FIXED AGENDA SOCIAL CHOICE CORRESPONDENCES*

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ABSTRACT

In this paper we analyze the explicit representation of fixed agenda social choice correspondence under different rationality assumptions (independence, neutrality, monotonicity, ...). It is well known in the literature that, under some of these assumptions, the existence of dictators, oligarchies or individuals with veto power can be proven ([7] and [10]); but no information about the social choice set is obtained. We now establish a relationship between the social choice set and the individual maximal sets which explicitly describes a fixed agenda social choice correspondence that satisfies these rationality assumptions. Some of the results in [2] about the explicit representation of social decision functions are then translated and reinterpreted in the fixed agenda framework.

Key words: Fixed Agenda; Choice Correspnodences; Explicit Representation.

1 Introduction

One of the main problems analyzed in Social Choice Theory is that of determining the social choice set from the different opinions that individuals have about the set of feasible alternatives. These opinions may be considered in terms of individual binary preference relations or in terms of individual choice correspondences. In the literature, there are different approaches to analyzing this problem, depending on the point of view considered. One approach analyzes the problem of aggregation of individual preferences into a social preference relation (*social decision functions*) by imposing different rationality assumptions. Once we know the social preference relation, the social choice problem is easily solved by maximizing it. Nevertheless, although the assumptions imposed are considered natural each one by itself, in most of the cases they produce impossibility results when considered together. The most famous and pioneering paper in this line is that of Arrow [3] in which the existence of a dictator is proven when the social decision function satisfies *Universal Domain, Independence of Irrelevant Alternatives, Weak Pareto Principle* and its *codomain* is restricted to being the family of *weak orders* (complete, reflexive and transitive binary relations). From Arrow's work, many papers have dealt with the problem of analyzing whether or not this result remains valid, when the different assumptions of the *arrovian framework* were weakened: by restricting the domain of preferences, by weakening the assumptions of Pareto Optimality and independence, or by weakening the kind of social relation required (semiorder, interval order, quasiorder, acyclic relation). With respect to this last possibility, we must mention the work of Gibbard [8], who proves the existence of an oligarchy when requiring a quasitransitive social binary relation, and that of Blair and Pollak [4], who prove the existence of individuals with veto power by considering acyclic social binary relations.

A different approach analyzes the aggregation of individual preferences into a social choice set *(social choice correspondences)*. That is, given a universal set of feasible alternatives, and given the individual preferences, we can determine the social choice set. Within this framework, one can distinguish between the *variable agenda* case and that of the *fixed agenda*. In the former case, every nonempty subset of the universal set could be presented for choice; so, it is possible to define the base relation R_b ($xR_by \Leftrightarrow x \in C(\{x,y\}, R_1, ..., R_n)$) which allows us to translate easily most of the results obtained in the context of social decision functions. In the case of a fixed agenda, however, there is a unique feasible set which is fixed a priori (given the restrictions of the particular problem which being analyzed), and the base relation cannot be defined. In spite of this, however, a one-to-one correspondence can be defined between the fixed agenda case and the context of aggregation of preferences (see [7] and [10]). These equivalence results allow us to establish the existence of dictators, oligarchies, or individuals with veto power in the fixed agenda framework.

We now focus on the case of fixed agenda social choice correspondences and analyze the implications of the existence of individuals with different degrees of power (dictator, oligarchy, veto power) over the social choice set. We study, in particular, the relationship between the maximal sets of these individuals and the social choice set under certain rationality assumptions, and we analyze whether we can perfectly describe the social choice set in terms of these maximal elements or not. Thus, if we know that, under certain assumptions, there exists a dictator, it is clear that the social choice set will be a subset of the maximal set of such an individual. We analyze the conditions under which the equality between both sets holds. The case of individuals with veto power, or oligarchies, is more complicated and, in general, it could be that the social choice set does not contain any of the maximal elements of these individuals. In order to obtain a little more information about the social choice set in these cases, it would be necessary to add other assumptions to those which ensure the existence of individuals with veto power or oligarchies. In particular, we introduce a like-*Condorcet consistency property* which proves to be very useful to our proposals.

2 Preliminaries

Let X be the universal and finite set of alternatives, fixed a priori, $\mid X \mid > 2$, and let N be the finite set of individuals. Each individual i is endowed with a weak order R_i on X (reflexive, transitive and complete binary relation). Let $W(X)$ denote the family of weak orders over X, while $A(X)$ denotes the family of acyclic binary relations. When working with binary relations, we can start with a complete and reflexive binary relation R which induces two natural associated relations (strict preference and indifference) defined as follows,

$$\begin{aligned} xPy &\Leftrightarrow xRy \textit{ and } not[yRx], \\ xIy &\Leftrightarrow xRy \textit{ and } yRx. \end{aligned}$$

Alternatively, we can start with an asymmetric binary relation P and define the complete and reflexive binary relation R,

$$xRy \Leftrightarrow not[yPx],$$

or the indifference relation I,

$$xIy \Leftrightarrow not[xPy] \ and \ not[yPx].$$

For our purposes, both ways of proceeding are equivalent, and both will be used indistinctly throughout the paper. A *profile* is any n-tuple of weak orders, $u = (R_1, ..., R_n) \in W^n(X)$, and for every subset of alternatives $Y \subset X$ we denote the restriction of u to Y by $u : Y$. Moreover, given a subset of alternatives Y and a binary relation R, we define the relation R^Y as follows:

$$\begin{aligned} if\ x, y &\in Y \text{ then } [xR^Y y \Leftrightarrow xRy], \\ if\ x, y &\notin Y \text{ then } xI^Y y, \\ if\ x &\in Y,\ y \notin Y \text{ then } xP^Y y, \end{aligned}$$

and we denote $u^Y = (R_1^Y, R_2^Y, ..., R_n^Y)$. In order to simplify the notation, we henceforth denote $u^{\{x,y\}} = u^{xy}$.
Given two complete binary relations R_1 and R_2, the relations $R^* = R_1 * R_2$ and $R^\cap = R_1 \cap R_2$ are defined as follows,

$$xP^*y \Leftrightarrow [xP_1y \quad or \quad (xI_1y \quad and \quad xP_2y)];$$

$$xP^\cap y \Leftrightarrow [xP_1y \quad and \quad xP_2y].$$

These definitions are immediately extended to any finite family of binary relations. Finally, given a profile $u = (R_1, ..., R_n)$, and a subset $A \subset X$ we denote by

$$M_j^u(A) = \{a \in A \mid aR_jy \quad \forall y \in A\},$$

that is, the set of maximal elements of individual j on A; whenever the whole space X is considered, we denote $M_j^u(X)$ by M_j^u.

We formally present the notion of a fixed agenda social choice correspondence, as well as the notion of a social decision function, in the next definitions.

Definition 1 *A* ***fixed agenda social choice correspondence (fixed agenda SCC)*** *is a correspondence* $C : W^n(X) \to X$ *that selects a nonempty subset of alternatives for each profile of individual preferences.*

Definition 2 *A* ***social decision function (SDF)*** $F : W^n(X) \to A(X)$ *is a mapping that associates an acyclic binary relation to each profile of individual preferences.*

In order to simplify the notation, we denote $F(u) = R_u$, while P_u and I_u respectively denote the associated social strict preference and social indifference relations.

Finally, we present the notions of a dictator, an oligarchy and an individual with veto power, as well as those of hierarchies, in the fixed agenda framework.

