COMPETITION AMONG AUDITIONEERS*

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A B S T R A C T

In this paper, we analyse a multistage game of competition among auctioneers. In a first stage, auctioneers commit to some publicly announced reserve prices, and in a second stage, bidders choose to participate in one of the auctions. We prove existence of Nash equilibria in mixed strategies for the whole game. We also show that one property of the equilibrium set is that when the numbers of auctioneers and bidders tend to infinity, almost all auctioneers with production cost low enough to trade announce a reserve price equal to their production costs. Our paper confirms previous results for some "limit" versions of the model by McAfee [9], Peters [13], and Peters and Severinov [18].

KEYWORDS: Auctions; Perfect Competition; Large Markets.

1 Introduction

In this paper, we study a multistage game of competition among auctioneers. In the first stage auctioneers compete for a common pool of bidders by means of credible announcements of the minimum price accepted in a second price auction. In a second stage, that we call the *entry game*, each bidder chooses an auction, if any, to participate. Finally, in the last stage, each of the announced auctions takes place among all those bidders that have chosen it. This time structure was originally suggested by McAfee [9] in his pioneer work on competition among auctioneers.

We provide two results in this model. First, we prove that there always exists an equilibrium of the whole game. Second, we show that almost all the auctioneers with types low enough to trade announce a reserve price equal to their respective production costs with probability arbitrary close to one in equilibrium when the numbers of auctioneers and bidders go to infinity.

Existence of an equilibrium is not at all obvious in this game. The reason is that there are no standard existence theorems for multistage games. Instead, we proceed analysing the game backwards. We shall show that the last stage, each of the games generated by the auctions, has a natural equilibrium. We assume this equilibrium and then characterise the equilibrium of the entry game. We shall show that the equilibrium of the entry game is unique among the equilibria in which all the bidders use the same entry strategy, i.e. among the symmetric equilibria. Moreover, this equilibrium changes continuously with respect to the vector of reserve prices announced by the auctioneers.

The above two points make straightforward the proof of existence of an equilibrium in the whole game. Uniqueness means that we can define in an obvious way the reduced game of competition among auctioneers: evaluating the auctioneers' payoff functions at the unique symmetric equilibrium of the entry game. Continuity assures that the auctioneers' payoff functions of the reduced game are continuous. Hence, we can apply standard theorems to prove existence of an equilibrium in the reduced game of competition among auctioneers. Our characterisation of the solution of the entry game also provides a tractable way of computing the limit payoffs in the reduced game of competition among auctioneers. We use these limit payoffs to prove our convergence result.

The intuition of our limit result is better understood considering first the single auctioneer case. Myerson [11] shows that in this case it is optimal for the auctioneer to fix a reserve price above his production cost with generality. This strategy has a cost because it means that the auctioneer does not trade with bidders that have valuations between the auctioneer's production cost and the reserve price. These losses are, however, more than offset by the increase in the price that bidders with valuation above the reserve price pay.

Two features of our model explain why this result does not hold when the numbers of auctioneers and bidders go to infinity. The first one is that the unique equilibrium of the entry game is such that bidders expect to pay the same price conditional on winning in all the auctions which they enter. The second one is that changes in a single reserve price have a negligible effect on the price that bidders expect to pay in the other auctions when the numbers of auctioneers and bidders approach to infinity.

Hence, when an auctioneer increases his reserve price the effect on the expected price that the bidders with valuation above the reserve price pay conditional on winning should vanish as the market increases to the limit. This means that the positive effect of increasing the reserve price above the production cost disappears in the limit. However, the negative effect of losing profitable trades still remains. Hence, the auctioneer does not have incentives to distort trade by announcing reserve prices above his production cost in the limit.

Note, however, that when there is a finite number of auctioneers and bidders the auctioneer can still have incentives to increase the reserve price above his production cost. In this case, an increase of the reserve price means that some bidders move to other auctions and, hence, it increases the expected price in these other auctions. This increase in the level of expected prices in the other auctions implies that in equilibrium our auctioneer's expected price must also increase. This explains why auctioneers can find profitable to fix a reserve price above the production cost.

Burguet and Sákovics [6] have shown that this is the case when there are only two auctioneers. Consequently, our result implies that this monopolistic distortion vanishes as the numbers of auctioneers and bidders tend to infinity. Note that Burguet and Sákovics also prove existence of an equilibrium in the whole game when there are only two auctioneers. Our proof supersedes their proof since we allow for more than two auctioneers.

Our limit result is aligned with those by McAfee [9], Peters [13], and Peters and Severinov [18]. They show that in a game of competition among auctioneers similar to ours but with infinite numbers of auctioneers and bidders assumptions,[1] there exist an equilibrium in which each auctioneer always fixes a reserve price equal to his production cost. But, there are two differences with respect to our model. First, our result is based on the convergence properties of the equilibrium set when the numbers of auctioneers and bidders tend to infinity. Whereas, these other papers study an equilibrium of limit games defined under the assumption that the numbers of auctioneers and bidders are infinite.

Second, we provide a kind of uniqueness equilibrium prediction. We show that the limit of the set of equilibria characterises uniquely the outcome of the game up to a negligible fraction of auctioneers. In these other papers, only existence results are provided. One exception is the paper by Peters and Severinov [18]. They also provide uniqueness results although they limit to equilibria in which all the auctioneers announce the same reserve price. Our uniqueness result consider equilibria in which different auctioneers announce different reserve prices.

Peters [15, 16] also looks to related models under some infinite number of agents assumptions. The first of the papers deals with the private value assumption with correlated types, and the second with the common value assumption. Our model only covers the private value assumption with independent types.

[1] McAfee [9] does not exactly assume that the numbers of auctioneers and bidders are infinite. Instead, McAfee assumes that an auctioneer does not take into account that when he changes his mechanism, the expected utility that bidders can get in other auction mechanisms changes. McAfee justifies this assumption conjecturing that it should be true in the limit with infinite number of auctioneers and bidders. We say in this sense that this is an infinite number of agents assumption.

All these papers study limit games, the problem is whether the equilibrium of the limit game approximates the equilibrium of the finite game for large enough numbers of auctioneers and bidders or not. Our paper shows that the answer to this question is yes. Note that this answer is quite important since a negative answer will mean that the analysis of the limit game would make much less sense.

The closest of these papers to our model is [18]. As they explain, their analysis implies that if there exists a Nash equilibrium in which all the auctioneers announce the same reserve price for large numbers of auctioneers and bidders, then this equilibrium should be close to the limit equilibrium that they propose. Nevertheless, even if we assume that all the auctioneers are identical as Peters and Severinov do, we can show that there exists no such Nash equilibrium for finite numbers of auctioneers and bidders with generality.

Another value added of our paper is that we are able to characterise the entry game through a set of equations and to prove using these equations that the solution of the entry game is well behaved in some sense. A similar analysis has been provided by Peters and Severinov [18] and Burguet and Sákovics [6], but in both cases the result limits to cases in which there are no more than two different reserve prices announced by the auctioneers. In fact, the complexities of the analysis of the entry game arise when we allow for more than two different reserve prices.

The importance of the former contribution is that the understanding of such entry games is the first step towards the analysis of models of decentralised trade in which there is heterogeneity in both market sides. This kind of models have presented severe difficulties in similar set-ups to ours. This is for instance the case of the model of price competition of Peters [17], or the model of contract competition of Peters [14]. In both cases, the complexity of the associated entry game has forced to allow for heterogeneity in one side of the market at most. Hopefully, our analysis can give new insights for such models.

>From a related perspective, it is interesting to remark that our model offers a very natural way of studying decentralised trade. Each auctioneer announces a supply curve for the only unit that he has. This supply curve is characterised by the reserve price of a second price auction. The bids of the bidders that participate in each auction form the demand curve for the only unit for sale. In this sense, we can say that each of the auctions constitutes a *local market* in which the auctioneer's supply curve is crossed with the bidders' demand curve to determine the associated price.

One of the features of the second price auction is that to announce the true value of the good is an equilibrium strategy for the bidders, i.e. to announce their true demand. Moreover, we show in this paper that as the numbers of auctioneers and bidders go to infinity, the fraction of auctioneers with production cost low enough to trade that announce a reserve price different to their production cost tends to zero, i.e. in the limit almost all relevant auctioneers announce their true supply curve. This means that in the limit almost all the relevant local markets converge to a competitive outcome.

Another interesting question is whether the *global market*, this is the market of all the auctioneers and bidders, converges to a competitive outcome or not. In this sense, we can say that the global market is not competitive in two senses. The first one is that the price does not converge to a competitive price. The price in each of the auctions is a random variable that it is not only determined by the global demand and supply curves but also by the random entry strategy of bidders. The second is that the allocations are not competitive. Since the equilibrium involves all the bidders randomising entry it could happen that some auctions do not receive any bidder, even if they have a relatively low production cost. Moreover, it could also be that some other auctions receive several high valuation bidders. Then, only one of the bidders wins the auction whereas the others are rationed.[2]

It is interesting to consider the source of this non competitive result contrasting with two other types of model of competition. Consider first models that study centralised institutions of trade with the structure of an auction. This is for instance the case of the paper by Satterthwaite and Williams [21], and the paper by Williams [25]. These authors have shown that the incentives to misrepresent the preferences in a double auction disappear when the numbers of auctioneers and bidders tend to infinity. Hence, we can infer that the frictions that preclude a competitive outcome are due to our assumption that trade happens in a decentralised fashion.

Consider now models of Bertrand competition, and more precisely, under the assumption that the sellers have capacity constraints as it is the case in our model in which each auctioneer has only one unit. Recall that this is the variation suggested by Edgewort. One characteristic of Bertrand-Edgeworth competition is that it does not imply that the competitive price prevails in equilibrium with generality. Note that Burguet and Sákovics [6] have shown that in our model this is also true. On the other hand, when the numbers of auctioneers and bidders tend to infinity, the market price and the allocation converge respectively to the competitive price and the competitive allocation with probability one, see the paper by Allen and Hellwig [1].

This perspective suggests that it is not only the decentralisation of trade what explains the inefficiency of the type of competition that we propose. However, note that the frictions of decentralised trade are usually minimised in the Bertrand-Edgeworth set-up by the assumption that if a buyer is rationed by one seller she can turn to other sellers. In fact, we can deduce from the analysis of Peters [17], and Burdett, Shi and Wright [5] that once we introduce our assumption that buyers can attend to no more than one buyer, the same inefficiencies as in our model arise in a model of Bertrand-Edgeworth competition, even with an infinite number of buyers and sellers.

Finally, notice that our paper also relates to another branch of papers, those that deal with mechanism design under common agency, see for instance Stole [23]. In these papers, several principals design simultaneously an optimal mechanism for the same agent. Although mechanism design is a more general set-up that includes auctions, in order to allow for such generality these models only consider one single agent. Our model differs in that we allow for more than one agent, this is we allow for more than one bidder.

We start with a description of the model in Section 2. In order to solve the game we proceed backwards. In Section 3, we solve the second stage, the entry game. We

[2] These inefficiencies in the global market have been already pointed out by Peters [13] for a limit version of our model with infinite numbers of auctioneers and bidders.

use the solution of the entry game to compute the reduced game of competition of auctioneers. We study this reduced game in Section 4. Section 5 provides the limit results of our model. Section 6 concludes. We also provide an Appendix with one proof of Section 3 and the main proofs of Section 5.

2 The Model

There are $J \in \mathbb{N}$ auctioneers and $kJ \in \mathbb{N}$ bidders. We shall later consider the limit $J \to \infty$. When doing this, we shall keep the ratio $k > 0$ of bidders to auctioneers fixed.

Each auctioneer has the ability to produce a single indivisible unit of output. We assume that each auctioneer j observers his own production cost w_j before the beginning of the game, whereas the other auctioneers (and bidders) only know that it is drawn independently from the set $[0, 1]$ according to a probability distribution function H which is the same for all auctioneers.

Each bidder wishes to purchase exactly one unit of the commodity. Each bidder i observes her reservation prices x_i privately before the beginning of the game. All other players only know that reservation prices are independently drawn from the set $[0, 1]$ according to the same distribution function F with a density[3] f and support[4] $[0, 1]$.

If an auctioneer j with production cost w_j trades with a bidder i with type x_i at a price p, they are assumed to obtain a von Neumann Morgenstern utility of $p - w_j$ and of $x_i - p$ respectively. In the case that there is no trade, both the auctioneer and the bidder get a von Neumann Morgenstern utility of 0. Notice that this assumption implies that the production occurs, and production costs are incurred, only once a trade has been agreed. The production cost could also be seen as an opportunity cost.

We consider a three stage game. In the first stage, each auctioneer announces an auction rule. For most of the paper we assume that auctioneers can only choose second price auctions without entry fees. Their only choice variable is the reserve price in their auction. Auctioneers make these choices simultaneously. Once each auctioneer has chosen his reserve price the choices are made public.

In the second stage, that we call the *entry game*, each bidder can either pick one and only one auction[5] in which she wants to participate, or she can choose to participate

[3] Note that we are imposing more structure to the distribution function of the bidders' types than to the distribution function of the auctioneers' types. More precisely we assume that F has a density and a convex support. We introduce these assumptions to simplify the auctioneers' payoff function that the corresponding bidding game induces. Note also that if F has no density, the revenue equivalence between the first price auction and the second price auction is not assured, even in the single auctioneer case. Hence, the results that we claim in the Subsection 3.1 could not hold.

