AN EVOLUTIONARY MODEL OF BERTRAND OLIGOPOLY*

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ABSTRACT

We analyze the long-run outcome of markets in which boundedly rational firms with a decreasing returns to scale technology compete in prices. The behavior of these firms is based on imitation of success and experimentation. In this framework, we introduce a new approach to model boundedly rational behavior, based on the idea of "behavioral principles," i.e. formal descriptions of how firms' decisions are triggered by specific market situations. Even with the simplest ones, the result is that the prices announced are a strict refinement of the set of Nash equilibria. With more sophisticated behavioral principles, the long-run outcome corresponds to the concept of "central prices" (which are also Nash equilibria) introduced here. This is a robust and clear-cut prediction which, under quadratic costs and arbitrary demand, essentially coincides with the Walrasian equilibrium.

KEYWORDS: Evolution; Mutation; Imitation; Bertrand Oligopoly.

1 Introduction

Evolutionary theory provides us with the tools to explicitly study the long-run outcome of markets in which agents repeatedly make short-run decisions on certain variables every period, according to a given type of behavior like best response or imitation.

In the study of oligopolistic markets, price is considered as a key decision variable for the firm in the short run, since it is one of the instruments that can be easily changed and adjusted. Only in the medium run are firms able to adjust their production capacities. The study of price competition becomes then a fundamental part in the study of a market. See e.g. Shapiro [10].

As a first approach, in the context of homogeneous product, the conclusion is drawn that price competition will lead firms facing constant-returns-to-scale technologies to set prices equal to marginal cost and thus earn zero profits. This constitutes the well-known Bertrand paradox. A different look to the models of price competition reveals that, if firms are not able to serve the whole market (because of capacity constraints for instance), there is room for equilibrium prices strictly above marginal cost. Nevertheless, price competition in the context of homogeneous product was regarded as uninteresting and mainly ignored as a topic of research for a long time due to both the paradoxical behavior and the fact that price and quantity competition were reconciled in the context of monopolistic competition.

Only recently, the set of Nash equilibria of a Bertrand game under decreasing returns to scale has been characterized (see Dastidar [3]), and it has been found that there is a continuum of pure strategy Bertrand equilibria, which contains marginal-cost pricing, i.e. Walrasian behavior, see Dastidar [4]. This could be regarded as the Bertrand paradox with convex costs. Not only that, but also average-cost pricing, i.e. zero profits, is a Bertrand equilibrium in this context.

A common ingredient of all these models is the assumption of full rationality on the part of firms. This is a very strong and even unrealistic assumption, especially if firms only have diffuse knowledge of market conditions and lack complete information.

Evolutionary models of imitation seem to be more realistic, simple, and appropriate for the representation of a market. Models of social learning by imitation have proven successful for the understanding of individuals' behavior under conditions of limited information (see e.g. Bjönerstedt and Weibull [2], or Schlag [9]).

Once one drops the assumption of full rationality, it is not clear anymore how behavior should be modeled. In contrast to profit maximization, which is always a well defined and very specific rule, there are many types of behavior which could be considered as "boundedly rational," even within the specific framework of imitation models.

Instead of pretending to have a unique description of "bounded rationality," we want to introduce here a new approach to the problem of describing social behavior.

The key to this new approach are what we call "behavioral principles." A behavioral principle is a formal description of a "rule of thumb" to be followed to make a decision when a certain social situation is faced. For instance a behavioral principle could require a firm to imitate every one of the firms which have obtained maximal profits with positive probability, or it could specify that only firms which have actually faced positive demand are worth imitating.

Each behavioral principle characterizes a whole family of imitation rules. We see our approach as an exploratory one: our aim is not to discuss a particular model of boundedly rational behavior, but rather to explore the properties of different families of imitation rules.

Here we apply this approach to the study of an oligopolistic market in which identical firms, producing a homogeneous good under decreasing returns to scale, decide on prices with the only information they can get from the market alone: all prices announced in the previous period and all profits obtained. Firms will then imitate those prices which have proven to be more successful in terms of profits – a clear indicator of success in a market. There are, though, many different types of imitation rules. As explained above, we will describe them by behavioral principles which capture the underlying rules of thumb.

We will find that, even under the simplest principles, the long-run prediction selects a set of Bertrand-equilibrium prices such that all firms make strictly positive profits, thus excluding the zero-profits equilibrium. Nevertheless, this set of prices could be quite big. We see this as a consequence of the crudeness of the behavioral principles, and thus turn to analyzing more detailed ones. Then, we obtain a clear-cut prediction which is a refinement of the former and which is related to a certain idea of "central prices." We compare this new concept to Walrasian behavior in order to understand to what extent the latter is a robust Bertrand equilibrium, finding that, under specific conditions, the Walrasian price is central, although, in general, there is no clear relation between "central prices" and the Walrasian ones.

An evolutionary model of price competition has already been studied by Qin and Stuart [7]. They use the deterministic, continuous-time replicator dynamics based on expected payoffs to model economic natural selection in a Bertrand game with constant marginal costs and show that the classical Bertrand equilibrium fails to be an evolutionarily stable strategy. To justify their use of expected payoffs, they postulate a very large number of different markets and invoke a law of large numbers.

In contrast to Qin and Stuart [7], here evolution is based on actual payoffs, and not on expected ones. Moreover, we model natural economic evolution through a true stochastic dynamical system, where random mutation is used to capture the idea of experimentation. In other models, this randomness reflects trembling errors or mutation in a more biological sense.

We will rely on the techniques for discrete-time Markov processes with finite state space as stated in Freidlin and Wentzell [5, Chap. 6], as applied in Kandori *et al.* [6] and Young [12]. Similar evolutionary techniques in the framework of Cournot competition have been used by Vega–Redondo [11], Alós–Ferrer, Ania, and Vega–Redondo [1], and Schenk–Hoppé [8].

Section 2 presents the model. Section 3 defines what imitation rules are and analyzes the most naive ones. Section 4 presents more sophisticated imitation rules and obtains the main result as a refinement of the analysis in the previous section. Section 5 compares this result with Walrasian behavior, and Section 6 concludes.