Definition 3 *Given a fixed agenda SCC,* $C : W^n(X) \to X$, *it is said that:*

i) individual $i \in N$ *is a* ***dictator*** *if for every* $x, y \in X$ *and* $u \in W^n(X)$ *if* xP_iy, *then* $y \notin C(u)$;

ii) individual $i \in N$ *has* ***veto power***[1], *if for every* $x, y \in X$ *and* $u \in W^n(X)$ *if* xP_iy, *then* $C(u^{xy}) \neq \{y\}$;

iii) subset $J \subset N$ *is an* ***oligarchy*** *if for every* $x, y \in X$ *and* $u \in W^n(X)$ *if* xP_iy $\forall i \in J$ *then* $y \notin C(u)$. *Furthermore, each individual in* J *has veto power.*

Definition 4 *Given a fixed agenda SCC,* $C : W^n(X) \to X$, *it is said that:*

i) ***subset*** $J = \{i_1, i_2, ..., i_j\} \subset N$ *is* ***a hierarchy of dictators*** *if individual* i_1 *is a dictator and for every* $x, y \in X$ *and* $u \in W^n(X)$, *if for all* $p = 1, 2, ..., j-1$ $[xI_{i_k}y$ $\forall k = 1, 2, ..., p$ *and* $xP_{i_{p+1}}y]$ *then* $y \notin C(u)$. *In general, it is said that there exists a* ***hierarchy of dictators*** *if there is a permutation* $\sigma : N \to N$ *such that subset* $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ *is a hierarchy of dictators.*

[1]The notion of veto power we use here corresponds to the one introduced in [10] for fixed agenda SCCs and which is proven to be equivalent, under standard assumptions, to the usual notion of veto power for SDFs.

ii) ***subset*** $J = \{i_1, i_2, ..., i_j\} \subset N$ *is* ***a hierarchy of individuals with veto power*** *if individual* i_1 *has veto power and for every* $x, y \in X$ *and* $u \in W^n(X)$ *if for all* $p = 1, 2, ..., j-1$ $[xI_{i_k}y\ \forall k = 1, 2, ..., p$ *and* $xP_{i_{p+1}}y]$, *then* $C(u^{xy}) \neq \{y\}$. *In general, it is said that there exists a* ***hierarchy of veto power*** *if there is a permutation* $\sigma : N \to N$ *such that subset* $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$ *is a hierarchy of individuals with veto power.*

iii) ***a family of subsets*** $\{J_1, J_2, ..., J_r\}$ *such that* $J_t \subset N\ \forall t = 1, 2, ..., r$ *is a* ***hierarchy of oligarchies*** *if* J_1 *is an oligarchy and for every* $x, y \in X$ *and* $u \in W^n(X)$, *if for all* $i = 1, 2, ..., r-1$ $[xI_ky\ \forall k \in J_1 \cup J_2 \cup ... \cup J_i$ *and* $xP_ky\ \forall k \in J_{i+1}]$, *then* $[y \notin C(u)$ *and each individual of* J_{i+1} *has veto power over the pair* $x, y]$.

3 Impossibility results in fixed agenda SCCs.

In this section we present some impossibility results (existence of dictators, oligarchies and individuals with veto power) in the context of fixed agenda SCCs. In the next section, we will discuss what these results imply with regard to the explicit representation of fixed agenda SCCs.

Early impossibility results in the fixed agenda context, are those of Denicolò [7], that prove the existence of dictators and oligarchies under independence and Pareto Optimality assumptions. In order to do so, he introduces two different notions of independence (*Independence* and *Weak Independence)* which are as follows:

Axiom 1 ***Independence:*** $\forall x, y \in X, \forall u \in W^n(X)$, *if* $x \in C(u)$, $y \notin C(u)$ *and* $u : \{x, y\} = v : \{x, y\}$, *then* $y \notin C(v)$.

Axiom 2 ***Weak Independence:*** $\forall x, y \in X, \forall u \in W^n(X)$, *if* $C(u) = \{x\}$ *and* $u : \{x, y\} = v : \{x, y\}$, *then* $y \notin C(v)$.

Furthermore, in order to obtain the impossibility results, he uses the *Weak Pareto Optimality* axiom.

Axiom 3 ***Weak Pareto Optimality:*** $\forall x, y \in X, \forall u \in W^n(X)$, *if* xP_iy $\forall i \in N$ *then* $y \notin C(u)$.

Denicolò [7] proves that the conjunction of *Independence* and *Weak Pareto Optimality* implies the existence of a dictator; while the conjunction of *Weak Independence* and *Weak Pareto Optimality* implies the existence of an oligarchy. The key to obtaining these results, is to establish a one-to-one correspondence between fixed agenda SCCs satisfying *Independence* (*Weak Independence*) and *Weak Pareto Optimality* and *transitive (quasitransitive)* SDFs satisfying the well known conditions of *Independence of Irrelevant Alternatives* and *Weak Pareto Principle.*

> *Independence of Irrelevant Alternatives (IIA)*: $\forall x, y \in X$, $\forall u, v \in W^n(X)$ if $u : \{x, y\} = v : \{x, y\}$, then $xR_u y \Leftrightarrow xR_v y$.
>
> *Weak Pareto Principle*: $\forall x, y \in X, \forall u \in W^n(X)$, if $xP_i y$ $\forall i \in N$, then $xP_u y$.

In [10], Sánchez and Peris introduce a weaker independence assumption (*Pseudo-Independence*) in order to obtain results on the existence of individuals with veto power in the fixed agenda context.

Axiom 4 ***Pseudo-Independence:*** $\forall x, y \in X, \forall u \in W^n(X)$, *if* $C(u) = \{x\}$ *and* $u : \{a, y\} = v : \{a, y\}$ $\forall a \in A_y(u) \cup \{x\}$ *where* $A_y(u) = \{w \in X \mid y \notin C(u^{yw})\}$, *then* $y \notin C(v)$.

In [10] it is proven that there exists a one-to-one correspondence between fixed agenda SCCs satisfying *Pseudo-Independence* and *Weak Pareto Optimality* and *acyclic SDFs* satisfying *Independence of Irrelevant Alternatives* and *Weak Pareto Principle.*

The rest of the present section is devoted to obtaining impossibility results that we will use in the following section in order to obtain explicit representations of fixed agenda SCCs. First, we introduce the following Pareto assumptions that will be used throughout the paper.

Axiom 5 ***Strong Pareto Optimality:*** $\forall x, y \in X, \forall u \in W^n(X)$, *if* $xR_i y$ $\forall i \in N$ *and there exists* $j \in N$ *such that* $xP_j y$, *then* $y \notin C(u)$.

Axiom 6 ***Pareto Indifference:*** $\forall x, y \in X, \forall u \in W^n(X)$, *if* $xI_i y$ $\forall i \in N$, *then* $[x \in C(u) \Leftrightarrow y \in C(u)]$.

These properties are the immediate translations, to the fixed agenda framework, of the corresponding ones for SDFs [6].

Strong Pareto Principle: $\forall x, y \in X, \forall u \in W^n(X)$, if xR_iy $\forall i \in N$ and there exists $j \in N$ such that xP_jy, then xP_uy.

Pareto Indifference: $\forall x, y \in X, \forall u \in W^n(X)$, if xI_iy $\forall i \in N$, then xI_uy.

Theorem 1 *If $C : W^n(X) \to X$ is a fixed agenda SCC satisfying* ***Independence*** *and* ***Strong Pareto Optimality****, then there exists a hierarchy of dictators.*

Proof. If C is a fixed agenda SCC satisfying the required axioms, we define the associated SDF as follows,

$$xP_uy \Leftrightarrow C(u^{xy}) = \{x\}.$$

We start by proving that it is a *transitive* binary relation. Consider $x, y, z \in X$ and $u \in W^n(X)$ such that xR_uy, yR_uz; that is, $C(u^{xy}) \neq \{y\}$ and $C(u^{yz}) \neq \{z\}$. By contradiction, assume that $C(u^{xz}) = \{z\}$. On the one hand, and by applying *Strong Pareto Optimality* we know that $C(u^{xyz}) \subset \{x, y, z\}$ and on the other hand, by *Independence,* that $x \notin C(u^{xyz})$. But then, if $y \in C(u^{xyz})$, by applying *Independence* we conclude that $x \notin C(u^{xy})$, a contradiction; and if $C(u^{xyz}) = \{z\}$, by *Independence* we obtain that $y \notin C(u^{yz})$, a contradiction. Therefore we can conclude that xR_uz.

Next, we will prove that the SDF satisfies *Independence of Irrelevant Alternatives* and *Strong Pareto Principle.*

1)*Independence of Irrelevant Alternatives:* This is obviously satisfied because of the definition of the SDF: if $u, v \in W^n(X)$ and $u : \{x, y\} = v : \{x, y\}$, then

$$xP_uy \Leftrightarrow \{x\} = C(u^{xy}) = C(v^{xy}) \Leftrightarrow xP_vy.$$

2)*Strong Pareto Principle:* Consider $x, y \in X$ and $u \in W^n(x)$ such that xR_iy $\forall i \in N$, and there exists $j \in N$ such that xP_jy. Then, by *Strong Pareto Optimality,* in particular we have $y \notin C(u^{xy})$. Therefore, since this assumption also implies that $C(u^{xy}) \subset \{x, y\}$ we obtain $C(u^{xy}) = \{x\}$, that is xP_uy.

