i.e., players in the set $B(s) \setminus \{i; j\}$. If there are no players who would like to switch actions then we repeat step above and arrive at a new state with a larger α-component. If there are players who wish to switch actions from β to α then we get them to move one at a time. Eventually, we arrive at either a new state $s' \in S^{\alpha\beta}$, or we arrive at a state $s' \in S^{\alpha}$.

In either case, we have shown that starting from a state $s \in S^{\alpha\beta}$, we can move with a single mutation to a state s' such that $A(s') \geq A(s) + 1$. Since $s \in S^{\alpha\beta}$ was arbitrary, the proof is complete for the first part. The second conclusion concerning some new equilibrium state s'' with $|A(s'')| \leq |A(s)| - 1$ is analogous. □

7 APPENDIX B

Proof of Proposition 4.1: We first show that $a_i = a_j$, if i and j belong to the same component. Suppose not and let $a_i = \alpha$ while $a_j = \beta$. Let there be k players in this component with k_α players choosing action α and k_β $(= k - k_\alpha)$ players choosing action β. The payoff to player i from action α is given by $(k_\alpha - 1)d + k_\beta e$. Similarly, the payoff to player i from action β is given by $(k_\alpha - 1)f + k_\beta b$. Since, in equilibrium, player i prefers action α it follows that $(k_\alpha - 1)(d - f) \geq k_\beta(b - e)$. Similar calculations show that since, in equilibrium, player j prefers action β it must be true that $(k_\beta - 1)(b - e) \geq k_\alpha(d - f)$. Given that $d > f$ and $b > e$, this generates a contradiction.

We next show that if an equilibrium network is non-empty then there is only one component, i.e., the network is connected. Fix some equilibrium and suppose g is the corresponding (non-empty) network, and g' is a non-singleton component in g. Suppose without loss of generality that $a_i = \alpha$ for every $i \in g'$. Suppose player $j \in g''$, with $g' \neq g''$, and let $a_j = \beta$. Let k' and k'' be, respectively, the cardinality of the two components and let $k' \geq k''$. Suppose, without loss of generality, that player i forms some links in g'. The payoff to i is given by $(k' - 1)d - l_i c$, where l_i is the number of links that she forms. Since g is part of an equilibrium, it follows that $(k' - 1)d - l_i c \geq k''b - c$. We note next that since $k' \geq 2$, and g is part of an equilibrium it must be the case that $k'' \geq 2$. Let $(k'' - 1)d - l_j c$ be the payoff to player j, where $l_j \geq 1$ is the number of links that she forms. It follows from the definition of equilibrium that $(k'' - 1)b - l_j c \geq k'd - c$. Putting these two inequalities together yields a contradiction. The argument is analogous in case $a_j = \alpha$. This proves that a non-empty equilibrium network is connected. The minimality of the equilibrium network follows from the assumption that $c > 0$. □

Proof of Proposition 4.2: From Proposition 4.1, we know that an equilibrium network is either empty or minimally connected. Consider a minimally connected equilibrium network g. Suppose that player i has a link with player j in this network, i.e. $g_{i,j} = 1$. We show that in a strict Nash equilibrium, this implies that player j does not have a link with any other player, i.e., $\bar{g}_{j,k} = 0$ for all $k \neq i$. Suppose there is some player k such that $\bar{g}_{j,k} = 1$. In this case, individual i can simply interchange her link with j for a link with k and get the same payoffs. Thus, the strategy of forming a link with j is not a strict best response. Hence g is not a strict Nash network. The above argument also implies that, since g is connected, player i must be linked to every other player directly. The resulting network is therefore a star. Moreover, it also follows that this link must be formed by player i himself. For otherwise, if there is a player k such that $g_{k;i}$ = 1; then this player is again indi®erent between the link with i and some other agent in the star. This implies that the star must be center-sponsored and completes the proof. 2

Proof of Proposition 4.3: First, consider case (a). We know from Proposition 4.1 that every player in a component chooses the same action. We also know that there are only two possible equilibrium architectures, g 2 G^{cs} and g^e. Clearly, the empty network cannot be part of a strict Nash equilibrium (see also arguments for part (c) below). Thus the only candidates for strict Nash equilibrium are s 2 G^{cs} £ f(®; ®; :::; ®)g or s 2 G^{cs} £ f(¯; ¯; :::; ¯)g. It is easily checked that any of those are indeed strict Nash equilibria.