2 The model

Consider a market for a homogeneous good where n firms compete in prices. All firms are equal, using a technology characterized by a cost function $C : R_+ \to R_+$, which is assumed increasing and strictly convex. We assume zero fixed costs, but all results hold true for the general case. The consumer side of the market is summarized by a strictly decreasing demand function $D : R_+ \to R_+$. Suppose each firm $i = 1, \dots, n$ announces some price p_i. Then, defining $\boldsymbol{p} = (p_1, \dots, p_n)$ and

$$M(\boldsymbol{p}) := \{i \mid p_i = \min_{j=1,\dots,n} p_j\}, \tag{1}$$

their profits $\Pi_i(\boldsymbol{p}) = \Pi_i(p_1, \dots, p_n)$, $i = 1, \dots, n$ are given by

$$\Pi_i(\boldsymbol{p}) = \begin{cases} p_i \frac{D(p_i)}{|M(\boldsymbol{p})|} - C\left(\frac{D(p_i)}{|M(\boldsymbol{p})|}\right) & \text{if } i \in M(\boldsymbol{p}) \\ 0 & \text{if } i \notin M(\boldsymbol{p}) \end{cases} \tag{2}$$

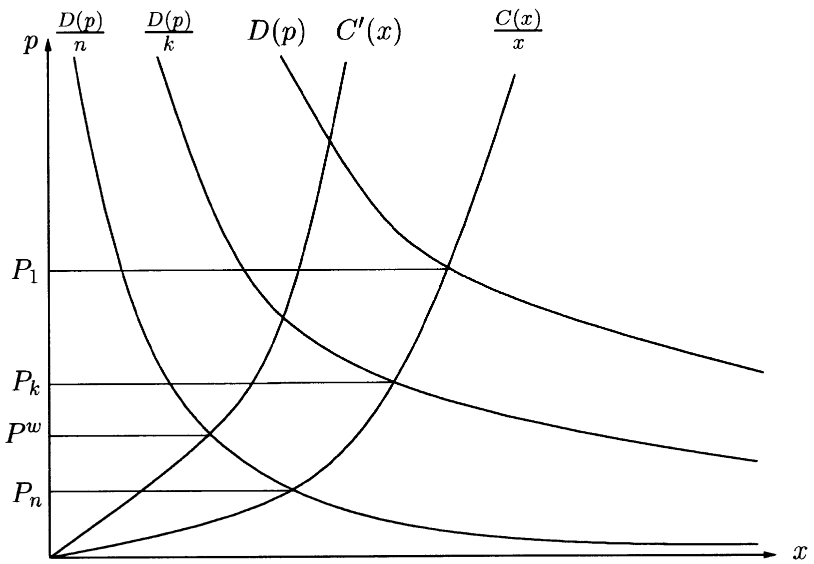

Figure 1:

Prices P_k and Walrasian equilibrium P^w

This defines a Bertrand game with the tie-breaking rule of *equal shares.* For each $k = 1, \ldots, n$, and $p \in R_+$ define

$$\Pi(p,k) := p\frac{D(p)}{k} - C\left(\frac{D(p)}{k}\right) \tag{3}$$

We introduce the following key concept. Define $P_k \in R_+$ through the condition

$$P_k \text{ is such that } \Pi(P_k, k) = 0 \text{ and } D(P_k) > 0.$$

Dastidar [3, Lem. 1 and 7] proves in quite generality[1] that these prices are well defined and unique and also that P_k is strictly decreasing in k. The prices P_k and the Walrasian price are depicted in figure 1.

[1] $D(p)$ is twice continuously differentiable with strictly negative first derivative, and there exist finite constants $P^{\max} \geq 0$ and $Q^{\max} \geq 0$ such that $D(P^{\max}) = 0$, $D(0) = Q^{\max}$. $C(q)$ is twice continuously differentiable, strictly convex, and increasing.

A first, static interpretation of P_k is that it corresponds to the price such that k firms make exactly zero profits, when they announce the same price and they are the only active firms in the market, i.e. all other $n-k$ firms announce a higher price and face therefore zero demand. A second, more dynamic interpretation could be: the lower bound of the set of prices to which k firms can jointly deviate, getting and sharing among them k the whole demand, leaving the other $n-k$ firms without any market share, and still making positive profits.

We assume that all prices that firms can announce belong to an *a priori* fixed and finite set Γ. The only restriction on this set is that all P_k belong to Γ, and that there exists at least one price strictly lower than P_n, i.e. one for which all n firms announcing the same price would make losses. This restriction is made for the sake of clarity, but it could be largely dispensed with.

Let us now turn to the description of the dynamic evolutionary model studied. We assume that a fixed finite number of firms play the Bertrand game at each discrete time period. When firms have to announce a price in period $t+1 \in N$, the prices and profits of all firms at the preceding time t are known. Based on these data, each firm chooses a price for time $t+1$. We consider three different mechanisms, determining the price announcement $p_i(t+1)$ of the ith firm, given the prices $\boldsymbol{p}(t) = (p_1(t), \ldots, p_n(t))$ and the profits $\Pi(\boldsymbol{p}(t)) = (\Pi_1(\boldsymbol{p}(t)), \ldots, \Pi_n(\boldsymbol{p}(t)))$. With certain probabilities, a firm upholds its price $p_i(t)$, imitates a firm by announcing some price $p_k(t)$ that yielded maximal profit at period t, or randomly chooses some price from Γ. The sequential structure of these events is as follows. First, a choice is made whether upholding or imitation takes place. Afterwards, it is decided whether experimentation occurs or not.

Fix an imitation probability $0 < \delta < 1$ and a experimentation probability $0 < \varepsilon < 1$. We call $1 - \delta > 0$ the upholding probability.

Upholding (occurs with probability $1 - \delta > 0$) $p_i(t+1) := p_i(t)$. That is, firm i does not change its price announcement.

Imitation (occurs with probability $\delta > 0$) $p_i(t+1)$ is some price $p_k(t)$ which yielded maximal profit at period t. This behavior is precisely specified by an imitation rule introduced in the next section.

After either upholding or imitation took place, it is decided whether the firm experiments with some price or whether the choice made before is kept. In the first case, we will say that a mutation occurs.

Experimentation (occurs with probability $\varepsilon > 0$) $p_i(t+1)$ is randomly chosen from Γ according to some fixed distribution which assigns positive measure to every state.

This completely determines the new vector of prices $\boldsymbol{p}(t+1)$, given a state $\boldsymbol{p}(t)$. Since firms are non-cooperative, we assume that each firm makes its decision independently of the others and, therefore, we obtain a Markov process $\boldsymbol{p}(t)$ in discrete time with finite state space Γ^n.

It is worth to point out that each single firm may have its own preferences and, for instance, may assign higher probability to imitation than to upholding. It may even have a tendency to experimenting with high prices and only very rarely with low ones. The same remark applies to the imitation case. All these possibilities are only limited by the assumption that certain events have to happen with positive probability and thus are not allowed to be excluded when fixing the preferences of firms as discussed above.

Lemma 2.1 *The discrete-time Markov process $\boldsymbol{p}(t)$ with state space Γ^n has a unique ergodic measure μ_ε for each fixed $\varepsilon > 0$. In particular, μ_ε is invariant under the Markov process $\boldsymbol{p}(t)$ and completely determined by the (finitely many numbers) $\mu_\varepsilon(\boldsymbol{p})$, $\boldsymbol{p} \in \Gamma^n$. μ_ε is often called stationary distribution.*

This statement follows immediately from the observation that the transition probability matrix of the finite state Markov process is irreducible, i.e. for any two states there is a positive probability to reach the second state in one step when starting at the first state. (This probability is bounded from below by $\varepsilon^n \prod_{i=1}^n \lambda_i(p_i) > 0$ for each state $(p_1, \ldots, p_n)$, where $\lambda_i(p_i)$ is the ith firm's probability of choosing p_i when experimenting.)