So, since the SDF is *transitive* and satisfies *Independence of Irrelevant Alternatives* and *Strong Pareto Principle,* we can ensure the existence of a hierarchy of dictators for the SDF (see [6]). Finally, it is now easy to prove, by the way in which the SDF has been defined, that it is a hierarchy of dictators for the fixed agenda SCC. ■

Aleskerov and Vladimirov [2], by making use of different assumptions, obtain a representation result for SDFs by means of a *hierarchical operator.* In fact, they prove that the power is distributed among a *subset* of individuals of the society, *which is a hierarchy of dictators.* This result can be also obtained in the fixed agenda framework. First, we introduce the different assumptions needed to establish the results, by starting with those relative to SDFs.

Monotonicity: $\forall x, y \in X, \forall u = (R_1, ..., R_n), v = (R'_1, ..., R'_n) \in W^n(X)$ if $[xP_iy \Rightarrow xP'_iy]$ and $[xI_iy \Rightarrow xR'_iy]$ then $[xP_uy \Rightarrow xP_vy]$.

Neutrality: For any $\sigma : X \to X$, *permutation of* X and for any $u \in W^n(X)$ if we denote by $\sigma(u) = (\sigma(R_1), ..., \sigma(R_n))$[2], then $[xR_{\sigma(u)}y \Leftrightarrow \sigma^{-1}(x)R_u\sigma^{-1}(y)]$.

Negative Pareto: $\forall x, y \in X$,if $xR_iy \quad \forall i \in N$, then xR_uy.

Positive non-imposedness: $\forall x, y \in X, \exists u \in W^n(X)$ *such that* xP_uy.

Negative non-imposedness: $\forall x, y \in X, \exists u \in W^n(X)$ *such that* yR_ux.

In particular, Aleskerov and Vladimirov's result [2] states that any transitive SDF satisfying the previous assumptions as well as *Independence of Irrelevant Alternatives,* can be represented by means of a *hierarchy operator* which is defined as follows: there exists a group of individuals J, such that (if we index it by $J = \{1, 2, ..., j\}$), the SDF can be expressed as follows,

$$R_u = *_{i=1}^{j} R_i.$$

The corresponding properties that we will use to state our result in the fixed agenda framework are listed below.

Axiom 7 ***Monotonicity:*** $\forall x, y \in X, \forall u = (R_1, ..., R_n), v = (R'_1, ..., R'_n) \in W^n(X)$ *if* $C(u) = \{x\}$, $[xP_iy \Rightarrow xP'_iy]$ *and* $[xI_iy \Rightarrow xR'_iy]$, *then* $y \notin C(v)$.

Axiom 8 ***Neutrality:*** $\forall \sigma$ *permutation of* $X, \forall u \in W^n(X)$, *then* $\sigma[C(u)] = C(\sigma(u))$.

Axiom 9 ***Negative Pareto:*** $\forall x, y \in X, \forall u = (R_1, ..., R_n) \in W^n(X)$ *if* $xR_iy \; \forall i \in N$ *then* $[y \in C(u) \Rightarrow x \in C(u)]$.

[2]Given a binary relation, R, defined over the set of alterantives X and a permutation of X, $\sigma : X \to X$, we define the binary relation $\sigma(R)$ as follows:

$$x\sigma(R)y \Leftrightarrow \sigma^{-1}(x)R\sigma^{-1}(y).$$

Axiom 10 ***Non-Imposedness:*** *$\forall x \in X, \exists u \in W^n(X)$ such that $C(u) = \{x\}$.*

In the next lemma we prove the existence of a one-to-one correspondence between transitive SDFs and fixed agenda SCCs satisfying the above-mentioned assumptions.

Lemma 1 *There exists a transitive SDF satisfying* ***Independence of Irrelevant Alternatives, Monotonicity, Neutrality, Negative Pareto, Positive Non-imposedness*** *and* ***Negative Non-imposedness*** *if and only if there exists a fixed agenda SCC satisfying* ***Independence, Monotonicity, Neutrality, Negative Pareto*** *and* ***Non-imposedness.***

Proof. Let C be a fixed agenda SCC satisfying the required axioms, and consider the associated SDF as in Theorem 1, that is,

$$xP_u y \Leftrightarrow C(u^{xy}) = \{x\}.$$

We are going to prove that it satisfies all of the properties mentioned above (we omit the proof of *Independence of Irrelevant Alternatives* since it is exactly the same as in Theorem 1).

1)*Monotonicity*: Let $x, y \in X, u = (R_1, ..., R_n), v = (R'_1, ..., R'_n) \in W^n(X)$ such that $[xP_i y$ implies $xP'_i y]$, and $[xI_i y$ implies $xR'_i y]$. If $xP_u y$, then $C(u^{xy}) = \{x\}$, so since the fixed agenda SCC satisfies *Monotonicity,* we obtain $a \notin C(v^{xy}),\ \forall a \in X - \{x\}$, so $C(v^{xy}) = \{x\}$,that is, $xP_v y$.

2)*Neutrality: Let σ be a permutation of $X, u \in W^n(X)$,* then

$$xR_{\sigma(u)}y \Leftrightarrow C(\sigma(u)^{xy}) \neq \{y\},$$

but, from *Neutrality* of the fixed agenda SCC, we know that

$$C(\sigma(u)^{xy}) = \sigma(C(u^{\sigma^{-1}(x)\sigma^{-1}(y)})),$$

therefore we obtain

$$\sigma(C(u^{\sigma^{-1}(x)\sigma^{-1}(y)})) \neq \{y\} \Leftrightarrow C(u^{\sigma^{-1}(x)\sigma^{-1}(y)}) \neq \{\sigma^{-1}(y)\} \Leftrightarrow \sigma^{-1}(x)R_u\sigma^{-1}(y).$$

3)*Negative Pareto:* Consider $x, y \in X$ such that $xR_i y \quad \forall i \in N$, and assume that $yP_u x$, that is $C(u^{xy}) = \{y\}$. But it implies $x \in C(u^{xy})$, a contradiction.

4)*Positive non-imposedness:* Given $x, y \in X$, we know that there exists $u \in W^n(X)$ *such that* $C(u) = \{x\}$; but then, by applying *Independence,* we know that $y \notin C(u^{xy})$ and so, by *Negative Pareto*, we know that $a \notin C(u^{xy})\ \forall a \in X - \{x, y\}$. Therefore $C(u^{xy}) = \{x\}$, and we conclude xP_uy.

5)*Negative non-imposedness:* Given $x, y \in X$, if for every $u \in W^n(X)$ it is satisfied that xP_uy, that is $C(u^{xy}) = \{x\}$, then, by applying *Independence,* $y \notin C(u)$ for every $u \in W^n(X)$, which contradicts *Non-imposedness.*

Finally we need to show that it is a transitive binary relation. In order to do so, let's consider $x, y, z \in X$ and $u \in W^n(X)$ such that xR_uy, yR_uz; that is, $C(u^{xy}) \neq \{y\}$ and $C(u^{yz}) \neq \{z\}$. By contradiction, assume that $C(u^{xz}) = \{z\}$; then, by applying *Independence,* we know $x \notin C(u^{xyz})$ and by *Negative Pareto,* $a \notin C(u^{xyz})$, $\forall a \in X - \{x, y, z\}$, so $C(u^{xyz}) \subset \{y, z\}$. But if $y \in C(u^{xyz})$, we can apply *Independence* and conclude that $x \notin C(u^{xy})$, which is a contradiction. Therefore $C(u^{xyz}) = \{z\}$ but, in such a case, and by reasoning in the same way, we obtain that $y \notin C(u^{yz})$, a contradiction. So we can conclude xR_uz.

Conversely, consider that we have a transitive SDF satisfying the required properties, and define the associated fixed agenda SCC by maximizing the social binary relation, that is:

$$C(u) = \{a \in X \mid aR_uz \quad \forall z \in X\}.$$

It is obviously well defined, so we only have to prove that it satisfies all of the required axioms.

1)*Independence:* Let $x, y \in X, u \in W^n(X)$, such that $x \in C(u)$, $y \notin C(u)$ and $u : \{x, y\} = v : \{x, y\}$. The definition of $C(u)$ implies that xP_uy and by applying *Independence of Irrelevant Alternatives* we obtain that xP_vy, therefore $y \notin C(v)$.