[4] The assumption that the support of F equals $[0, 1]$ implies that we do not consider situations in which the production cost of an auctioneer is below the minimum valuation of the bidders. We believe that our model could be extended to cover this case. The only required modification would be that in the limit, as J tends to infinity, auctioneers with production costs below the lower bound of the support of F would set reserve prices between their production costs and this lower bound, rather than equal to their production costs. This fact has already been mentioned by Peters [13].

[5] We believe that our results could be easily extended to the case in which bidders can participate in more than one auction under the following additional assumptions. Each bidder has a constant marginal utility for a finite number of units and zero for additional units. The number of units from which the bidder obtains strictly positive utility is a finite number greater than the maximum number

in no auction. In the final stage those bidders who have chosen to participate in some particular auction make their bids in that auction.

Notice that it is a weakly dominant strategy in the final stage to bid one's true value. This is independent of the number of other bidders in the auction. Therefore, it is unimportant whether the outcome of the second stage is observed before the third stage begins.

The most obvious restrictive assumption in our model is that auctioneers can only choose second price auctions without entry fees. We make that assumption for simplicity. However, we shall show later that our results extend to the case in which auctioneers cannot only choose second price auctions but also first price auctions. Note that in this case it might matter whether the outcome of the second stage is observable or not. This is because optimal bidding behaviour in a first price auction depends on the number of other bidders participating in that auction. Hence, we shall also consider that each auctioneer can choose whether the number of bidders in his auction becomes common knowledge before the third stage begins or not. We shall explain why the main results which we show for the basic version of our model also hold for this extended version.

Obviously, it would be desirable to analyse a model in which the auctioneers' strategy space is even larger. For example, one would like to allow the auctioneers to announce other standard auctions which treat all bidders symmetrically, such as all pay auctions, or second price auctions with entry fees, for instance as McAfee [9] and Peters [13] do. In addition, one could allow auctioneers to choose auctions which treat bidders asymmetrically, for example by allowing only some but not all bidders to participate. Finally, it is potentially important to consider mechanisms which condition on the mechanism choice by other auctioneers, for example by including rules which are similar to "price matching clauses", see for instance Epstein and Peters [7]. We do not know whether our results extend to the case in which auctioneers are allowed to choose from these more general classes of mechanisms.

The reason why it is easy to introduce first price auctions into the auctioneers' strategy space, but difficult to extend the strategy space further, is somewhat subtle. If a second price auction with reserve price is replaced by a first price auction with the same reserve price, then the equilibrium entry pattern and allocation rule remain unchanged.[6] Therefore, by the revenue equivalence theorem, the auctioneer's expected revenue stays the same. Now suppose that we allowed the auctioneers to choose in addition second price auctions with entry fees. We could still find an entry fee which generates the same entry pattern and allocation rule as a second price auction with reserve price, and hence yields the same expected revenue by the revenue equivalence theorem. However, the appropriate entry fee would now depend on the choices of all

of auctions that the bidder can enter. Under these assumptions it is still true that it is weakly dominant for the bidder to bid her true value of the good. If these assumptions are not met then there is no straightforward solution for the bidding game, and hence, we cannot extend easily our analysis.

[6] If auctioneers offer symmetric mechanisms, we shall restrict attention to symmetric equilibria of the entry game and of the bidding stage. If we allowed asymmetric equilibria in either of these two stages, the revenue equivalence theorem would not even allow us to generalise our analysis to first price auctions.

other auctioneers. As soon as there is uncertainty about the other auctioneer's choices, for example because of private information about their production costs, we cannot rule out that an auction with entry fee yields higher expected revenue than an auction with reserve price, and thus that the choices which constitute equilibria in the restricted strategy space are no longer equilibria in the extended strategy space. Obviously, asymmetric auctions might generate asymmetric entry patterns or allocation rules, and hence the revenue equivalence theorem does not apply, and we cannot be certain of any relation between the equilibria which we identify here, and the equilibria of a game in which auctioneers are allowed to choose asymmetric auctions.

The fact that we assume that the bidders know their own type at zero cost is also restrictive. With this assumption we disregard situations in which information acquisition is an issue. In a more general model we could distinguish two kinds of information acquisition costs: those due to an external information acquisition technology, and those which the auctioneer can influence. Peters and Severinov [18] analyse a model of competition among auctioneers in which they allow for the latter type of information acquisition cost.

Our assumption that types are known from the beginning of the game implies that each bidder can condition her entry decision on her type. The fact that these types are privately known implies that the entry game is a game of incomplete information. The same entry game has been studied previously under the assumption that it is common knowledge that bidders are identical at the stage of choosing an auction (Peters and Severinov [18]).

We study the game using backward induction. Since we have restricted the selling mechanisms that can be used in the third stage game to second price auctions, the solution of this game is trivial. We assume that bidders play the unique weakly dominant strategy, to bid their true value. Hence, in equilibrium the bidder with highest valuation among those that have entered the auction and bid above the auctioneers' reserve price, wins the auction and pays a price equal to the maximum of the second highest valuation and the reserve price announced by the auctioneer. This fully determines the bidders' expected utility of participating in an auction given the entry decisions of the other bidders. With these bidders' payoffs we can define the reduced game that bidders play in the second stage, the entry game. We solve this game in the next section.

3 The Entry Game

In this section we study the second stage game. In this game, bidders choose the auction that they will attend, if any, after observing the auctioneers' announced reserve prices. We shall show that this game has a unique symmetric Nash equilibrium and that this equilibrium is continuous in the auctioneers' reserve prices. We shall use the first result to define the reduced game that the auctioneers play in the first stage in a straightforward manner and the second result will assure the continuity of the auctioneers' payoffs in this auctioneers' game. We shall also characterise the symmetric equilibrium of the entry game in a way which facilitates the proof of the convergence result in Section 5.

Bidders take their entry decision conditioning on the vector of reserve prices announced by the auctioneers, $\vec{r} \in [0,1]^J$, and on their private types. For notational convenience we shall assume that the elements of the vector of reserve prices are ordered increasingly. The expected utility of entering an auction given the entry decisions of the other bidders are computed assuming that the bidders bid the true value of the good. We restrict attention to equilibria in which all the bidders play the same entry strategy, possibly mixed. This means that two bidders with the same type assign in equilibrium the same probability of entering to a given auction.

Although the restriction to symmetric equilibrium is a standard practice, it is clearly restrictive in this game. To understand these restrictions it is useful to consider the following example. Assume that there are two second price auctions with no reserve price and two bidders both with the same valuation $x > 0$. It is trivial to show that this game has three Nash equilibria: a symmetric equilibrium in which each bidder enters each of the auctions with the same probability, and two asymmetric equilibria in which each bidder enters a different auction.

This example shows in particular that the symmetric equilibria of the entry game may be Pareto dominated by the asymmetric equilibria.[7] On the other hand, the asymmetric equilibria seem to require that bidders co-ordinate their entry behaviour. Therefore, by restricting attention to the symmetric equilibria of the entry game we are implicitly assuming that frictions prevent bidders from co-ordinating their entry decisions. This is probably a reasonable assumption for many markets, mainly those in which the numbers of auctions and bidders are large. This assumption has also been made in other papers like those by McAfee [9], Peters and Severinov [18], and Peters [13] that have studied similar models of competition among auctioneers.

We characterise the (possibly random) entry decision of the bidders with a function $\pi : [0,1] \times [0,1]^J \to [0,1]^J$. This function gives a vector of probabilities of entering each of the auctions $\pi(x;\vec{r})$ for a bidder with type x given the announcement of reserve prices $\vec{r}$. We denote the j-th component of this vector by $\pi_j(x,\vec{r})$. Define the set E_j to be the closure of the interior of the set $\{x : \pi_j(x,\vec{r}) > 0\}$. Moreover, define the cut-off valuation for a given auction j to be equal to $y_j = \min\{x : x \in E_j\}$. Then:

Lemma 1. *A symmetric Nash equilibrium of the entry game must satisfy for all auctions j, l:*

(a) $E_j = [y_j, 1]$.

(b) If $r_l > r_j$, *then* $y_l \geq y_j$ *and if* $r_l = r_j$, *then* $y_l = y_j$.

(c) If $r_l \geq r_j$. *then for almost all* $x \geq y_l$, $\pi_j(x,\vec{r}) = \pi_l(x,\vec{r})$.

Proof. See [13, Lemma 2]. ■

[7] Peters [13] provide examples of other asymmetric equilibria when there are many bidders with different types and many auctioneers announcing different reserve prices.

In the following, we shall call strategies of the type described in Lemma 1 "cut-off strategies". Lemma 1 thus says that any symmetric Nash equilibrium of the entry game must be an equilibrium in cut-off strategies. One surprising feature of cut-off strategies is that bidders who enter several auctions with positive probability always randomise uniformly among these auctions. We shall provide some intuition for this feature, later, following Lemma 3.

Our next goal is to derive necessary and sufficient conditions for cut-off strategies to constitute an equilibrium. We begin with the following lemma:

Lemma 2. *If all the bidders play the same cut-off strategy, the probability that a bidder with type $x \geq y_j, y_l$ wins if she enters auction j is the same as the probability that this bidder wins if she enters auction l.*

Proof. A bidder with type x can win in equilibrium a given auction if and only if each of the other bidders either has a valuation below x, or enters any of the other auctions. The first condition is trivially the same in auction l and in auction j. The second condition holds with the same probability for both auctions given that each bidder randomises uniformly among all the auctions that she enters. ■

Now recall the following standard result:

Lemma 3. *The expected utility of a bidder with type x in a second price auction is a continuous convex function (of x) which is almost everywhere differentiable with first derivative equal to the probability of winning that auction for a bidder with type x.*

Proof. See [11] and also [13, Lemma 1]. ■

Suppose that all the bidders play the same cut-off strategies. Then, the last two lemmas imply that if a bidder with a type $x \geq y_j, y_l$ is indifferent between auction j and auction l, she will also be indifferent between auction j and auction l for all types above x. This result depends on the bidders randomising uniformly among all the auctions that they attend. Otherwise, both the probability of winning and the expected utility will differ in auction l and auction j. This also explains why in equilibrium if bidders randomise among the auctions that they attend, they must randomise uniformly. Otherwise, bidders will have incentives to deviate.

The above paragraph suggests a way in which the task of checking whether a given cut-off strategy constitutes a symmetric Nash equilibrium can be simplified. One of the conditions which one needs to check for this is that bidders who enter different auctions with positive probability are indifferent between these auctions. The above paragraph indicates that it is sufficient to check this condition for the smallest type which enters two auctions, and that then all bidders with higher type will automatically also be indifferent.

Typically, the smallest type which enters two auctions with positive probability will be a cut-off point. In fact, we shall show in the next Lemma that necessary and sufficient conditions for a cut-off strategy to constitute a symmetric Nash equilibrium can be constructed which refer only to the incentives of bidders with cut-off types.

Our conditions will compare the expected price paid by a bidder with a type equal to an arbitrary cut-off y_j conditional on winning in auction j with the same conditional expected price in auction $j-1$. Given that Lemma 2 says that the probability of winning is the same in both auctions we are in fact comparing the expected utility of entering auction j and auction $j-1$.

In order to formalise these conditions we first introduce a function Ψ_{j-1}, where $\Psi_{j-1}(x, y_{j-1}, y_j, ..., y_J)$ is the expected price paid by a bidder with type x conditional on winning auction $j-1$, and given that all the other bidders play some cut-off strategies represented by $y_1, ..., y_J$. Note that we only allow Ψ_{j-1} to depend on the cut-offs $y_{j-1}, ..., y_J$. The reason is that changes in the other cut-offs do not affect entry in auction $j-1$ and hence, do not affect the expected price in that auction. We shall restrict the domain of Ψ_{j-1} to $x \geq y_{j-1}$, $y_l \geq r_l$ for all $l \geq j-1$, and $y_{j-1} \leq y_j \leq ... \leq y_J$. Other values do not make sense in an equilibrium in cut-off strategies.

In the following Lemma, the first condition has an obvious meaning. Point (ii) says that a bidder i with type $x_i = y_j$ is indifferent between auction j and auction $j-1$ if $y_j < 1$. Recall, that y_j is the minimum type that enters auction j. Hence, a bidder with type y_j only wins auction j if no other bidder enters auction j and then she pays r_j. Similarly, point (iii) says that a bidder i with type $x_i = y_j$ weakly prefers auction $j-1$ to auction j if $y_j = 1$ this is that bidder i enters with zero probability auction j.

Lemma 4. *A necessary and sufficient condition for a Nash equilibrium in cut-off strategies is that each cut-off y_j is greater than or equal to r_j and satisfies that:*

(i) If $r_j = r_1$, then $y_j = r_1$.

(ii) If $r_j \neq r_1$ and $y_j < 1$, then $r_j = \Psi_{j-1}(y_j, y_{j-1}, y_j, ..., y_J)$.

(iii) If $r_j \neq r_1$ and $y_j = 1$, then $r_j \geq \Psi_{j-1}(y_j, y_{j-1}, y_j, ..., y_J)$.

