CONTINUOUS-TIME EVOLUTIONARY DYNAMICS: THEORY & PRACTICE*

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A B S T R A C T

This paper surveys some recent developments in the literature which studies continuous-time evolutionary dynamics in the context of economic modeling.

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Jel Classification Number: C73, C62.

1. Evolutionary Dynamics and Equilibrium Analysis

Since Adam Smith (1776) introduced the notion of *natural price* as ì*...the central price, to which the prices of all commodities are continually gravitating...*î it is difficult to think of an idea more extensively applied in economics than the idea of an *equilibrium.* Yet, it is even more difficult to name another concept about which interpretation has been more controversial. Compare, for example, the alternative equilibrium notions proposed by Walras (1874), Cournot (1838), Marshall (1916), Keynes (1936), Arrow and Debreu (1954), Hahn (1973), Lucas (1972) and Cass and Shell (1983). In other words, despite its pervasive use in economic modeling, many questions concerning the foundational aspects of equilibrium analysis remain. Questions that are too serious to be dismissed as mere academic puzzles. Market fluctuations and imperfections are endemic in real-life economics. From the *unemployment equilibria* of Keynesian memory to the various *real business cycle* and *disequilibrium* theories proposed in later literature, the focus on equilibrium analysis has been continuously challenged on various grounds and from different perspectives.

Clearly, game theory cannot avoid considering the foundational aspects of equilibrium analysis, since it is an equilibrium concept, namely *Nash equilibrium,* which has made its fortune. In this respect, we follow Binmore (1987-8) in distinguishing between two alternative justifications of equilibrium analysis which have been maintained by game-theoretic tradition:

ï an *eductive* justification, which relies on the agentsí ability to reach equilibrium through careful reasoning. Since agents are *fully rational,* they can always correctly predict (and optimally respond to) their opponentsí behavior;
ï an *evolutive* justification, which relies on the possibility that *boundedly rational* agents reach equilibrium by means of some adjustment process.

The aim of this paper is to survey this latter methodological approach. In particular, we focus on those papers that follow the evolutive justification using *evolutionary dynamics* to describe how imperfectly rational players adjust their behavior in reaction to a changing environment. In other words, this evolutive approach tries to answer the (supposedly simple) question:

how do people learn to play?

We shall break this grand question into smaller pieces. By doing so, we will introduce the main assumptions on which the evolutionary paradigm followed by this literature is based.

ï *Where do we learn?* The environment in which agents operate is modeled by an infinitely repeated game. Moreover, the set of feasible behaviors coincides with the strategy set of the stage game. In this respect, evolutionary games differ from other strategic frameworks like *differential games* (in which the current payoff is a function of time) or *supergames* (in which strategies are defined over time-paths). Clearly, any justification for such a drastic assumption, apart from mathematical tractability, contains serious weaknesses. It is almost impossible to consider two situations as being absolutely identical,[1] or in which future consequences are completely neglected. However, when interaction is *anonymous* (i.e. it takes place among a large population of agents who have no prior knowledge of the identity, the history, or any other relevant characteristics of their opponents) this framework appears to be more justifiable. This is why the literature has focused almost entirely on this case.

ï *What do we learn (from)?* We follow Selten (1991) in distinguishing three classes of learning models.

(i) *Rote* (individual) learning models, in which success and failure directly influence choice probabilities.

[1]Models of learning in which games are *similar* are those of Li Calzi (1995) and Romaldo (1995).

(ii) Imitation (social) learning models, in which success and failure *of others* directly influence choice probabilities.

(iii) Belief learning models, in which experience has only a direct effect on playersí beliefs.

This survey deals with models of the first two categories, in which players need not know (or care) much about the game they play, other than the payoff they (or other agents in the population) obtain.[2]

ï *How does memory affect learning?* A distinguishing feature of this literature is that *agents have no memory*. All the quantitative features of the adjustment process are completely characterized by the current state of the system.[3]

In their standard form, evolutionary models are based on the assumption that agentsí behavior is genetically encoded in the genes which characterize each agentís *type.* The evolution of a population of competing types is subject to *natural selection,* which links game payoffs (*ìfitnessî)* to growth rates of each type in the population. This evolutionary paradigm has a long tradition in the history of economic thought, from the seminal contributions of Marshall (1916) and Schumpeter (1936) to the work of Nelson and Winter (1982) and their followers. However, despite their intuitive appeal, the interest on evolutionary dynamics has been mostly confined to specialists. There are at least three reasons for this state of affairs:

[2]See Battigalli *et al.* (1992) and Fudenberg and Levine (1998) for comprehensive surveys on belief learning models.

[3]Some might regard this assumption also as unreasonable. Look at the case in which the dynamics exhibit *limit cycles,* as in Ponti (forthcoming). This would imply that agents are not able to recognize the cycle and modify their response accordingly. As Fudenberg and Levine (1998: p. 3) argue: ì*we suspect that if cycles persisted long enough the agent would eventually use more sophisticated inference rules that detected them; for this reason we are not convinced that models of cycles in learning are useful descriptions of actual behavior...î* .

_ lack of *microfoundation,* that is, a formal link between this biological metaphor and explicit models of social interaction;[4]

_ lack of *generality,* because the formal analysis is often restricted to very specific dynamics (namely, the so-called *Replicator Dynamics*);

_ lack of *convergence* results, that is, lack of characterization of the asymptotic properties of the adjustment process. This, essentially, translates into lack of *predictive power.*

The potential of evolutionary dynamics outside the narrow bounds of specialistic interest lies in the extent to which these theoretical gaps are being filled. The aim of this survey is to acknowledge a more general audience of some new theoretical results on these matters which, in our opinion, improve our understanding on the working of evolutionary dynamics and substantially enlarge the feasible fields of applications of evolutionary techniques.[5]

The remainder of this survey is arranged as follows. In section 2 we choose the notation and set up the relation between dynamics and game payoffs. In section 3 we review the literature which derives some evolutionary dynamics starting from explicit models of social interaction. Section 4 deals with convergence results, considering more general adjustment processes (namely, monotonic dynamics) than the classic Replicator Dynamics. Finally, section 5 explores the recent literature on evolutionary dynamics *with drift.* This approach allows more flexibility in describing the dynamics by introducing arbitrarily small perturbations. In consequence, we can study evolutionary models in terms of their *structural stability* properties.

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[4] This is how Bˆrgers (1996) describes the difficulties in applying pure evolutionary techniques to study social evolution. First, it is not practically feasible, given the state-of-the art knowledge in genetics, to derive predictions of human behavior by appealing to its genetic determination. Moreover, the way in which genes affect behavior appears to be very complicated. Finally, the adaptation of human genes occurs too slowly to derive predictions of some interest for the social scientist.

[5]A clear signal of this renewed interest is the growing literature which reviews the state-of-the-art of the discipline. Among others, see Fudenberg and Levine (1998), Samuelson (1997), Vega-Redondo, (1996) and Weibull (1995,1997).

2. CONTINUOUS-TIME EVOLUTIONARY DYNAMICS

Let $\Gamma \equiv \{\Im, S_i, u_i\}$ be a normal form game, where $\Im \equiv \{1,\ldots,I\}$ is the finite set of players with generic element i; $S_i \equiv \{1,\ldots,K_i\}$ is the finite set of player i ís pure strategies with generic elements h and k; $u_i: S \to \Re$ is player $i$ís (VNM) payoff function, with $S \equiv \times_{i\in\Im} S_i$ denoting the set of pure strategy profiles with generic element s. Thus, the set of player $i$ís mixed strategies is the $|K_i - 1|$-dimensional unit simplex $\Delta_i \equiv \{\sigma_i \in \Re_+^{K_i}: \sum_{k\in S_i} \sigma_i^k = 1\}$, with $\Delta \equiv \times_{i\in\Im} \Delta_i$ ($\Delta_{-i} \equiv \times_{j\neq i} \Delta_j$) denoting the set of mixed strategy profiles (of i's opponents). Generic elements of Δ_{-i} and Δ are denoted by σ_{-i} and $\sigma \equiv (\sigma_i, \sigma_{-i})$ respectively. Finally, let Δ^0 (Δ^0_{-i}) be the relative interior of Δ (Δ_{-i}), that is, the set of completely mixed strategy profiles (of i's opponents).