2)*Monotonicity:* Consider $x, y \in X, u = (R_1, ..., R_n), v = (R'_1, ..., R'_n) \in W^n(X)$ such that $C(u) = \{x\}$, [xP_iy implies xP'_iy], and [xI_iy implies xR'_iy]. Since $y \notin C(u)$, then xP_uy, and then xP_vy, which implies $y \notin C(v)$.

3)*Neutrality:* If σ is a permutation of X, then for every $u \in W^n(X)$ it is satisfied that

$$x \in C(\sigma(u)) \Leftrightarrow xR_{\sigma(u)}y \quad \forall y \in X,$$

but then,

$$xR_{\sigma(u)}y \quad \forall y \in X \Leftrightarrow \sigma^{-1}(x)R_u\sigma^{-1}(y) \quad \forall \sigma^{-1}(y) \in X \Leftrightarrow$$
$$\Leftrightarrow \sigma^{-1}(x) \in C(u) \Leftrightarrow x \in \sigma[C(u)].$$

4)*Negative Pareto:* Consider $x, y \in X$ and $u = (R_1, ..., R_n) \in W^n(X)$ such that $xR_iy \ \forall i \in N$ and assume that $y \in C(u)$. Then, we can ensure xR_uy, and so, from the definition of the associated SCC, $x \in C(u)$.

5)*Non-imposedness:* Let $x \in X$; in order to show that there exists a profile $u \in W^n(X)$ such that $C(u) = \{x\}$, consider $y \neq x$. By applying *Positive Non-imposedness* we know that there exists $u \in W^n(X)$ such that xP_uy, so $y \notin C(u)$. But, by *Independence of Irrelevant Alternatives,* we know that $xP_{u^{xy}}y$, that is $y \notin C(u^{xy})$, and *Negative Pareto* and the transitivity of the SDF imply $xP_{u^{xy}}a \ \forall a \neq x, y$ therefore, by definition of C, we obtain that $C(u^{xy}) = \{x\}$. ■

The next result states the existence of a hierarchy of dictators under the above assumptions.

Theorem 2 *If $C : W^n(X) \to X$ is a fixed agenda SCC satisfying* ***Independence, Monotonicity, Neutrality, Negative Pareto*** *and* ***Non-imposedness,*** *then there exists a subset $J \subset N$ which is a hierarchy of dictators.*

Proof. By applying Lemma 1 we know that there exists a transitive SDF satisfying the corresponding assumptions and defined by

$$xP_uy \Leftrightarrow C(u^{xy}) = \{x\}.$$

So, we can apply Aleskerov and Vladimirov's result [2] and obtain the existence of a group of individuals J such that (if we index it by $J = \{1, 2, ..., j\}$) the SDF can be represented as follows,

$$R_u = *_{i=1}^{j} R_i.$$

Thus, if xP_1y, since individual 1 is a dictator for the SDF, we know that xP_uy, so $C(u^{xy}) = \{x\}$, which implies, by applying *Independence,* that $y \notin C(u)$, that is, individual 1 is also a dictator for the fixed agenda SCC. By following a similar argument for the rest of individuals of J, we obtain that J is a hierarchy of dictators. ■

Finally, we translate into the fixed agenda framework the following representation result for quasitransitive SDFs obtained by Aleskerov and Vladimirov [2]. This result provides a particular distribution of power among individuals of the society. In fact, it does not only imply the existence of an oligarchy, but of a hierarchy of oligarchies. Informally speaking, it states that the society is structured in such a way that individuals are divided in different groups and, within each of these groups, the decision is made by means of a hierarchical operator; the final decision is then arrived at by means of the intersection of the different opinions provided by all of the groups.

Formally they prove that if we consider a quasitransitive SDF satisfying *Independence of Irrelevant Alternatives, Monotonicity, Neutrality, Negative Pareto, Positive Non-imposedness* and *Negative Non-imposedness*, then it is possible to prove the existence of a hierarchical operator that represents the social binary relation, that is,

$$R_u = \bigcap_{i=1}^{p} *_{j=1}^{\alpha_i} R_{ij}.$$

In order to obtain the corresponding result in the fixed agenda framework, first we establish the corresponding equivalence result between both contexts (SDFs and fixed agenda SCCs) by imposing these assumptions.

Lemma 2 *There exists a quasitransitive SDF satisfying* ***Independence of Irrelevant Alternatives, Monotonicity, Neutrality, Negative Pareto, Positive Non-imposedness*** *and* ***Negative Non-imposedness*** *if, and only if, there exists a fixed agenda SCC satisfying* ***Weak Independence, Monotonicity, Neutrality, Negative Pareto*** *and* ***Non-imposedness.***

Proof. We only prove the quasi-transitivity of the SDF associated with the fixed agenda SCC, since the rest of the proof runs parallel to that of Lemma 1.

Consider $x, y, z \in X$, $u \in W^n(X)$ such that xP_uy, yP_uz; that is $C(u^{xy}) = \{x\}$, $C(u^{yz}) = \{y\}$. By applying *Weak Independence* we obtain that $y \notin C(u^{xyz})$ and $z \notin C(u^{xyz})$, and *Negative Pareto* implies $a \notin C(u^{xyz})$, $\forall a \in X - \{x, y, z\}$; therefore the only possibility is $C(u^{xyz}) = \{x\}$, which implies, by *Weak Independence,* that $z \notin C(u^{xz})$. But then, by applying *Negative Pareto* we obtain $C(u^{xz}) = \{x\}$, that is xP_uz. ■

Theorem 3 *If* $C : W^n(X) \to X$ *is a fixed agenda SCC satisfying* ***Weak Independence, Monotonicity, Neutrality, Negative Pareto*** *and* ***Non-imposedness,*** *then there exists a hierarchy of oligarchies.*

Proof. By reasoning in a similar way to Theorem 2, first note that Lemma 2 implies that the associated SDF ($xP_uy \Leftrightarrow C(u^{xy}) = \{x\}$) satisfies the corresponding assumptions. Then, Aleskerov and Vladimirov's result [2] implies

$$R_u = \bigcap_{i=1}^{p} *_{j=1}^{\alpha_i} R_{ij}.$$

From this representation form of the SDF, we consider the following subsets of individuals:

$$J_1 = \{a_{11}, a_{21}, ..., a_{p1}\},\ J_2 = \{a_{12}, a_{22}, ..., a_{p2}\}, ...$$

where a_{ij} represents the individual of the society whose preference relation is R_{ij} in the representation form of R_u. It is not difficult to prove now that $J_1, J_2, ..., J_\alpha$, with $\alpha = \max\ \{\alpha_1, \alpha_2, ..., \alpha_p\}$, constitute a hierarchy of oligarchies for the fixed agenda SCC. Thus, for instance, if we consider $x, y \in X$ and $u \in W^n(X)$ such that $xP_iy\ \forall i \in J_1$, by the representation form of the SDF we know that xP_uy, that is, $C(u^{xy}) = \{x\}$, which in turn implies by *Weak Independence* that $y \notin C(u)$. Moreover if xP_ky for some $k \in J_1$, then it is also clear that xR_uy, so $C(u^{xy}) \neq \{y\}$. Therefore, subset J_1 is an oligarchy. By applying a similar way of reasoning, we obtain that $J_1, J_2, ..., J_\alpha$ is a hierarchy of oligarchies. ■

4 Explicit representation of fixed agenda SCCs.

In this section we present the results obtained on the explicit form of fixed agenda SCCs in the presence of dictators, oligarchies and individuals with veto power. In order to present the results in a clear way, we divide this section in different subsections concerning these different cases.

4.1 Dictators and Hierarchy of Dictators.

It is clear, from the definition itself, that if individual i is a *dictator*, then $C(u) \subseteq M_i^u$; that is, the social choice set is a subset of the maximal elements of the dictator. However, in general, we can not ensure that the equality holds, since, in the cases in which the dictator is indifferent between a pair of alternatives, we have no information about the social relation between such elements. So, the social choice set depends on the opinion and influence on the choice process of the rest of individuals of the society.

If instead of a dictator we assume the existence of a hierarchy of dictators, we will show how the social choice set can be described in terms of the maximal elements of the individuals of the hierarchy. In particular, in the following result, we prove that whenever a hierarchy of dictators exists, the social choice set is contained in the result of a maximization process defined from it.