Proof. We start by showing that our conditions are sufficient. Since we impose that $y_j \geq r_j$ for all j, all bidders who enter an auction get non-negative expected utility. Hence they do not have incentives to stay out of the market. Point (i) guarantees that the minimum type that participates in any auction is r_1. Since bidders with types below r_1 cannot profitably trade in the market, point (i) assures that these types do not have incentives to deviate and enter an auction. Hence, we only need to show that points (ii), and (iii), imply that: (*) a bidder with a given type is indifferent among all the auctions which she enters with positive probability conditional on her type; and (**), a bidder with a given type does not gain from entering auctions which she does not enter with positive probability conditional on her type.

By the definition of cut-off strategies (*) says that bidder i with a type $x_i \geq y_j$ (if $y_j < 1$) must be indifferent between all auctions l such that $l \leq j$. Point (ii) implies the indifference of bidder i conditional on a type $x_i = y_j$ between auction j and auction $j-1$. Lemma 2 and Lemma 3 thus say that bidder i with type $x_i \geq y_j$ is indifferent between auction j and auction $j-1$. We can apply the same argument to show that bidder i conditional on type $x_i \geq y_j$ is indifferent between auction $j-1$ and auction

$j-2$. Repeating this argument, we can show that bidder i is indifferent among all the auctions $l \leq j$.

If we take account of the definition of cut-off strategies, condition (**) says that a bidder i with type $x_i \in [y_l, y_j)$ cannot improve by deviating and entering auction j. This claim holds trivially if $x_i \leq r_j$. Consider the case $x_i > r_j$. If x_i were equal to y_j and $y_j < 1$ (the case $y_j = 1$ is considered below), the expected utility of entering auction j would be the same as the expected utility of entering auction l because of (*). Hence we only need to prove that the derivative of the expected utility of entering auction l with respect to the type is not larger than the derivative of the expected utility of entering auction j with respect to the type for bidder i with type $x_i < y_j$. If bidder i deviates and enters auction j, she cannot do better than bidding x_i. In this case, she only wins if no other bidder enters auction j, and then she pays the reserve price r_j. This implies that the derivative with respect to the type of the expected utility that bidder i can achieve in auction j equals the probability that no other bidder enters auction j. According to Lemma 2 this probability equals the probability that bidder i wins auction l if she had type $x_i = y_j$. On the other hand, Lemma 3 says that the derivative of the expected utility of entering auction l is the probability that bidder i wins auction l with her true type x_i. Since this type is lower than y_j, the probability of winning is lower with this type. This proves that the derivatives verify the required condition.

If $y_j = 1$, then point (iii) implies that a bidder i with type $x_i = y_j$ weakly prefers auction $j-1$ to auction j. We can show as in the above paragraph that this implies that bidder i with type $x_i = y_j$ weakly prefers auction l, for $l < j$, to auction j. Hence, we can repeat the argument above.

Finally, we show that the points (i)-(iii) are necessary. Point (i) is trivial. Suppose that there is a cut-off $y_j < 1$ (and $r_j \neq r_1$) for which (ii) does not hold, this is that bidders with type y_j strictly prefer entering auction $j-1$ to entering auction j. Then, the continuity of the bidder's expected utility in the bidder's type, implied by Lemma 3, means that there must exist a non-empty interval of types $[y_j, y')$ that strictly prefer entering auction $j-1$ to entering auction j. Therefore, these types have incentives to deviate. We can proceed symmetrically in the case that bidders with type y_j ($y_j < 1$ and $r_j \neq r_1$) strictly prefer entering auction j to entering auction $j-1$.

We prove that (iii) is necessary in a similar fashion. Suppose that bidders with type $y_j = 1$ strictly prefer entering auction j to auction $j-1$. First, note that this can only be if $y_{j-1} < 1$, otherwise, types $y_j = 1$ would prefer auction $j-1$ because by assumption $r_{j-1} < r_j$. Then, the continuity of the bidders' expected utility guaranties that there exists a set of types $(y', 1]$, such that $y' \geq y_{j-1}$, that strictly prefer auction j to auction $j-1$. Again, these types would have incentives to deviate. ■

In order to solve the condition in Lemma 4 for the cut-offs, we first give an explicit formula for Ψ_{j-1} and derive some of this function's properties. We begin by introducing the following notation. Consider a bidder i who follows a cut-off strategy π, and a type x with $x \geq y_1$. Let auction l be the auction which has the highest index among all auctions in which a bidder with type x participates with positive probability. Then we denote by $z(x;\pi)$ the probability that the bidder i either does not submit a bid in auction l or that she has a type below x. This probability is given by:

$$z(x;\pi) = 1 - \frac{F(y_{l+1}) - F(x)}{J\,\bar{G}^J(l)} - \sum_{q=l+1}^{J} \frac{F(y_{q+1}) - F(y_q)}{J\,\bar{G}^J(q)} \tag{1}$$

where $\bar{G}^J(l)$ is the fraction of auctioneers that announce a reserve price equal or below the l-th highest reserve price, and where[8] $y_{J+1} \equiv 1$.

We can now construct the conditional distribution function of the price paid by bidder i with type x conditional on winning in auction $j-1$, supposing, of course, that $x \geq y_{j-1}$. If all bidders other than some bidder i follow the same cut-off strategy π, then the probability that bidder i with type $x \geq y_{j-1}$ wins auction $j-1$ is $z(x;\pi)^{kJ-1}$. This implies that for $x \neq 0$ and weakly above y_{j-1}, and $\tilde{x} \in [y_{j-1}, x]$ the probability that the price in auction $j-1$ is below $\tilde{x}$ given that bidder i with type x wins auction $j-1$ is $z(\tilde{x},\pi)^{kJ-1}/z(x,\pi)^{kJ-1}$. It also implies that the probability that no other bidder enters auction $j-1$ conditional on bidder i winning that auction equals $z(y_{j-1},\pi)^{kJ-1}/z(x,\pi)^{kJ-1}$. In this last case bidder i pays the reserve price r_{j-1}.

Denote by $\nu_{j-1}|_x$ the conditional distribution function of the price paid by bidder i with type $x \neq 0$ conditional on winning in auction $j-1$. Then we can summarise the arguments in the preceding paragraph with the following formal description of $\nu_{j-1}|_x$:

- If $\tilde{x} < r_{j-1}$, then $\nu_{j-1}|_x(\tilde{x}) = 0$.
- If $r_{j-1} \leq \tilde{x} < y_{j-1}$, then $\nu_{j-1}|_x(\tilde{x}) = \frac{z(y_{j-1};\pi)^{kJ-1}}{z(x;\pi)^{kJ-1}}$.
- If $y_{j-1} \leq \tilde{x} < x$, then $\nu_{j-1}|_x(\tilde{x}) = \frac{z(\tilde{x};\pi)^{kJ-1}}{z(x;\pi)^{kJ-1}}$.
- Otherwise, $\nu_{j-1}|_x(\tilde{x}) = 1$.

Hence for $x \geq y_{j-1}$:

$$\Psi_{j-1}(x, y_{j-1}, y_j, ..., y_J) = \int_{-\infty}^{+\infty} \tilde{x}\, d\nu_{j-1}|_x(\tilde{x}). \tag{2}$$

Using this formula, we can now obtain some useful properties of Ψ_{j-1}.

Lemma 5. *The function Ψ_{j-1} is continuous in all its arguments, strictly increasing in x, and in all cut-offs y_j, y_{j+1}, ..., y_J, and strictly decreasing in y_{j-1}.*

Proof. In order to prove the continuity of Ψ_{j-1} with respect to a parameter that affects the distribution function $\nu_{j-1}|_x(.)$ we only need to show that this distribution function $\nu_{j-1}|_x(.)$ changes continuously with respect to the parameter of interest in all the points of continuity of the distribution function $\nu_{j-1}|_x(.)$ [4, Theorem 25.8, p. 335]. The continuity of this distribution function in these parameters follows from the continuity of F.

[8] Note that the formula which we have given has on the right hand side one minus the probability of the event which is complementary to the event described in the text.

We prove that Ψ_{j-1} is monotonic with respect to the parameters by showing that changes in the parameters produce shifts of the distribution function $\nu_{j-1}|_x(.)$ in the sense of first order stochastic dominance. It is easy to see that a decrease in y_{j-1} or an increase in x shifts the distribution function $\nu_{j-1}|_x(.)$ in the sense of first order stochastic dominance downwards. An increase in y_l for $l > j-1$ decreases the ratio $z(\tilde{x};\pi)/z(x;\pi)$, as one can verify through differentiation, and hence it also shifts the distribution function $\nu_{j-1}|_x(.)$ in the sense of first order stochastic dominance downwards. ■

It seems worthwhile to explain the intuition behind the monotonic properties in Lemma 5. That Ψ_{j-1} is increasing in x is self-explanatory. Next, Lemma 5 says that an increase in the minimum type that enters auction $j-1$, say from y_{j-1} to y'_{j-1}, keeping other things constant, decreases the price that a bidder i with type x expects to pay conditional on winning auction $j-1$. To understand this result note that the price that i pays only changes if the maximum type of the other bidders that enters auction $j-1$ with cut-off y_{j-1} is between y_{j-1} and y'_{j-1}. If the cut-off is y'_{j-1}, then the price is fixed by this maximum type of the other bidders whereas, if the cut-off is y'_{j-1}, the price equals the reserve price r_{j-1}. Since y'_{j-1} is strictly above r_{j-1}, it explains the decrease in the expected price.

The effect of an increase in a cut-off associated to another auction l to which bidders with type x enters, say $y_l < x$ to y'_l, is slightly different. Then, the only difference in the price that bidder i with type x pays when she wins occurs under the following event: a bidder with type $\tilde{x} \in (y_l, y'_l)$ is the bidder with maximum type among those bidders that enter auction $j-1$ when the cut-off is y'_l, and this bidder enters auction l when the cut-off is y_l. This means that the price that i pays when the cut-off is y'_l is $\tilde{x}$, and the price that i pays when the cut-off is y_l is below $\tilde{x}$.

More subtle is the effect of an increase in a cut-off associated to another auction l to which bidders with type x do not enter, say y_l ($y_l \geq x$) to y'_l. A bidder i with type x does not win auction $j-1$ under the event that there is another bidder with type between y_l and y'_l that enters auction $j-1$. But, the probability of this event is higher when the cut-off is y'_l than when the cut-off is y_l. The reason is that bidders with these types enter with higher probability to auction $j-1$ when the cut-off is y'_l than when the cut-off is y_l because in the latter case these types also enter auction l. As a consequence, the probability that the other bidders have a type between y_l and y'_l conditional on the event that bidder i wins with a type x is lower when the cut-off is y'_l than when the cut-off is y_l. Hence, the probability that the other bidders have types between y_{j-1} and x conditional on the event that i wins with a type x is higher when the cut-off is y'_l than when the cut-off is y_l. This implies that moving y_l to y'_l should produce a downwards shift in the sense of first order dominance to the distribution of number of entrants in auction $j-1$ conditional on bidder i wins with a type x. This increase of entry explains why the expected price that i pays increases.

We apply the results of last lemma to show that there is a unique solution to the conditions of Lemma 4. We start by proving the existence of an implicit function that relates y_J and y_{J-1}.

Lemma 6. *If $r_J > r_1$, then for each $y_{J-1} \in [r_{J-1}, 1]$, there exists a unique func-*

tion $\psi_J(y_{J-1}) \in [y_{J-1}, 1]$ *such that* $y_J = \psi_J(y_{J-1})$ *satisfies condition (ii) and (iii). Moreover,* $\psi_J(y_{J-1})$ *is continuous and strictly increasing if* $\psi_J < 1$*, and satisfies* $\psi_J(y_{J-1}) = y_{J-1}$*, if* $r_{J-1} = r_J$*.*

Proof. Define the function $\Delta(y_J) \equiv \Psi_{J-1}(y_J, y_{J-1}, y_J) - r_{J-1}$ for a given value $y_{J-1} \in [r_{J-1}, 1]$. Lemma 5 says that $\Psi_J(x, y_{J-1}, y_J)$ is continuous and strictly increasing in x and in y_J. This implies that $\Delta(.)$ must be continuous and strictly increasing. Since $\Psi_{J-1}(y_{J-1}, y_{J-1}, y_{J-1}) = r_{J-1} \le r_J$, then $\Delta(y_{J-1}) \le 0$, with equality when $r_{J-1} = r_J$. Hence, either: (*) $\Delta(1) > 0$ and then there exists a unique $\psi_J(y_{J-1}) \in [y_{J-1}, 1)$ such that $\Delta(\psi_J(y_{J-1})) = 0$; or (**) $\Delta(1) \le 0$. In case (*), $y_J = \psi_J(y_{J-1})$ satisfies condition (ii), and in case (**) let $\psi_J(y_{J-1}) \equiv 1$, then $y_J = \psi_J(y_{J-1})$ verifies condition (iii). Note also that if $r_{J-1} = r_J$, then $\psi_J(y_{J-1}) = y_{J-1}$. The monotonic properties of Ψ_{J-1} stated in Lemma 5 also imply that ψ_J is strictly increasing under case (*), this is when $\psi_J < 1$. ■

Now, assume that there exist some functions $\{\psi_l\}_{l=j+1}^{J}$ where $y_l = \psi_l(y_{l-1})$ and that have the same properties as ψ_J. The next lemma shows that then there exists a function ψ_j such that $y_j = \psi_j(y_{j-1})$ that relates y_j with y_{j-1} with the same properties.