Recall that ergodicity of μ_ε means that $\lim_{s\to\infty} \frac{1}{s} \sum_{t=0}^{s-1} f(x(t)) = \int_{\Gamma^n} f(x)\, \mu_\varepsilon(dx)$ for each integrable function f (and for almost all sequences of realizations of the random variables). In particular, if $f = 1_A$ is the characteristic function, then this reads $\lim_{s\to\infty} \frac{1}{s} \sum_{t=0}^{s-1} 1_A(x(t)) = \mu_\varepsilon(A)$, i.e. the average amount of time a typical sample path of the model spends in a set A equals the μ_ε-measure of this set. Note that the process may start at an arbitrary state. Thus the unique ergodic measure μ_ε completely summarizes the long-run behavior of the model from a statistical point of view.

We next discuss the role of the experimentation mechanism built into the evolutionary model. First, consider the Markov process above in the absence of experimentation ($\varepsilon = 0$). Then each monomorphic state (i.e. each vector with identical components) is obviously a fixed point under the previous dynamics. Further, from each non-monomorphic state a monomorphic state is reached with probability one as time goes by. Thus the system settles down on the diagonal of the state space Γ^n. The experimentation introduced forces the system to occasionally move away from each state and in particular perturbs all steady states of the system lacking experimentation.

The occurrence of experimentation is controlled by the value of ε. If $\varepsilon = 0$, then we obtain the model discussed in the preceding paragraph. If ε is large, then the system fluctuates fast through the entire state space Γ^n. If ε is small, then upholding and imitation become important "pushing" the price vector toward monomorphic states. The smaller ε is, the more important these two mechanisms become. In the following, we will study the ergodic measure μ_ε (and thus the long-run behavior of the dynamic evolutionary model) as the occurrence of experimentation vanishes, i.e. as $\varepsilon \to 0$.

Two different interpretations of experimentation (and the method to (slowly) decrease the probability for its occurrence) from a dynamical point of view can be given.

On the one hand, experimentation is the source of new information in our model. Since each monomorphic state is a steady state of the Markov process introduced above for $\varepsilon = 0$, the only way the system can move away from any of these states is via experimentation. For small $\varepsilon > 0$, the impact of imitation becomes better observable than for large ε, and, in particular, the system will mainly rest on monomorphic states. Thus, as $\varepsilon \to 0$, the process can be expected to settle down on monomorphic states which are stronger attracting than others.

On the other hand, one can interpret experimentation as an artificially introduced stochastic perturbation of a deterministic model which is used to move the system away from any steady state. A motivation for this comes from the fact that most real-world systems are subject to (small) random shocks. Then, letting $\varepsilon \to 0$, the system will settle down on those steady states which are more stable – and thus more important in the corresponding real-world system – than others. In particular, this singles out unstable steady states, and it helps to detect those states which are more likely to be observed in the system modeled.

3 Imitation rules

Imitation is a kind of replicating behavior which enables firms to learn from other firms' experience, without them needing to have full knowledge of market conditions, or to be able to make complicated calculations in order to make a decision. We believe that imitation is a major determinant of actual behavior in social and economic systems. In a market framework, this can be attributed to the lack of information about market conditions that firms must face when making a decision.

In particular, in the context of an oligopolistic market, if a firm has to decide optimally what price to charge, it needs precise information about the market demand function, the cost function, and the prices that other firms are going to charge. Not only that, but also important requirements of common knowledge of rationality, and computation capabilities are needed in order for it to reply optimally to its competitors behavior.

There are a number of markets with only a small number of firms, but due to the structure of these firms they are unable to fulfill all these information and computation requirements. Even in markets with a very small number of big firms, concerned with the relevant data of the market, it is very restrictive to assume that they have perfect or even sensible knowledge of market conditions (demand and cost functions).

A simple way out for a firm that has to make a decision on what price to charge is to imitate prices which other successful firms in the industry have charged in previous periods. In that sense, profits obtained could be regarded as the clearest indicator of success.

In the following, we will formally define what an imitation rule is. Then, we will explore different sets of imitation rules characterized through specific behavioral principles, and then study the implications of these principles in our model.

Every period, we will consider the vector of prices that each of the n firms in the market have set. Given a price announcement $\boldsymbol{p} = (p_1, \ldots, p_n)$ which yielded profits $(\Pi_1(\boldsymbol{p}), \ldots, \Pi_n(\boldsymbol{p}))$, define

$$B(\boldsymbol{p}) := \{p_k \mid \Pi_k(\boldsymbol{p}) = \max_{j=1,\ldots,n} \Pi_j(\boldsymbol{p})\} \tag{4}$$

Definition 3.1 *An imitation rule of firm i is a family of random variables $I_i(\boldsymbol{p})$ taking values in $B(\boldsymbol{p})$, i.e. $Prob\{I_i(\boldsymbol{p}) \in B(\boldsymbol{p})\} = 1$ for all $\boldsymbol{p} \in \Gamma^n$. A collection of independent random variables $\{I_i(\boldsymbol{p})\}_{\boldsymbol{p}\in\Gamma^n, i=1,\ldots,n}$ is called imitation rule.*

Imitation will have no effect on those states in which all firms are announcing the same price. Such states will be called *monomorphic*. We will denote them by $\mathrm{mon}(p) = (p, \ldots, p)$. We can extend the definition of monomorphic states to sets.

Definition 3.2 *We define for each set A, the set of all corresponding* monomorphic states *by*

$$\mathrm{mon}(A) := \{(p, \ldots, p) \mid p \in A \cap \Gamma\}.$$

Our definition of imitation rules is not sufficient to fully characterize the behavior of firms. It is necessary to specify how firms will choose among elements of $B(\boldsymbol{p})$. This is done through the introduction of behavioral principles. The first and most straightforward principle that we will consider simply states that not only a firm imitates firms with highest profits, but also any of them with positive probability.

(all-best) principle An imitation rule is said to satisfy the (all-best) principle, if

$$Prob\{I_i(\boldsymbol{p}) = p_k\} > 0 \text{ for all } p_k \in B(\boldsymbol{p}) \text{ and for all } i.$$

Theorem 3.3 *Let $n \geq 3$. Given any imitation rule which satisfies the (all-best) principle, we have that in the long run, as the probability of experimentation tends to zero, all firms announce identical prices $p \in]P_{n-1}, P_1]$, i.e.*

$$\lim_{\varepsilon \to 0} \mu_\varepsilon\{\mathrm{mon}(]P_{n-1}, P_1])\} = 1$$

Moreover, we have that $\lim_{\varepsilon \to 0} \mu_\varepsilon\{\mathrm{mon}(p)\} > 0$ *for all* $p \in]P_{n-1}, P_1] \cap \Gamma$.

Proof. We will rely on the characterization of the stationary distribution μ_ε given by Freidlin and Wentzell [5]. This characterization involves the key concept of a *tree*. Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ be a state of the system. A $\boldsymbol{p}$-tree is a directed graph without cycles, such that each state different from $\boldsymbol{p}$ is the origin of exactly one arrow (denoted by $\rightarrow$). A *path* is a sequence of compatible arrows (for notational simplicity, we will also denote a path by $\rightarrow$, when there is no possible confusion about the corresponding sequence of arrows.) Note that in a $\boldsymbol{p}$-tree there always exists a path from any state to $\boldsymbol{p}$. The cost of an arrow between two states is the minimal number of mutations needed to directly reach the second one when starting at the first one. In this sense, imitation and upholding have zero cost. The cost of a path or a tree is the sum of all the costs of its arrows.