Consider case (b) next. Again, the empty network is not sustainable by a strict Nash equilibrium. Then the only candidates are s 2 G^{cs} £ f(®; ®; :::; ®)g or s 2 G^{cs} £ f(¯; ¯; :::; ¯)g. It is immediate to see that none of the latter is sustainable as an equilibrium since c > b, which implies that the central player does not have an incentive to form a link with isolated players. Thus the only remaining candidates are the former states, which are easily checked to be strict Nash equilibria.

Finally, consider case (c). If c > d; then the center-sponsored star cannot be an equilibrium network. Thus, the only candidate for a strict equilibrium network is the empty one. However, if a network is empty, the choice of actions is irrelevant. This means that there is no strict Nash equilibrium in this case. The proof is complete. 2

Proof of Lemma 15: Let $s^®$ and $s^¯$ be generic states in $S^®$ and $S^¯$; respectively.

Step 1: First, we focus on the transitions from s^{β} and s^{α}. Let g be the network associated to s^{β} and choose any given player $i \in N$. Consider the network $g - g_i$ derived from g by the deletion of all of player i's links. Suppose that this latter network has L components, $C_1; C_2; \ldots; C_L$; with C_1 corresponding to the component of player i. Furthermore, denote by $x(h)$ the total number of players in C_1 who choose action $h = \alpha; \beta$ in s^{β}: Similarly, let $y(h)$ stand for the total number of players in $N \backslash C_1 = \cup_{l=2}^{L} C_l$ who choose action $h = \alpha; \beta$ in s^{α}: Assume now that, starting from s^{β}; player i is given a revision opportunity. With the above notation in hand, we may write her maximum payoff from choosing α as follows:

$$\pi_{\alpha} = x(\alpha)d + x(\beta)e + y(\alpha)d + y(\beta)e - (L - 1)c; \tag{28}$$

where we use the fact that $c < e$ and, therefore, player i must find it optimal to link to all components. On the other hand, the maximum payoff to choosing β is given by:

$$\pi_{\beta} = x(\alpha)f + x(\beta)b + y(\alpha)f + y(\beta)b - (L - 1)c: \tag{29}$$

To initiate a transition towards s^{α}; we must have that player i prefers action α. This may be written as follows:

$$\pi_{\alpha} - \pi_{\beta} = (x(\alpha) + y(\alpha))(d - f) - (x(\beta) + y(\beta))(b - e) > 0: \tag{30}$$

We are interested in a network structure which requires the minimum number of players who are choosing α. Let $x(\alpha) + y(\alpha) = k$ and, therefore, $x(\beta) + y(\beta) = n - k - 1$. From the above expression it follows that the minimum value of k for which (30) holds is insensitive to the particular network structure and only depends on the number of players choosing different actions. This leads to the conclusion that the minimum number $m^{\beta\alpha}$

of simultaneous mutations required to move from any $s^{\beta} \in S^{\beta}$ to some state $s^{\alpha} \in S^{\alpha}$ must satisfy:

$$\tilde{m}^{\beta\alpha} \geq \frac{b - e}{(d - f) + (b - e)}(n - 1). \qquad (31)$$

In fact, the above considerations also imply that any such number of mutations is sufficient to trigger the desired transition.

Step 2: Consider, on the other hand, the transition from s^{α} to s^{β}. Using again the expressions (28) and (29), we can deduce that the minimum number of mutations required (also sufficient) is given by:

$$\tilde{m}^{\alpha\beta} \geq \frac{d - f}{(d - f) + (b - e)}(n - 1). \qquad (32)$$

Combining (31) and (32), the desired conclusion follows. □

Proof of Lemma 16: First, we extend former notation. Let g be some arbitrarily given network and $i \in N$ a given player in the population. Again, we focus on the network $g - g_i$ derived from g by the deletion of all of player i's links, and let C_1 be the component of player i in $g - g_i$, denoting by $x(h)$ the number of players who choose action h in C_1. Now, however, it is useful to classify the remaining $L - 1$ components, $C_2, C_3, ..., C_L$, into different categories depending on the mix of actions they display. Specifically, let C_l^h ($h = \alpha, \beta$) stand for a generic α-component in network $g - g_i$ (i.e. a component in which every player chooses action h) and, similarly, let $C_l^{\alpha\beta}$, refer to an $\alpha\beta$-component in which some players choose α while others choose β. These components C_l^h are indexed by $l = 1, 2, ..., L^h$, where $h = \alpha, \beta, \alpha\beta$ – naturally, $L^{\alpha} + L^{\beta} + L^{\alpha\beta} = L - 1$. And, for each these components C_l^h, the number of players choosing action h' ($h' = \alpha, \beta$) is denoted by $y_l^h(h')$ (hence, for example, $y_l^{\alpha}(\beta) = 0$ for all l). Finally, we aggregate across different components and make $y^h(h') \equiv \sum_{l=1,2,...,L^h} y_l^h(h')$ and $y(h') \equiv y^{h'}(h') + y^{\alpha\beta}(h')$

Let s^α and s^β be generic states in S^α and S^β, respectively.