Player $i$ís behavior is described by the mixed strategy she adopts at each point in time, $r_i(t) \in \Delta_i$, with $r(t) \equiv (r_i(t))$ denoting the vector collecting such probabilities.

ASSUMPTION 1. For any given $r(0) \in \Delta$, $r(t)$ evolves according to the following system of continuous-time differential equations:

$$\dot{r}_i^k(t) = f_i^k(r(t)). \tag{2.1}$$

We refer to the autonomous system (2.1) as the *selection dynamics,* i. e. the term that captures the relevant forces governing the playersí strategy revision. Taylor and Jonker (1978) propose two alternative interpretations for the dynamics (2.1).

ï There is a single agent for each playerís position i. At each time t, player i randomly selects a pure strategy $k \in S_i$ using a probability distribution, $r_i(t) \in \Delta_i$. This probability distribution evolves according to (2.1) as a result of some (unmodeled) learning adjustment process.

ï There are I populations of agents, one for each playerís position i. Each agent is genetically programmed to play a pure strategy $k \in S_i$. An (unmodeled) natural selection process adjusts the relative frequencies of each type in each population according to (2.1).

The latter interpretation follows more closely the biological metaphor on which these dynamics have been originally proposed, whilst the former considers these adjustment processes as mimicking some form of *individual learning*. Both these interpretations will be formally derived in section 3.[6]

DEFINITION 1. The function $f \equiv (f_i)$ is said to yield a *regular* dynamic if the following conditions are satisfied:

i) $f_i^k : \Delta \to \Re$ is Lipschitz continuous[7] for all i and k;

ii) $\sum_{k \in S_i} f_i^k(\sigma) = 0$ for all and i and σ;

iii) $\lim_{\sigma_i^k \to 0} \frac{f_i^k(\sigma)}{\sigma_i^k}$ exists for all i and k.

These regularity assumptions imply that growth rates $\gamma_i^k(\sigma) \equiv \frac{f_i^k(\sigma)}{\sigma_i^k}$ are continuous on Δ^0 and that the system (2.1) has a unique solution $\rho(r(0),t)$ through any initial state $r(0)$ which leaves Δ, as well as Δ^0, invariant. In other words, all trajectories starting from Δ (Δ^0) do not leave Δ (Δ^0). This can be interpreted as a ìno creation/no extinctionî property. A pure strategy which is played with positive probability at time zero will also be played in any finite time. On the other hand, if a strategy is not played at time zero, it will never be used.

[6]One of the aims of Taylor and Jonker (1978) was indeed to establish a formal link between the equilibrium concept of Evolutionary Stable Strategy (Maynard Smith and Price (1973) and some evolutionary dynamics which did converge to it. Cressman (1992) provides an excellent survey of this research field.

[7]Sometimes (e. g., in Cressman (1997)) the condition of Lipschitz continuity is replaced by the stronger requirement of continuous differentiability. In this case the dynamics are defined as *smooth regular*.

To complete the description of the dynamics we also need to establish a formal link between the selection process and the game payoffs. We do so by introducing the *Replicator Dynamics* (RD hereafter), that is, a regular dynamic (2.1) where, $\forall k \in S_i, \forall \sigma \in \Delta$,

$$f_i^k(\sigma) = \sigma_i^k \left(u_i(k, \sigma_{-i}) - u_i(\sigma_i, \sigma_{-i})\right). \tag{2.2}$$

For the RD *success breeds success*, since only strategies which pay off ìmore than averageî have a positive growth rate.[8] Nachbarís (1990) *monotonicity* condition should be interpreted in the same spirit.

DEFINITION 2. A function f is said to yield a monotonic dynamic (MD) if $\forall h,k \in S_i, \forall \sigma \in \Delta$,

$$u_i(h, \sigma_{-i}) \leq u_i(k, \sigma_{-i}) \Leftrightarrow \gamma_i^h(\sigma) \leq \gamma_i^k(\sigma) \ . \tag{2.3}$$

Condition (2.3) generalizes an appealing property of the RD. For any given pair of pure strategies, the relative frequency of the more successful grows at a higher rate. Samuelson and Zhang (1992) extend the condition of monotonicity to mixed strategies introducing the notion of *aggregate monotonicity.*

DEFINITION 3. A function f is said to yield an aggregate monotonic dynamic(AMD) if, $\forall \sigma_i, \sigma_i' \in \Delta_i, \forall \sigma^* \in \Delta$,

$$u_i(\sigma_i, \sigma_{-i}^*) \leq u_i(\sigma_i', \sigma_{-i}^*) \Leftrightarrow \sum_{k \in S_i} \gamma_i^k(\sigma^*) \sigma_i^k \leq \sum_{k \in S_i} \gamma_i^k(\sigma^*) \sigma_i'^k \ . \tag{2.4}$$

According to (2.4), if σ_i yields a lower expected payoff than σ_i' against σ_{-i}^*, then the vector $f(\sigma^*)$ should point ìmoreî in the direction of σ_i' than σ_i.

[8]This dynamic was first introduced by Taylor and Jonker (1978). While equation (2.2) refers to the multi-population case (i.e. assigns one set of differential equations to each player position), Taylor and Jonker (1978) deal with the single-population case only. The multi-population RD was first introduced by Taylor (1979). See also Cressman (1992) for a multi-population model in which agents can play mixed strategies.

It follows from the above definitions that $RD \subset AMD \subset MD$.[9] Some useful properties of MD are listed in the following

[9]See Weibull (1995), chapter 5.

PROPOSITION 1. Suppose that f satisfies condition (2.3) and consider the associated MD (2.1).[10]

i) If $\sigma \in \Delta$ is stable,[11] then σ is a Nash equilibrium of Γ.
ii) If $\sigma \in \Delta$ is the limit for some interior solution, then σ is a Nash equilibrium of Γ.

PROOF. See Weibull (1995), Theorem 5.2 *(b,c)*.

3. ECONOMIC MICROFOUNDATION

There are two basic ìstoriesî which have been proposed to justify the use of evolutionary dynamics in the context of economic learning. Not surprisingly, each story follows closely one of Taylor and Jonkerís (1978) interpretations mentioned earlier in this paper. We shall look at each story more in detail in the remainder of this section.

3.1. THE ÌINDIVIDUAL LEARNINGÎ STORY: LEARNING BY REINFORCEMENT

To present this first class of models, we use as a reference the paper by Bˆrgers and Sarin (1997). In their model, agents use *mixed strategies.* A very

[10]As it turns out, Proposition 1 holds for a larger class of evolutionary dynamics than MD, namely *weakly payoff positive dynamics.* The condition of weak payoff positivity requires that at least one pure strategy which yields a payoff above average (provided that such a strategy exists) has a positive growth rate. We restrict our attention to MD, since the results surveyed in this paper refer to this class.
[11]Loosely speaking, a state σ is called (Lyapunov) stable if trajectories starting arbitrarily close stay sufficiently close. If σ is also *attracting,* i.e. is the limit point for all trajectories which start sufficiently close, then σ satisfies the stronger condition of *asymptotic stability.* For more formal definitions, see Weibull (1995).

simple rule links the current payoff with the mixed strategy which will be used in the subsequent round. In particular, pure strategies which perform well against the opponentsí actions are *reinforced,* and the probability with which they are selected grows accordingly.