Freidlin and Wentzell [5, Chap. 6, § 3] prove that the stationary distribution is well defined and that it assigns positive weight to and only to those states with minimal-cost trees among all trees of all states.

Step 1. Non-monomorphic states cannot have minimal-cost trees. To see this, suppose that a non-monomorphic state $\boldsymbol{p}$ were to have a minimal-cost tree. Consider this tree, and choose some $p' \in B(\boldsymbol{p})$. We could then construct a $\text{mon}(p')$-tree by connecting $\boldsymbol{p}$ to $\text{mon}(p')$ and deleting the arrow exiting $\text{mon}(p')$. The former can be done at zero cost, since there is positive probability that all firms imitate the same price at the same time, while the latter saves at least cost one, since no monomorphic state can be left without any mutation. The $\text{mon}(p')$-tree so constructed has lower cost, which yields a contradiction.

Step 2. We will now consider all possible transitions (paths) between monomorphic states in order to see what the minimal-cost trees are.

In a first step we will focus on downward transitions, i.e. undercutting of prices. Consider all transitions of the form $\text{mon}(p) \to \text{mon}(p')$ such that $p' < p$ and $p' < P_1$. We will see that one mutation is not enough for this transition to occur, but two mutations are enough. Start with $\text{mon}(p)$ and consider one mutation from p to p'. After mutation, non-mutants make zero profits, since they do not get any demand, but the mutant makes losses, since $p' < P_1$ implies that a firm alone in the market cannot make positive profits. Pure imitation can only lead the process back to $\text{mon}(p)$. If $p' = \min \Gamma$, then obviously the transition requires n mutations. Otherwise two mutations suffice. Start again with $\text{mon}(p)$ and consider two simultaneous mutations, one to p', and another one to $p'' < p'$. After mutation, the p'' mutant makes losses since $p'' < P_1$, but both the p' mutant and the non-mutants make zero profits. Therefore, by the (all-best) principle, there is a positive probability that the p' mutant will be imitated by all other firms.

Consider now all downward transitions of the form $\text{mon}(p) \to \text{mon}(p')$ such that $p' < p$ and $p' \geq P_1$. Note that this implies $p > P_1$. We will see that one mutation is enough for this transition to occur. Start with $\text{mon}(p)$ and consider one mutation from p to p'. After mutation, the non-mutants make zero profits, but the mutant makes a positive profit since $p' \geq P_1$. Pure imitation can lead the process to $\text{mon}(p')$, relying on the (all-best) rule if necessary.

Let us now focus on upward transitions. Consider all possible transitions of the form $\text{mon}(p) \to \text{mon}(p')$ such that $p' > p$ and $p \leq P_{n-1}$. We will see that one mutation is enough for the transition to occur. Start with mon(p) and consider one mutation from p to p'. After mutation, the mutant makes zero profits, since it loses any possible share of the market it had before, but non-mutants make losses or zero profits since $p \leq P_{n-1}$. Notice that if $p = P_{n-1}$, this transition relies on the (all-best) principle.

Consider now all possible upward transitions of the form mon(p) $\rightarrow$ mon(p') such that $p' > p$ and $p > P_{n-1}$. We will see that one mutation is not enough for this transition to occur, but two mutations suffice. Start with mon(p) and consider one mutation from p to p'. After mutation, the mutant makes zero profits, since it gets no demand, but the non-mutants make strictly positive profits since $p > P_{n-1}$. Then imitation can only lead back to mon(p). Start again with mon(p) and consider two simultaneous mutations, one to p' and another one some $p'' < p$, $p'' < P_1$. After mutation, the p'' mutant makes losses, since $p'' < P_1$, but both the p' mutant and the non-mutants make zero profits. Therefore by the (all-best) rule there is positive probability that the p' mutant will be imitated by all other firms.

Step 3. We have explored the minimal number of mutations required to leave any monomorphic state. Any tree which uses exactly this minimal number of mutations to leave each monomorphic state will be a minimal-cost tree. Consider the mon(P_1)-tree given by mon(p) $\rightarrow$ mon(P_1) for all $p \neq P_1$. These transitions have cost one for all $p \leq P_{n-1}$ and also for all $p > P_1$, but they have cost two for all p such that $P_{n-1} < p < P_1$. Note that these transitions use disjoint paths through the non-monomorphic states. This proves that mon(P_1) has a minimal-cost tree.

We will argue now that all states mon(p) such that $P_{n-1} < p < P_1$ also have minimal-cost trees. Construct a mon(p)-tree from the mon(P_1)-tree we have just constructed reversing the transition from mon(p) to mon(P_1). This new transition also needs two mutations. Therefore the new mon(p)-tree has exactly the same cost as the old mon(P_1)-tree. This proves that all the states mon(p) such that $P_{n-1} < p < P_1$ have minimal-cost trees.

Let $S = \text{mon}(]P_{n-1}, P_1])$. We have just shown that all states in S have minimal-cost trees. Consider now any other monomorphic state mon(p') not contained in S. Any mon(p')-tree includes a link from some state in S to another one out of S. Any transition of this sort requires two mutations. The only cost that can be saved in this tree with respect to the previous ones comes from the fact that no arrow leaving mon(p') is needed, but such an arrow was only at cost one. So any mon(p')-tree would require at least one more mutation than the minimal-cost trees. Thus mon(p') cannot have a minimal-cost tree. □

Remark 3.4 *The previous theorem holds for $n \geq 3$ firms. If $n = 2$, then the limit of the stationary distributions μ_ε assigns positive weight to and only to states in $\{\text{mon}(p) \mid p \geq P_1\}$. This is because all transitions downwards between monomorphic states corresponding to prices greater than or equal to P_1 take only one mutation, all transitions upwards from other monomorphic states take also only one mutation. The transitions upwards from* mon(P_1) *take only one mutation by the (all-best) principle, and the rest of the transitions take two mutations.*

Remark 3.5 *Under constant marginal costs and any number of firms, the argument in the previous remark would yield the same conclusion. Note that in this case P_1 is equal to the marginal cost.*

It is important to notice that Theorem 3.3 already provides us with some refinement of the set of Nash equilibria in the underlying static game. Dastidar [3] proves that the set of Bertrand-Nash equilibria is formed by (in our notation) all states mon(p) (symmetric Nash equilibria) such that $P_n \leq p \leq P'_n$ where P'_n is such that $\Pi(P'_n, n) = \Pi(P'_n, 1)$. Dastidar also shows that $P'_n > P_1$. Our prediction is obviously a strict subset of those. Moreover, mon(P_n) is a focal Nash equilibrium (there, price is equal to average costs) where all n firms make exactly zero profits. In our framework, the limit of the stationary distributions assigns no weight to this state as the probability of experimenting tends to zero. All states in the support of this limit yield strictly positive profits.

