MINIMAL RIGHTS IN CLAIMS PROBLEMS*

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A B S T R A C T

This paper focuses on some well-known properties for claims problems, as well as on duality. Our main result is that the Constrained Equal-Losses Rule is the only rule satisfying equal treatment of equals, composition from minimal rights, and path independence.

1. Introduction

A group of agents have claims over an estate, and the estate is insufficient to cover all the claims. How should the estate be divided among the agents? In this paper we deal with such *claims problems,* and explore some rules to solve them. As examples of claims problems, we may think of bankruptcy situations, when the net worth of a firm is not enough to cover the debts, or inheritance situations, when the estate is not enough to cover the bequeathed amounts. Thomson (1995) is an excellent survey on the literature on claims problems.

We focus on two well-known rules: the *constrained equal-awards rule* and the *constrained equal-losses rule.* They both fulfill some appealing properties: *equal treatment of equals, composition,* and *path independence.* The *constrained equal-awards rule* also satisfies *independence of claims truncation,* a property stating that any claim exceeding the estate can be substituted by the estate value without affecting the distribution. Dagan (1996) characterizes the *constrained equal-awards rule* as the only rule satisfying *equal treatment of equals, composition,* and *independence of claims truncation.*

In solving a claims problem, any part of the estate which is left after fully honoring all agents' claims but one, can be interpreted as a *minimal right of the remaining agent.* This idea appears in the *contested garment problem,* one of the examples of claims problems discussed in the Talmud (Aumann and Maschler, 1985). The idea to fully honoring at least minimal rights underlies the property of *composition from minimal rights.* It recommends to assign any agent *her minimal right* as a first step, and then to divide the remainder after adjusting the claims down by these amounts. The *constrained equal-losses rule* satisfies *composition from minimal rights.*

Our main result is that the *constrained equal-losses rule* is the only rule satisfying *equal treatment of equals, path independence* and *composition from minimal rights.*

A rule provides a division method. A new rule, *its dual,* can be obtained by first assigning to everyone his claim, and then applying that method to allocate losses. The constrained equal-losses and the constrained equal-awards rules are dual rules. Dual rules satisfy *dual properties.* By noting so, we obtain our main result from Dagan's characterization of the *constrained equal-awards rule* and duality.

The interest of this characterization is twofold. On the one hand, it is made out of well-known and widely accepted properties, something of particular interest in order to support rules from an axiomatic prespective. On the other hand, it is an example of the powerness of the idea of dual properties. Dual rules were introduced by Aumann and Maschler (1985), but duality of properties is a novel idea no used so far.

The paper is organized as follows. Section 2 describes the model, the rules and the properties. Section 3 introduces dual rules and dual properties, and by exploiting these ideas, provides with a characterization of the constrained equal-losses rule. An alternative (direct) proof of our main result is presented in an Appendix..

2. The Model

Let $N = \{1, 2, \ldots, n\}$ be a set of agents. A **claims problem** (O'Neill, 1988) is a pair (c, E), where $E \in \mathbb{R}_+$, $c \in \mathbb{R}^n_+$ and $\sum_i c_i > E$. These data are interpreted as a list of claims, c, where c_i is the claim of agent $i \in N$, over an estate E. Let $\mathbb{C}$ denote the class of all problems.

A **rule** is a mapping $F : \mathbb{C} \to \mathbb{R}^n$, that associates with every $(c, E) \in \mathbb{C}$ a unique point of $\mathbb{R}^n$, $F(c, E)$ such that: (i) For all $i \in N$, $0 \leq F_i(c, E) \leq c_i$, and (ii) $\sum F_i(c, E) = E$. The point $F(c, E)$ is interpreted as a desirable way of dividing E. Requirement (i) is that each agent receives an award that is non-negative and bounded above by her claim. Requirement (ii) is that the entire estate must be allocated. Let $\mathcal{F}$ be the set of all rules on $\mathbb{C}$.