In contrast with a typical evolutionary biological model, here alternative strategies compete in the agentsí minds as *populations of ideas.* As Bˆrgers and Sarin (1997: p. 3) observe: ì*Decision makers are usually not completely committed to just one set of ideas, or just one way of behaving. Rather, several systems of ideas, or several possible ways of behaving are present in their mind simultaneously. Which of these predominate, and which are given less attention, depends on the experiences of the individual...î.* This approach is not new, as it follows the tradition of Estesí (1950) ìstimulus sampling theoryî of behavioral psychology, subsequently formalized by Bush and Mostellerís (1951,1955) stochastic learning theory, and by the theory of ìadaptive economic behaviorî proposed by Cross (1973, 1983).[12]

In describing Bˆrgers and Sarinís results on the relationship between the stochastic process they analyze and the deterministic dynamics studied in this paper, we modify slightly their assumptions to allow a higher degree of generality. Anna and Beppe are two individuals playing an infinitely repeated game. At each point in discrete time $n \in (0,1,2,\ldots)$, each player selects an action using a probability distribution, $r_i(n)$. It is assumed that, at each round, Anna/Beppe knows only about the action s/he plays and the payoff s/he obtains. Suppose that, at round *n,* Anna has played her pure strategy $k \in S_A$ and Beppe has played his pure strategy $k^* \in S_B$. Under these circumstances, Anna will update her mixed strategy as follows:

$$r_A^k(n+1) = v_A\left((k,k^*),r(n)\right) + \left(1 - v_A\left((k,k^*),r(n)\right)\right)r_A^k(n), \tag{3.1}$$

[12]Papers on reinforcement learning are also those by Bendor *et al.* (1991), Bˆrgers and Sarin (forthcoming), Sarin (1995) and the experimental studies conducted by Roth and Erev (1995), Mookherjee and Sopher (1997) and Erev and Roth (1998).

$$r_A^h(n+1) = \left(1 - v_A\left((k,k^*), r(n)\right)\right) r_A^h(n) \text{ for all } h \neq k, \tag{3.2}$$

with $v_i : S \times \Delta \to (0,1)$ satisfying $v_i(s,\sigma) = \alpha_i(\sigma) + \beta_i(\sigma) u_i(s)$, where $\alpha_i : \Delta \to \Re$ and $\beta_i : \Delta \to \Re_+$ are Lipschitz continuous functions.[13] In words: the change in probability $\Delta r_A^k(n) \equiv \left(r_A^k(n+1) - r_A^k(n)\right)$ is proportional to a given increasing linear transformation of the payoff Anna received in the stage game, with coefficients which may depend on the state variable $r(n)$, up to a rescaling that constrains $r_A(n+1)$ to be in the unit simplex.[14]

Denote by $E\left[\Delta r_A^k | r\right]$ the expected value of $\Delta r_A^k(n)$, given that the state at time n is $r(n) = r$. From (3.1-2) we derive the following :

$$\begin{aligned} E\left[\Delta r_A^k | r\right] &= \left(E\left[v_A\left((k,k^*),r\right)|r\right] + \left(1 - E\left[v_A\left((k,k^*),r\right)|r\right]\right) r_A^k\right) r_A^k + \sum_{h \neq k}\left(\left(1 - E\left[v_A\left((k,k^*),r\right)|r\right]\right) r_A^k\right) r_A^h \\ &= E\left[v_A\left((k,k^*),r\right)|r\right]\left(1 - r_A^k\right) r_A^k - r_A^k \sum_{h \neq k} E\left[v_A\left((k,k^*),r\right)|r\right] r_A^h \\ &= \left(\alpha_A(r) + \beta_A(r) u_A(k, r_B)\right)\left(1 - r_A^k\right) r_A^k - r_A^k \sum_{h \neq k}\left(\alpha_A(r) + \beta_A(r) u_A(k, r_B)\right) r_A^h \\ &= r_A^k \beta_A(r)\left(u_A(k, r_B) - u_A(r)\right), \end{aligned} \tag{3.3}$$

which implies

PROPOSITION 2. For each player $i \in \Im$, the expected motion of the discrete-time dynamics (3.1-2) is aggregate monotonic.[15]

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[13]A similar expression holds for Beppe.

[14]Notice that the probability of the selected action at time n is *always increasing,* i.e. $\Delta r_A^k(n) \equiv v_A\left((k,k^*), r(n)\right)\left(1 - r_A^k(n)\right) > 0$. In other words, for the learning dynamic (3.1-2) every experience is positively reinforced. Also notice that v_i is a function of the state variable σ. Following Bˆrgers *et al.* (1998), this accounts for environmental conditions that may affect the individual learning process.

[15]Since the stochastic process is defined in discrete time, aggregate monotonicity is defined substituting $\frac{\Delta r_i^k(n)}{r_i^k(n)}$ for γ_i^k in (2.4).

3.2. THE ÌCULTURAL EVOLUTIONÎ STORY: LEARNING BY IMITATION

We now move on to the literature which identifies *cultural* (or *social)* evolution with the ability to observe and successfully imitate other agents. These models follow the biological metaphor more closely, since they look at the *aggregate* behavior of a population.

To present this alternative approach, we refer to Schlag (1998). Suppose there are two large populations, one population of Annas and one population of Beppes, playing an infinitely repeated game. Within each round, agents select an action before they play against a randomly matched opponent. Between rounds, each agent knows about the strategy and the payoff of another agent in the same player position, randomly selected by symmetric sampling.[16] Each agent then updates her current strategy using a rule which maps from current payoffs and actions of both sampling and sampled agent to the action which is to be played by the sampling agent in the following round.

Schlag (1998) begins by proposing a class of updating rules that agents might eventually select if required to choose a rule, once and for all, before entering into the matching and sampling scenario. He defends this class of rules on the basis of a set of axioms by which he characterizes bounded rationality. Such rules exhibit the following properties:

ï they are *imitative,* in the sense that an agent never switches to an action that has not been observed in the current period;
ï the probability of switching to an action which performed better is proportional to the difference in payoffs between the action of the sampled agent compared with the action currently used by the sampling agent.

We refer to these rules as *proportional imitation rules.* Under these rules, imitation occurs only if the realized payoff of the sampled agent was *higher*.

[16]By ìsymmetricî sampling the author means a matching scheme in which the probability with which agent *x* samples agent *y* must be equal to the probability with which *y* samples *x*.

More formally, let $X_i \equiv \max_{\substack{h,k\in S_A \\ h^*,k^*\in S_B}} \left|u_i(h,h^*)-u_i(k,k^*)\right|$ be the maximal payoff difference for player *i.* Assume that, at round *n,* an agent in Annaís position, after having played strategy $h \in S_A$ against $h^* \in S_B$, samples an agent who has played $k \in S_A$ against $k^* \in S_B$. Under these circumstances, the sampling agent will revise her strategy from h to k with probability

$$\tilde{\beta}_A\left(u_A(k,k^*)-u_A(h,h^*)\right) \text{ if } u_A(h,h^*)\leq u_A(k,k^*), \text{ and}$$
$$0 \text{ otherwise,}$$

where $\tilde{\beta}_A \in \left(0,\frac{1}{X_A}\right]$ is a fixed constant.

Next step is to consider an environment in which agents of the same population use the same proportional imitation rule and to look at the expected motion of the frequencies with which the various actions are played. By analogy with (3.3), we obtain

$$E\left[\Delta r_A^k | r\right]=\tilde{\beta}_A\left(u_A\left(k,r_B\right)-u\ \left(r\ \right)\right)r_A^k. \tag{3.4}$$

This, in turn, implies

PROPOSITION 3. For each population $i \in \Im$, the expected motion of the discrete-time dynamics (3.4) is aggregate monotonic.