Lemma 7. *SUPPOSE that there exist some functions* $\{\psi_l\}_{l=j+1}^{J}$ *such that* $\psi_l : [r_{l-1}, 1] \to [y_{l-1}, 1]$ *and that each function* ψ_l *gives* y_l *as a function of* y_{l-1}*. Assume also that these functions are continuous, and strictly increasing if* $\psi_l < 1$*.*

THEN, if $r_j > r_1$*, for each* $y_{j-1} \in [r_{j-1}, 1]$*, there exists a unique function* $\psi_j(y_{j-1}) \in [y_{j-1}, 1]$ *such that* $y_j = \psi_j(y_{j-1})$ *satisfies condition (ii) and (iii). Moreover,* $\psi_j(y_{j-1})$ *is continuous and strictly increasing if* $\psi_j < 1$*, and satisfies* $\psi_j(y_{j-1}) = y_{j-1}$*, if* $r_{j-1} = r_j$*.*

Proof. The sequence of functions $\{\psi_l\}_{l=j+1}^{J}$ defines each y_l (l>j) as a continuous and increasing function $\omega : [r_j, 1] \to [r_l, 1]$ of y_j where $\omega_l(y_j) \equiv \psi_l \circ \psi_{l-1} \circ ... \circ \psi_{j+1}(y_j)$. The properties of each function ψ_l assure that $y_J \ge y_{J-1} \ge ... \ge y_j$. Then, we can substitute these functions ω_l in the conditions (ii) and (iii), and we get the following two conditions:

- If $r_j \ne r_1$ and $y_j < 1$, then $r_j = \Psi_{j-1}(y_j, y_{j-1}, \omega_{j+1}(y_j), ..., \omega_J(y_j))$.
- If $r_j \ne r_1$ and $y_j = 1$, then $r_i \ge \Psi_{j-1}(y_j, y_{j-1}, y_j, \omega_{j+1}(y_j), ..., \omega_J(y_j))$.

We can apply the arguments in the proof of Lemma 6 to show that these conditions define the required function with the properties stated in the lemma. ■

Corollary 1. *The equilibrium cut-off strategy computed in the lemma above is invariant to changes in the indexes of the reserve prices of the vector* $\vec{r}$*.*

One direct implication of Lemmas 6 and 7 is that we can prove by induction that there exists a set of increasing functions $\psi_2, ..., \psi_J$ that give y_j as a function of y_{j-1} for all $j > 1$. Hence, according to Lemma 4, the first part of the next proposition follows (and so we omit the proof). The second part proves continuity of the equilibrium with respect to the auctioneers' reserve prices.

Proposition 1. *There exists a unique symmetric Nash equilibrium of the entry game. The associated cut-offs of this equilibrium are defined by* $y_j = \psi_j \circ \psi_{j-1} \circ ... \circ \psi_2(r_1)$*. These equilibrium cut-offs change continuously with respect to the vector of announced reserve prices* $\vec{r}$*.*

Proof. The equilibrium cut-offs are the unique solution of a set-of equations (conditions (i), (ii), and (iii)). The functions that form these equations (i), (ii), and (iii) are continuous and have some monotonic properties (see Lemma 5). Moreover, these functions change continuously with respect to changes of the reserve prices. These continuity and monotonic properties of the equations that define the equilibrium cut-offs imply that the map from reserve prices to equilibrium cut-offs is invertible. Since these map is also unique, this implies that we can prove that the map is continuous. We provide this proof in Appendix A. We show that if we take a sequence of vectors of reserve prices $\{\vec{r^n}\}_{n=1}^{\infty}$ that converges to vector of reserve prices $\vec{r}$, then the sequence of equilibrium cut-offs associated to $\{\vec{r^n}\}_{n=1}^{\infty}$ converges to the equilibrium cut-offs associated to $\vec{r}$. This result implies continuity of the equilibrium cut-offs with respect to the vector of reserve prices. ■

The importance of this result is that shows that the continuation game that the bidders play after the auctioneers announce their reserve prices is well behaved. This point is crucial to obtain continuous auctioneers' payoff functions. This result shows that the worries expressed by Peters [13] that the equilibrium selection of the entry game could have discontinuities when the number of agents is finite when auctioneers offer mechanisms from a wider class does not hold if we restrict to second price auctions with reserve prices.

Note that the continuity of the equilibrium bidders' entry behaviour can seem paradoxical if we think of our model as an extension of the Bertrand model. In the standard Bertrand model, buyers attend with probability one to the seller with minimum price. This produces a discontinuity in the buyers' equilibrium entry decision when we decrease one sellers' price slightly below the minimum price offered by the other sellers. The difference of our model with the entry game that it is induced by the standard Bertrand competition is that in our model it is not obvious that the bidder should enter the auction with minimum reserve price. The reason is that low reserve prices can be associated with a higher probability of rationing or even with a higher expected price. In fact, Peters [12] has shown that once we modify the Bertrand game introducing capacity constraints and restricted mobility in a similar fashion to our model, the entry decisions of buyers are continuous with respect to the sellers price offers.

To understand the originality of our result is important to remark that both uniqueness and continuity can be deduced from the analysis of Peters and Severinov [18] in two cases, when the vector of reserve prices $\vec{r}$ is restricted to have no more than two different reserve prices and when all the bidders are identical at the stage of choosing auction. We generalise this result in some sense by eliminating these restrictions and considering general heterogeneity in both market sides.

3.1 First Price Auctions

In the main text of this section we have assumed that each of the auctioneers uses a second price auction to allocate the good among those buyers that match with him. We relax this assumption in this subsection and we study entry games in which some or all of the auctioneers conduct a first price auction and the other auctioneers a second price auction. Note that in the case of a first price auction, it is relevant whether the number of bidders that enters the auction is observable or not. The reason is that the bidder's optimal behaviour depends on the number of other bidders.

We shall show that from the point of view of both the bidders and the auctioneers, the second price auction and the first price auction, with or without observable entry, are equivalent. In order to prove this, we proceed in two steps. First, we verify that for a given entry strategy bidders get in the equilibrium associated to each auction format the same expected utility. Next, we show that the set of symmetric equilibria of the entry game is invariant to changes in the auction format of some of the auctions.

We shall refer to the different auction formats with a set $\mathcal{F} \equiv \{$second price auction, first price auction with observable entry, first price auction with unobservable entry$\}$. Let $f_j \in \mathcal{F}$ be the auction format of a generic auction j, and $\vec{f} \in \mathcal{F}^J$, the vector of auction formats of all the auctions.

Conditional on an entry strategy π that it is used by all the bidders,[9] each auction format specifies a continuation game, that we call bidding game. The strategy for a bidder when auction j's format is a first price auction with unobservable entry is a bid function that maps types that enter auction j according to π into bids. If the auction format is a first price auction with observable entry, then the strategy is a bid function that maps types that enter auction j according to π, and number of bidders that enter auction j into bids. For the second price auction we shall assume that each bidder bids her true value of the good. Note that this is the unique symmetric equilibrium of a second price auction.

Lemma 8. *Consider the different bidding games generated in an auction j for a fixed entry strategy π, a fixed reserve price r_j, and for each $f_j \in \mathcal{F}$. Then:*

- *There exists a unique symmetric equilibrium for the induced bidding game associated to each auction format $f_j \in \mathcal{F}$. These equilibria are in strictly increasing strategies. They are such that the bidder with highest type that enters auction j wins auction j if her type is weakly above r_j. Otherwise, the auctioneer keeps the good.*

- *Suppose that all the bidders follow the unique symmetric equilibrium of each bidding game. Then, the bidder's expected utility of participating in auction j, conditional on a type x that enters with positive probability auction j according to π, is independent of j's auction format.*

[9] Note that we do not study continuation games induced by asymmetric entry strategies, i.e. when not all the bidders use the same entry strategy. They are technically complex since they imply asymmetric first price auctions.

Proof. The first of the points can be proved with an adaptation of the proofs given by Matthews [8, Section 6] to our model.

We prove the second point starting with a bidder with type y_j, i.e. the infimum of the closure of the set of types that enters with positive probability auction j according to π. If y_j is less than r_j, independently of the auction format, the bidder will bid below r_j and hence, will get zero expected utility. Suppose now that $y_j \geq r_j$. As stated in point one of this lemma, the unique symmetric equilibrium associated to each auction format is such that a bidder with type y_j only wins if no other bidder enters auction j. We next show that a bidder with type y_j pays price r_j conditional on winning in each of the auction formats. This remark is obvious for the second price auction and also for the first price auction with observable entry. In the case of the first price auction with unobservable entry, the result follows because the equilibrium bid of a type y_j in auction j is r_j. This is a consequence of the parallel analysis to [8] that we suggested above. This result follows for other types because given the results above, the revenue equivalence theorem implies that the second price auction and the first price auction with observable entry are equivalent in terms of expected utility for bidders conditional on types and conditional on the number of bidders that enter the auction. The same implication holds but unconditional on the level of entry for the second price auction and the first price auction with unobservable entry. Hence, the three auctions are equivalent for the bidders unconditional on the level of entry. ■

We next show that a given entry strategy π is a symmetric equilibrium for a given vector $\vec{r}$ that describes the auctioneers' reserve prices, independently of the auction format that it is used by each auctioneer.

Lemma 9. *Consider a family of entry games defined by $\{\vec{r}, \vec{f^l}\}_{\vec{f^l} \in \mathcal{F}^J}$, and the associated bidding games. Assume that bidders play the unique symmetric equilibrium associated to each auction when all the bidders play the same entry strategy. Then, the set of symmetric equilibrium of the entry game is invariant with respect to the auction format of each auction, i.e. with respect to $\vec{f^l}$.*

Proof. Lemma 8 says that the bidders' expected utility is invariant across auction formats if the bidders play a symmetric entry strategy. Hence, we only need to show the following. Consider a given entry strategy π and the symmetric equilibrium of the induced bidding games. If one bidder conditional on a type x_i deviates and enters an auction j that she does not enter according to π, the maximum payoffs in auction j that the bidder can get are independent of the auction format $f_j \in \mathcal{F}$.

The case in which $x_i < r_j$ is trivial. For the other cases, note that in a first price auction, the incentives to increase the bid for a type x_i are weakly below the incentives to increase the bid for types above x_i. Similarly, the incentives to increase the bid for a type x_i are weakly above the incentives to increase the bid for types below x_i. Hence, if $x_i \in [r_j, y_j]$, the optimal bid for the bidder must be between r_j and the optimal bid of y_j. The same reasoning that we use in the proof of Lemma 8 for a bidder with type y_j can be used here to show that the optimal bid gives the same expected utility in auction j across auction formats. Finally, if $x_i > y_j$, the consequence of the above argument is that x_i's optimal bid must lie between the equilibrium bid of the maximum type below x_i that enters auction j with positive probability, say x_-, and the equilibrium bid of the minimum type above x_i that enters auction j with positive probability if defined, say x_+. If the auction is a first price auction, it is clear that a bidder with type x_+ will submit the same bid as a bidder with type x_-. The reason is that if x_+'s bid were above x_-'s bid, a bidder with type x_+ would have incentives to deviate and decrease her bid. This implies that x_i's optimal bid in auction j must be x_-'s bid in a first price auction. If the auction format is a second price auction x_i's optimal bid is x_i. But note that bidding x_i in a second price auction gives the same expected utility as bidding x_-. The reason is that in a second price auction a bid x_i wins under the same circumstances than a bid x_-. Since the revenue equivalence theorem we proof in Lemma 8 implies that a bidder with type x_- pays the same expected price and wins with the same probability in the three auction formats, the maximum expected utility that a bidder with type x_i can get in the three auction formats is the same. If x_+ is not defined the proof is similar. Note only that in a first price auction, a bidder with type x_i does not have incentives to bid above x_-'s bid if x_+ is not defined. ■

Corollary 2. *Consider a family of entry games defined by $\{\vec{r}, \vec{f^l}\}_{\vec{f^l} \in \mathcal{F}^J}$, and the continuation bidding games. Suppose that bidders play the unique symmetric equilibrium associated to each auction when all the bidders play the same entry strategy. Then, bidders' expected utility conditional on the type and the auctioneers' expected profit, are independent of $\vec{f^l}$ in the unique symmetric equilibrium of the entry game.*

4 The Auctioneers' Game

In this section, we study the reduced game of competition among auctioneers. This reduced game is defined by the auctioneers' payoffs evaluated at the unique symmetric Nash equilibrium of the entry game. This equilibrium was characterised in the previous section.

We first describe the expected profit of a generic auctioneer j. For this, we assume that the auctioneer j announces a reserve price r_j, the other auctioneers announce $\vec{r}_{-j} \in [0,1]^{J-1}$, and these announcements of reserve prices generate an equilibrium π in the entry game characterised by the cut-offs $\{y_j\}_{j=1}^J$.