Next we introduce two well-known rules. The **constrained equal-awards rule** makes awards as equal as possible, subject to the condition that no agent receives more than her claim. The **constrained equal-losses rule** makes losses as equal as possible, subject to the condition that no agent ends up with a negative award.[1] Formally:

Constrained equal-awards rule, CEA: For all $(c, E) \in \mathbb{C}$ and all $i \in N$, $CEA_i(c, E) = \min\{c_i, \lambda\}$, where λ solves $\sum \min\{c_i, \lambda\} = E$.

Constrained equal-losses rule, CEL: For all $(c, E) \in \mathbb{C}$ and all $i \in N$, $CEL_i(c, E) = \max\{0, c_i - \lambda\}$, where λ solves $\sum \max\{0, c_i - \lambda\} = E$.

[1] The principle underlying this rule, the *equal-loss* principle, has been applied to other distribution problems, such as cost-sharing, taxation or axiomatic bargaining [see for instance Young (1987), (1988), Chun (1988b), Herrero and Marco (1993)].

Next, we formulate several properties of rules. The first one is a basic equity requirement: agents with identical claims should be treated identically. Formally:

Equal treatment of equals: For all $(c, E) \in \mathbb{C}$ and all $i, j \in N$, if $c_i = c_j$, then $F_i(c, E) = F_j(c, E)$.

To motivate the next two properties, suppose that a tentative distribution is made by first forecasting the value of the estate. Assume that, once the tentative division is done, the actual value of the estate is greater than initially thought. Then, two options are open: either the tentative division is cancelled altogether and the actual problem is solved, or the rule is applied to the problem of dividing the incremental value of the estate, after adjusting the claims down by the amounts assigned in the tentative division. **Composition** states that the rule should be invariant with respect to the chosen option. Alternatively, assume that, once the tentative division is done, it turns out that the actual value of the estate falls short of what was expected. **Path independence** requires that the solution of the actual problem should be the same as that of the problem resulting from adjusting the claims down to the tentative solution. Formally:

Composition (Young, 1988): For all $(c, E,) \in \mathbb{C}$ and all $E_1, E_2 \in \mathbb{R}_+$ such that $E_1 + E_2 = E$, $F(c, E) = F(c, E_1) + F[c - F(c, E_1), E_2]$.

Path Independence (Moulin, 1987): For all $(c, E) \in \mathbb{C}$, and all $E' > E$, $F(c, E) = F[F(c, E'), E]$.

Remark. It is easy to see that if a rule satisfies either *composition* or *path independence* it is *monotonic* and *continuous with respect to estate.*

The *constrained equal-awards* and the *constrained equal-losses* rules satisfy *equal treatment of equals, composition* and *path independence.*

The next property states that a rule should not consider any claim that is greater than the estate: namely, replacing c_i by E if $c_i > E$ should not affect the recommendation.

Independence of claims truncation (Dagan, 1996): For all $(c, E) \in \mathbb{C}$, $F(c, E) = F(c^T, E)$, where, for all $i \in N$, $c_i^T = \min\{E, c_i\}$.

Finally, consider a property related to *composition*, but obtained by requiring first the rule to respect **minimal rights.** The minimal right of an agent corresponds to that part of the estate which is left after fully honoring the claims of all other agents, or zero, if the previous amount is negative.Composition from minimal rights establishes that each agent should receive first her minimal right, whereas the remainder should be divided after adjusting the claims down by these amounts. Formally, let $(c, E) \in \mathbb{C}$. For all $i \in N$, let $m_i(c, E) = \max\{0, E - \sum_{j \neq i} c_j\}$.

Composition from minimal rights: For all $(c, E) \in \mathbb{C}$, $F(c, E) = m(c, E) + F[c - m(c, E), E - \sum_N m_i(c, E)]$.