As for the related literature on social evolution, Binmore and Samuelson (1997) propose another model in which agents base their strategy revision on imitation. However, their model has also some similarities with the ìindividual learningî approach of section 3.1, since switching occurs only if the current payoff is lower than the payoff received in the previous round. In other words, the updating rule is based upon an endogenous *aspiration level* equal to the previous round payoff.[17] Under these assumptions, they show that the expected motion of the frequencies with which each pure strategy is played follows the RD. Bjˆrnerstedt and Weibull (1995) consider a model in which agents receive a ìnoisyî signal on the realized payoffs of a sample of other agents in the population. In this case, if the support of the noise is sufficiently large, then the resulting dynamic is monotinic.[18]

3.3. EXPECTED MOTION VS. ASYMPTOTIC BEHAVIOR

We have just derived evolutionary dynamics as expected motions of two alternative stochastic processes based on different models of social interaction. As it turns out, for both models, the same evolutionary dynamics also approximate the stochastic process as the time scale gets to its continuous limit, since both stochastic processes converge in probability to the corresponding (aggregate monotonic) deterministic dynamics. To show this, we shall focus on Bˆrgers and Sarinís (1997) learning model, although a similar result can be proved also for Schlagís (1998) imitation dynamics.[19]

To construct its continuous-time limit, we modify the system (3.1-2) as follows: conditional of any realization (k,k^*) at time n

$$r_A^k(n+1) = \theta v_A\big((k,k^*),r(n)\big) + \big(1-\theta v_A\big((k,k^*),r(n)\big)\big)r_A^k(n), \tag{3.1í}$$

$$r_A^h(n+1) = \big(1-\theta v_A\big((k,k^*),r(n)\big)\big)r_A^h(n), \text{ for all } h \neq k, \tag{3.2í}$$

where $\theta \in (0,1]$ measures the ìreal timeî interval between two repetitions of the game. Let $R^\theta(n) \in \Delta$ define the state of the system at time $n\theta$ for a given initial

[17]Se also the related works on aspiration learning by Bjˆrnerstedt (1993), Banerjee and Fudenberg (1995) and Ponti and Seymour (1997).

[18]See also Cabrales (forthcoming).

[19]See Schlag (1998), Theorem 3.

condition $R^{\theta}(0) \in \Delta$. Consider now the following system of differential equations:

$$f_i^k(\sigma) = \beta_i(\sigma)\left(u_i(s_i^k, \sigma_{-i}) - u_i(\sigma)\right) \tag{3.5}$$

with β_i as in (3.1-2). Since β_i is Lipschitz continuous, the associated dynamic (2.1) is AMD.[20]

To establish the relationship between the discrete-time stochastic dynamics (3.1í-2í) and the continuous-time deterministic dynamics induced by (3.5), we evaluate the continuous-time limit of $R^{\theta}(n)$ at some time $t \geq 0$ for any sequence of θs and ns with the property that $\theta \to 0$ and $n\theta \to t$. In other words, we take limits keeping fixed the ratio between the rate at which players adjust their mixed strategy and the rate at which the time interval shrinks.

PROPOSITION 4. Suppose $R^{\theta}(0) = r(0)$ almost surely. Then, for all $t \geq 0$, $R^{\theta}(n)$ converges in probability to $\rho(r(0), t)$ as $\theta \to 0$ and $n\theta \to t$, where $\rho(\cdot,\cdot)$ is the solution mapping of (3.5).

PROOF. See Bˆrgers and Sarin (1997), Proposition 1.[21]

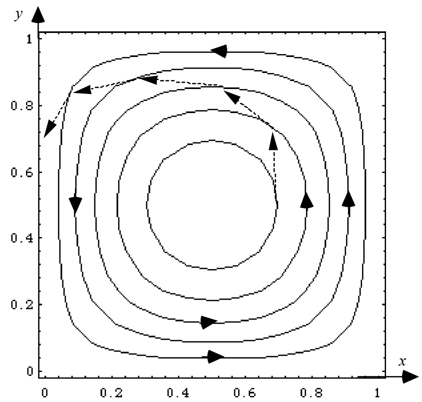
Proposition 4 holds only for any finite time $t \geq 0$. This is to say that the asymptotic properties may differ, depending on whether we consider the stochastic process or its (either continuous or discrete time) deterministic approximation. To clarify this point, consider the asymptotic behavior of Bˆrgers and Sarinís (1997) learning dynamics in the case of zero-sum games.

[20]See Samuelson and Zhang (1992), Theorem 3.

[21]Although Bˆrgers and Sarinís (1997) proof refers to the RD only, its generalization to AMD follows directly from Lipschitz continuity of β_i.

		$1-x$	x
		L	R
y	T	1,0	0,1
$1-y$	B	0,1	1,0

i)



ii)

FIGURE A

Discrete-vs. continuous -time RD and zero -sum-games

Figure A*ii)* traces orbits of the continuous-time RD for the game of Figure A*i)*. The dotted arrows of Figure A*ii)* represent the expected jumps of the discrete time dynamics (3.1-2). As the diagram shows, the continuous time process cycles around the (unique) equilibrium in mixed strategies, whereas the discrete time (deterministic) dynamics does not converge, approaching the boundaries of the state space. Moreover, we also know that the stochastic process (3.1-2).will eventually settle down on one of the four pure strategy profiles, which constitute the set of absorbing states.[22] In consequence, even if the discrete-time stochastic process is well approximated by the continuous-time deterministic dynamics within any finite time interval, the asymptotic properties of the two processes may significantly differ.

4. SOME CONVERGENCE RESULTS

This section reviews some recent results on the convergence properties of dominance solvable games. We also frame this literature by introducing a new concept (which we call τ-dominance) to help the reader understand how these results have been established.

4.1. MD and Strictly Dominated Strategies

We begin by considering the evolutionary properties of strictly dominated strategies. In this respect, it turns out to be crucial how the concept is formally

[22]By (3.1-2) only pure strategy profiles are absorbing states and they are reachable in finite time with positive probability from any interior state.

defined. Conventionally, we say that a pure strategy $h \in S_i$ is strictly dominated if there exists another (pure or mixed) strategy $\sigma_i \in \Delta_i$ which yields a (strictly) higher payoff against all the opponentsí mixed strategy profiles:

$$u_i(h, \sigma^*_{-i}) < u_i(\sigma_i, \sigma^*_{-i}), \forall \sigma^*_{-i} \in \Delta_{-i}. \quad (4.1)$$

Otherwise, to consider strategy h as strictly dominated, we might ask for the stronger requirement of $\sigma_i \in \Delta_i$ being a *pure strategy* itself. If strict dominance is interpreted in this more restrictive sense, we then know that, for all MD, not only strategies which are strictly dominated,[23] but also strategies which do not survive the iterated deletion of strictly dominated strategies, will eventually vanish.

PROPOSITION 5. Suppose that f satisfies condition (2.3) and consider the associated MD (2.1). If $h \in S_i$ does not survive the iterated deletion of pure strategies strictly dominated by pure strategies, then $\lim_{t \to \infty} \rho_i^h(r(0), t) = 0$ for all $r(0) \in \Delta^0$.

PROOF. See Samuelson and Zhang (1992), Theorem 1.

Things are different if we consider strict dominance with respect to mixed strategies. In this case, to obtain the same result as in Proposition 5 we then need to impose some more stringent condition on the dynamic than monotonicity alone.[24] To clarify this point, we provide an example adapted from Dekel and Scotchmer (1992).

[23]This result is due to Nachbar (1990).

[24]For example, Akin (1980) shows that strictly dominated pure strategies vanish along any interior solution of the single-population RD. However, as noted by Akin and Hofbauer (1982), this result does not hold for pure strategies *in the support* of a mixed strategy which is strictly dominated by another mixed strategy.

		$1-x$	x
		L	R
z	T	1,0	0,1
y	M	.4,.4	.4,.4
$1-y-z$	B	0,1	1,0

FIGURE B

An adaptation of Dekel and Scotchmerís (1992) counterexample

The game of Figure B differs from the game of Figure A*i)* only by the fact that Anna (the row player) has an additional strategy (*M)* which yields a payoff of .4 to both players, regardless of what Beppe (the column player) does. Strategy *M* is not strictly dominated by a pure strategy, although it is strictly dominated by any mixed strategy sufficiently ìcloseî to the (unique Nash equilibrium) strategy which attaches probability .5 to strategies *T* and *B*.