In any case, the prediction of Theorem 3.3 is not very sharp, i.e. a wide range of prices can arise as a result of such a simple rule of price imitation. This is a characteristic of the family of imitation rules described by the (all-best) principle. This principle allows for very naive behavior, e.g. with two mutations it is possible to lead the process almost anywhere. From any monomorphic state, think of one mutation to a very low price and a second mutation to a price only slightly higher than the first one, then there is positive probability, according to the (all-best) principle, that all other firms will imitate the second mutant without any particular good reason to do so. In the next section we want to propose other, less naive behavioral principles and study the long-run outcomes for the families of imitation rules which they describe.

4 Alternative imitation rules

In this section, we introduce alternatives to the previous imitation rules which were characterized by the requirement to satisfy the (all-best) principle. As explained above this simple principle may lead to very naive behavior. The behavioral principles proposed below provide a more sophisticated model for the imitation considerations of firms.

In what follows, we focus on the imitation of a particular type of successful behavior displayed by firms. First, we assume that firms imitate only *active* competitors among those who obtained maximal profits, i.e. firms which produced in the preceding time period because they had some customers and which obtained profits. This means that firms which did not produce at all are not considered successful. We will call this the (activity) principle.

This principle is relevant only in the states in which all firms earn zero profits, but some of them are producing and the rest are facing zero demand. The (all-best) principle required that both active and inactive firms had strictly positive probability of being imitated. In general, it seems more reasonable to imitate firms with customers rather than firms without. However, when the (activity) principle is relevant, it is more difficult to justify such type of behavior, since all firms are obtaining zero profits. Nevertheless, two types of arguments should be considered. On the one hand, firms that are not active in the market may even not be observable. On the other hand, there is no hope for the inactive firms to have positive profits, but slight changes of the market conditions could give profits to the active ones, and ultimately firms aim at positive profits.

Second, it may occur that the firms which maximize profits are precisely the ones that face no demand, and this is because all active firms, in that case, have losses due to very low prices. Thus there may be no active firms with profits. Then different ways of plausible behavior can be proposed. In the sequel we will analyze two alternative behavioral principles for this case. One will be called the (caution) principle, and the other one will be called the (frequency) principle. Under the (caution) principle, firms which are avoiding losses will refrain from revision of prices, upholding their current prices. This can be seen as if they were imitating themselves, since they are among the profit maximizers. Under the (frequency) principle, firms which find themselves in the situation of choosing between low prices which give losses and higher prices which yield zero profits will conform with the most extended behavior, i.e. the most frequent price adopted by other firms. We think these are two general and reasonable simple rules of thumb, but of course they are not the only ones. Nevertheless, the same conclusions are reached under both of them. On the one hand, these imitation principles are simple and thus keep the problem tractable. On the other hand, they describe a less naive imitation behavior.

Let us turn to the formal definitions. The set of prices announced by active firms obtaining maximal profits is given by

$$A(\boldsymbol{p}) := B(\boldsymbol{p}) \cap \{p_i \mid i \in M(\boldsymbol{p})\}. \tag{5}$$

The tie-breaking rule of equal shares implies that $A(\boldsymbol{p})$ either contains exactly one price or is the empty set. Following the first consideration of imitating active firms when possible, we define the

(activity) principle An imitation rule is said to satisfy the (activity) principle, if whenever $A(\boldsymbol{p}) \neq \emptyset$

$$Prob\{I_i(\boldsymbol{p}) \in A(\boldsymbol{p})\} = 1 \text{ for all } i.$$

The second consideration comes into play, if $A(\boldsymbol{p}) = \emptyset$. Let us first analyze the (caution) principle.

(caution) principle An imitation rule is said to satisfy the (caution) principle, if whenever $A(\boldsymbol{p}) = \emptyset$

$$Prob\{I_i(\boldsymbol{p}) = p_i\} = 1 \text{ for all } i \text{ such that } \Pi_i(\boldsymbol{p}) = 0.$$

With the refined imitation rule we can make a more precise statement about the long-run outcome as in Theorem 3.3. This is related to the following key concept.

Definition 4.1 *The set of* central prices $CP(\Gamma)$ *is*

$$CP(\Gamma) = \left[P_{\lceil \frac{n}{2} \rceil}, P_{\lceil \frac{n-1}{2} \rceil}\right] \cap \Gamma$$

This set will prove to be very resistant to experimentation, and will turn to be our long-run prediction. Notice that it reduces to the single price $P_{n/2}$ when n is even, and it is a proper interval when n is odd, because the concept of center changes with parity. The prices in this set have low risk attached in the sense that more than half of the population would have to coordinate in a joint deviation to destabilize a monomorphic situation in which all firms announce a price in this set, while it only takes half of the population or less to destabilize any other monomorphic situation and to reach one of the former. Note that if at least half of the population coordinates at these prices, they will be able to make positive profits. It is precisely the cost structure what underlies this phenomenon.

Theorem 4.2 *Let $n \geq 2$. Given any imitation rule which satisfies the (activity) principle and the (caution) principle, then in the long-run, as the probability for experimentation tends to zero, all firms announce identical prices $p \in CP(\Gamma)$, i.e.*

$$\lim_{\varepsilon \to \mathbf{0}} \mu_\varepsilon \{\text{mon}(CP(\Gamma))\} = 1.$$

Moreover, $\lim_{\varepsilon \to \mathbf{0}} \mu_\varepsilon \{\text{mon}(p)\} > 0$ *for all* $p \in CP(\Gamma)$.

As we have said before, the "precision" of the result depends to a certain extend on the fact whether there is an even or an odd number of firms. The crucial difference (which plays a role in the proof) is that an even number of firms can be divided in two groups of the same size.

Proof. We are going to apply the method of Freidlin and Wentzell [5], as introduced in the proof of Theorem 3.3, showing that only states $\text{mon}(p)$ with $p \in CP(\Gamma)$ possess minimal-cost trees.

It follows analogously to Step 1 of the proof of Theorem 3.3 that non-monomorphic states cannot have minimal-cost trees, because each arrow starting at some monomorphic state has at least cost one, but the proposed imitation rule leads from each non-monomorphic state to some monomorphic one at cost zero. Let us show the latter fact. In any non-monomorphic state $\boldsymbol{p}$, if $A(\boldsymbol{p}) \neq \emptyset$, there is positive probability that all firms will imitate the same price by the (activity) principle. If $A(\boldsymbol{p}) = \emptyset$, then by the (caution) principle non-active firms will uphold their prices, but by the definition of an imitation rule there is positive probability that all active firms will imitate some other price, thus yielding a new state where strictly less different prices than in $\boldsymbol{p}$ are observed. Applying the same reasoning iteratively we will arrive either to a monomorphic state or to a state $\boldsymbol{p}'$ such that $A(\boldsymbol{p}') \neq \emptyset$. We next determine the less costly paths between different monomorphic states.