The following theorem provides a characterization of the *constrained equal-awards rule* on the basis of some of the properties just defined:

Theorem 1. (Dagan, 1996): *The constrained equal-awards rule is the only rule satisfying equal treatment of equals, composition and independence of claims truncation.*

3. Duality and our Main Result.

Let us now consider an operator on $\mathcal{F}$. Given a rule $F \in \mathcal{F}$, we define **its dual**, F^*, as follows (Aumann and Maschler, 1985): For all $b = (c, E) \in \mathbb{C}$, $F^*(c, E) = c - F(c, \sum c_i - E)$.

Note that for all $b = (c, E) \in \mathbb{C}$,we have that $\sum c_i - E \in \mathbb{R}_+$, and $\sum c_i > (\sum c_i - E)$, and consequently, the problem $(c, \sum c_i - E) \in \mathbb{C}$. Additionally, $0 \leq F(c, \sum c_i - E) \leq c$ and $\sum F_i(c, \sum c_i - E) = C - E$, and thus, $0 \leq F^*(c, E) \leq c$ and $\sum F_i^*(c, E) = E$, that is, $F^* \in \mathcal{F}$.

Rules F and F^* are related in a simple way: F^* divides what is available in the same way as F divides what is missing (see Aumann and Maschler, 1985). It is immediate that $CEL = CEA^*$.

A rule F is **self-dual** if $F^* = F$. Examples of self-dual rules are the *proportional rule* and the

Talmud rule, T (Aumann and Maschler, 1985)**:** For all $N \in \mathbb{F}$, all $(c, E) \in \mathbb{C}$, and all $i \in N$, $T_i(c, E) = \begin{cases} \min\{\frac{1}{2}c_i, \lambda\} & \text{if } E \leq \frac{1}{2}\sum_{i \in N} c_i \\ \max\{\frac{1}{2}c_i, c_i - \mu\} & \text{if } E \geq \frac{1}{2}\sum_{i \in N} c_i \end{cases}$

where λ and μ are chosen so that $\sum_{i \in N} T_i(c, E) = E$.*Talmud rule.*[2]

Given two properties $\mathcal{P}$, $\mathcal{P}^*$, we say that $\mathcal{P}^*$ is the ***dual property of*** *$\mathcal{P}$ if for all $F \in \mathcal{F}$, it happens that F that satisfies $\mathcal{P}$ iff its dual rule, F^*, satisfies $\mathcal{P}^*$. A property $\mathcal{P}$ is* ***self-dual*** *if $\mathcal{P}^* = \mathcal{P}$. It is immediate that equal treatment of equals is self-dual. We also have the following results:*

Lemma 1. *Composition and path independence are dual properties.*

Proof: *For all $F \in \mathcal{F}$, all $(c, E) \in \mathbb{C}$, and all $E_1, E_2 \in \mathbb{R}_+$ such that $E_1 + E_2 = E$, $F^*(c, E) = c - F(c, \sum c_i - E) = c - F(c, \sum c_i - E_1 - E_2)$.*

Let $z = F(c, \sum c_i - E_1)$. If F satisfies path independence, $F(c, \sum c_i - E_1 - E_2) = F(z, \sum c_i - E_1 - E_2) = F(z, \sum z_i - E_2) = z - F^*(z, E_2)$. Thus, $F^*(c, E_1 + E_2) = c - z + F^*(z, E_2) = c - F(c, \sum c_i - E_1) + F^*(z, E_2) = F^*(c, E_1) + F^*(c - F^*(c, E_1), E_2)$. Therefore, F^* satisfies composition.

Similarly, $F(c, E_1 + E_2) = c - F^*(c, \sum c_i - E_1 - E_2)$.

Let $y = F^*(c, \sum c_i - E_1)$. If F^* satisfies composition, $F(c, E_1 + E_2) = c - F^*(y, \sum c_i - E_1 - E_2) = c - z + F(z, E_2) = F(c, E_1) + F(z, E_2)$. Therefore, F^* satisfies path-independence. $\square$

Lemma 2. *Composition from minimal rights and independence from claims truncation are dual properties.*

Proof: *Let F be a rule satisfying composition from minimal rights. For all $(c, E) \in \mathbb{C}$ we have $F^*(c, E) = c - F(c, \sum c_i - E) =$*

$c - m(c, \sum c_i - E) - F[c - m(c, \sum c_i - E), \sum c_i - E - \sum m_i(c, \sum c_i - E)]$.