Figure C*i)* traces some trajectories of the RD for the game of Figure B. The trajectories of Figure A*ii)* are now *limit cycles*[25] for those of Figure C*i)*, once the strictly dominated strategy *M* has been eliminated. Figure C*ii)* shows trajectories of an MD in which growth rates are as follows:

$$\gamma_i^k(\sigma) = \sqrt{u_i(k, \sigma_{-i})} - \sum_{h \in S_i} \sigma_i^h \sqrt{u_i(h, \sigma_{-i})} \tag{4.2}$$

As the diagram shows, the face $\Phi \equiv \{(x, y, z) \in \Delta | y + z = 1\}$ is an attractor for some interior trajectories (e.g. those of Figure C*ii)*) of the dynamics induced by (4.2). In consequence, the strictly dominated strategy *M* fails to be eliminated.

[25]By limit cycle we mean a periodic solution of (2.1) which attracts some interior trajectory starting sufficiently close to it. For a more formal definition, see Hofbauer and Sigmund (1988).

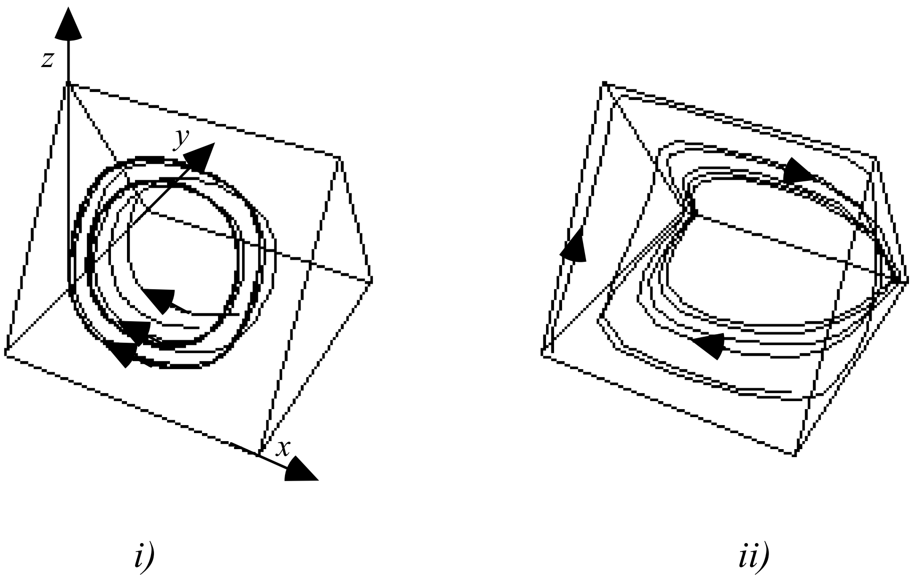


FIGURE C
MD and strictly dominated strategies

Hofbauer and Weibull (1996) show how this behavior is not specific of the functional form (4.2). They consider a class of regular evolutionary dynamics (which they call *functional selection dynamics)* in which growth rates are as follows:

$$\gamma_i^k(\sigma) = \alpha_i(\sigma) + \beta_i(\sigma)\varphi\left[u_i(k, \sigma_{-i})\right], \tag{4.3}$$

with α_i and β_i as in (3.1-2), and φ Lipschitz continuous. The asymptotic behavior of strictly dominated strategies for functional selection dynamics (4.3) is summarized in the following

PROPOSITION 6. Suppose that γ satisfies condition (4.3) and consider the associated functional selection dynamics (2.1). If h does not survive the iterated deletion of pure strategies strictly dominated by mixed strategies and φ is *strictly increasing and convex* then, for all $r(0) \in \Delta^0$, $\lim_{t\to\infty} \rho_i^h(r(0), t) = 0$.

PROOF. See Hofbauer and Weibull (1996), Theorem 2.

If the difference in growth rates is *exactly* proportional to the difference in payoffs, then we have an AMD. In this respect, Proposition 6 generalizes an earlier result of Samuelson and Zhang (1992), showing that aggregate monotonicity is a sufficient condition for the extinction of strategies (iteratively) strictly dominated by mixed strategies.

4.2. MD AND WEAKLY DOMINATED STRATEGIES

We now move on to weak dominance. We restrict our attention to the case of pure strategies which are weakly dominated by other pure strategies, that is, strategies $h \in S_i$ for which there exists another pure strategy $k \in S_i$ which never yields a (strictly) lower payoff against all the opponentsí mixed strategy profiles:

$$u_i(h, \sigma^*_{-i}) \le u_i(k_i, \sigma^*_{-i}), \forall \sigma^*_{-i} \in \Delta_{-i},$$

with $u_i(h, \sigma^*_{-i}) < u_i(k_i, \sigma^*_{-i})$ for some $\sigma^*_{-i} \in \Delta_{-i}$ and, *a fortiori,* $\forall \sigma^*_{-i} \in \Delta^0_{-i}$.

Consider the extensive form game of perfect information of Figure D, known in the literature as the *Entry Game*.[26]

FIGURE D
The Entry Game

In this game, Anna (the potential entrant) has to decide whether to challenge Beppe (playing strategy *D*) under the threat that Beppe (the incumbent) may fight back (playing *d* in return). This would lead to an inferior outcome for both players. She also know that Beppeís threat to fight back is not credible, since her action is observed by Beppe before he has to move and he has no incentive to carry out the threat. The game of Figure D has a Nash (subgame-perfect) equilibrium in pure strategies, namely *(D,c),* and a *component* (that is, a closed

[26]See Selten (1978).

and connected set) of Nash equilibria with the common property that Anna plays

C with probability 1 and Beppe plays his weakly dominated strategy *d* with probability $x \geq 1/3$. Let the symbol NE denote this component, which signals the presence of *alternative best replies* to the equilibrium strategy *C*.[27]

Figure E traces some interior trajectories of the RD for the Entry Game.

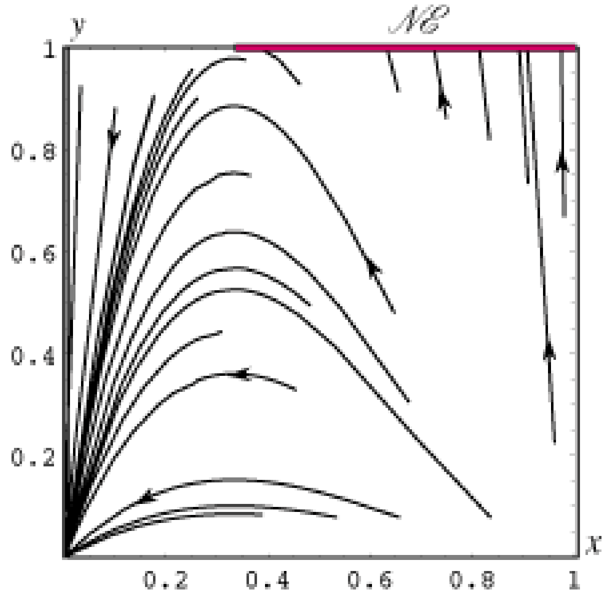


FIGURE E
The RD and the Entry Game

As the diagrams show, there are interior trajectories leading to NE. In other words, for some interior solutions, the playersí limiting behavior may fail to eliminate weakly dominated strategies.

PROPOSITION 7. The subgame perfect-equilibrium (D,c) is the unique asymptotically stable restpoint for the RD. All Nash equilibria in the relative interior of NE are stable and are the limit point of some interior trajectory.
PROOF. See Gale *et al.* (1995), Proposition 1. ð

This result, which contrasts standard game-theoretic analysis, seems counterintuitive also from an evolutionary perspective. In fact, if initial

[27]A detailed account of the dynamic properties of games with alternative best replies is

conditions lie in Δ^0, as it is commonly assumed by the evolutionary literature,[28] weakly dominated strategies will *always* yield strictly lower payoffs than those strategies which dominate them, at least in any finite time. This is essentially because the system will never reach in finite time one of the faces of Δ in which the dominant and the dominated strategy yield the same payoff.