Transitions which use different mutations. Because of the (caution) principle, any transition between two monomorphic states driven by mutations to two or more different prices will have cost higher than or equal to that of some transition between the same two states which uses only mutations to one price. Thus, we only need to study the second type of transitions. Let us show this fact. Suppose we want to study the transition $\text{mon}(p) \to \text{mon}(p')$. If the transition uses mutations to two different prices p' and p'', then the process will go by an intermediate, non-monomorphic state where at least three different prices are present. Let us call this state $\boldsymbol{q} = (p, \dots, p, p', \dots, p', p'', \dots, p'')$. If $A(\boldsymbol{q}) \neq \emptyset$, then, by the (activity) principle it has to be the case that $A(\boldsymbol{q}) = \{p'\}$ if the process has to reach $\text{mon}(p')$. But then, the same result would be achieved if only the mutations to p' occurred. If $A(\boldsymbol{q}) = \emptyset$, then the active firms are making losses. Obviously, it cannot be the case that they are announcing p' if the process has to reach $\text{mon}(p')$. If they are announcing p'', then by the definition of an imitation rule, these firms can only imitate p or p' while the non-active firms uphold their prices because of the (caution) principle. This leads to another intermediate state which could have been reached directly with the same number of mutations, but all of them to p'. Finally, if active firms are announcing p, then they will be forced to imitate either p' or p''. But then, it is clear that having all the mutants announce p' instead of p'' will give us a different transition with the same cost where the non-mutants will be forced to imitate p'.

Obviously, these arguments turn even stronger if mutations to more than two different prices are invoked. This way, we can discard the "naive transitions" where some mutant firms announce a very low price and some other mutants are imitated only because the former made huge losses.

Mutations to lower prices. Consider some fixed state $\text{mon}(p)$, $p > \min \Gamma$. The minimal number of mutations needed to reach $\text{mon}(p')$ where $\min \Gamma < p' < p$ and $P_j \leq p' < P_{j-1}$ is precisely j. This can be seen as follows.

Let us analyze how many mutations are needed to obtain the state $\text{mon}(p')$ by applying the (activity) principle. In order to make the price p' appear in the set $A(\cdot)$ it takes at most and at least j mutations. After j firms mutated to p', these firms obtain non-negative profits and thus $A(p, \dots, p, p', \dots, p') = \{p'\}$. (Note that the number of mutations needed to reach $\min \Gamma \leq p < P_n$ is equal to n.)

The (caution) principle can only be used if the set $A(\cdot)$ is empty. But such a situation can not lead to the state $\text{mon}(p')$ unless mutations to at least two different prices are invoked. It is clear that no combination of the (caution) and the (activity) principle yields less costly paths.

Mutations to higher prices. Consider some fixed state $\text{mon}(p)$, $p < \max \Gamma$ with $P_i \leq p < P_{i-1}$. The minimal number of mutations needed to reach mon(p') with $p' > p$ when starting in mon(p) is $n - i + 1$. This can be proved as follows.

It is clear that the (activity) principle cannot be applied to obtain imitation of higher prices with less than n mutations. Let us now analyze the (caution) principle. The firms announcing p only obtain losses (and thus make the set $A(\cdot)$ empty) if at least $n - i + 1$ firms move to the higher price p'. Then, the mutants will uphold their prices by the (caution) principle and the non-mutants will imitate them by definition of an imitation rule.

Minimal-cost trees. We next show that all states mon(p) with $p \in CP(\Gamma)$ have minimal-cost trees by using only the minimal-cost paths obtained above.

Connect all states mon(p') with $p' < p$ to mon(p) taking the minimal-cost path given in the "mutations to higher prices" case. Connect all states mon(p') with $p' > p$ according to the following rule. Each price $p' > p$ is connected to the next lower price in Γ by taking the minimal-cost path given in the "mutations to lower prices" case.

If n is even, then the interval consists only of one element. If n is odd, then one has to show in addition that all states mon(p) with $p \in CP(\Gamma)$ have a minimal tree of the same cost. But this is due to the fact that transitions between different monomorphic states corresponding to this set are of cost $j = \lceil \frac{n}{2} \rceil = (n+1)/2$ to lower prices, and that they are also of cost $n - i + 1 = n - (n+1)/2 + 1 = (n+1)/2$ to higher prices.

We finally show that all trees of the remaining monomorphic states are of strictly higher costs. Let mon(p') be some monomorphic state not contained in mon($CP(\Gamma)$). On the one hand, each mon(p') tree has to contain a path from some state in mon($CP(\Gamma)$) which gives an additional cost of at least $n - \lceil \frac{n}{2} \rceil + 1 = \lceil (n+1)/2 \rceil$. On the other hand, one saves the cost of some path by not having to connect p', but this is at most $\lceil (n-1)/2 \rceil$. Hence this less costly mon(p')-tree is of strictly higher cost. Note that all minimal-cost paths are non-intersecting. □

Remark 4.3 *Let us analyze the case of constant marginal costs.*

Following the procedure of the previous theorem, we do a quick consideration. Transitions from states mon(p) *with* $p \geq P_1$ *to states* mon(p') *with* $p' < P_1$ *have cost* n*, but in the other direction they have cost one. Downward mutations in* $[P_1, \max \Gamma]$ *have cost one. Upward mutations have cost* n*. This yields*

$$\lim_{\varepsilon \to 0} \mu_\varepsilon \{\text{mon}(P_1)\} = 1.$$

giving a quite clear-cut prediction which simply reproduces the Bertrand-Nash equilibrium.

Theorem 4.2 provides us with a significant refinement of both the set of Nash equilibria and the prediction of Theorem 3.3. The prediction of central prices not only yields strictly positive profits for all firms, but also it is quite clear-cut, reducing to a single price in the case of an even number of firms. Also, as the previous remark shows, this prediction reduces to the classical Bertrand-Nash equilibrium in the particular case of constant marginal costs.

We are going to consider now a second possible behavioral principle for the case when there are no active firms with positive profits in the market. This new principle specifies that, in this case, instead of upholding their prices, non-active firms will try to mimic some market price, i.e. one of the most frequent price among those which yield maximal profits. The set of such prices is

$$F(\boldsymbol{p}) := \{p_k \mid k \in \underset{m=1,\dots,n}{\operatorname{argmax}} |\{j \mid p_j = p_m,\ p_m \in B(\boldsymbol{p})\}|\}. \tag{6}$$

The corresponding assumption on the imitation behavior is as follows.

(frequency) principle An imitation rule is said to satisfy the (frequency) principle, if whenever $A(\boldsymbol{p}) = \emptyset$

$$Prob\{I_i(\boldsymbol{p}) \in F(\boldsymbol{p})\} = 1 \text{ for all } i$$

and

$$Prob\{I_i(\boldsymbol{p}) = p_k\} > 0 \text{ for all } p_k \in F(\boldsymbol{p}) \text{ and for all } i.$$

Note that $A(\boldsymbol{p}) = \emptyset$ implies $F(\boldsymbol{p}) \neq \emptyset$ and thus the above principle is well defined.