Let $c' = c - m(c, \sum c_i - E)$. Thus, $\sum c'_i = \sum c_i - \sum m_i(c, \sum c_i - E)$. Then, $F^*(c, E) = c' - F(c', \sum c'_i - E) = F^*(c', E)$.

Note that $c'_k = c_k - m_k(c, \sum c_i - E) = c_k - \max\{0, \sum c_i - E - \sum_{j \in N \setminus \{k\}} c_j\} = c_k - \max\{0, c_k - E\} = \min\{c_k, E\}$.and thus, F^* satisfies independence from claims truncation.

Similarly, assume that F^* satisfies independence from claims truncation. Then,

$F(c, E) = c - F^*(c, \sum c_i - E) = c - F^*(c', \sum c_i - E)$,

where $c'_k = \min\{c_k, \sum c_i - E) = c_k - \max\{0, c_k - \sum c_i + E\} = c_k - m_k(c, E)$.

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[2]**Proportional rule, P**: For all $(c, E) \in \mathbb{C}$ and all $i \in N$, $P_i(c, E) = \lambda c_i$, where λ solves $\sum \lambda c_i = E$.

Talmud rule, T (Aumann and Maschler, 1985)**:** For all $N \in \mathbb{F}$, all $(c, E) \in \mathbb{C}$, and all $i \in N$,

$$T_i(c, E) = \begin{cases} \min\{\frac{1}{2}c_i, \lambda\} & \text{if } E \leq \frac{1}{2}\sum_{i \in N} c_i \\ \max\{\frac{1}{2}c_i, c_i - \mu\} & \text{if } E \geq \frac{1}{2}\sum_{i \in N} c_i \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} T_i(c, E) = E$.

Thus, $F(c, E) = c - c' + F(c', \sum c'_i - \sum c_i + E) = m(c, E) + F[c - m(c, E), E - \sum m_i(c, E)]$*. Therefore,* F *satisfies composition from minimal rights.* $\square$

Theorem 2. *The constrained equal-losses rule is the only rule satisfying equal treatment of equals, composition from minimal rights, and path independence.*
Proof: *Is a direct consequence of Theorem 1, Lemmas 1 and 2 and self-duality of equal treatment of equals.*$\square$

Remark. *The properties in Theorem 2 are independent. We provide examples of rules fulfilling all but one property at any time. We mention in each case the property that is not fulfilled:*
Equal treatment of equals: Choose an agent $i \in N$*. Now, for all* (c, E)*, take* $F_i(c, E) = m_i(c, E)$*, and for all* $j \in N \backslash \{i\}$*,* $F_j(c, E) = \max\{0, c_j - \lambda\}$*, where* λ *solves* $\sum_{j \in N \backslash \{i\}} \max\{0, c_j - \lambda\} = E - m_i(c, E)$.
Composition from minimal rights: The constrained equal-awards rule.
Path independence: The Talmud rule.

References

[1] Aumann, R. and Maschler, M. (1985), Game Theoretic Analysis of a Bankruptcy Problem from the Talmud, Journal of Economic Theory, 36 : 195–213.

[2] Chun, Y. (1988), The Equal-Loss Principle for Bargaining Problems, Economics Letters, 26 : 103-106.

[3] Dagan, N. (1996), New Characterizations of Old Bankruptcy Rules, Social Choice and Welfare, 13 : 51–59.

[4] Herrero, C. and Marco, M.C. (1993), Rational Equal Loss Solutions for Bargaining Problems, Mathematical Social Sciences, 26 : 273–286.

[5] Moulin, H. (1987), Equal or Proportional Division of a Surplus, and other Methods, International Journal of Game Theory, 16 (3) : 161-186.