We compute the auctioneer's expected profit as the expected price that the bidder that wins the auction pays minus the production cost w_j whenever there is a sale. The probability that at least one bidder enters auction j equals $1 - z(y_j;\pi)^{kJ}$. Conditional on the former event, the probability that the bidder that wins has a type below x, for $x \in [y_j, 1]$, equals $z(x;\pi)^{kJ}/(1 - z(y_j;\pi)^{kJ})$. Hence, the auctioneer's expected profit equals:

$$\int_{y_j}^{1} \left(\Psi_j(x; y_{j-1}, y_j, ..., y_J) - w_j\right) \, dz(x;\pi)^{kJ}. \tag{3}$$

Lemma 10. *The auctioneer's payoff function is continuous in w_j, r_j, and r_{-j}.*

Proof. The continuity with respect to w_j is trivial. In order to prove continuity with respect to the vector of reserve prices, let $\Psi'_j : [0,1] \rightarrow [0,1]$ be a function such that $\Psi'_j(x) = \Psi_j(x; y_{j-1}, y_j, ..., y_J)$ if $x \geq y_j$, and $\Psi'_j(x) = \Psi_j(y_j; y_{j-1}, y_j, ..., y_J)$ otherwise. Let also z' be a measure defined on the measurable space $([0,1], \mathcal{B})$, where $\mathcal{B}$ is the class of Borel sets, and generated by a function equal to $z(x;\pi)^{kJ}$ if $x \in [y_j, 1]$, and equal to $z(y_j;\pi)^{kJ}$ if $x \in [0, y_j]$. Hence, the auctioneer j's expected profits equal $\int_0^1 \Psi'_j \, dz'$. Next, note that the function $z(x;\pi)$ for a fix x changes continuously with respect to changes in the cut-offs. Moreover, the equilibrium cut-offs change continuously with respect to changes in the vector of reserve prices, see Proposition 1. Hence, the function $z(x;\pi)$ for a fix x changes continuously with respect to the vector of reserve prices. This is sufficient for set-wise continuity of the measure z' with respect to the vector of reserve prices. We have also shown that the function Ψ_j is continuous with respect to the cut-offs, see Lemma 5, hence, Ψ'_j changes continuously with respect to the vector of reserve prices. As a consequence, we can apply the generalised Lebesgue bounded convergence theorem (see [19, Proposition 18, p. 270]) to prove the continuity of the integral with respect to the vector of reserve prices. ■

The reader can find this result surprising because other papers that study similar models have suggested that the auctioneers' payoff functions could be discontinuous. Peters [13] argues that in a game in which auctioneers are allowed to choose auctions from a wider class of mechanisms the equilibrium selection of the entry game could be discontinuous in the reserve prices. We have already discussed in the previous section why this is not the case in our model. Peters and Severinov [18] proves that the auctioneers' payoffs are discontinuous in the limit game with infinite numbers of auctioneers and bidders. But, the discontinuity that they prove depends crucially on the assumption of infinite number of agents. In this case, even if the bidder's individual behaviour is continuous in the auctioneers' reserve prices, the aggregate behaviour can produce a discontinuity on the level of entry to the auctions. This will be the case when an infinite number of bidders change their entry behaviour with respect to a finite number of auctions.

The continuity result given above allows us to use standard theorems to prove existence of an equilibrium. For this, we consider the mixed extension of the strategy space of the auctioneers. We use Milgrom and Weber's [10] notion of *distributional strategy*. Milgrom and Weber shows that a distributional strategy is simply another way of representing mixed strategies.[10] Let Π_H be the support of the distribution of auctioneers' types H, then j's distributional strategy is a probability measure μ_j on the set $\Pi_H \times [0,1]$, such that the marginal distribution on Π_H is the distribution of the auctioneers' types H.

Proposition 2. *The auctioneers' reduced game has at least one Nash equilibrium in distributional strategies.*

[10] More precisely, Aumann [2] shows that there is a many-to-one mapping from mixed to behavioral strategies that preserves the players' expected payoffs, and Milgrom and Weber [10] show that there is another many-to-one payoff-preserving mapping from behavioral strategies to distributional strategies.

Proof. We use Milgrom and Weber's (1985) existence theorem (Theorem 1). This theorem can be used because the set of actions (reserve prices) and types (production costs) are compact metric spaces, auctioneers' types are statistically independent across auctioneers, and the auctioneer's payoff function is continuous in the auctioneer's production cost and the vector of reserve prices (Lemma 10). ■

With this proposition we complete the analysis of the finite game.

5 Limit Results

In this section we study the convergence properties of the equilibrium set of the reduced game of competition among auctioneers when the numbers of auctioneers and bidders go to infinity. As we explained in the last section, this reduced game is obtained by substituting into the auctioneers' payoff functions the unique symmetric equilibrium strategies of the bidders' game.

We shall proceed in four steps. First, we compute the limit of the cut-offs that characterise the unique symmetric equilibrium of the entry game. Second, we shall use these limits to compute the limit of the auctioneers' payoff functions. Third, we show that in the limit game defined by these payoff functions, for each auctioneer the unique best response to most of the other auctioneers' announcements of reserve prices is to set a reserve price equal to the auctioneer's production cost. In fact, we shall show that this is the unique weakly dominant strategy in the game defined with the limit payoff functions. More importantly, we show that the limit payoff functions allow for strict payoff comparisons. We use this limit strict payoff comparisons to deduce that we can rule out certain strategies in the finite game, provided that J is large enough. We then show that this process gives a precise equilibrium prediction: as J tends to infinity, almost all auctioneers with production costs low enough to get positive surplus from trade announce a reserve price equal to their production cost with probability arbitrary close to one.

Note that by working with the limit payoffs we avoid dealing with the more complex payoff functions of the finite game. Payoffs in the finite game are complex because the change of an auctioneer's reserve price produces not only a direct effect on the cut-off associated to this auction but also a complex indirect effect on the other cut-offs. To see why note the following argument. When an auctioneer changes his reserve price he affects the entry decisions of some types of the bidders. This change will be associated to a change in the entry decisions of the same types with respect to some other auctions. These are all the auctions with reserve prices below our auction reserve price. This has an impact on the expected price in such auctions, but it does not affect the expected price in the other auctions. Remember that a feature of the equilibrium is that bidders are indifferent among all the auctions in which they participate. Hence, if a bidder was indifferent between the auctions that have been affected by the change in our auction reserve price and the other auctions, she will no longer be indifferent between both groups of auctions after the change in the reserve price. The indifference conditions required by the equilibrium of the entry game are restored through a complex change in the level of entry to the different auctions, this is, a change in all the equilibrium cut-offs.

In the limit game, with infinite numbers of auctioneers and bidders, the indirect effect that we pointed out in the last paragraph should be negligible. The change in the entry decisions of types with respect to one single auction should have no effect on the level of entry in each of the other auctions. In other words, in the limit, changes of the reserve price of one single auctioneer should not affect to the expected utility that bidders could get in other auctions.[11]

In order to simplify the characterisation of the limit of the equilibrium cut-offs when the numbers of auctioneers and bidders go to infinity we shall discretise the auctioneer's strategy space. Under this assumption, we guarantee that in the limit when the numbers of auctioneers and bidders go to infinity there is only a finite number of different reserve prices. We can thus use a finite number of conditions similar to conditions (i), (ii), and (iii) in Section 3 to characterise the limit of the equilibrium cut-off associated to each reserve price. In fact, we can show that these limit conditions are the limit of a reformulation of the original conditions (i), (ii), and (iii). This approach is more complex when we allow for a continuum of different reserve prices. Since our conditions compare the expected price in two auctions with two adjacent reserve prices, in the limit they typically turn into a complicated differential equation.

In the following we thus assume that the auctioneers choose the reserve price from a given finite subset Π of $[0, 1]$. We also assume that the distribution of the auctioneers' production cost H has support Π_H contained in the set Π. Under this assumption, we can prove that in the limit of the equilibrium of the game, the auctioneers announce reserve prices equal to their production costs with probability one. Otherwise we could only prove that the auctioneers' equilibrium randomisation puts positive probability on the two reserve prices closest to their production costs.

STEP 1: Convergence of the Equilibrium Cut-offs

Our first aim is to prove that the equilibrium cut-offs converge when J goes to infinity under some conditions and to characterise their limits. For this, we consider a sequence of entry games in which J is the number of auctioneers and kJ is the number of bidders. Along the sequence we keep $k > 0$ fixed and let J take values in an infinite subset of the natural numbers, $\mathbb{N}^*$, such that if $J \in \mathbb{N}^*$ then kJ is a natural number. Then we let J tend to infinity, and consider the limit behaviour of the equilibrium strategies.

To formalise this approach we need additional notation. Instead of referring explicitly to the vector of reserve prices chosen by J auctioneers, it is sufficient to refer to the frequency distribution of reserve prices. Lemma 1 shows that this frequency distribution alone determines bidders' equilibrium entry behaviour. For every $J \in \mathbb{N}^*$, we thus denote by $\mathcal{G}^J$ the set of probability distributions that can describe the announcement of reserve prices of J auctioneers. A probability distribution $G^J \in \mathcal{G}^J$ must satisfy the following conditions:

[11] This result was originally conjectured by McAfee [9]. See after Lemma 15.

- supp $G^J \subset \Pi$, # supp $G^J \leq J$; and
- for all $x \in [0,1]$, $G^J(x) = j/J$ for some $j = 0, 1, ...J$.

We also denote by $\mathcal{G}$ the set of probability distributions with support contained in Π. Note that each set $\mathcal{G}^J$ is a compact subset of $\mathcal{G}$ which is itself compact.

For the sake of clarity, we shall concentrate in the main text of the paper on sequences of entry games such that in each entry game each of the reserve prices in Π is announced by at least one auctioneer. These are games in which the support of the associated distribution function G^J is Π. We define some functions for each of these games. Next, we use these functions to re-formulate in the notation of this section conditions (i), (ii), and (iii) in Lemma 4. Remember that these are the conditions that characterise the set of equilibrium cut-offs. We shall show in the Appendix that the re-formulated conditions converge in an appropriate sense when J tends to infinity to some limit conditions. These limit conditions will be used to prove that the equilibrium cut-offs converge and to characterise their limits. We show in the Appendix (see the proof of Lemma 12) how to extend our analysis to general convergence sequences of entry games, i.e. sequences whose elements do not have necessarily support Π.

In the following we denote by R the number of elements of Π and by $\{\hat{r}_l\}_{l=1}^R$ an increasing sequence that describes Π itself. As we mention above, we shall focus on distributions $G^J \in \mathcal{G}^J$ that have support $\{\hat{r}_l\}_{l=1}^R$. According to Lemma 1, in order to describe a given cut-off strategy π for a given entry game we only need to specify two things: an increasing sequence of cut-offs $\hat{\pi} \equiv \{\hat{y}_l\}_{l=1}^R$, where $\hat{y}_l$ is the cut-off associated to auctions with reserve price $\hat{r}_l$; and the distribution of reserve prices G^J. We shall denote with $\mathcal{P}$ the set of increasing sequences of R elements in the interval $[0,1]$. Note that $\mathcal{P}$ is a compact set.

Hence, for all $\hat{\pi} \in \mathcal{P}$ and $x \geq \hat{y}_1$, we can define the function $\tilde{z}^J(x; \hat{\pi}, G^J) \equiv z(x; \pi)$ for a given entry game described by G^J. Remember that this function specifies the probability that a given bidder either has a type below x, or that she does not submit a bid in a given auction with associated cut-off $\hat{y}_l \leq x$. Hence, let $\hat{r}_l$ be the maximum reserve price of the auctions which type x enters and $\hat{y}_l$ its associated cut-off, then:

$$\tilde{z}^J(x; \hat{\pi}, G^J) = 1 - \frac{F(\hat{y}_{l+1}) - F(x)}{J\, G^J(\hat{r}_l)} - \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1}) - F(\hat{y}_q)}{J\, G^J(\hat{r}_q)}, \tag{4}$$

where $\hat{y}_{R+1} \equiv 1$.

We also provide a new function for the expected price that a bidder pays in an auction $l-1$ conditional on winning with a type $x \geq \hat{y}_{l-1}$. We define this function as $\tilde{\Psi}_{l-1}^J(x; \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J) \equiv \Psi_{j-1}(x; y_{j-1}, y_j, ...y_J)$, where $\hat{r}_{l-1} = r_{j-1}$ and hence $\hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R$ and G^J are sufficient to describe $y_{j-1}, y_j, ...y_J$. This function can also be computed as the integral:

$$\tilde{\Psi}_{l-1}^J(x; \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J) = \int_{-\infty}^{+\infty} \tilde{x} d\hat{\nu}_{l-1}^J|_x, \tag{5}$$

where $\hat{\nu}^J_{l-1}|_x(.)$ is a different way of referring to the measure $\nu_{l-1}|_x(.)$, i.e. using the new notation:

$$\hat{\nu}^J_{l-1}|_x(\tilde{x}) \equiv \begin{cases} 0 & \text{if } \tilde{x} < \hat{r}_{l-1} \\ \frac{\tilde{z}^J(\hat{y}_{l-1};\hat{\pi},G^J)^{kJ-1}}{\tilde{z}^J(x;\hat{\pi},G^J)^{kJ-1}} & \text{if } \hat{r}_{l-1} \leq \tilde{x} < \hat{y}_{l-1} \\ \frac{\tilde{z}^J(\tilde{x};\hat{\pi},G^J)^{kJ-1}}{\tilde{z}^J(x;\hat{\pi},G^J)^{kJ-1}} & \text{if } \hat{y}_{l-1} \leq \tilde{x} < x \\ 1 & \text{otherwise.} \end{cases} \tag{6}$$

We use the above functions to re-formulate the conditions that characterise the equilibrium cut-offs in Lemma 4. The unique symmetric equilibrium strategy of an entry game $G^J \in \mathcal{G}^{\mathcal{J}}$ where G^J has support Π is characterised by the unique sequence of cut-offs $\hat{\pi}$ such that for all $\hat{y}_l \in \hat{\pi}$:

(I) If $\hat{r}_1 = \hat{r}_1$, then $\hat{y}_l = \hat{r}_1$.