Proposition 1 *If C is a fixed agenda SCC and there exists a hierarchy of dictators $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$, then for every profile $u \in W^n(X)$,*

$$C(u) \subset M^u_{\sigma(n)}(M^u_{\sigma(n-1)}(...(M^u_{\sigma(2)}(M^u_{\sigma(1)}))...).$$

Proof. From the definition of dictator, we know that $C(u) \subset M^u_{\sigma(1)}$. Since elements in $M^u_{\sigma(1)}$ are considered indifferent by individual $\sigma(1)$, if there exists $z \in M^u_{\sigma(1)}$ such that $z \notin M^u_{\sigma(2)}(M^u_{\sigma(1)})$, then there exists $w \in M^u_{\sigma(1)}$ such that $wP_{\sigma(2)}z$. Since $w, z \in M^u_{\sigma(1)}$ we know that $wI_{\sigma(1)}z$, so, from the definition of a hierarchy of dictators, $z \notin C(u)$, that is $C(u) \subset M^u_{\sigma(2)}(M^u_{\sigma(1)})$. By successively applying this argument, we obtain the required inclusion. ■

In the next example we show that under the assumptions which ensure the existence of a dictator for the fixed agenda SCC (that is, *Independence* and *Weak Pareto Optimality*) the social choice set is undetermined.

Example 1 *Let $X = \{x_1, x_2, ..., x_n\}$ be the set of alternatives with an order fixed a priori. We define the fixed agenda SCC as follows:*

$$C(u) = \{x_i\} \textit{ such that } i = \min\{\ j : \ x_j \in M^u_1\}.$$

It is now easy to show that this SCC satisfies Independence *and* Weak Pareto Optimality *and, therefore, there exists a dictator (individual 1). However, it is clear that, in general, $C(u)$ is strictly contained in M^u_1 (except for when M^u_1 is a singleton).*

Furthermore, we can consider a modification of Example 1, in which the fixed agenda SCC selects the first alternative, according to the established order, within the set $M^u_n(M^u_{n-1}(...(M^u_2(M^u_1))...)$, that is

$$C(u) = x_i \qquad \text{such that} \quad i = \min\{\ j : \ x_j \in M^u_n(M^u_{n-1}(...(M^u_2(M^u_1))...)\}.$$

This SCC satisfies *Strong Pareto Optimality* and *Independence*, but the social choice set is, generally, strictly contained in $M_n^u(M_{n-1}^u(...(M_2^u(M_1^u))...)$. Therefore, these assumptions, which are sufficient conditions in order to ensure the existence of a hierarchy of dictators, are not enough to characterize the social choice set either.

In the following result, we prove that by imposing *Strong Pareto Optimality, Pareto Indifference* and *Independence*, we can determine the social choice set.

Theorem 4 *If σ is a permutation of N, then the fixed agenda SCC, $C^* : W^n(X) \longrightarrow X$, defined by*

$$C^*(u) = M_{\sigma(n)}^u(M_{\sigma(n-1)}^u(...(M_{\sigma(2)}^u(M_{\sigma(1)}^u))...), \tag{1}$$

satisfies ***Independence, Strong Pareto Optimality*** *and* ***Pareto Indifference****. Conversely, every fixed agenda SCC satisfying these properties takes this form.*

Proof. Let us show that C^* satisfies the required properties (without loss of generality, consider $\sigma(i) = i$, for all $i \in N$):

1) *Independence:* Consider $x, y \in X$ and $u = (R_1, ..., R_n), v = (R_1', ..., R_n') \in W^n(X)$ such that $x \in C^*(u)$, $y \notin C^*(u)$ and $u : \{x, y\} = v : \{x, y\}$. Since $y \notin C^*(u)$, there exists $j \in \{1, 2, ..., n\}$ such that $y \in M_{j-1}^u(M_{j-2}^u(...(M_2^u(M_1^u)...)$ but $y \notin M_j^u(M_{j-1}^u(M_{j-2}^u(...(M_2^u(M_1^u)...)$. So, we know that $xI_ky \quad \forall k \in \{1, 2, ..., j-1\}$ and xP_jy; moreover since u and v coincide over alternatives x and y, we also know that $xI_k^{'}y \quad \forall k \in \{1, 2, ..., j-1\}$ and $xP_j^{'}y$. Therefore $y \notin M_j^v(M_{j-1}^v(...(M_2^v(M_1^v)...)$, so $y \notin C^*(v)$.

2) *Strong Pareto Optimality:* Consider $x, y \in X$ and $u \in W^n(X)$, such that xR_iy $\forall i \in N$ and xP_jy, for some $j \in N$. Then, by the definition of C^*, it is obvious that $y \notin C^*(u)$.

3) *Pareto Indifference*: Consider $x, y \in X$ and $u \in W^n(X)$, such that xI_iy $\forall i \in N$, and suppose $x \in C^*(u)$. Then $x \in M_1^u$ and, being R_1 a weak-order, $y \in M_1^u$. By repeating this reasoning, we obtain that $y \in C^*(u)$.

Conversely, given a fixed agenda SCC satisfying *Independence, Strong Pareto Optimality* and *Pareto Indifference*, we know by applying Theorem 1 that there exists a hierarchy of dictators, namely $\{\sigma(1), \sigma(2), ..., \sigma(n)\}$.

So, we only need to prove that the social choice set is given by $M^u_{\sigma(n)}(M^u_{\sigma(n-1)}(...(M^u_{\sigma(2)}(M^u_{\sigma(1)}))...)$. From Proposition 1, whenever this set is a singleton, we have the required equality. In general, if we assume that there are two different alternatives $x, y \in M^u_{\sigma(n)}(M^u_{\sigma(n-1)}(...(M^u_{\sigma(2)}(M^u_{\sigma(1)}))...)$ such that $x \in C(u)$ but $y \notin C(u)$, by applying *Independence*, we know that $y \notin C(u^{xy})$, but then, *Strong Pareto Optimality* implies $C(u^{xy}) = \{x\}$, which contradicts *Pareto Indifference*, since $xI_{\sigma(i)}y$ $\forall i \in N$. ■

If instead of Theorem 1 we make use of Theorem 2, we can obtain the following representation result for fixed agenda SCCs (note that, in this case, the hierarchy of dictators does not necessarily include all the individuals).

Theorem 5 *If $J = \{i_1, i_2, ..., i_j\} \subset N$, then the fixed agenda SCC $C^* : W^n(X) \longrightarrow X$, defined by*

$$C^*(u) = M^u_{i_j}(M^u_{i_{j-1}}(...(M^u_{i_2}(M^u_{i_1}))...),$$

satisfies ***Independence, Monotonicity, Neutrality, Negative Pareto*** *and* ***Non-imposedness****. Conversely, every fixed agenda SCC satisfying these properties takes this form.*

Proof. Let us show that C^* satisfies the required properties. Without loss of generality, consider $J = \{1, 2, ..., j\}$.

1)*Independence:* Analogous to the proof in Theorem 4.

2)*Monotonicity:* Consider $x, y \in X$, $u = (R_1, ..., R_n), v = (R'_1, ..., R'_n) \in W^n(X)$ such that $x \in C^*(u)$, $y \notin C^*(u)$, [xP_iy implies xP'_iy], and [xI_iy implies xR'_iy]. Since $y \notin C^*(u)$, there exists $i \in \{1, 2, ..., j\}$ such that $y \in M^u_{i-1}(M^u_{i-2}(...(M^u_2(M^u_1)...)$ but $y \notin M^u_i(M^u_{i-1}(M^u_{i-2}(...(M^u_2(M^u_1)...)$. So we know that xI_ky $\forall k = 1, 2, ..., i-1$ (and therefore that xR'_ky $\forall k = 1, 2, ..., i-1$), and that xP_iy (and therefore xP'_iy). Thus, we can conclude that $y \notin M^v_i(M^v_{i-1}(M^v_{i-2}(...(M^v_2(M^v_1)...)$, that is $y \notin C^*(v)$.