Step 1: We start by focusing on the transitions from s^β to s^α. Let g be the network associated to s^β and consider any given player i. Suppose that, starting at s^β, player i receives a revision opportunity. Then note that, since we assume that $c > e$, there exists some number $z \geq 2$ such that $(z-1)e < c \leq ze$ and, therefore, if player i chooses action α, she will not form any links with components $C_l^\beta \neq C_1$ whose cardinality $|C_l^\beta| < z$. This motivates dividing the set of C_l^β components into two groups, small and large, depending on whether their cardinality falls or not below the number z. We number the small C_l^β components from 1 to L^β, while the large components are indexed from 1 to $\hat{L}^\beta$. Furthermore, we define $y^\beta(\beta) = \sum_{l=1,\ldots,L^\beta} y_l^\beta(\beta)$ and $\hat{y}^\beta(\beta) = \sum_{l=1,\ldots,\hat{L}^\beta} y_l^\beta(\beta)$. With this notation in place, the payoff to player i of choosing α may be written as follows:

$$\pi_\alpha = x(\alpha)d + x(\beta)e + [y^\alpha(\alpha) + y^{\alpha\beta}(\alpha)]d + [y^{\alpha\beta}(\beta) + \hat{y}^\beta(\beta)]e - [L^\alpha + L^{\alpha\beta} + \hat{L}^\beta]c. \quad (33)$$

On the other hand, the payoff from choosing β is equal to:

$$\pi_\beta = x(\alpha)f + x(\beta)b + [y^\alpha(\alpha) + y^{\alpha\beta}(\alpha)]f + [y^{\alpha\beta}(\beta) + \hat{y}^\beta(\beta) + y^\beta(\beta)]b - [L^\alpha + L^{\alpha\beta} + \hat{L}^\beta + L^\beta]c. \quad (34)$$

To initiate the transition towards s^α, player i must prefer action α to β; i.e. $\pi_\alpha - \pi_\beta > 0$. Using (33)-(34) and making $k = x(\alpha) + y^\alpha(\alpha) + y^{\alpha\beta}(\alpha)$, this inequality can be rewritten as follows:

$$k(d-f) - x(\beta)(b-e) - [y^{\alpha\beta}(\beta) + \hat{y}^\beta(\beta)](b-e) - [y^\beta(\beta)b - L^\beta c] > 0. \quad (35)$$

As before we wish to minimize the value of k, conceived as the number of simultaneous mutations towards action α that perturb the state s^α. This in turn means that we aim at minimizing the value of the negative terms in (35). We begin by noting that, for a fixed value of $y^\beta(\beta)$, the value of the term $[y^\beta(\beta)b - L^\beta c]$ is minimized when $L^\beta = y^\beta(\beta)$, i.e. when each of the small components is a singleton. This allows us to rewrite (35) as follows:

$$k(d-f) - x(\beta)(b-e) - [y^{\alpha\beta}(\beta) + \hat{y}^\beta(\beta)](b-e) - y^\beta(\beta)[b-c] > 0. \quad (36)$$

We next note that for any fixed value of $x(\beta) + y^{\alpha\beta}(\beta) + \hat{y}^\beta(\beta) + y^\beta(\beta)$, the value of k is minimized when we set the number $x(\beta) + y^{\alpha\beta}(\beta) + \hat{y}^\beta(\beta) = 0$; i.e. if $y^\beta(\beta) = n - k - 1$. This follows from the fact that $b - e > b - c$. Combining these observations, we find that the minimum number of mutations $\bar{m}^{\beta\alpha}$ required for the contemplated transition must satisfy:

$$\bar{m}^{\beta\alpha} \geq \frac{b-c}{(d-f)+(b-c)}(n-1). \quad (37)$$

We now show that a number of mutations satisfying the above inequality is also sufficient, if appropriately chosen. Recall that $s^\beta \in S^\beta$ is a center-sponsored star. Let player n be the center of the star. and suppose that the following simultaneous mutations occur. On the one hand, $\bar{m}^{\beta\alpha} - 1$ players at the spokes switch their action from β to α. On the other hand, player n's strategy also undergoes a mutation: she switches to α and retains her links with the $\bar{m}^{\beta\alpha} - 1$ players who have switched actions but deletes all her links with the remaining $n - \bar{m}^{\beta\alpha}$ players (who are still playing action β). This pattern of $\bar{m}^{\beta\alpha}$ mutations results in a network where the players choosing action α form a center-sponsored star, while all the players choosing β are rendered as singleton components. If these players are then picked for a revision opportunity, the computations leading to (37) imply that they will all choose action α and become linked to the α-component. Subsequently, by Proposition 13, the unperturbed dynamics alone is enough to lead the process a.s. to a center-sponsored star with everyone choosing action α. Thus, in sum, we conclude that $\bar{m}^{\beta\alpha}$ mutations satisfying (37) are sufficient for a transition from any $s^\beta \in S^\beta$ to some $s^\alpha \in S^\alpha$.

Step 2: Consider next the transition from s^{α} to s^{β}. Then, we would like to have the player i who receives a revision opportunity to choose β: Again, her payoffs π_{α} and π_{β} from choosing either action are given by (33) and (34). Thus, the required inequality $\pi_{\beta} - \pi_{\alpha} > 0$ can be rewritten as follows:

$$-[x(\alpha)+y^{\alpha}(\alpha)+y^{\alpha\beta}(\alpha)](d-f)+[x(\beta)+y^{\alpha\beta}(\beta)+\tilde{y}^{\beta}(\beta)](b-e)+[y^{\beta}(\beta)b-L^{\beta}c] > 0. \tag{38}$$

Let $k = x(\beta)+y^{\alpha\beta}(\beta)+\tilde{y}^{\beta}(\beta)+y^{\beta}(\beta)$. As before, we wish to minimize the value of k. This, on the one hand, amounts to the minimization of the first negative term in (38). But, since the value of $[x(\alpha)+y^{\alpha}(\alpha)+y^{\alpha\beta}(\alpha)] = n-k-1$ is insensitive to the precise links of the α-players, these considerations are irrelevant and may be simplified by setting $x(\alpha) = y^{\alpha\beta}(\alpha) = 0$. Next, we take up the other terms, for which we must identify the "best distribution" of the β-players leading to a minimum k in (38). First, we focus on the number L^{β} of small β-components. Since only if player i chooses β will she link to any of these components, the net payoff gain she would enjoy through each of them by choosing β rather than α is $rb-c$, where r stands for the cardinality of the (small) component in question. On the other hand, if those r players were instead part of a large component, player i would link to them both if she plans to play α or β: Consequently, the net gain obtained through them by choosing β rather than α would be $r(b-e)$. Combining both considerations, we find that the difference between these net gains (corresponding to the alternative possibilities that the r players under consideration belong to either a small or a large β-component) is $re-c$: Since, by definition, $r \leq z-1$ and $(z-1)e < c$, we conclude that the latter difference is negative and therefore the sought-after distribution of β-players involves no small components, i.e. $L^{\beta} = 0$. Hence, introducing this fact in (38), the minimum number of mutations $\tilde{m}^{\alpha\beta}$ is found to satisfy :

$$\tilde{m}^{\alpha\beta} \geq \frac{d - f}{(d - f) + (b - e)}(n - 1). \tag{39}$$

Finally, we show that this number of mutations is also sufficient for the transition. Suppose that, starting from s^{α} (whose associated network is a center-sponsored star), there is a simultaneous mutation in the strategy of some $\tilde{m}^{\alpha,\beta}$ players whereby they switch their action from α to β without altering their links. Since, in particular, the central player retains her links, those mutations result in a network where all players are still connected through a center-sponsored star. Thus, if those who are still playing action α are subsequently provided with a revision opportunity, the computations leading to (39) imply that they will choose action β. Thereafter, by Proposition 13, the unperturbed dynamics will lead the system (almost surely) to a center-sponsored star with everyone choosing action β, i.e. a state in $\tilde{S}^{\beta}$. In sum, therefore, we confirm that $\tilde{m}^{\alpha\beta}$ mutations are sufficient for a transition from any state $s^{\alpha} \in \tilde{S}^{\alpha}$ to some state $s^{\beta} \in \tilde{S}^{\beta}$. $\square$

Proof of Lemma 17: The proof is analogous to that of Lemma 16 and is thus omitted. $\square$

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