(II) If $\hat{r}_l \neq \hat{r}_1$ and $\hat{y}_l < 1$, then $\hat{r}_l = \tilde{\Psi}^J_{l-1}(\hat{y}_l, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J)$.

(III) If $\hat{r}_l \neq \hat{r}_1$ and $\hat{y}_l = 1$, then $\hat{r}_l \geq \tilde{\Psi}^J_{l-1}(\hat{y}_l, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J)$.

Note that these conditions in general imply less restrictions than those imposed by conditions (i), (ii), and (iii) in Lemma 4. The reason is that we have eliminated those conditions that relate auctions with the same reserve prices. We can do so, because as Lemma 1 says, auctions with the same reserve price have the same equilibrium cut-off.

Next, we compute the limit of these conditions when J tends to infinity. For this, we consider a sequence of entry games described by a sequence of distributions of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) that converges to a limit distribution of reserve prices $G \in \mathcal{G}$. This will give us some limit conditions that we shall use to prove convergence of the equilibrium cut-offs and to characterise their limit. In order to state the limit of conditions (I), (II), and (III) we first define three functions. We show in the proof of Lemma 12 in the Appendix that these functions are the limit of the functions $\tilde{z}^J$, $\tilde{\Psi}^J_{l-1}$, and $\hat{\nu}^J_{l-1}|_x$ in an appropriate sense.

We denote with $\underline{r}(G)$ the minimum reserve price in the support of a given distribution $G \in \mathcal{G}$. Note that we need to define the lower bound of the support of G because we do not restrict G to have support Π. We also denote with $\underline{y}$ the cut-off associated to a reserve price $\underline{r}(G)$. Then, we define the function $\bar{z}$ for a given increasing sequence $\hat{\pi}$ with R elements in $[0,1]$ (i.e. $\hat{\pi} \in \mathcal{P}$), and a type $x \in [\hat{y}_l, \hat{y}_{l+1}]$ and $x \geq \underline{y}$, as follows:

$$\bar{z}(x;\hat{\pi},G) \equiv e^{-k\left[\frac{F(\hat{y}_{l+1})-F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G(\hat{r}_q)}\right]},$$

where recall that $\hat{y}_{R+1} \equiv 1$. We also define the function $\bar{z}$ to be equal to zero for $x \in [\hat{y}_1, \underline{y})$.

We define the function $\bar{\Psi}_{l-1}$ for a sequence $\hat{\pi} \in \mathcal{P}$, and $x \geq \hat{y}_{l-1}$, as follows,

$$\bar{\Psi}_{l-1}(x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G) \equiv \int_{-\infty}^{\infty} \tilde{x}\, d\bar{\nu}_{l-1}|_x(\tilde{x}),$$

where the probability measure $\bar{\nu}_{l-1}|_x(.)$ is defined below:

$$\bar{\nu}_{l-1}|_x(\tilde{x}) \equiv \begin{cases} 0 & \text{if } \tilde{x} < \hat{r}_{l-1} \\ \frac{\bar{z}(\hat{y}_{l-1};\hat{\pi},G)}{\bar{z}(x;\hat{\pi},G)} & \text{if } \hat{r}_{l-1} \leq \tilde{x} < \hat{y}_{l-1} \\ \frac{\bar{z}(\tilde{x};\hat{\pi},G)}{\bar{z}(x;\hat{\pi},G)} & \text{if } \hat{y}_{l-1} \leq \tilde{x} < x \\ 1 & \text{otherwise,} \end{cases} \tag{7}$$

if $x \geq \underline{y}$, and it is defined by a single point with mass one at x for all $x < \underline{y}$.

With these equations we can define the following set of conditions, similar in spirit to conditions (i), (ii), and (iii) but defined for a limit game G.

(i') If $\hat{r}_l \leq \underline{r}(G)$, then $\hat{y}_l = \hat{r}_l$.

(ii') If $\hat{r}_l > \underline{r}(G)$ and $\hat{y}_l < 1$, then $\hat{r}_l = \bar{\Psi}_{l-1}(\hat{y}_l, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G)$.

(iii') If $\hat{r}_l > \underline{r}(G)$ and $\hat{y}_l = 1$, then $\hat{r}_l \geq \bar{\Psi}_{l-1}(\hat{y}_l, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G)$.

Lemma 11. *There is a unique sequence of values $\{\hat{y}_l\}_{l=1}^R \in \mathcal{P}$ that satisfies conditions (i), (ii), and (iii').*

Proof. We could show with a similar approach to Lemma 5 that the functions $\bar{\Psi}_{l-1}$ are continuous in $x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R$, and that verify some monotonocities similar to those proved in Lemma 5 for Ψ_{l-1}. Hence, we can use the same method as in Section 3 to show that conditions (i'), (ii'), and (iii') define implicitly a unique sequence $\hat{\pi} \in \mathcal{P}$. ■

We next provide the central result of this step that we prove in the Appendix. The proof of this lemma involves some technical steps to show that the conditions (I), (II), and (III) converge in some sense to the conditions (i'), (ii'), and (iii'). Then, we use the same arguments as in the proof of Proposition 1 to show that the unique solution of conditions (I), (II), and (III) also converges to the unique solution of conditions (i'), (ii'), and (iii'). It is important to note that we do not restrict in this lemma to sequences of distribution functions that have support Π.

Lemma 12. *Consider a reserve price $\hat{r}_l \in \Pi$ that it is announced infinitely often in a sequence of entry games $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) that converges to $G \in \mathcal{G}$ when J tends to infinity. Then, the equilibrium cut-off associated to this reserve price converges when J tends to infinity and its limit is the l-th entry of the R-dimensional solution of conditions (i'), (ii'), and (iii').*

Corollary 3. *Consider two sequences of entry games whose associated sequences of distributions of reserve prices converge to the same limit distribution function $G \in \mathcal{G}$ when J tends to infinity. If the reserve price $\hat{r}_l \in \Pi$ is announced infinitely often in both sequences, then, the equilibrium cut-off associated to $\hat{r}_l$ converges when J tends to infinity to the same value for the two sequences.*

STEP 2: The Limit Auctioneers' Payoff Function

The next step is to use the limits of the equilibrium cut-offs to compute the limit of the auctioneer's expected profit. In what follows we shall denote by $\hat{\pi}^J \equiv \{\hat{y}_l^J\}_{l=1}^R$ the unique solution of conditions (I), (II), and (III) for a given entry game $G^J \in \mathcal{G}^J$. Although in the main text these conditions are defined only for distribution functions with support Π, the reader can find in the Appendix (proof of Lemma 12) how to generalise these conditions to general distribution functions in $\mathcal{G}^J$. According to this extension, conditions (I), (II), and (III) define a value associated to each reserve price in Π. This value play no role but when the reserve price, say $\hat{r}_j$, is announced by at least one auctioneer. In this case, the element $\hat{y}_j^J$ denotes the equilibrium cut-off associated to auctions with a reserve price $\hat{r}_j$, where $\hat{r}_j$ denotes without loss of generality the j-th lowest reserve price in Π. We also denote with $\hat{\pi}^*$ the limit of $\hat{\pi}^J$ when J tends to infinity. Let also $\hat{y}_{R+1}^J \equiv 1$ and $\hat{y}_{R+1}^* \equiv 1$.

Using this new notation we can write the expected profit of an auctioneer with production cost w_j and that announces a reserve price $\hat{r}_j \in \Pi$, given that the auctioneers' announcements are described by $G^J \in \mathcal{G}^J$ as:

$$\tilde{\Phi}^J(\hat{r}_j, G^J|w_j) \equiv \int_{\hat{y}_j^J}^1 \left(\tilde{\Psi}_j^J(x; \hat{y}_{j-1}^J, \hat{y}_j^J, ..., \hat{y}_R^J, G^J) - w_j\right) dz^J(x; \hat{\pi}^J, G^J)^{kJ}. \quad (8)$$

The next result gives us the limit of the auctioneer's expected profit in terms of a function $\bar{\Phi}$. This function is defined for $w_j \in \Pi_H$, $\hat{r}_j \in \Pi$, and $G \in \mathcal{G}$, if $\hat{r}_j \geq \underline{r}(G)$, then:

$$\bar{\Phi}(\hat{r}_j, G|w_j) \equiv \int_{\hat{y}_j^*}^1 \left(\bar{\Psi}_j(x; \hat{y}_{j-1}^*, \hat{y}_j^*, ..., \hat{y}_R^*, G) - w_j\right) d\bar{z}(x; \hat{\pi}^*, G),$$

and if $\hat{r}_j < \underline{r}(G)$:

$$\bar{\Phi}(\hat{r}_j, G|w_j) \equiv \int_{\underline{r}(G)}^1 \left(\bar{\Psi}_j(x; \hat{y}_{j-1}^*, \hat{y}_j^*, ..., \hat{y}_R^*, G) - w_j\right) d\bar{z}(x; \hat{\pi}^*, G) + (\underline{r}(G) - w_j)\bar{z}(\underline{r}(G); \hat{\pi}^*, G).$$

Note that this definition implies that if $\hat{y}_j^* = 1$, the above limit expected profit equals zero. This means that the auctioneer's expected profit converges to zero when the associated cut-off converges to one. Intuitively this means that the auctioneer gets no expected profit when in the limit he attracts no bidder with probability one.

Lemma 13. *Consider a sequence of distribution functions $\{G^J\}_{J\in\mathbb{N}^*}$, where $G^J \in \mathcal{G}^J$ for all $J \in \mathbb{N}^*$, that converges to $G \in \mathcal{G}$ when J tends to infinity. Then, for $w_j \in \Pi_H$, and $\hat{r}_j \in \Pi$,*

$$\tilde{\Phi}^J(\hat{r}_j, G^J|w_j) \xrightarrow{J\to\infty} \bar{\Phi}(\hat{r}_j, G|w_j).$$

See the proof in the Appendix.

STEP 3: Properties of the Limit of the Auctioneers' Payoffs

Let $\overline{r}(G)$ be the minimum reserve price in Π that has an associated limit equilibrium cut-off one. Then, we can state the following result:

Lemma 14. *The limit of the auctioneers' expected profit verifies,*

$$\bar{\Phi}(w_j, G|w_j) \geq \bar{\Phi}(r_j, G|w_j), \tag{9}$$

for all $r_j \in \Pi \setminus \{w_j\}$. Moreover, the inequality is strict but in the following cases:

(1) When $r_j < \underline{r}(G)$ and $w_j \leq \underline{r}(G)$.

(2) When $r_j, w_j \geq \overline{r}(G)$.

Proof. Corollary 3 says that changes in the reserve price r_j do not affect to the limit of the equilibrium cut-offs associated to the other auctions. Hence, the sequence $\hat{\pi}^* = \{\hat{y}_i^*\}_{i=1}^R$ is invariant with respect to changes in one single reserve price. This means that the change in the reserve price only changes the lower bound of the integral that constitutes the auctioneer's limit expected profits. The function that we integrate is strictly increasing in x, and equals zero at the limit equilibrium cut-off that corresponds to the reserve price w_j. This implies our first result. It is a bit tedious, but mechanical, to show using the results in Lemma 13 that the inequality is strict but in the cases that we mention. ■

Corollary 4. *In the limit game defined by the limit payoff functions $\bar{\Phi}$, each auctioneer has a unique weakly dominant strategy to announce a reserve price equal to his production cost.*

As we explain in the introduction the intuition underlying this result is based on two properties of the entry game. The first one is a direct conclusion from the fact that bidders randomise entry among a set of auctions: the bidders must be indifferent among entering any of these auctions. The second property is that in the limit when the number of auctioneers and bidders go to infinity changes in one auction's reserve price should not affect to the expected utility that bidders can get in other auctions. This is shown in the next Lemma, see the Appendix for the proof.

Lemma 15. *Consider a family $\mathcal{S}$ of convergent sequences of entry games such that each of them only differs from the others in the reserve price announced by a given auctioneer j. Then, the expected utility that a generic bidder i with type x_i gets in an auction $l \neq j$ converges to the same value for all the sequences in $\mathcal{S}$ when the numbers of auctioneers and bidders go to infinity.*

This lemma proves a property of the limit game that was conjectured by McAfee [9] to solve his pioneer model. McAfee assumed that each auctioneer computes the payoffs of changing the design of his auction assuming that the expected utility that bidders can get in the mechanisms offered by the other auctioneers is unaffected by the change in his auction design. McAfee admits that in general this assumption is not consistent with the Nash equilibrium analysis of the entry game of bidders when the numbers of auctioneers and bidders are finite. However, McAfee conjectures that it should be true when there are infinite numbers of auctioneers and bidders.