3)*Neutrality:* Let σ be a permutation of the alternatives; we must now prove that

$$\sigma[C^*(u)] = C^*(\sigma(u)),$$

that is,

$$\sigma[M^u_j(M^u_{j-1}(...(M^u_2(M^u_1))...)] = M^{\sigma(u)}_j(M^{\sigma(u)}_{j-1}(...(M^{\sigma(u)}_2(M^{\sigma(u)}_1))...).$$

step 1: $\sigma[M^u_1] = M^{\sigma(u)}_1$.

$$\begin{aligned} x &\in M_1^{\sigma(u)} \Leftrightarrow x\sigma(R_1)y \; \forall y \in X \Leftrightarrow \sigma^{-1}(x)R_1\sigma^{-1}(y) \; \forall y \in X \Leftrightarrow \\ &\Leftrightarrow \sigma^{-1}(x) \in M_1^u \Leftrightarrow x \in \sigma(M_1^u). \end{aligned}$$

step 2: $\sigma[M_2^u(M_1^u)] = M_2^{\sigma(u)}(M_1^{\sigma(u)})$.

$$\begin{aligned} x &\in M_2^{\sigma(u)}(M_1^{\sigma(u)}) \Leftrightarrow x\sigma(R_2)y \quad \forall y \in M_1^{\sigma(u)} \Leftrightarrow \\ &\Leftrightarrow \sigma^{-1}(x)R_2\sigma^{-1}(y) \quad \forall y \in M_1^{\sigma(u)} = \sigma(M_1^u) \Leftrightarrow \\ &\Leftrightarrow \sigma^{-1}(x)R_2\sigma^{-1}(y) \quad \forall \sigma^{-1}(y) \in M_1^u \Leftrightarrow \\ &\Leftrightarrow \sigma^{-1}(x) \in M_2^u(M_1^u) \Leftrightarrow x \in \sigma[M_2^u(M_1^u)]. \end{aligned}$$

By repeating this reasoning, we obtain the required result.

4)*Negative Pareto:* Consider a profile $u = (R_1, ..., R_n) \in W^n(X)$ such that $xR_iy \quad \forall i \in N$, and assume $y \in C^*(u)$. But then, and by the way in which C^* has been defined, it is clear that $x \in C^*(u)$.

5)*Non-imposedness:* Given $x \in X$, it is sufficient to consider a profile in which the preference of the first individual is given by the relation R_1 defined in such a way that $xP_1z \quad \forall z \in X - \{x\}$; then it is clear that $C^*(u) = M_j^u(M_{j-1}^u(...(M_2^u(M_1^u))...) = \{x\}$.

Conversely, given a fixed agenda SCC satisfying all of these assumptions, we can apply Theorem 2 and conclude that there exists a *subset* $J \subset N$ which is a *hierarchy of dictators* for the fixed agenda SCC. We index it by $J = \{1, 2, ..., j\}$. In order to show that $C(u) \subset M_j^u(M_{j-1}^u(...(M_2^u(M_1^u))...)$, we start by proving that $C(u) \subset M_1^u$. If $y \notin M_1^u$, then there exists $x \in X$ such that xP_1y. But then, since J is a hierarchy of dictators, this implies that $y \notin C(u)$. Moreover, since all the elements in M_1^u are indifferent for individual 1, they will be included or excluded from the social choice set depending on the opinion that the second individual has about them. Assume that there exists $z \in M_1^u$ such that $z \notin M_2^u(M_1^u)$; then, there exists $w \in M_1^u$ such that wP_2z. Since $w, z \in M_1^u$ we know that wI_1z so, from the hierarchical structure of J, $z \notin C(u)$, that is, $C(u) \subset M_2^u(M_1^u)$. By successively applying the same reasoning, we obtain the required inclusion.

Finally, if $M_j^u(M_{j-1}^u(...(M_2^u(M_1^u))...)$ is a singleton, then the equality holds. Assume that there exists $x, y \in M_j^u(M_{j-1}^u(...(M_2^u(M_1^u))...)$ such that $x \in C(u)$ and $y \notin C(u)$. On the one hand, we can apply *Independence* and conclude that $y \notin C(u^{xy})$; on the other hand, since $xI_iy \quad \forall i = 1, 2, ..., j$, we know that xI_uy, which means $C(u^{xy}) = \{x, y\}$, a contradiction. Therefore, $M_j^u(M_{j-1}^u(...(M_2^u(M_1^u))...) \subset C(u)$. ■

4.2 Oligarchies and individuals with veto power.

The analysis of the explicit representation of fixed agenda SCCs whenever oligarchies or individuals with veto power exist, is more difficult than in the case of the previous section. Basically, the difference is due to the definition of individual with veto power: while in the case of a dictator, if he decides on xP_iy, it implies that y is not socially chosen ($y \notin C(u)$), in the case of an individual with veto power we only know that $C(u^{xy}) \neq \{y\}$, and it does not give much information about what the social choice set $C(u)$ is like. That is why the implications of the existence of veto power over the social choice set are not so clear. Moreover, although individuals of an oligarchy act together as a dictator, we only can ensure that some alternative y is not socially chosen if all of the individuals in the oligarchy agree on that some other alternative x is strictly better than y. In the rest of the cases, we have no information about what the social choice set is like.

So, although the existence of oligarchies in the context of aggregation of preferences (quasitransitive SDFs) implies that *"an agent has no power at all or full veto power"* (see [9]), in terms of choice functions we will show that the individuals in the oligarchy don't have so much power with respect to the final social choice set in the fixed agenda framework.

In order to analyze the explicit representation of fixed agenda SCCs in the presence of oligarchies, we first consider those that satisfy *Weak Independence* and *Weak Pareto Optimality* (since, as we mentioned in the previous section, these are sufficient conditions to ensure the existence of an oligarchy in the fixed agenda framework[7]). First of all note that, under these assumptions and from the definition of an oligarchy, it is clear that if J is an oligarchy then

$$C(u) \subset \{a \in X \mid \nexists\, b \in X : bP_ia \ \forall i \in J\}.$$

However, it is not possible to know much more about the social choice set in the presence of the oligarchy by imposing these assumptions alone. The following example shows that, in fact, it is possible that the intersection of the social choice set, and the maximal sets of the individuals in the oligarchy, would be empty; that is, the social choice set could not select any of the maximal alternatives of the individuals of the oligarchy.

Example 2 *Let* $X = \{x_1, x_2, x_3, x_4, x_5\}$ *be the set of alternatives and* $N = \{1, 2, 3, 4, 5\}$ *the set of individuals. For each profile* $u = (R_1, R_2, ..., R_5) \in W^5(X)$ *we define the following binary relation,*

$$xP_uy \Leftrightarrow xP_iy \ \forall i \in \{1, 2, 3\},$$

and a fixed agenda SCC which selects the first and second alternatives (with respect to the order fixed in the set X) within the maximal set provided by this relation. If there is only one alternative in this set, then the fixed agenda SCC selects it. It is clear that individuals $\{1, 2, 3\}$ *form an oligarchy and that* C *satisfies* Weak Independence *and* Weak Pareto Optimality*. But if we consider the following individual preferences,*

$$\begin{array}{ll} R_1: & x_5P_1x_1P_1x_3P_1x_2P_1x_4, \\ R_2: & x_3P_2x_1P_2x_5P_2x_4P_2x_2, \\ R_3: & x_4P_3x_2P_3x_1P_3x_5P_3x_3, \end{array}$$

then, $C(u) = \{x_1, x_2\}$*; so the social choice set selects alternatives which are not maximal for any of the individuals in the oligarchy.*

We introduce an additional assumption which in a way translates the property of "*Generalized Condorcet Winner*" [11] (at least, whenever we impose *Weak Pareto Optimality*), to the context of fixed agenda SCCs. Informally speaking, it means that whenever there exists an alternative which is better than or equal to any other alternative of the set, in a "pairwise comparison"[3], then it has to be chosen from the total set.

Axiom 11 ***Condorcet Property:*** $\forall u \in W^n(X)$, $\forall x \in X$, *if* $x \in C(u^{xy})$ $\forall y \in X$, *then* $x \in C(u)$.

The following result states that under the assumptions of *Pseudo-Independence* , *Weak Pareto Optimality* and *Condorcet Property*, we can ensure that at least some of the maximal elements of the individuals with veto power belong to the social choice set. Therefore, this proposition provides an alternative definition of an individual with veto power in terms of the individual choice set (maximal set): an individual has veto power if he can

[3] Note that in this framework, since the set of alternatives is fixed a priori, we can not state that an alternative is better than another one in a pairwise comparison by choosing over the set $\{x, y\}$, but by analyzing the social choice set when the profile considered is u^{xy}, which is in some sense equivalent in the presence of Weak Pareto Optimality.

always ensure that some of his maximal elements are in the social choice set. Note that, although it seems a "natural" definition of veto power, it does not correspond, in general, to the well-known notion of veto power in the context of aggregation of preferences.