[6] O'Neill, B. (1982), A Problem of Rights Arbitration from the Talmud, Mathematical Social Sciences, 2 : 345–371.

[7] Thomson, W. (1995), Axiomatic Analyses of Bankruptcy and Taxation Problems: A Survey, Mimeo, University of Rochester.

[8] Young, P. (1987), On Dividing an Amount According to Individual Claims or Liabilities, Mathematics of Operations Research, 12 : 398–414.

[9] Young, P. (1988), Distributive Justice in Taxation, Journal of Economic Theory, 43 : 321–335.

4. Appendix: A Direct Characterization of the constrained equal-losses rule.

Theorem 2. *The constrained equal-losses rule is the only rule satisfying equal treatment of equals, composition from minimal rights, and path independence.*

Proof: *The CEL rule satisfies all the properties.*

Conversely, let F be a rule that satisfies all the properties. Let us show that $F = CEL$.

Let $(c, E) \in \mathbb{C}$. Let $C(c) = \sum_i c_i$, $\delta(c) = \min_i c_i$, and $D(c) = C(c) - \delta(c) = \max_{i \in N}\{\sum_{j \in N \setminus \{i\}} c_j\}$. Also, let $m_i(c, E) = \max\{0, E - \sum_{j \neq i} c_j\}$

Case 1. *$C(c) - \delta(c) \leq E$.*

By composition from minimal rights, $F(c, E) = m(c, E) + F[c - m(c, E), E - \sum_i m_i(c, E)]$.

Note that for all $i \in N$, $m_i(c, E) = E - C(c) + c_i$ and $c_i - m_i(c, E) = C(c) - E$. Thus, by equal treatment of equals, for all $k \in N$, we have $F_k[c - m(c, E), E - \sum_i m_i(c, E)] = \frac{n-1}{n}[C(c) - E]$. Thus, for all $i \in N$, $F_i(c, E) = c_i + E - C(c) + \frac{n-1}{n}[C(c) - E] = c_i - \frac{C(c)-E}{n} = CEL_i(c, E)$.

Case 2. *$C(c) - n\delta(c) < E \leq C(c) - \delta(c)$.*

Without loss of generality, assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. Thus, $\delta(c) = c_1$, and $D(c) = c_2 + \cdots + c_n$.

Step 1. Note that $(c, D(c)) \in \mathbb{C}$ and by construction, it is covered by Case 1. Thus, for all $i \in N$, $F_i(c, D(c)) = CEL_i(c, D(c)) = c_i - \frac{\delta(c)}{n} = c_i^1$.

By path independence, $F(c, E) = F(c^1, E)$.

Now, note that $c_1^1 \leq c_2^1 \leq \cdots \leq c_n^1$. Thus, $\delta(c^1) = c_1^1 = \frac{n-1}{n}\delta(c)$, and $D(c^1) = C(c^1) - \delta(c^1) = C(c) - \delta(c) - \frac{n-1}{n}\delta(c)$.

Now, two possibilities are open: either $D(c^1) \leq E$, or $D(c^1) > E$.

If $D(c^1) \leq E$,then (c^1, E) is covered by Case 1. Thus, $F(c^1, E) = CEL_i(c^1, E)$. Since CEL satisfies path independence, $CEL(c^1, E) = CEL(c, E)$, and thus, $F(c, E) = CEL(c, E)$.

If $D(c^1) > E$, go to step 2.

Step k. Note that $(c^{k-1}, D(c^{k-1})) \in \mathbb{C}$ and by construction, it is covered by Case 1. Thus, for all $i \in N$, $F_i(c^{k-1}, D(c^{k-1})) = CEL_i(c^{k-1}, D(c^{k-1})) = c_i^{k-1} - \frac{\delta(c^{k-1})}{n} = c_i - \frac{\delta(c)}{n}\left[1 + \frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^{k-1}\right] = c_i^k$.