If this principle is used instead of the (caution) one, then we obtain exactly the same prediction $CP(\Gamma)$, thus revealing that the prediction of central prices is robust with respect to the specification of the behavior in the situation $A(\boldsymbol{p}) = \emptyset$.

Theorem 4.4 *Let $n \geq 2$. Given any imitation rule which satisfies the (activity) principle and the (frequency) principle, then in the long-run, as the probability for experimentation tends to zero, all firms announce identical prices $p \in CP(\Gamma)$, i.e.*

$$\lim_{\varepsilon \to 0} \mu_\varepsilon\{\mathrm{mon}(CP(\Gamma))\} = 1.$$

Moreover, $\lim_{\varepsilon\to 0} \mu_\varepsilon\{\mathrm{mon}(p)\} > 0$ *for all* $p \in CP(\Gamma)$.

Proof. It follows analogously to Step 1 of the proof of Theorem 3.3 that non-monomorphic states cannot have minimal-cost trees, so attention can be restricted again to monomorphic states.

The only change with respect to Theorem 4.2 is that, now, it is possible to construct transitions $\text{mon}(p) \to \text{mon}(p')$ by having one mutant announce a very low price p'', obtaining the whole demand and making huge losses, and also having half of the remaining firms mutate to p' and making zero profits. Such transitions cost $\lceil (n+1)/2 \rceil$. Taking them into account and following the lines of the previous proof, it is straightforward to see that transitions to lower prices p' with $P_j \le p' < P_{j-1}$ cost $\min\{j, \lceil (n+1)/2 \rceil\}$ and transitions to higher prices from p such that $P_i \le p < P_{i-1}$ cost $\min\{n-i+1, \lceil (n+1)/2 \rceil\}$. The minimal-cost trees are then constructed exactly as in the previous proof. □

Remark 4.5 *Again, let us make a quick consideration of the case of constant marginal costs. Transitions from states* $\text{mon}(p)$ *with* $p \ge P_1$ *to states* $\text{mon}(p')$ *with* $p' < P_1$ *have cost* $\lceil (n+1)/2 \rceil$*, but in the other direction they have cost one. Downward mutations in* $[P_1, \max \Gamma]$ *have cost one. Upward mutations cost* $\lceil (n+1)/2 \rceil$*. This yields again the Bertrand-Nash equilibrium*

$$\lim_{\varepsilon \to 0} \mu_\varepsilon \{\text{mon}(P_1)\} = 1.$$

5 Central prices and Walrasian behavior

In order to understand the concept of central prices better, we want to provide with some examples and compare our prediction with the Walrasian price. Dastidar [4] shows that the Walrasian price is always in the set of Nash equilibria when costs are convex. We would like to know to what extent our prediction of central prices departs from the Walrasian one.

To this purpose, we first prove a proposition which states that under quadratic costs and arbitrary demand central prices essentially coincide with the Walrasian one. However there is not a general relation between these two concepts and we provide with two counterexamples which show that central prices can be either lower or greater than the Walrasian price, essentially depending on the curvature of the cost function.

Example 5.1 *Consider a market with linear demand and quadratic costs.*

$$\begin{aligned} D(p) &:= 10 - p \\ C(q) &:= \frac{1}{2} \cdot q^2 \end{aligned}$$

Elementary calculations yield $P_1 = 10/3$, $P_2 = 2$, $P_3 = 10/7$.

Then, if $n = 3$ the Walrasian price $P^w = 5/2 \in [P_2, P_1]$, i.e. the Walrasian price is a central price. Also, if $n = 5$, then $P^w = 5/3 \in [P_3, P_2]$, which is the set of central prices in this case.

If n is even, there is only one central price, $P_{n/2}$. Consider, for the sake of illustration, the case $n = 4$. In this case, the Walrasian price coincides exactly with the central price, i.e. $P^w = 2 = P_2$.

The next proposition shows that this property is independent of the particular demand function, provided that costs are quadratic.

Proposition 5.2 *For every quadratic cost function, i.e. $C(q) = c \cdot q^2 + b \cdot q$, the Walrasian price is a central price. More precisely:*

(i) If n is even, then $P^w = P_{n/2}$

(i) If n is odd, then $P^w \in [P_{(n+1)/2}, P_{(n-1)/2}]$

Proof. P^w is a root of the function $p - C'\left(\frac{D(p)}{n}\right)$. However, as this function is strictly increasing when costs are convex, it follows that P^w is also its unique root. Under quadratic costs

$$P^w - C'\left(\frac{D(P^w)}{n}\right) = 0 \iff P^w - 2c \cdot \left(\frac{D(P^w)}{n}\right) - b = 0.$$

Notice that $P_{n/2}$ can be defined, even if n is odd, by the equation

$$\begin{aligned} & \Pi(P_{\frac{n}{2}}, n/2) \equiv P_{\frac{n}{2}} \cdot \frac{D(P_{\frac{n}{2}})}{n/2} - C\left(\frac{D(P_{\frac{n}{2}})}{n/2}\right) = 0 \\ \iff \quad & P_{\frac{n}{2}} \cdot \left(\frac{D(P_{\frac{n}{2}})}{n/2}\right) - c\left(\frac{D(P_{\frac{n}{2}})}{n/2}\right)^2 - b\left(\frac{D(P_{\frac{n}{2}})}{n/2}\right) = 0 \\ \iff \quad & P_{\frac{n}{2}} - 2c \cdot \left(\frac{D(P_{\frac{n}{2}})}{n}\right) - b = 0 \end{aligned}$$

where the last equivalence holds because $D(P_{n/2}) \neq 0$.

Since P^w is a unique solution of this equation, $P^w = P_{n/2}$. If n is even, this completes the proof. If n is odd, $P^w = P_{n/2} \in [P_{(n+1)/2}, P_{(n-1)/2}]$. □

This result suggests that, in the particular case of quadratic costs, marginal-cost pricing is very robust in evolutionary terms, i.e. it takes many mutations to destabilize it. It is interesting to find that, under quadratic costs there is a reconciliation of quantity and price competition. An evolutionary model of quantity competition also based on imitation of successful behavior has been studied by Vega-Redondo [11].

However, this property depends crucially on the fact that costs are quadratic. With different cost functions, one can build counterexamples where $P^w > P_{n/2}$ or $P^w < P_{n/2}$.

Example 5.3 *Consider an economy with the demand and cost functions*

$$\begin{aligned} D(p) &:= 10 - p \\ C(q) &:= q^3 \end{aligned}$$

We will show that, in this example, the Walrasian price is essentially lower than the central prices.

1. $P^w = C'(q^w) = 3 \cdot (q^w)^2$ *and* $q^w = D(P^w)/n = (10 - P^w)/n$.

   The only solution of this system of equations which is lower than the maximum possible price $P^{\max} = 10$ *is the lowest root of the polynomial*

   $$f(x) := x^2 - \left(20 + \frac{n^2}{3}\right) \cdot x + 100$$

   i.e. $P^w(n) = 10 - \frac{n}{6} \cdot (\sqrt{n^2 + 120} - n)$.