As we know from Proposition 7, this is still not sufficient to ensure the extinction of a weakly dominated strategy. However, if a weakly dominated strategy does not vanish, then all the opponentsë pure strategies against which the dominated strategy yields a lower payoff than the dominant strategy are bound to get eliminated. This result, first proved by Nachbar (1990) in the case of MD which converge to a Nash equilibrium, has been substantially generalized in subsequent works.[29] As it turns out, the same result can be fruitfully applied to analyze the convergence properties of weakly dominance solvable games, once its implications are suitably translated into an alternative notion of dominance.

DEFINITION 4. Fix some regular dynamic (2.1). A pure strategy $h \in S_i$ is said to be strictly τ-dominated by some pure strategy $k \in S_i$ ($h <_\tau k$ hereafter) if we can identify a time τ and a non-empty compact set $C_{-i} \subseteq \Delta_{-i}$ such that

$$\rho_{-i}(r(0),t) \in C_{-i}, \forall r(0) \in \Delta^0, \forall t > \tau, \tag{4.4}$$

$$u_i(h,\sigma_{-i}) < u_i(k,\sigma_{-i}),\ \forall \sigma_{-i} \in C_{-i}. \tag{4.5}$$

Moreover, h is weakly τ-dominated by k ($h \leq_\tau k$ hereafter), if (4.4) holds and we replace (4.5) by the following conditions:

$$u_i(h,\sigma_{-i}) \leq u_i(k,\sigma_{-i}), \forall \sigma_{-i} \in C_{-i}, \tag{4.6}$$

$$u_i(h,\sigma_{-i}) < u_i(k,\sigma_{-i}), \forall \sigma_{-i} \in C^0_{-i}, \tag{4.7}$$

provided by Samuelson (1994).

[28]This assumption is justified by the fact that a strategy that has zero weight at time zero would also have zero weight at all subsequent times. Thus, if initial conditions were not completely mixed, the dynamics would then operate on a different game.

[29]See Cressman (1996), Proposition 3.1 and Weibull (1995), Proposition 5.8.

where $C^0_{-i} \equiv C_{-i} \cap \Delta^0_{-i}$.

Definition 4 establishes a weaker condition of dominance which is defined only with reference to the dynamics under consideration.[30] This definition is based on the existence of a finite point in time, τ, after which (independently of the initial conditions) the system is confined into a compact subspace in which the usual conditions of dominance hold.

Some interesting properties of the asymptotic behavior of τ-dominated strategies are contained in the following propositions.

PROPOSITION 8. Suppose that f satisfies condition (2.3) and consider the associated MD (2.1). If $h <_\tau k$ then, for all $r(0) \in \Delta^0$,

$$\lim_{t\to\infty} \frac{\rho_i^h(r(0),t)}{\rho_i^k(r(0),t)} = \lim_{t\to\infty} \rho_i^h(r(0),t) = 0 \quad .$$

PROOF. See Ponti (forthcoming), Proposition 4.2.ð

Let $\omega(r(0))$ define the $\omega-$limit set of $\rho(r(0),t)$; i.e. $\omega(r(0)) \equiv \left\{ \sigma \in \Delta \middle| \rho(r(0),t_m) \to \sigma \text{ for some sequence } \langle t_m \rangle_{m=1}^{\infty} \right\}$.

PROPOSITION 9. Suppose that f satisfies condition (2.3) and consider the associated MD (2.1). If $h \leq_\tau k$ then

i) $\lim_{t\to\infty} \frac{\rho_i^h(r(0),t)}{\rho_i^k(r(0),t)} \equiv L_i^{(h,k)}(r(0)) \geq 0$ for all $r(0) \in \Delta^0$;

*ii)*if $L_i^{(h,k)}(r(0)) > 0$ then $u_i(h,\sigma_{-i}) = u_i(k,\sigma_{-i})$, for all $\sigma_{-i} \in \omega_{-i}(r(0))$;

iii) if $j \leq_\tau h$ then $j \leq_\tau k$.

PROOF. See Ponti (forthcoming), Proposition 4.1.ð

[30]If h is strictly (weakly) dominated by k, then h is also strictly (weakly) τ-dominated by k. In this case, $\tau = 0$ and $C_{-i} = \Delta_{-i}$.

By (4.8) the ratio $\frac{\rho_i^h(r(0),t)}{\rho_i^k(r(0),t)}$ must converge, whether player i's mixed strategy converges or not. By (4.9), if $\frac{\rho_i^h(r(0),t)}{\rho_i^k(r(0),t)}$ converges to a positive constant, this implies that both pure strategies *h* and *k* must yield the same payoff against all mixed strategy profiles in $\omega_{-i}(r(0))$. Finally, (4.10) ensures that the weak τ-dominance relation is transitive, as is the ìclassicalî definition of dominance.

Proposition 9 better explains to which extent the intuition ìdomination implies extinctionî holds and how this is related to the performance of strategies *h* and *k in the limit*. In particular, the extinction of a weakly dominated strategy *h* is guaranteed only if, in the limit, its relative performance is *uniformly worse* (i.e. *h* is strictly τ-dominated).

We provide the reader with an application of Propositions 8-9 to prove convergence in the case of the Entry Game

PROPOSITION 10. Suppose that f satisfies condition (2.3) and consider the associated MD (2.1) in the case of the game of Figure D. For all $r(0) \in \Delta^0$, $\rho(r(0), t)$ converges to a Nash equilibrium.
PROOF. In the Appendix.

In the remainder of this section, we review some theoretical results which apply similar techniques to study the convergence properties of other weakly dominance solvable games.

FINITELY REPEATED PRISONERÍS DILEMMA. Cressman (1996) shows that, in the finitely repeated Prisonerís Dilemma, all interior trajectories of the RD converge to a Nash equilibrium, that is, an outcome equivalent to the unique subgame-perfect equilibrium by which both players to defect at all stages.

TWO-PLAYER EXTENSIVE FORM GAMES OF PERFECT INFORMATION WITH DISTINCT PAYOFFS. Also for these games the use of backward-induction (or the iterative deletion of weakly dominated strategies) selects a unique subgame-perfect Nash equilibrium outcome. Cressman and Schlag (1998) restrict their analysis to the RD and prove (among other properties) the following

THEOREM 1. every interior path converges to a Nash equilibrium.

THEOREM 2. For "simpleî games, (games of perfect information in which at most three consecutive decisions are made), the Nash equilibrium component which contains (i.e. is outcome-equivalent to) the backward induction solution is the unique interior asymptotically stable set.

Theorem 1 identifies a class of games for which an equilibrium notion (namely, Nash equilibrium) accurately describes the asymptotic play of a particular evolutionary dynamic (namely, the RD). However, this result does not support more stringent equilibrium requirements like, for example, subgame-perfection. Non subgame-perfect Nash equilibria may be limit points of a non-zero measure set of interior trajectories, as we already know from Proposition 7.

We also learn from Theorem 2 that the Nash equilibrium component NE of Figure E cannot be *asymptotically stable,* although it is in the limit set of the RD. Trajectories starting arbitrarily close to NE move away from it and never come back, where the same phenomenon does not occur when we consider the subgame-perfect equilibrium *(D,c).* Finally, asymptotic stability of the backward induction solution is guaranteed by Theorem 2 only for games that are *simple* in Cressman and Schlagís terminology. For more complex games such an asymptotically stable set may even fail to exist.

MD AND THE CENTIPEDE GAME. The results we just reviewed have been proved for the RD only . However, in his evolutionary analysis of the Centipede Game (a game of perfect information with distinct payoffs) Ponti (forthcoming) shows how Propositions 8-9 can be used to generalize all the results above to MD. As we noticed in the introduction, this generalization allows more flexibility in the use of continuous-time dynamics outside the field of evolutionary biology, where the specific form of the RD is used to mimic a stylized reproductive process.

5. EVOLUTIONARY DYNAMICS WITH DRIFT

This section deals with some recent papers that employ continuous-time dynamics to approximate *perturbed* adjustment processes. Here the evolutionary dynamics (2.1) are slightly modified to account for the imperfections that may interfere with the selection process.

