ADJUSTING CORRELATION MATRICES

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A B S T R A C T

This article proposes a new algorithm for adjusting correlation matrices and for comparison with Finger's algorithm, which is used to compute Value-at-Risk in RiskMetrics for stress test scenarios. The solution proposed by the new methodology is always better than Finger's approach in the sense that it alters as little as possible those correlations that we do not wish to alter but they change in order to obtain a consistent Finger correlation matrix .

Keywords: correlation matrix, Kuhn-Tucker conditions, eigenvalue, Value-at-Risk.

1 INTRODUCTION

An important problem, arising when using RiskMetrics (RM) for Value-at-Risk (VaR), is that sometimes it is desirable to alter the correlation matrix in order to reflect a view of markets that differs from the traditional one. An arbitrary alteration in the correlation matrix however can breakdown the required consistency of the methodology since the new correlation matrix may be indefinite.

Finger (1997) introduces a methodology in RM to alter some correlations from a correlation matrix such that the new matrix is still consistent. A problem that arises in using this algorithm is that the new matrix indicates more correlations to be altered than the desired ones.

This paper introduces a new algorithm to adjust the correlation matrix and compare it with Finger's. In particular, this new methodology uses the Finger's correlation matrix and then modifies, as little as possible, those correlations that we do not wish to alter by minimizing the distance to the original ones, subject to the restriction of the correlation matrix being consistent.

The remainder of the paper is organized as follows. Finger's algorithm is reviewed in Section 2. In Section 3, a new algorithm is proposed with the proof given in the Appendix. Finally, in Section 4, both methodologies are applied to the hypothetical currency correlation matrix example taken from Finger (1997).

2 FINGER'S ALGORITHM

Let X be a random vector in R^n that represents n asset returns with a mean μ and a covariance matrix $\Omega = [\sigma_{ij}]_{n \times n}$. Let $C = [c_{ij}]_{n \times n}$ denote the correlation matrix of X; i.e. $C = \Gamma \Omega \Gamma$ where $\Gamma = \mathrm{diag}\left(\sigma_{11}^{-1/2}; \ldots; \sigma_{nn}^{-1/2}\right)$. Then, a 2×1

partition of X, according to the assets whose correlations we wish to change and the ones we do not, the two subsets being denoted as I and J respectively, is

$$X = \begin{bmatrix} X_I \\ X_J \end{bmatrix}$$

where $X_I \in \mathbb{R}^m$ and $X_J \in \mathbb{R}^{n-m}$. We can express C as

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}$$

where C_{11} is a $m \times m$ matrix containing the correlations of I, C_{22} is a $(n-m) \times (n-m)$ matrix containing the correlations of J and C_{12} denotes the correlations between both I and J. Finally, by M^T we denote the transpose of a matrix M.

Let $\tilde{C}_{11}$ be the matrix containing the new correlations of I. If C_{11} is replaced by $\tilde{C}_{11}$ in C, a new matrix $\tilde{C}$ is obtained, which, at times, can produce undesirable results; i.e. $\tilde{C}$ may be indefinite and is therefore not a true correlation matrix. Finger's algorithm, denoted by F, is a method for altering correlations consistently, such that a new correlation matrix $C_F \neq \tilde{C}$ is obtained when $\tilde{C}$ is indefinite, verifying that C_F is non-negative definite. The algorithm is defined as follows:

Let Z be the random vector in $\mathbb{R}^n$ such that $Z = \Gamma(X - \mu)$, then $Z \sim (0, C)$. Let $\overline{Z} = \frac{1}{m} \sum_{i \in I} Z_i$. The random variables (rv's) X_i^F, $i = 1, \ldots, n$ are now defined as

$$X_i^F = \begin{cases} (1 - \theta_i) Z_i + \theta_i \overline{Z} & \text{if } i \in I \\ Z_i & \text{otherwise} \end{cases}$$

where $\mu_i \in [0,1]$. We can express the new rv's[1], in a matrix way, by $X_F = AZ$ where A is a $n \times n$ matrix defined as