2. $\Pi(P_k, k) = P_k \cdot \frac{D(P_k)}{k} - C\left(\frac{D(P_k)}{k}\right) = 0 \Leftrightarrow P_k \cdot \left(\frac{10-P_k}{k}\right) - \left(\frac{10-P_k}{k}\right)^3 = 0$.

   The only solution of this equation which is lower than the maximum possible price $P^{\max} = 10$ *is the lowest root of the polynomial*

   $$g_k(x) := x^2 - \left(20 + k^2\right) \cdot x + 100$$

   i.e. $P_k = 10 - \frac{k}{2} \cdot (\sqrt{k^2 + 40} - k)$.

3. *The above expressions imply that* $f(x) < g_{n/2}(x)$ *for all* $0 < x < 10$. *In particular,* $f(P_{n/2}) < g_{n/2}(P_{n/2}) = 0$. *Thus* $P^w(n) < P_{n/2}$ *for all* n, *because* $P^w(n)$ *is the lowest root of* $f(x)$. *This implies, in particular, that our prediction is strictly greater than the Walrasian price in the case where* n *is even.*

4. *Analogously, with* $0 < x < 10$, $f(x) < g_{(n+1)/2}(x)$ *for all* $n \geq 6$, *which implies that in the case where* n *is odd* $P^w(n) < P_{\lceil n/2 \rceil}$ *for all* $n \geq 7$. *One can also check that* $P^w(n) \in [P_{\lceil n/2 \rceil}, P_{\lceil (n-1)/2 \rceil}]$ *for all* $n \leq 5$.

Example 5.4 *Consider an economy with the demand and cost functions*

$$\begin{aligned} D(p) &:= 10 - p \\ C(q) &:= q^{\frac{3}{2}} \end{aligned}$$

In this example, it is not difficult to see that $P^w(n) = \frac{9}{8}(\sqrt{1 + (160/9)n} - 1)$ *and, if* n *is even,* $P_{n/2} = \sqrt{1 + 20n} - 1$. *Then, e.g. with* $n = 6$, *we have that the Walrasian price exceeds the central one:* $P^w \cong 11.5483 > 11 = P_{n/2}$.

All the previous examples use power cost functions, $C(q) = c \cdot q^a$. In fact, it is not difficult to fully characterize the relationship between the Walrasian price for n firms, P^w, and the central price $P_{n/2}$ (which can also be defined if n is odd). Figure 2 depicts both Walrasian and central price for a particular case.

Proposition 5.5 *Under power costs, i.e.* $C(q) = c \cdot q^a$, *with* $a > 1$ *one has:*

(i) If $a = 2$, *then* $P^w = P_{n/2}$

(ii) If $a > 2$, *then* $P^w < P_{n/2}$

(iii) If $a < 2$, *then* $P^w > P_{n/2}$

Proof. P^w is the unique root of the strictly increasing function $f(p) = p - C'(D(p)/n) = p - ac \cdot (D(p)/n)^{a-1}$. $P_{n/2}$ is defined through

$$\Pi\left(P_{\frac{n}{2}}, \frac{n}{2}\right) = P_{\frac{n}{2}} \cdot \frac{D(P_{\frac{n}{2}})}{n/2} - C\left(\frac{D(P_{\frac{n}{2}})}{n/2}\right) = 0$$

Since $D(P_{n/2}) \neq 0$, it follows that $P_{n/2} = c \cdot 2^{a-1} \cdot (D(p)/n)^{a-1}$. Thus $f(P_{n/2}) = c \cdot (2^{a-1} - a) \cdot (D(p)/n)^{a-1}$.

If the last expression is positive, $P_{n/2}$ has to be greater than P^w, and vice versa. But the sign of this expression depends exclusively on the sign of the function $g(a) = 2^{a-1} - a$, which is strictly negative for all $a \in (1, 2)$, strictly positive for all $a \in (2, +\infty)$, and zero if $a = 2$. This completes the proof. □

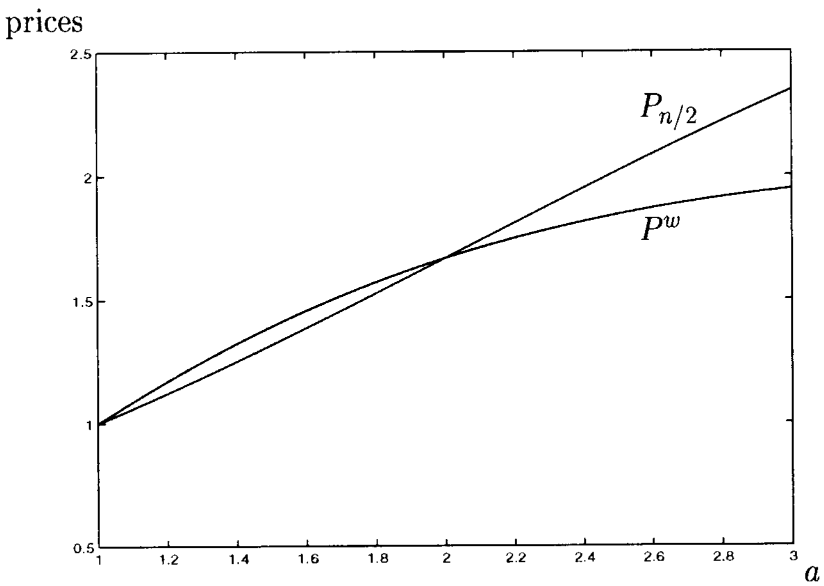

Figure 2:

Walrasian price P^w and central price $P_{n/2}$ for$D(p) = 10 - p, n = 10$, and power costs $C(q) = q^a$

We should note that extending the analysis of the previous proposition to the case of linear costs, the Walrasian price again coincides with the central one, since then $g(1) = 0$. Therefore, this proposition is consistent with the previous result (see Remark 4.3) for the case of constant returns to scale, given that $P_1 = P_{n/2}$. In fact, both the central price $P_{n/2}$ and the Walrasian one will be very close when costs are almost linear or almost quadratic, but they will differ (with the latter being higher) under intermediate returns to scale.

How should we interpret this last result? If returns to scale are not too decreasing, then there is room for even tougher competition. In other words, as long as the cost structure allows for it, it seems that competition as tough as possible will be evolutionarily successful.

6 Conclusions

The introduction of a dynamics of economic natural selection based on imitation and experimentation provides us with new insights in the theory of price competition.

First, very low prices are very unstable in an evolutionary sense, because unilateral deviations to higher prices would leave active firms making losses. In particular this is the case of the price equal to average cost. This instability is reflected by the fact that, even under very simple imitation rules, such low prices have zero weight in the long-run prediction.

Second, if one introduces sensible refinements of the imitation rules, it is possible to single out a price (or a very narrow set of prices) that has the property of being central in the sense that it would take more than half of the firms present in the market to destabilize it.

This last result resolves to a great extent the problem of the multiplicity of equilibria present in models of oligopolistic markets for homogeneous goods with price competition and decreasing returns to scale analyzed from a classical game-theoretic point of view. Moreover, it allows us to reconsider the question of "paradoxical," competitive price-setting behavior in such markets.

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