This methodology has been prompted by the vast literature on discrete-time stochastic processes *with noise.*[31] In these models, the stochastic process takes the form of an ergodic Markov chain. Ergodicity is obtained by introducing a noise term, which makes every state reachable with some positive probability within a finite time. In a biological context, this noise may be interpreted as a *mutation,* i. e. a random alteration of the agentsí genetic code. In a learning context, this noise can be interpreted as a *mistake,* i. e. a random alteration of the agentsí behavior, or an effect of the playersí experimentation.

The formal steps to obtain a continuous-time deterministic dynamic starting from a discrete-time stochastic process with noise involve approximation techniques similar to those we already used in section 3. Consider *I* populations of fixed size *P* whose members occasionally revise their strategy according to some (unmodeled) learning process. Let us further assume that the expected state of the system at time $n+\theta$, given that the state at time n is $r(n)=r$ can be written as follows:

$$E\left[r(n+\theta)|r\right]=F(r)+\lambda G(r). \tag{5.1}$$

To interpret (5.1), we can think of F as the selection dynamics and G as the noise term. Samuelson (1997: p. 172). justifies the presence of this perturbation on the ground that: ì...*like any model, the selection process is an approximation,*

[31]See, for example, Kandori *et al.* (1993) and Young (1993). Vega Redondo (1996) provides a comprehensive survey on this research field.

designed to capture the important features of a problem, while excluding other considerations...î.[32] By analogy with (3.3), if the derivatives of F and G are Lipschitz continuous, we can take a Taylor expansion of (5.1) to obtain

$$E[\Delta r | r] = \theta(f(r) + \lambda g(r)) + o(\theta^2), \tag{5.2}$$

where f and g are the derivatives of F and G. Divide both sides by θ, taking limits for $P \to \infty$ and $\theta \to 0$, such that $\frac{\Delta r}{\theta} \to c > 0$ as $\theta \to 0$ to get

$$\dot{r}(t) = f(r(t)) + \lambda g(r(t)), \tag{5.3}$$

that is, a ìperturbedî version of the dynamics (2.1), provided λ (the *drift level)* is sufficiently small.[33]

Gale *et al.* (1995) use a special case of (5.3) to study the evolutionary properties of the *Ultimatum Game*. In this game Anna offers Beppe a share of some fixed cake. If Beppe accepts the offer, then the pie is shared as agreed; if Beppe rejects the offer, nobody gets anything. This game has a unique subgame-perfect equilibrium in which Anna offers (an ε more than) nothing and Beppe accepts. The intuition is the same as in the Entry Game: if Anna knows that Beppe is rational, she can rely on the fact that Beppe will accept anything, no matter how little it is. In fact, there is a clear analogy between the two games. If we restrict the possible offers to *high or low,* assuming that a high offer is automatically accepted by Beppe, then the Ultimatum Game is strategically equivalent to the Entry Game of Figure D.[34]

The Ultimatum Game is a game for which the backward induction hypothesis is universally rejected by the experimental evidence, although the various

[32]See also Boylan (1995) and Seymour (1994) for a more detailed account of the technicalities presented in this section.

[33]This reflects the fact that all the major forces governing the dynamics should be captured by f. The terminology of *drift* (as opposed to *noise*) highlights the fact that the latter is a random variable, whereas the former is a purely deterministic dynamic.

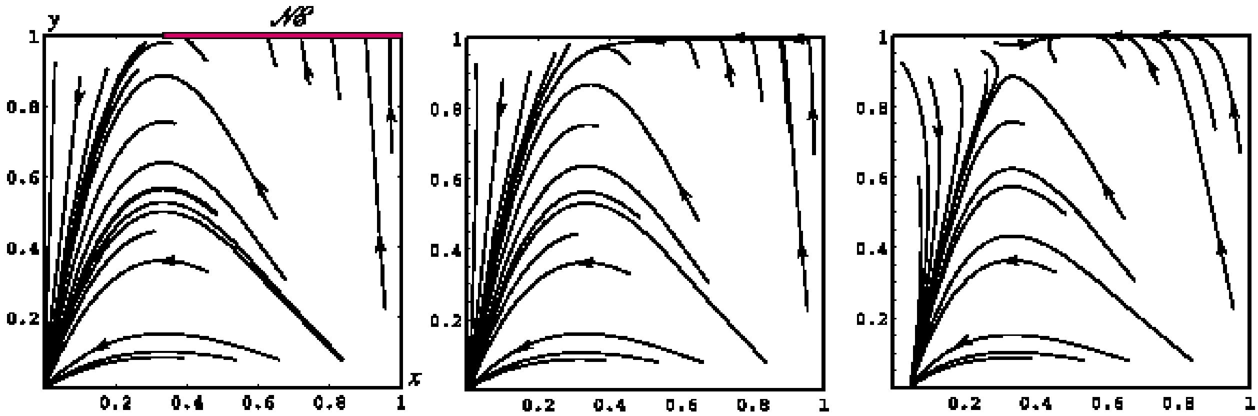
[34]This is why Gale *et al.* (1995) refer to the game of Figure D as the *Ultimatum Minigame.*

experimental results provide no clear alternative hypothesis.[35] To explain the fundamental weaknesses of backward induction in the context of the Ultimatum Game, Gale *et al.* (1995) propose the following dynamics:

$$\dot{r}_i^k(t) = r_i^k(t)\left(u_i\left(k, r_{-i}(t)\right) - u_i(r(t))\right) + \lambda_i\left(\mu_i^k - r_i^k(t)\right), \lambda_i \geq 0, \mu_i^k = \frac{1}{|K_i|}. \quad (5.4)$$

In Gale *et al.* (1995), the dynamic (5.4) is derived from a population game in which agents die (or leave the game, or experiment new ways of playing) at a fixed rate $\lambda_i dt$. Those who die are replaced by *novices* (or *experimenters)* who play each strategy k with equal probability $\frac{1}{|K_i|}$, while the aggregate behavior of the rest of the population follows the RD.

Figure F traces some trajectories of (5.4) for the Entry Game with different drift levels.



i) $\lambda_i = 0$ *ii)* $\lambda_i = 0.001$ *iii)* $\lambda_A = 0.001; \lambda_B = 0.1$

FIGURE F
RD with drift and the Entry Game

Figure F*i)* shows trajectories of the RD without drift that mimic the behavior already shown in the phase diagram of Figure E. Figure F*ii)* shows trajectories of (5.4) when both λ_A and λ_B are ìnegligibleî. In this case, the drift against

[35] A detailed accoun on the experimental evidence on the Ultimatum Game is provided by Roth (1995).

Beppeís weakly dominated strategy *d* is sufficient to push the system away from NE. In Figure F*iii)* λ_B is substantially higher than λ_A. In this case, the system (5.4) has two restpoints close to NE, one of which is asymptotically stable. In other words, although the drift points toward the relative interior of the state space, this may not be sufficient to destabilize the Nash equilibrium component in which a suboptimal action is played with positive probability.

It is possible to show that this behavior is not specific of the drift parametrization of Figure F*iii)*. To show this, we replicate Gale *et al.* (1995) results fixing $\mu_B^C \equiv \mu$, $\lambda_A = \lambda_B = \lambda$ and letting $\lambda \to 0$. In other words, we prove that their conclusions are robust to a different specification of the drift term in which we do not fix the mixed strategy μ played by the mutants, but let the drift level be arbitrarily small.

PROPOSITION 11. Let $\hat{RE}(\mu)$ be the set of restpoints of (5.4) for λ sufficiently close to 0.

a) For all $\mu \in (0,1)$, $\hat{RE}(\mu)$ contains the subgame-perfect equilibrium *(D,c)*, which is also asymptotically stable.
b) When μ is sufficiently large, $\hat{RE}(\mu)$ contains also two additional restpoints, both belonging to NE, one of which is asymptotically stable.

PROOF. In the Appendix.