$$\begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}$$

such that A_{11} is a $m \times m$ matrix whose elements a_{ij} are

$$a_{ij} = \begin{cases} 1 - \mu_i + \dfrac{\mu_i}{m} & \text{if } i = j; \quad i \in I \\ \dfrac{\mu_i}{m} & \text{if } i \neq j; \quad i, j \in I \end{cases} \tag{1}$$

and I is the identity matrix. Then, $E\left(X_F X_F^T\right) = ACA^T$ gives us the covariance matrix of X_F, denoted as Ω_F, which can be partitioned as

$$\begin{bmatrix} \Omega_{11}^F & \Omega_{12}^F \\ \Omega_{21}^F & \Omega_{22}^F \end{bmatrix} = \begin{bmatrix} A_{11}C_{11}A_{11}^T & A_{11}C_{12} \\ C_{12}^T A_{11}^T & C_{22} \end{bmatrix}.$$

Consider

$$\Gamma_F = \begin{bmatrix} \Gamma_{11}^F & 0 \\ 0 & I \end{bmatrix}$$

where $\Gamma_{11}^F = \operatorname{diag}\left(\delta_{11}^{-1/2}, \ldots, \delta_{mm}^{-1/2}\right)$ and δ_{ii} denotes the (i,i) element of the matrix Ω_{11}^F. Let Z_F be a new random vector in R^n such that $Z_F = \Gamma_F X_F$. Then $E\left(Z_F Z_F^T\right) = \Gamma_F \Omega_F \Gamma_F$, denoted as C_F, is the covariance matrix of Z_F, which is well defined[2]. By splitting matrix C_F in the same way as we did with

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[1] We introduce, here, a different version of Finger's algorithm, the only difference is about defining X_i^F. In Finger's paper, $\mu_i = \mu \ \forall i \in I$ while this new version allows μ_i to be different to a better adjusting of the new C_{11} matrix, obtained through F and denoted as C_{11}^F, to the matrix $\tilde{C}_{11}$.

[2] It is easy to prove that C_F is non-negative definite.

the previous matrices, we can write

$$C_F = \begin{bmatrix} C_{11}^F & C_{12}^F \\ C_{21}^F & C_{22}^F \end{bmatrix} = \begin{bmatrix} \Pi C_{11} \Pi^T & \Pi C_{12} \\ C_{12}^T \Pi^T & C_{22} \end{bmatrix}$$

where $\Pi = \Gamma_{11}^F A_{11}$. Notice that when computing C_F there may be more than $\left(m^2 - m\right)/2$ new correlations in C_F as was expected at first. At most, there can be $\left[\left(m^2 - m\right)/2 + m(n - m)\right]$ new correlations; i.e., those belonging to C_{12}^F.

Finally, note that A_{11} is a function of the parameter vector $\theta \in R^m$; i.e. $A_{11} = A_{11}(\theta)$; so that the first step to computing C_F is to solve the following constrained minimization program:

$$\min_{\{\theta_i\}_{i=1}^m \in [0;1]} \left\| \left[A_{11}(\theta) C_{11} A_{11}^T(\theta)\right] - \tilde{C}_{11} \right\|^2$$

where $\|\cdot\|$ denotes the euclidean norm[3].

3 NEW ALGORITHM

Now, we shall try to obtain a better correlation matrix than the one presented in the previous section. Note that $C_{12} \neq C_{12}^F$; so that our goal is to modify as little as possible the elements from C_{12} in the new adjusted correlation matrix. In order to do so, we present an algorithm which is composed of a two-step procedure. The first step consists of computing C_F and the second is to obtain

[3] The euclidean norm of a matrix A is defined as:

$$\|A\| = \left(\sum_{i=1} \sum_{j=1} a_{ij}^2 \right)^{1/2} ; \quad A = [a_{ij}]_{p \times q}.$$

a new correlation matrix C^*, defined as $C^* = C_F + B$, where

$$B = \begin{bmatrix} 0 & B_{12} \\ B_{12}^T & 0 \end{bmatrix}$$

being $B_{12} = [b_{ij}]$ a known $m \times (n - m)$ matrix. We can now write

$$C^* = \begin{bmatrix} C_{11}^F & C_{12}^* \\ (C_{12}^*)^T & C_{22} \end{bmatrix}$$

with $C_{12}^* = C_{12}^F + B_{12}$. If we denote the elements in C_{12}^F by $[d_{ij}]$, then $C_{12}^* = [d_{ij} + b_{ij}]$. We try to choose b_{ij} such that C_{12}^* be approximately equal to C_{12}. In order to guarantee that the new matrix C^* is a non-negative definite matrix, we apply the following known result[4]:

Theorem 1 Let γ be an eigenvalue of $C^* = C_F + B$, then

$$\gamma \in \bigcup_{k=1}^{n} \{z \in \mathbb{R} : |z - \lambda_k| \leq r\}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of C_F and $r = \|B\|$.

Let $\lambda = \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then if $\lambda \geq r$ we know that $\lambda_k \geq r$, $k = 1, \ldots, n$. So, the following condition is sufficient to ensure that C^* is a non-negative definite matrix

$$\lambda^2 \geq 2 \sum_{i=1}^{m} \sum_{j=1}^{n-m} b_{ij}^2.$$

[4] See, for instance, Lancaster and Tismenetsky (1985), Chapter 11, p. 388-9.

We are interested in choosing C_{12}^{*} such that it minimizes $\|C_{12}^{*} - C_{12}\|^2$ subject to C_{12}^{*} being non-negative definite. We know that

$$C_{12}^{*} - C_{12} = C_{12}^{F} - C_{12} + B_{12}.$$

Let us call $E = C_{12}^{F} - C_{12}$, which is a $m \times (n - m)$ matrix whose elements, denoted as $[e_{ij}]$, are known. To obtain the new correlation matrix C^{*} we must solve the following constrained minimization program where the parameters are the elements of B_{12}, i.e. $[b_{ij}]$:

$$\begin{aligned} &\min_{\{b_{ij}\}} \sum_{i=1}^{m} \sum_{j=1}^{n-m} (e_{ij} + b_{ij})^2 \\ &\text{s.t.} \quad 2 \sum_{i=1}^{m} \sum_{j=1}^{n-m} b_{ij}^2 \leq \xi^2 \\ &\qquad -1 \leq d_{ij} + b_{ij} \leq 1 \end{aligned} \tag{2}$$

where the number of constraints is $m(n - m) + 1$. Note that the second restriction guarantees that the elements of C^{*}, specifically the elements of C_{12}^{*}, must belong to the interval $[-1, 1]$ since they are correlations.

Remark 1 It must be noted that, since a feasible possibility in the above problem consists of considering $B = 0$, that is $C^{*} = C_F$, the solution proposed by the new algorithm is always better than the one proposed by Finger's approach. Moreover, the feasible set being closed, bounded and non-empty, program (2) always has a solution, which is unique due to the strict convexity of the objective function.

An important feature of this algorithm is that it is possible to find conditions that ensure that matrix C_{12} does not change under the computations of matrix C^*. Thus, from the Kuhn-Tucker (K-T) conditions of (2) given in the Appendix we know that, if

$$\lambda^2 \geq 2 \sum_{i=1}^{n} \sum_{j=1}^{m} e_{ij}^2$$

then, we do not need to apply the second step, since we obtain that the solution for program (2) is $b_{ij} = -e_{ij}$ and we directly have:

$$C^* = \begin{bmatrix} C_{11}^F & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}.$$

In other cases the solution is, $b_{ij} = -e_{ij}/\varphi$, being $\varphi > 1$ the value obtained from the K-T conditions (see Appendix) so that $C_{12}^* - C_{12} = E + B_{12} = (1 - 1/\varphi)E$. Thus, we directly have:

$$C^* = \begin{bmatrix} C_{11}^F & C_{12} + (1 - 1/\varphi)E \\ (C_{12} + (1 - 1/\varphi)E)^T & C_{22} \end{bmatrix}.$$

4 FINGER'S EXAMPLE

We now apply the new algorithm (N-Ag) to an example in the hypothetical currency correlation matrix taken from Finger[5] (1997) and compare it to Finger's algorithm (F). Let us consider the following currency correlation matrix:

[5] See Table 1 from page 4, though shifting the currencies so that the submatrix C_{11} contains the Asian currencies whose correlations we wish to alter.