Proposition 2 *If C is a fixed agenda SCC satisfying* ***Pseudo-Independence, Weak Pareto Optimality*** *and* ***Condorcet Property****, then individual i has veto power if, and only if,*

$$C(u) \cap M_i^u \neq \emptyset \quad \forall u \in W^n(X).$$

Proof. Assume that $C(u) \cap M_i^u \neq \emptyset \quad \forall u \in W^n(X)$. In order to show that individual i has veto power consider alternatives $x, y \in X$ such that xP_iy; then, since $C(u^{xy}) \cap M_i^{u^{xy}} \neq \emptyset$, we obtain that $x \in C(u^{xy})$; therefore individual i has veto power.

Conversely, assume that individual i has veto power and consider a profile $u \in W^n(X)$. For every $z \in M_i^u$ and $a \notin M_i^u$, it is satisfied that zP_ia, which implies (since i is a vetoer) $C(u^{az}) \neq \{a\}$, that is

$$z \in C(u^{az}) \quad \forall a \notin M_i^u.$$

We will now show that there exists at least one alternative $b \in M_i^u$ such that $b \in C(u^{bz}) \quad \forall z \in M_i^u$. By contradiction assume that for every $z_k \in M_i^u$ there always exists another alternative $z_s \in M_i^u$ such that $z_k \notin C(u^{z_s z_k})$. So, if we start by $z_1 \in M_i^u$, we know that there exists $z_2 \in M_i^u$ such that $z_1 \notin C(u^{z_1 z_2})$, that is $C(u^{z_1 z_2}) = \{z_2\}$; by applying the same argument to z_2 we know that there exists $z_3 \in M_i^u$ such that $z_2 \notin C(u^{z_2 z_3})$, that is $C(u^{z_2 z_3}) = \{z_3\}$; by repeating this process, since the set of alternatives is finite, we know that, at some point in this reasoning, we will obtain the following situation:

$$\begin{aligned}
z_i &\Rightarrow \exists z_{i+1} : C(u^{z_i z_{i+1}}) = \{z_{i+1}\}, \\
z_{i+1} &\Rightarrow \exists z_{i+2} : C(u^{z_{i+1} z_{i+2}}) = \{z_{i+2}\}, \\
z_{i+2} &\Rightarrow \exists z_{i+3} : C(u^{z_{i+2} z_{i+3}}) = \{z_{i+3}\}, \\
& \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
z_k &\Rightarrow \exists z_i : C(u^{z_k z_i}) = \{z_i\}.
\end{aligned}$$

But then, if we consider profile $u^{z_i z_{i+1} \ldots z_k}$, by *Pseudo-Independence*, we obtain $C(u^{z_i z_{i+1} \ldots z_k}) = \emptyset$, which is a contradiction. This last statement is obtained since, on the one hand, by *Weak Pareto Optimality* it is clear that $C(u^{z_i z_{i+1} \dots z_k}) \subset \{z_i, ..., z_k\}$; while, on the other hand, and by applying *Pseudo-Independence* with profiles $(u^{z_i z_{i+1} \dots z_k})$ and $(u^{z_i z_{i+1}})$, $(u^{z_i z_{i+1} \dots z_k})$ and $(u^{z_{i+1} z_{i+2}})$, ... and so on, we obtain $z_j \notin C(u^{z_i z_{i+1} \dots z_k})\ \forall j = i, i+1, ..., k$.

Therefore, we can ensure the existence of an alternative $b \in M_i^u$ such that $b \in C(u^{bx}) \quad \forall x \in X$ and finally, by applying *Condorcet Property,* we obtain $b \in C(u) \cap M_i^u$. ∎

So, under the assumptions of the previous result, we can ensure that at least one maximal alternative of each of the individuals in the oligarchy is socially chosen. However, it may be that elements which are not maximal for any of such individuals belong to the social choice set.

In the following result, we analyze the case in which there exists a subset of individuals which constitutes a hierarchy of veto power. This result is similar to that in Proposition 2, and as in that case, under the appropriate axioms, it provides an alternative definition of a hierarchy of veto power.

Proposition 3 *If C is a fixed agenda SCC satisfying* ***Pseudo-Independence, Weak Pareto Optimality*** *and* ***Condorcet Property****, then subset $J = \{i_1, i_2, ..., i_j\}$ is a veto hierarchy if, and only if,*

$$C(u) \cap M_{i_j}^u(M_{i_{j-1}}^u(....(M_{i_1}^u)...) \neq \emptyset \quad \forall u \in W^n(X).$$

Proof. Without loss of generality, consider $J = \{1, 2, ..., j\}$. Assume that $C(u) \cap M_j^u(M_{j-1}^u(....(M_1^u)...) \neq \emptyset \quad \forall u \in W^n(X)$. In order to show that $J = \{1, 2, ..., j\}$ is a veto hierarchy, consider alternatives x, y such that xP_1y. As $C(u^{xy}) \cap M_j^{u^{xy}}(M_{j-1}^{u^{xy}}(....(M_1^{u^{xy}})...) \neq \emptyset$, and $M_1^{u^{xy}} = \{x\}$, then we can ensure that $C(u^{xy}) \cap M_j^{u^{xy}}(M_{j-1}^{u^{xy}}(....(M_1^{u^{xy}})...) = \{x\}$ and, therefore, that $C(u^{xy}) \neq \{y\}$. So, individual 1 has veto power over any pair of alternatives. Assume now that xI_1y and xP_2y; then, $M_2^{u^{xy}}(M_1^{u^{xy}}) = \{x\}$ and $C(u^{xy}) \cap M_j^{u^{xy}}(M_{j-1}^{u^{xy}}(....(M_1^{u^{xy}})...) = \{x\}$. So, individual 2 has veto power whenever individual 1 is indifferent. By repeating this reasoning we obtain the result.

Conversely, let us assume that J is a veto hierarchy. By reasoning as in the proof of Proposition 2, it is not difficult now to prove that there is an alternative $b \in M_j^u(M_{j-1}^u(....(M_1^u)...)$, such that $b \in C(u^{bx}) \quad \forall x \in X$; therefore, by applying *Condorcet Property,* $b \in C(u) \cap M_i^u(M_{i-1}^u(....(M_1^u)...)$. ■

The next result provides a bit more information about the representation of the social choice set in terms of the maximal elements of an oligarchy and proves that, under *Weak Independence, Weak Pareto Optimality* and *Condorcet Property,* if all of the individuals in the oligarchy agree on some maximal elements, the social choice set is a selection of the union of their maximal elements and, moreover, some of the maximal elements on which they all agree are socially chosen.

Proposition 4 *If C is a fixed agenda SCC satisfying **Weak Independence** and **Weak Pareto Optimality**, then there exists a subset of individuals J (an oligarchy), such that for every $u \in W^n(X)$, if $\bigcap_{i \in J} M_i^u \neq \emptyset$ then,*

1) $C(u) \subset \bigcup_{i \in J} M_i^u$.

*2) If C also satisfies **Condorcet Property**, then $\bigcap_{i \in J} M_i^u \cap C(u) \neq \emptyset$.*

Proof. We know that, under these assumptions, an oligarchy $J \subseteq N$ exists (see [7]). Let us now show that 1) and 2) are fulfilled by such an oligarchy.

1) Assume that $x \notin \bigcup_{i \in J} M_i^u$; then $x \notin M_i^u \; \forall i \in J$ and therefore by considering $b \in \bigcap_{i \in J} M_i^u$, we know that $bP_ix \; \forall i \in J$, which implies that $x \notin C(u)$.

2) By reasoning in the same way as in Proposition 2, it is not difficult to prove that there is some $z \in \bigcap_{i \in J} M_i^u$, such that $z \in C(u^{zx}) \; \forall x \in X$, so, by applying *Condorcet Property,* we can conclude that $z \in C(u)$. Therefore $\bigcap_{i \in J} M_i \cap C(u) \neq \emptyset$. ■

Finally, in the following results, we present a characterization of the social choice set, when we know of the existence of oligarchies. In particular, Proposition 5 states that the social choice set is defined in the following way: society is divided into different groups and within each of these groups they make choices in the same way as a hierarchy of dictators; then the different choices of all of these groups are socially chosen. Moreover, if all of them agree on some alternatives, then these alternatives completely determine the social choice set.