Similar considerations apply when we consider (an appropriate finite normal form of) the full Ultimatum Game. In this case, Gale *et al.* (1995) show, with the aid of simulations, how the dynamic (5.4) yields as constant prediction one of the Nash equilibria in which Anna offers a positive share of the cake and Beppe accepts. In other words, the system converges to an outcome in which the first-mover advantage is not fully exploited by the proposer (and, therefore, the subgame perfect prediction is violated).

As Binmore and Samuelson (1999) put it, in the Ultimatum Game *drift matters,* as arbitrarily small perturbations yield dramatic changes in the dynamic properties of the game. In particular, the existence of an asymptotically stable equilibrium belonging to NE for a non-negligible set of admissible perturbations weakens the subgame-perfect prediction. If initial conditions are sufficiently close, Beppeís ìincredible threatî may be *sustainable* even in the presence of perturbations.[36]

6. CONCLUSION

Although promising, the literature reviewed in this paper leaves many questions unanswered, challenging the discipline with new puzzles. For example, further theoretical work (as in Ritzberger and Weibull (1997)) is needed on the convergence properties of evolutionary dynamics outside the class of dominance solvable games. Similar considerations hold for the literature on perturbed evolutionary dynamics, whose results (with the sole exception of Binmore and Samuelson (1999)) still refer to specific classes of games and dynamics.

Above all, now that a formal (although preliminary) microfoundation of these dynamics has been established, its empirical relevance remains open to discussion. That is, to which extent the behavioral models presented in section 3 are capable of solving the *grand questions* from which we started.

[36]See also Cabrales and Ponti (forthcoming) and Ponti (1998) for the evolutionary properties of Nash equilibrium refinements, such as subgame-perfection or iterated deletion of weakly dominated strategies, in the context of implementation theory.

APPENDIX

PROOF OF PROPOSITION 10. To prove the proposition, it is enough to show that all interior trajectories converge. This is because, once convergence has been proved, convergence to a Nash equilibrium follows directly from Proposition 1*ii)*.

Fix a generic initial condition $r(0) \in \Delta^0$. First note that Beppe has a weakly dominated strategy (namely, *d)*. Thus, by Proposition 9*i)*, $\frac{\rho_B^d(r(0),t)}{\rho_B^c(r(0),t)} \to L_B^{(d,c)}(r(0)) \geq 0$ as $t \to \infty$. This already implies convergence of $\rho_B(r(0),t)$, since S_B contains only two strategies. Two alternatives need be discussed:

i) $L_B^{(d,c)}(r(0)) \geq 1/2$. That is, $L_B^{(d,c)}(r(0))$ is at least as high as the threshold value for $\frac{\rho_B^d(r(0),t)}{\rho_B^c(r(0),t)}$ that makes Anna indifferent between her pure strategies *C* and *B*. This implies $D \leq_\tau C$ (fix $\tau = 0$ and $C_B = \{x \in [0,1] | x \geq 1/3\}$), which in turn implies, by Proposition 9*i)*, convergence of $\rho(r(0),t)$ to a Nash equilibrium.[37] More precisely: if $L_B^{(d,c)}(r(0)) \geq 1/2$, then $\rho(r(0),t) \to \mathrm{NE}$. This is because, by Proposition 9*ii)*, $L_B^{(d,c)}(r(0)) > 0$ implies $\rho_A^D(r(0),t) \to 0$.

ii) $L_B^{(d,c)}(r(0)) < 1/2$ (i.e. $L_B^{(d,c)}(r(0)) = 1/2 - \varepsilon$). This implies $C <_\tau D$ (fix $\tau = \left\{ t \geq 0 \middle| x = \frac{1-\varepsilon/2}{3} \right\}$ and $C_B = \{x \in [0,1] | x \leq \frac{1-\varepsilon/2}{3}\}$) and, by Proposition 8, $\rho_A^C(r(0),t) \to 0$ (i.e. $\rho_A^D(r(0),t) \to 1$). This in turn implies $d <_\tau c$ (i.e. convergence to the subgame-perfect equilibrium (D,c)).

Since this exhausts all cases, the result follows.

[37] $D \leq_\tau C$ also when $L_B^{(d,c)}(r(0)) = 1/2$. This is because, by weak dominance of *d*, $\frac{\rho_B^d(r(0),t)}{\rho_B^c(r(0),t)}$ is decreasing in t for all $t > 0$. In consequence, $u_A(D, \rho_B(r(0),t)) < u_A(C, \rho_B(r(0),t))$ for all $t > 0$.

PROOF OF PROPOSITION 11. For the game of Figure D, the RD with drift is as follows.

$$\dot{y} = y(1-y)(3x-1) + \lambda(\frac{1}{2} - y), \tag{A.1}$$

$$\dot{x} = x(1-x)(y-1) + \lambda(\mu - x). \tag{A.2}$$

Denote by $RE(\Gamma)$ the set of restpoints of (A.1-2) when $\lambda = 0$, that is, the set of restpoints of the RD. It is straightforward to show that $RE(\Gamma)$ contains (together with all the pure strategy profiles) only the component $RE^1 = \{(x,y) \in \Delta | y = 1, x \in [0,1]\}$.

We know, from Binmore and Samuelson (1999), Proposition 1, that every limiting rest point of (A.1-2) as $\lambda \to 0$ must lie in $RE(\Gamma)$. Only two cases need be discussed.

CASE 0: $\lambda \to 0$ and $y \to 0$. This yields (0,0) and (1,0) as possible candidates for the limit points in $\hat{RE}(\mu)$. The first (second) point is (not) a limiting restpoint of (A.1-2) since it is a sink (source) of the unperturbed dynamics. We also know, from Binmore and Samuelson (1999), Proposition 2, that (0,0) must be asymptotically stable, since it is a sink of the unperturbed dynamics. This completes part *a)* of the proof.

CASE 1: $\lambda \to 0$ and $y \to 1$. Setting $\dot{y}=0$ in (A.1) yields the following:

$$\frac{1-y}{\lambda} = \frac{y - 1/2}{y(3x-1)}. \tag{A.3}$$

Denote by x^1 a limiting value for x in a rest point, if a limit exists, when $y \to 1$. It must be

$$\lim_{\substack{y \to 1 \\ \lambda \to 0}} \frac{1-y}{\lambda} = \frac{1}{2(3x^0 - 1)} \tag{A.4}$$

Setting $\frac{\dot{y}}{\lambda}$=0, substituting $\frac{(1-y)}{\lambda}$ with the right hand side of (A.3) and taking limits leads to the following solutions for x^1:

$$\hat{x}^1 = \frac{1+6\mu+\sqrt{1-28\mu+36\mu^2}}{10} \quad \text{and} \quad \check{x}^1 = \frac{1+6\mu-\sqrt{1-28\mu+36\mu^2}}{10}.$$

We know from (A.2) that x^1 must be a real, positive number, with $1/3 < x^1 < \mu$. For the expression under the square root of the numerator to be nonnegative, it must be $\mu \in [(7+2\sqrt{10})/18, 1]$. To study the stability properties of $\hat{x}^1$ and $\check{x}^1$ we look at he Jacobian matrix for the dynamic (A.1-2):

$$J(x,y,\lambda) = \begin{bmatrix} (3x-1)(1-2y)-\lambda & 3y(1-y) \\ x(1-x) & (1-2x)(y-1)-\lambda \end{bmatrix}.$$

We evaluate trace and determinant of $J(x,y,\lambda)$, factorizing for λ and substituting λ, y, $\frac{(1-y)}{\lambda}$ with their limiting values. The limiting trace of $J(x,y,\lambda)$ equals to $1-3x^1$, which is negative for all feasible x^1 . The sign of the limiting determinant of $J(x,y,\lambda)$ coincides with the sign of the following expression:

$$\psi(x^1) = (3x^1-1)(1-2x^1) + 2(3x^1-1)^2 - 3x^1(1-x^1), \tag{A.5}$$

which is positive only in the feasible domain of $\hat{x}^1$. In consequence, $\hat{x}^1$ is asymptotically stable whereas $\check{x}^1$ is not. This completes part *b)* of the proof.ð

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