By path independence, $F(c^{k-1}, E) = F(c^k, E)$. Note that $c_1^k \leq c_2^k \leq \cdots \leq c_n^k$. Thus, $\delta(c^k) = c_1^k = \left(\frac{n-1}{n}\right)^k \delta(c)$ and $D(c^k) = C(c^k) - \delta(c^k) = C(c) - \delta(c)\left[1 + \frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots\right.$

Now, two possibilities are open: either $D(c^k) \leq E$, or $D(c^k) > E$.

If $D(c^k) \leq E$, then (c^k, E) is covered by Case 1. Thus, $F(c^k, E) = CEL_i(c^k, E)$. Since CEL satisfies path independence, $F(c, E) = CEL(c, E)$.

If $D(c^k) > E$, go to step k+1...

We claim that for some $k \in \mathbb{N}$, $D(c^k) \leq E$. Suppose not. Then, for all $k \in \mathbb{N}$, $D(c^k) > E$. Thus, $\lim_{k\to\infty} D(c^k) \geq E$. That is,

$E \leq C(c) - \delta(c) \lim_{k\to\infty} \left[1 + \frac{n-1}{n} + \left(\frac{n-1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^k\right] = C(c) - n\delta(c)$, *which contradicts the fact that (c, E) is covered by Case 2. Thus, $F(c, E) = CEL(c, E)$.*

Case 3. *$E = C(c) - n\delta(c)$.*

Let $\{E_k\}$ be a sequence such that $E_k > E_{k+1}$, and $\{E_k\} \to E$. Thus, the sequence of problems $\{(c, E^k)\}$ converges to (c, E). All problems (c, E^k) in the sequence are covered by Case 2. Thus, for all $k \in \mathbb{N}$, $F(c, E^k) = CEL(c, E^k)$. By path independence, F is continuous with respect to the estate. Thus, $F(c, E) = \lim_{k\to\infty} F(c, E^k) = CEL(c, E)$.

Case 4. *$E < C(c) - n\delta(c)$.*

Let $N_1(c) = \{i \in N \mid c_i = \delta(c)\}$, and let $n_1 = |N_1(c)|$. By path independence, $F(c, E) = F(F(c, C(c) - n\delta(c)), E)$. For all $i \in N_1(c)$, $F_i(c, C(c) - n\delta(c)) = 0$, and for any other $i \in N \backslash N_1(c)$, $F_i(c, C(c) - n\delta(c)) = c_i - \delta(c)$. Let $d = F(c, C(c) - n\delta(c))$. Then, $F(c, E) = F(d, E)$. Let $\delta_2(d) = \min\{d_i \mid d_i > 0\}$. We consider several subcases:

4.a. *$C(d) - \delta_2(d) \leq E < C(c) - n\delta(c) = C(d)$.*

For all $i \in N \backslash N_1(c)$, $m_i(d, E) = d_i + E - C(d)$, and thus, $E - \sum_{i\in N} m_i(d, E) = (n - n_1 - 1)[C(d) - E]$. By equal treatment of equals, for all $i \in N \backslash N_1$, $F_i(d, E) = d_i - C(d) + E + \frac{n-n_1-1}{n_1}[C(d) - E] = CEL_i(d, E)$.

4.b. *$C(d) - (n - n_1)\delta_2(d) < E < C(d) - \delta_2(d)$.*

Let $D_2(d) = C(d) - \delta_2(d)$, and consider the problem $(d, D_2(d))$. Note that $F(d, D_2(d)) = CEL(d, D_2(d)) = d^1$. By path independence, $F(d, E) = F(d^1, E)$. Now, two options are open: either $C(d^1) - \delta_2(d^1) \leq E$, or the opposite. Then, we may repeat the procedure of Case 2, by only considering the agents in $N \backslash N_1$.

4.c. *$E = C(d) - (n - n_1)\delta_2(d)$. Repeat the procedure in Case 3 only considering the agents in $N \backslash N_1$.*

4.d. *From then on, repeat the procedure, considering at any step only the agents in $N \backslash N_1 \cup \cdots \cup N_k$, until all possible values of E are covered.* □