Peters and Severinov [18] have proved this claim when auctioneers offer second price auctions and when there are no more than two different reserve prices announced by the auctioneers. They have proved McAfee's conjecture as we do. They look to the unique equilibrium of the entry game with finite number of auctioneers and bidders and compute its limit when the numbers of auctioneers and bidders go to infinity. Then, they show that the limit of the unique equilibrium verifies McAfee's conjecture. Our result supersedes Peters and Severinov analysis in the sense that we study entry games in which there are more than two different reserve prices announced by the auctioneers.

Peters [13] also proves McAfee's conjecture for more than two reserve prices, but his analysis is quite different. Peters considers a non generic sequence of entry games and a sequence of equilibria associated to that sequence of entry games. Then, Peters shows that in the limit of this sequence of equilibria when the numbers of auctioneers and bidders go to infinity McAfee's conjecture holds. Our analysis improves Peters' approach in the sense that we show that McAfee's conjecture holds for the limit of the equilibrium of the entry game for all sequences of entry games that converge to a given limit entry game. On the other hand, Peters' analysis is more general than us in the sense that he allows for a continuum of different reserve prices in the limit, whereas we only consider entry games with finitely many different reserve prices. Moreover, Peters proves McAfee's conjecture when we allow our auctioneer to choose from a wider set of mechanisms than second price auctions with reserve price. Although Peters also restricts to the case in which the other auctioneers offer second price auctions with reserve price.

STEP 4: Properties of the Limit of the Equilibria Set

In order to study the limit properties of the equilibria set we use the limit payoff comparisons in Lemma 14. With these payoff comparisons we eliminate certain strategies that cannot belong to the equilibrium set when J is large enough. This procedure allows us to determine the equilibrium strategies up to a negligible fraction of auctioneers when J goes to infinity. Note that although the payoffs comparisons in Lemma 14 are for the limit game we can use them for payoff comparisons in the finite game. The reason is that they provide strict comparisons. Since the payoff functions of the limit game are the limit of payoff functions of the finite game (see the convergence results in Lemma 13, and Lemma 20 in the Appendix) the strict payoff comparisons in the limit game should also hold for the finite game for J large enough.

This procedure differs from the method by McAfee [9], by Peters [13], and by Peters and Severinov [18]. They instead compute the equilibrium of the limit game. Peters and Severinov [18] suggest that a similar method to ours implies that if there exists an equilibrium in the finite game in which all the auctioneers announce the same reserve price for J finite but large, it must be closed to the equilibrium of the limit game that they propose. However, since it is not clear that such equilibria exists, they cannot provide any limit result as we do.

Note that the strict payoff comparisons in Lemma 14 extend only up to the thresholds $\underline{r}(G)$ and $\overline{r}(G)$. We next explain why we cannot extend our strict payoff comparisons out of these bounds.

The boundary $\overline{r}(G)$ specifies the minimum reserve price that has a limit equilibrium cut-off equal to one. All reserve prices above $\overline{r}(G)$ will have an equilibrium cut-off equal to one, thus they will attract bidders with probability zero in the limit and give zero limit payoffs. The auctioneer achieves his maximum expected revenue in the limit fixing a reserve price equal to his production cost (Lemma 14). This means that if the production cost is weakly above $\overline{r}(G)$, then the maximum limit payoff of the auctioneer is zero. Moreover, the auctioneer can achieve this maximum payoff with all the reserve prices weakly above $\overline{r}(G)$. Since we do not have strict payoff comparisons in the limit for reserve prices weakly above $\overline{r}(G)$, we cannot use the limit payoffs to get a single equilibrium strategy in the finite game.

Note, however, that the limit payoffs establishes that an auctioneer with production cost weakly above $\overline{r}(G)$ gets strictly higher expected utility with reserve prices weakly above $\overline{r}(G)$ than with reserve prices strictly below $\overline{r}(G)$. Hence, the limit payoffs can be used to rule out such strategies in the finite game. This means that although we cannot determine the announcement of auctioneers with production costs weakly above $\overline{r}(G)$, we can assure that in equilibrium they announce reserve prices weakly above $\overline{r}(G)$ when J is large enough, and hence trade with probability zero in the limit.

Next definition determines the set of types of the auctioneers that we are going to be able to determine their limit equilibrium reserve price.

Definition: Consider an arbitrary auctioneer j and an infinite sequence of reduced games in which all the auctioneers but j announce a reserve price equal to their production cost. We say that a given production cost w_j in the support of the distribution of the auctioneers' types is *tradable in the limit* if and only if the probability that auctioneer j attracts no bidder announcing w_j is bounded away from zero when the numbers of auctioneers and bidders tend to infinity, i.e. all types $w_j < \overline{r}(H)$, where H is the auctioneers' distribution of types.

Lemma 16. *There exist a unique set of production costs tradable in the limit.*

Proof. The Lemma follows because the value $\overline{r}(H)$ is uniquely defined according to Lemma 11. ∎

The other important boundary is $\underline{r}(G)$. This is the minimum reserve price that is announced by a strictly positive fraction of auctioneers in the limit when J goes to infinity. The limit auctioneers' payoff function in Lemma 14 is flat for types strictly below $\underline{r}(G)$. The reason is that an auction with a reserve price strictly below $\underline{r}(G)$ attracts an infinite number of bidders with the valuation immediately below $\underline{r}(G)$. This fixes the minimum price in such auction to the valuation immediately below $\underline{r}(G)$, producing the same effect in the limit auctioneer's payoffs as if the auctioneer announces a reserve price equal to this valuation.

This last problem makes the task of computing the limit of the equilibrium of the auctioneers' game more tedious. Moreover, it limits the reach of our results. We shall not be able to show that the minimum of the support of the equilibrium auctioneers' randomisation converges to the minimum of the support of the auctioneers' production costs.

We can now state our main limit result that it is proved in the Appendix.

Proposition 3. *For all $\epsilon > 0$, the fraction of auctioneers that announce in equilibrium a reserve price different to his production cost with probability greater than ϵ and conditional on having a production cost tradable in the limit goes to zero as J tends to infinity.*

6 Conclusions

In this paper we have analysed a multistage game of competition among auctioneers with a finite number of auctioneers and bidders. First, we have proved that the second stage game, the bidder's entry game, has a unique symmetric Nash equilibrium and we have provided a characterisation of the solution. With the unique solution of the entry game we have been able to compute the auctioneers' reduced game. We have shown that this reduced game is nice behaved and hence, we have used standard game theory theorems to show that the game always has an equilibrium (possibly in mixed strategies).

The originality of our approach is that we have been able to provide these results allowing for heterogeneity in both market sides. Similar models to ours have faced technical difficulties in dealing with this extension. In this sense, we think that our method to prove the existence of an equilibrium of the game can have two implications. First, it can give light on how to solve similar models of decentralised trade with heterogeneity in both market sizes. Second, it can suggest either how to construct models or how to modify existing models in order to assure the existence of equilibrium even under heterogeneity in both market sides.

We have also connected our results for the finite version of the game with the limit model in which there is a continuum of auctioneers and bidders. In this sense, we have given a result in the spirit of upper-hemicontinuity of the equilibrium correspondence. More precisely, we have shown a kind of convergence of the equilibrium set when the numbers of auctioneers and bidders go to infinity to the equilibrium already computed for limit versions of our model by Peters and Severinov [18], and Peters [13]. But, our result is more than a mere upper-hemicontinuity proof, it also shows that in the limit the equilibrium set contains an almost unique prediction.

The convergence that we have proved connects the results of imperfect competition by Burguet and Sákovics [6] for two auctioneers with the competitive results provided by McAfee [9], Peters [13], and Peters and Severinov [18], for the limit with infinite numbers of auctioneers and bidders. It proves the intuitive idea that the larger is the market the less monopolistic distortions will exist. Nevertheless, our convergence result has been provided only for a given class of equilibria of the entry game, the symmetric equilibria. It still remains unclear whether our results are robust when we allow for asymmetric equilibria of the entry game.

Our paper has one technical inconsistency. We proof the first result, existence of an equilibrium of the whole game, assuming that the auctioneers' strategy space is continuous, whereas in the second result, convergence of the equilibrium set, we assume that the auctioneers' strategy space is discrete. We believe that the existence result is more interesting when we allow for a continuous strategy space. Restricting to the case of a discrete strategy space would not clear up whether the existence result is a consequence of the finiteness of the game, or a consequence of the internal consistency of the game. On the other hand, the study of the convergence properties of the game assuming a discrete auctioneers' strategy space allows us to extend naturally the analysis of the game with a finite number of auctioneers and bidders to the limit game.

Had we wanted to prove our convergence result under the assumption that auctioneers' strategy space is a continuum, the main difficulty would be to prove the convergence of the equilibrium cut-offs and to characterise their limit. We could follow at least two approaches. The first one is to prove the convergence of the conditions that we provide to characterise the equilibrium cut-off in this more general set-up. The second one could be to use an approach similar to Peter's [13]. He studies the map from equilibrium cut-offs to reserve prices instead of the map from reserve prices to equilibrium cut-offs. The former map is more simple, and hence, allow for more simple proofs. The only difficulty is that this map is not in general a 1-1 map, mainly due to the fact that all high reserve prices have the same associated equilibrium cut-off, i.e. maximum cut-off. Moreover, it would imply to use a method completely different to the one we use to prove existence of an equilibrium.

Appendix

A Proof of Proposition 1

In order to study continuity of the equilibrium cut-offs we consider a sequence of vectors of reserve prices $\{\vec{r}^n\}_{n=1}^{\infty}$ that converges to a given limit vector of reserve prices $\vec{r}$. We assume without loss of generality that the elements of each of these vectors are ordered increasingly. We shall show that the sequence of equilibrium cut-offs associated to the sequence of vectors of reserve prices always converges to the equilibrium cut-offs associated to the limit vector of reserve prices.

To simplify the notation we shall only consider the case in which all the equilibrium cut-offs are interior, i.e. strictly less than one. In this case, the equilibrium cut-offs are defined by condition (ii) plus condition (i). The proof can be generalised to the case in which there are some equilibrium cut-offs that equal one, i.e. when condition (iii) also matters.

The equilibrium cut-offs $\{y_j^n\}_{j=1}^{J}$ associated to a given vector of reserve prices $\vec{r}^n$ are the unique solution of the following set of equations:

$$\begin{array}{l} \Psi_{J-1}^n(y_J^n, y_{J-1}^n, y_J^n) - r_J^n = 0 \\ \Psi_{J-2}^n(y_{J-1}^n, y_{J-2}^n, y_{J-1}^n, y_J^n) - r_{J-1}^n = 0 \\ \ldots \\ \Psi_1^n(y_2^n, y_1^n, y_2^n, ..., y_J^n) - r_2^n = 0 \\ y_1^n = r_1^n, \end{array} \qquad (10)$$

where that Ψ_j^n $(j \in \{2, 3, ..., J\})$ plays the same role as Ψ_j in the main text, but associated to the vector of reserve prices $\vec{r}^n$. Recall that each of the functions Ψ_j^n have compact domain and each is continuous in all the arguments, strictly increasing in the first argument, strictly decreasing in the second argument, and strictly increasing in the other arguments (see Lemma 5).

We can use a recursive argument similar to that in Lemmas 6 and 7 to show that the above equations define implicitly some functions ψ_j^n such that $y_j^n = \psi_j^n \circ \psi_{j-1}^n \circ ... \psi_2^n(r_1^n)$ for all $j \in \{2, 3, ..., J\}$.

Similarly, the equilibrium cut-offs associated to the limit vector of reserve prices $\vec{r}$ are the unique solution of:

$$\begin{array}{l} \Psi_{J-1}(y_J, y_{J-1}, y_J) - r_J = 0 \\ \Psi_{J-2}(y_{J-1}, y_{J-2}, y_{J-1}, y_J) - r_{J-1} = 0 \\ \ldots \\ \Psi_1(y_2, y_1, y_2, ..., y_J) - r_2 = 0 \\ y_1 = r_1, \end{array} \qquad (11)$$

or applying again the recursive argument of Lemmas 6 and 7, $y_j = \psi_j \circ \psi_{j-1} \circ ... \psi_2(r_1)$ for all $j \in \{2, 3, ..., J\}$.

It is easy to see that the sequence of functions Ψ_j^n associated to the sequence of $\vec{r}^n$ converges point-wise to the function Ψ_j associated to the limit vector of reserve prices $\vec{r}$, when $\vec{r}^n$ tends to $\vec{r}$. Hence, we can apply recursively Lemma 17 (see below) starting from the top equation of equations (10) to show that each function ψ_j^n converges uniformly to ψ_j. This implies that the sequence of equilibrium cut-offs associated to the sequence of vectors of reserve prices $\vec{r}^n$ converges to the equilibrium cut-offs associated to the vector of reserve prices $\vec{r}$.

Finally, we state and prove the lemma that we have used above to prove the convergence of the solution of equations (10). Basically, the next lemma says that the sequence of implicit functions defined by a convergence sequence of equations converges to the implicit function defined by the limit equation if some continuity properties hold and our equations are invertible. Note that we can show that the convergence is in fact uniform because the domain of our functions is compact.