Proposition 5 *Let* $C : W^n(X) \to X$ *be a fixed agenda SCC satisfying* ***Weak Independence, Monotonicity, Neutrality, Negative Pareto*** *and* ***Non-imposedness****. Then, there exists a hierarchical structure, given by* p *groups of individuals:* $A_1 = \{a_{11}, a_{12}, ..., a_{1\alpha_1}\}; A_2 = \{a_{21}, a_{22}, ..., a_{2\alpha_2}\}, ..., A_p = \{a_{p1}, a_{p2}, ..., a_{p\alpha_p}\}$
such that if we denote by

$$B_i^u = M_{i\alpha_i}^u(M_{i\alpha_i-1}^u(...(M_{i2}^u(M_{i1}^u)...) \quad \forall i = 1, 2, ...p,$$

then,
1) whenever $B_1^u \cap B_2^u \cap .... \cap B_p^u \neq \emptyset$*, the following inclusion is fulfilled*

$$C(u) \subset B_1^u \cup B_2^u \cup .... \cup B_p^u;$$

2) in any case, if C *also satisfies* ***Condorcet Property****, the converse inclusion holds, that is,*

$$B_1^u \cup B_2^u \cup .... \cup B_p^u \subset C(u).$$

Proof. 1) Since C is a SCC satisfying *Weak Independence, Monotonicity, Neutrality, Negative Pareto* and *Non-imposedness*, by applying Lemma 2 we know that there exists a quasitransitive SDF satisfying *Independence of Irrelevant Alternatives, Monotonicity, Neutrality, Negative Pareto, Positive Non-imposedness* and *Negative Non-imposedness*. We can then apply Aleskerov and Vladimirov's result (see [2]) and obtain that the associated SDF (which is defined as usual: xP_uy if, and only if, $C(u^{xy}) = \{x\}$) can be represented as follows:

$$R_u = \bigcap_{i=1}^{p} *_{j=1}^{\alpha_j} R_{ij},$$

that is, it is defined by means of the intersection of the opinions of the different groups of individuals ($A_1 = \{a_{11}, a_{12}, ..., a_{1\alpha_1}\}$, $A_2 = \{a_{21}, a_{22}, ..., a_{2\alpha_2}\}$, ..., $A_p = \{a_{p1}, a_{p2}, ..., a_{p\alpha_p}\}$ where a_{ij} represents the individual of the society whose preference relation is R_{ij} with respect to the representation form of R_u) and, within each of these groups, the decision is made as a hierarchy of dictators.

Consider $w \in B_1^u \cap B_2^u \cap .... \cap B_p^u$ and $x \notin B_1^u \cup B_2^u \cup .... \cup B_p^u$. Then we know that the opinion with respect to alternatives x and w in all of the groups $A_1, A_2, ..., A_p$ is as follows: $wP_{A_i}x$ (where $R_{A_i} = *_{j=1}^{\alpha_i} R_{ij}$). So, wP_ux, that is $C(u^{xw}) = \{w\}$, and by applying *Weak Independence*, we obtain $x \notin C(u)$.

2) Let $x \in B_1^u \cup B_2^u \cup .... \cup B_p^u$. If $x \in B_i^u$, we know that $xI_{A_i}w, \quad \forall w \in B_i^u$, and $xP_{A_i}w, \quad \forall w \notin B_i^u$. Therefore we can conclude that $xR_u z$ $\forall z \in X$, and *Condorcet Property* implies $x \in C(u)$. ■

By means of this result we have a complete characterization of the social choice set in the case in which the intersection of the opinions of the different groups is non-empty.

Corollary 1 *If C is a fixed agenda SCC satisfying* ***Weak Independence, Monotonicity, Neutrality, Negative Pareto, Non-imposedness*** *and* ***Condorcet Property****, then there exists a hierarchical structure $A_1, A_2, ..., A_p$ such that:*

$$if\ B_1^u \cap B_2^u \cap .... \cap B_p^u \neq \emptyset \Rightarrow C(u) = B_1^u \cup B_2^u \cup .... \cup B_p^u.$$

Remark. Under the hypotheses of the previous result and as we have mentioned, each group of individuals

$$A_i = \{a_{i1}, a_{i2}, ..., a_{i\alpha_i}\}$$

acts as a hierarchy of dictators by choosing B_i^u. If we now consider the set of individuals who appear at the top of each hierarchy, that is,

$$J_1 = \{a_{11}, a_{21}, ..., a_{p1}\}$$

then, only the maximal elements of such individuals can be in the social choice set. If we consider the set of individuals who are in the second place in each group,

$$J_2 = \{a_{12}, a_{22}, ..., a_{p2}\}$$

then, the opinion of such individuals is relevant only over the maximal elements of the individuals in J_1. Moreover, only maximal elements of the individuals in J_2 on this set can belong to the choice set. In general, we can define the following sets the same way we did in Theorem 3, that is,

$$J_i = \{a_{1i}, a_{2i}, ..., a_{pi}\} \qquad i = 1, 2, ..., \alpha, \text{ where } \alpha = \max\ \{\alpha_1, \alpha_2, ..., \alpha_p\}$$

If each of these subsets of individuals contains just one individual, then $\{J_1, J_2, ..., J_\alpha\}$ is a hierarchy of dictators. Otherwise, each J_i acts as an oligarchy, which is why we have called such sets of individuals a *hierarchy of*

oligarchies (see Theorem 3)
.

Now, by using the hierarchy of oligarchies, we consider the following subsets associated to each alternative: given $x \in X$ and a profile $u = (R_1, R_2, ..., R_n) \in W^n(X)$, we define the following sets,

$$\begin{aligned} D^u(x, J_1) &= \{a \in X \mid aP_ix \ \forall i \in J_1\}, \\ D^u(x, J_2) &= \{a \in X \mid aR_ix \ \forall i \in J_1 \text{ and whenever } aI_{j1}x \text{ then } aP_{j2}x\}, \\ D^u(x, J_3) &= \{a \in X \mid aR_ix \ \forall i \in J_1 \cup J_2 \text{ and whenever } aI_{j1}x, aI_{j2}x \text{ then } aP_{j3}x\}. \end{aligned}$$

and so on. The underlying idea about these subsets is that if, for some k, $D^u(x, J_k)$ is nonempty, alternative x is rejected from choice by the oligarchy J_k. By making use of this notation, we prove that, under the assumptions in the previous theorem, the social choice set is given by those alternatives for which the associated subsets $D^u(X, J_i)$ are empty.

Proposition 6 *Let $C : W^n(X) \to X$ be a fixed agenda SCC satisfying* ***Weak Independence, Monotonicity, Neutrality, Negative Pareto, Non-imposedness*** *and* ***Condorcet Property;*** *then there exists a hierarchy of oligarchies $J_1, J_2, ..., J_\alpha$ such that*

$$C(u) = \{x \in X \mid D^u(x, J_i) = \emptyset \qquad \forall i = 1, 2, ..., \alpha\}$$

Conversely, each SCC defined in that way satisfies all of the properties we mentioned above.

Proof. By applying Theorem 3, we obtain the existence of a hierarchy of oligarchies. Let us denote

$$Q^u = \{x \in X \mid D^u(x, J_i) = \emptyset \qquad \forall i = 1, 2, ..., \alpha\}$$

If $x \notin Q^u$, then there exists some i such that $D^u(x, J_i) \neq \emptyset$, that is, there exists some alternative $w \in X$ such that:

$$wR_jx \qquad \forall j \in J_1 \cup J_2 \cup ... \cup J_i,$$

and

$$wI_{kj}x \qquad \forall j \in J_1 \cup J_2 \cup ... \cup J_{i-1} \text{ implies } wP_{ki}x.$$

Then, by the way in which the social binary relation is obtained, this implies wP_ux, that is $C(u^{xw}) = \{w\}$, which in turn implies (by applying *Weak Independence*) that $x \notin C(u)$.

Consider now $x \in Q^u$; since $D^u(x, J_1) = \emptyset$, it implies that for any other alternative $w \in X$, it is not possible that all of the individuals in J_1 agree on $wP_{i1}x$. If one of them decides on $xP_{i1}w$, then xR_uw and we obtain $x \in C(u^{xw})$. If there exists an alternative w such that $wR_jx\ \forall j \in J_1$, $wI_{k1}x$, being $D^u(x, J_2) = \emptyset$, we know that $xR_{k2}w$, and if the strict preference holds, we would obtain xR_uw. By applying this argument repeatedly, we obtain xR_uw, that is $x \in C(u^{xw})$ for every $w \in X$, and *Condorcet Property* implies $x \in C(u)$.

Finally, it is not difficult to see that the fixed agenda SCC defined by $C(u) = Q^u$ satisfies all of the required properties. ■

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