HKD	MYR	PHP	THB	ARS	DEM	GBP	
1.0000	−0.2100	0.1400	−0.1500	−0.2600	−0.1400	0.0600	HKD
	1.0000	0.2200	0.1000	0.1900	0.3100	−0.0800	MYR
		1.0000	0.0700	−0.2500	0.1600	0.0400	PHP
			1.0000	−0.1200	0.0900	0.0400	THB
				1.0000	0.1800	−0.1300	ARS
					1.0000	0.2200	DEM
						1.0000	GBP

where the currencies are Argentine Peso (ARS), German Mark (DEM), British Pound (GBP), Hong Kong Dollar (HKD), Malaysian Ringgit (MYR), Philippine Peso (PHP) and Thai Baht (THB). Let I denote the Asian currencies in the matrix, i.e. $I \equiv \{HKD, MYR, PHP, THB\}$. Following Finger (1997), C_{11} is changed to $\tilde{C}_{11}$, whose correlations are set to 0.85, so that the new correlation sub-matrix for Asian currency markets properly describes the market behavior.

By changing only the l⁰s coe¢cients, the new correlation matrix © is

HKD	MYR	PHP	THB	ARS	DEM	GBP	
1:0000	0:8500	0:8500	0:8500	¡0:2600	¡0:1400	0:0600	HKD
	1:0000	0:8500	0:8500	0:1900	0:3100	¡0:0800	MYR
		1:0000	0:8500	¡0:2500	0:1600	0:0400	PHP
			1:0000	¡0:1200	0:0900	0:0400	THB
				1:0000	0:1800	¡0:1300	ARS
					1:0000	0:2200	DEM
						1:0000	GBP

It is shown that © is not a true correlation matrix since its minimum eigenvalue is -0.04. The next step consists of introducing an algorithm, F or N-Ag, to adjust the above correlation matrix in a consistent way. The solution[6] of µ for our example is

$$[0{:}8199; 0{:}7786; 0{:}7026; 0{:}7956]^T :$$

[6] In order to obtain the μ_i 's vector that corresponds to our example, we have used the GAUSS library "Constrained Optimization". In Finger's algorithm, a unique parameter µ is estimated, whose value is 0:7874:

Then, the correlation matrix C_F is

HKD	MYR	PHP	THB	ARS	DEM	GBP	
1:0000	0:8357	0:8661	0:8521	¡ 0:2625	0:1166	0:0443	HKD
	1:0000	0:8520	0:8656	¡ 0:0784	0:2705	¡ 0:0108	MYR
		1:0000	0:8348	¡ 0:2487	0:1990	0:0368	PHP
			1:0000	¡ 0:2058	0:1872	0:0369	THB
				1:0000	0:1800	¡ 0:1300	ARS
					1:0000	0:2200	DEM
						1:0000	GBP

N-Ag provides the following matrix $C^{¤}$ for our example

HKD	MYR	PHP	THB	ARS	DEM	GBP	
1:0000	0:8357	0:8661	0:8521	¡ 0:2620	0:0668	0:0473	HKD
	1:0000	0:8520	0:8656	¡ 0:0263	0:2781	¡ 0:0243	MYR
		1:0000	0:8348	¡ 0:2489	0:1915	0:0374	PHP
			1:0000	¡ 0:1891	0:1684	0:0375	THB
				1:0000	0:1800	¡ 0:1300	ARS
					1:0000	0:2200	DEM
						1:0000	GBP

Comparisons between both algorithms are made by using the mean absolute error (MAE) and the root mean square error (RMSE) as summary statistics, where the error is defined as $\hat{Y}_{12} - Y_{12}$ for $Y_{12} = C_F, C^*$. The summary statistics are:

	MAE	RMSE
F	0.0735	0.1165
N-Ag	0.0592	0.0939

We can observe that N-Ag scores better than F, as expected, under both statistics.

Appendix

Since the objective function and the constraints from program (2) are convex, if $\{b^*_{ij}\}$ verifies Kuhn-Tucker (K-T) conditions then $\{b^*_{ij}\}$ is a global minimum. Moreover, strict convexity of the objective function implies the unicity of such a minimum and, since the feasible set is compact and non-empty, and the objective function continuous, the existence of a solution is always ensured. We can rewrite (2) as

$$
\begin{aligned}
\min_{\{b_{ij}\}} & \sum_{i=1}^{n}\sum_{j=1}^{m} (e_{ij} + b_{ij})^2 \\
s.t. \quad & 2\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}^2 \leq \xi^2 \\
& d_{ij} + b_{ij} - 1 \leq 0 \\
& -(d_{ij} + b_{ij} + 1) \leq 0.
\end{aligned}
$$

The Lagrangian of this problem is given by

$$
\begin{aligned}
L = & \sum_{i=1}^{n}\sum_{j=1}^{m} (e_{ij} + b_{ij})^2 + \lambda_1 \left(2\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}^2 - \xi^2 \right) \\
& + \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_2^{ij} (d_{ij} + b_{ij} - 1) - \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_3^{ij} (d_{ij} + b_{ij} + 1).
\end{aligned}
$$

K-T conditions are:

[i] $\quad 2(e_{ij} + b_{ij}) + 4\lambda_1 b_{ij} + \lambda_2^{ij} - \lambda_3^{ij} = 0; \quad \forall i; j;$

[ii.1] $\quad \lambda_1 \left(2\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}^2 - \xi^2 \right) = 0;$

[ii.2] $\quad \lambda_2^{ij} (d_{ij} + b_{ij} - 1) = 0; \quad \forall i; j;$

[ii.3] $\quad \lambda_3^{ij} (d_{ij} + b_{ij} + 1) = 0; \quad \forall i; j;$

$[iii]\quad \lambda_1 \geq 0; \quad \lambda_2^{ij} \geq 0; \quad \lambda_3^{ij} \geq 0; \quad \forall i,j;$

$[iv.1]\quad 2\sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}^2 - \xi^2 \leq 0;$

$[iv.2]\quad d_{ij} + b_{ij} - 1 \leq 0; \quad \forall i,j;$

$[iv.3]\quad -(d_{ij} + b_{ij} + 1) \leq 0; \quad \forall i,j.$

Solution from K-T conditions:

Note that from $[ii.2]$ and $[ii.3]$, λ_2^{ij} and λ_3^{ij} cannot be different from zero at the same time for any given $i,j.$

We study the following cases according to the possible values of $[iii]$:

Case 1: If for some i,j we have $\lambda_2^{ij} > 0$ (which implies $\lambda_3^{ij} = 0$), then from $[ii.2]$ we obtain that $b_{ij} = 1 - d_{ij} \geq 0$ and, by substituting in $[i]$:

$$2(1 - c_{ij}) + 4\lambda_1 b_{ij} + \lambda_2^{ij} = 0;$$

which is not possible, since all elements are non-negative and at least one, λ_2^{ij}, is strictly positive.

Case 2: If for some i,j we have $\lambda_3^{ij} > 0$, by reasoning in a similar way as in Case 1, we prove that this is not possible.

So, we know that, for all i,j we must necessarily have:

Case 3: for all $i,j \quad \lambda_2^{ij} = 0; \quad \lambda_3^{ij} = 0.$

Case 3.1: If $\lambda_1 = 0$, from $[i]$ we obtain that

$$b_{ij}^* = -e_{ij}; \quad \forall i,j$$

is the solution if and only if $2\sum_{i=1}^{n}\sum_{j=1}^{m} e_{ij}^2 \leq \xi^2.$

Case 3.2: If $\lambda_1 > 0$, this corresponds to the fact that $2\sum_{i=1}^{n}\sum_{j=1}^{m} e_{ij}^2 > \xi^2$

and then, the solution is

$$b^*_{ij} = -e_{ij}/\varphi, \quad \forall i, j$$

where

$$\varphi = \frac{\sqrt{2 \sum_{i=1}^{n} \sum_{j=1}^{m} e^2_{ij}}}{\xi^2} > 1.$$

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