Lemma 17. *Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of continuous functions with compact domain in $\mathbb{R}^2$ that converges point-wise to a function Y. If each of the functions Y_n and Y are increasing in the first argument and decreasing in the second argument, then the sequence of functions y_n uniquely defined by $Y_n(y_n(x), x) = 0$ converges uniformly to the function y uniquely defined by $Y(y(x), x) = 0$.*

Proof. We start taking an $\epsilon > 0$. Note next that the monotonic properties and continuity of Y imply that y must be continuous. Hence, for each x in the domain of y, there exists a $\delta(x) > 0$ such that if $x' \in (x - \delta(x), x + \delta(x))$, then $y(x') \in \left(y(x) - \frac{\epsilon}{4}, y(x) + \frac{\epsilon}{4}\right)$. We denote by $J(x)$ the set of such x', i.e. $J(x) \equiv (x - \delta(x), x + \delta(x))$. Since by definition $Y(y(x'), x') = 0$, and $y(x) - \frac{\epsilon}{2} < y(x) - \frac{\epsilon}{4} < y(x')$ and $y(x) + \frac{\epsilon}{2} > y(x) - \frac{\epsilon}{4} > y(x')$, the monotonic properties of Y imply that for all $x' \in J(x)$, $Y(y(x) - \frac{\epsilon}{2}, x') < 0$, and $Y(y(x) + \frac{\epsilon}{2}, x') > 0$.

Point-wise convergence of Y_n to Y implies that there exists a $n_0(x) \in \mathbb{N}$ such that if $n \geq n_0(x)$, then $Y_n(y(x) - \frac{\epsilon}{2}, x') < 0$, and $Y_n(y(x) + \frac{\epsilon}{2}, x') > 0$, for all $x' \in J(x)$. Hence, the continuity of Y_n implies that for all $x' \in J(x)$ and $n \geq n_0(x)$,

$$y_n(x') \in \left(y(x) - \frac{\epsilon}{2}, y(x) + \frac{\epsilon}{2}\right) \subset \left(y(x') - \epsilon, y(x') + \epsilon\right).$$

Note that $x \in J(x)$, thus the domain of y, say D, is a subset of $\cup_{x \in D} J(x)$. Since D is compact, the Heine-Borel theorem (see [19, Theorem 15, p. 44]) implies that there exists a finite collection of sets in $\{J(x)\}_{x \in D}$ that covers D, i.e. $D \subset \cup_{m=1}^{M} J(x_m)$, for M finite. Take $n_0 = \max\{n_0(x_1), n_0(x_2), ..., n_0(x_M)\}$, then for all $n \geq n_0$,

$$y_n(x') \in (y(x') - \epsilon, y(x') + \epsilon),$$

for all $x' \in D$, this is for all x' in the domain of y. This proves uniform convergence of y_n to y. ■

B Proofs of the Results in Section 5

Proof of Lemma 12.

We first prove the statement of the Lemma assuming that the distribution functions $\{G^J\}_{J \in \mathbb{N}^*}$ have support Π for all J. For this, we start with the next result.

Lemma 18. *Consider a sequence of entry games described by a sequence of distribution functions* $\{G^J\}_{J\in\mathbb{N}^*}$ *that converges to* $G \in \mathcal{G}$*, and where* G^J *belongs to* $\mathcal{G}^J$ *and has support* Π *for all* $J \in \mathbb{N}^*$*. Then, for any* $\hat{\pi} \in \mathcal{P}$ *and* $x \in [\hat{y}_1, 1]$*:*

- $\tilde{z}^J(x;\hat{\pi}, G^J)^{kJ-1} \xrightarrow{J\to\infty} \bar{z}(x;\hat{\pi}, G)$*, where recall that for* $x < \underline{y}$*,* $\bar{z}(x;\hat{\pi}, G) = 0$*.*
- $\hat{\nu}^J_{l-1}|_x(\tilde{x}) \xrightarrow{J\to\infty} \bar{\nu}_{l-1}|_x(\tilde{x})$*, for all* $\tilde{x} \in \mathbb{R}$*. Recall that if* $\tilde{x} < x < \underline{y}$*,* $\bar{\nu}_{l-1}|_x(\tilde{x}) = 0$*.*
- *if* $x \geq \hat{y}_{l-1}$*, then:*

$$\tilde{\Psi}^J_{l-1}(x;\hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G^J) \xrightarrow{J\to\infty} \bar{\Psi}_{l-1}(x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R, G).$$

Where remember that $\underline{y}$ *is the cut-off associated to the reserve price* $\underline{r}(G)$*.*

Proof. We start with the following mathematical result:[12] for any sequence $a_J \xrightarrow{J\to\infty} a$, then $(1 + a_J/J)^J \xrightarrow{J\to\infty} e^a$. Thus, for any sequence of cut-offs $\hat{\pi} \in \mathcal{P}$ and x such that $x \in [\hat{y}_l, \hat{y}_{l+1})$ and $x \geq \underline{y}$,

$$\left(1 - \frac{\frac{F(\hat{y}_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G^J(\hat{r}_q)}}{J}\right)^{kJ-1} \xrightarrow{J\to\infty} e^{-k\left[\frac{F(\hat{y}_{l+1})-F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G(\hat{r}_q)}\right]}, \quad (12)$$

where recall that $\hat{y}_{R+1} = 1$.

If $x \in [\hat{y}_1, \underline{y})$, then $1 - \frac{\frac{F(\hat{y}_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G^J(\hat{r}_q)}}{J}$ is bounded away from one. This is because for all $\hat{r}_q < \underline{r}(G)$, $\lim_{J\to\infty} J\, G(\hat{r}_q)$ is finite and non zero. Hence,

$$\left(1 - \frac{\frac{F(\hat{y}_{l+1})-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^{R} \frac{F(\hat{y}_{q+1})-F(\hat{y}_q)}{G^J(\hat{r}_q)}}{J}\right)^{kJ-1} \xrightarrow{J\to\infty} 0. \quad (13)$$

The second convergence result in the lemma follows directly from the first result but in the case in which $x < \underline{y}$. In this last case, note that for $\tilde{x} < x$, the quotient $\tilde{z}^J(\tilde{x}, \hat{\pi}, G^J)/\tilde{z}^J(x, \hat{\pi}, G^J)$ is bounded away from one. As a consequent, $\hat{\nu}^J_{l-1}|_x(\tilde{x})$ goes to zero when J tends to infinity.

The last convergence result of the lemma follows because the second result proves convergence of the probability distribution function with respect to which we integrate. Convergence in probability distribution is sufficient for convergence in expectations (see [4, Theorem 25.8, p. 335]). ■

[12] This result can be proved using $\left(1 + \frac{a-\epsilon}{J}\right)^J \leq \left(1 + \frac{a_J}{J}\right)^J \leq \left(1 + \frac{a+\epsilon}{J}\right)^J$, for J large enough, and $\left(1 + \frac{a}{J}\right)^J \xrightarrow{J\to\infty} e^a$ for a rational. The last result is provided for instance by White [24, Exercise 14, p. 93]. Continuity assures that the last convergence result is valid for all $a \in \mathbb{R}$.

This lemma states a kind of point-wise convergence of the conditions (I), (II), and (III) to conditions (i'), (ii'), and (iii') for a sequence of entry games. Note also that the limit functions $\bar{\Psi}_j$ are also continuous with respect to the cut-offs values, and they have similar monotonistic properties to the functions Ψ_j in Section 3. Hence, we can use a similar proof to that of Proposition 1 (see Appendix A) to show that the equilibrium cut-offs converge when J goes to infinity and that their limit is characterised by the unique solution of conditions (i'), (ii'), and (iii'). Note that in this case, the vector of different reserve prices is fixed, but the functions $\tilde{\Psi}^J_{l-1}$ change because the distribution of reserve prices changes.

We next show how to modify the above proof to allow for general sequences of distribution of reserve prices. We can extend in a trivial way the method above to characterise the equilibrium cut-offs of an arbitrary entry game $G^J \in \mathcal{G}^J$. In fact, if the support of G^J is constant with respect to J, we can generalise in a trivial way the method above to prove convergence of the associated equilibrium cut-offs. The problem is that in general the support of G^J will change with respect to J. In this case, our method does not work because of two reasons. First, it is based on the convergence of some functions $\tilde{\Psi}^J_{l-1}$, and these functions have the same domain along J with generality only if the support of G^J is constant with respect to J. Second, the number of such functions changes if the support of G^J changes.

Our approach in this case is to introduce some conditions similar to (I), (II), and (III), defined with respect to some functions similar to $\tilde{\Psi}^J_{l-1}$, but with two important differences. The first one is that we shall introduce a condition associated to each of the reserve prices in Π and not only to the reserve prices in the support of G^J. The second is that the functions $\tilde{\Psi}^J_{l-1}$ will depend on a number of parameters independent of the support of G^J.

The solution of these conditions will give us a value associated to each reserve price in Π. These values will be such that the equilibrium cut-off associated to a reserve price $\hat{r}_l \in \Pi$ that belongs to the support of G^J is the l-th entry of the R-dimensional solution of our conditions, where recall that R is the cardinality of Π. We shall also show that these new conditions converge to conditions (i'), (ii'), and (iii'). Hence, these conditions will also characterise the limit of the associated cut-offs.

Basically, we extend the definition of the functions $\bar{\Psi}^J_{l-1}(x; \hat{\pi}, G^J)$ for distribution functions $G^J \in \mathcal{G}^J$ that do not have support Π. In order to do so, we first extend the definition of the function $\tilde{z}^J$. Note that for $x \geq \hat{y}_j$, where $\hat{y}_j$ is the cut-off associated to the minimum reserve price in the support of G^J, the definition given above for $\tilde{z}^J$ does not depend on the fact that G^J has support Π. We thus use this definition to extend the domain of $\tilde{z}^J$ to all $G^J \in \mathcal{G}^J$, and for $x \in [\hat{y}_j, 1]$. With the new definition of $\tilde{z}^J$ we define the measure $\hat{\nu}_{l-1}|_x(.)$ as above and we use this measure to extend the definition of $\tilde{\Psi}_{l-1}$ to all G^J. Note that we can only extend the domain of $\tilde{\Psi}_{l-1}$ with such an approach if $l-1 \geq j$. For $l-1 < j$ we need to evaluate $\tilde{z}^J(x; \hat{\pi}, G^J)$ at $x < \hat{y}_j$ in order to construct $\tilde{\Psi}^J_{l-1}$, and we have not defined this function for such points. We thus complete the extension of the definition of $\tilde{\Psi}^J_{l-1}$ by letting $\tilde{\Psi}_{l-1}(x; \hat{\pi}, G^J) \equiv x$ for all $l-1 < j$. This extension makes condition (II) ($\hat{r}_l = \tilde{\Psi}^J_{l-1}(\hat{y}_l; \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R)$) consistent with condition (i) ($y_1 = r_1$) for the cut-off $\hat{y}_j$. Remember that $\hat{y}_j$ is the cut-off associated to the minimum reserve price in the support of G^J, this is the cut-off y_1 according to the notation in Section 3. For other values, note that for finite J they do not represent any cut-off, and for the limit, they are consistent with condition (i').

We can now apply conditions (I), (II), and (III) to an arbitrary entry game described by $G^J \in \mathcal{G}^J$. We next argue that these conditions define a unique sequence $\hat{\pi} \in \mathcal{P}$, and that this sequence is such that the elements that correspond to reserve prices in the support of G^J are in fact the equilibrium cut-offs associated to these reserve prices.

The uniqueness proof is quite similar to that of conditions (i), (ii), and (iii) given in Section 3. The only difference is that we only need to apply the inductive argument to construct the solution for reserve prices above $\hat{r}_j$, this is the minimum reserve price in the support of G^J, instead of reserve prices above r_1 as in Section 3. Note that we can repeat the arguments in Section 3 because for $l-1>j$, $\tilde{\Psi}_{l-1}(x, \hat{y}_{l-1}, \hat{y}_l, ..., \hat{y}_R)$ is continuous in all the variables, strictly increasing in x, strictly decreasing in $\hat{y}_{l-1}$ and weakly increasing in all the other variables.

In order to prove that conditions (I), (II), and (III) applied to an arbitrary distribution function $G^J \in \mathcal{G}^J$ define the actual equilibrium cut-offs, we deduce from these conditions new conditions. We shall show that these conditions are essentially equivalent to conditions (i), (ii), and (iii) in Lemma 4, i.e. the conditions that characterise the equilibrium cut-offs.

Conditions (I), and (II) imply that with generality $\hat{y}_i = \hat{r}_i$, this is that the cut-off associated to auctions with the minimum reserve price in the support of G^J equals this minimum reserve price. Note that with different notation, this condition is essentially the same as condition (i).

Consider next two consecutive reserve prices in the support of G^J that are also consecutive in the increasing sequence $\{\hat{r}_l\}_{l=1}^R$ that describes Π. Then condition (II), and condition (III) applied to these reserve prices are essentially the same as conditions (ii) and (iii), respectively. The only difference is that in conditions (II) and (III) the functions $\tilde{\Psi}_{l-1}^J$ depend on values associated to all the reserve prices in Π, whereas conditions (ii) and (iii) only depend on cut-offs associated to reserve prices in the support of G^J. This does not imply any difference because the functions $\tilde{\Psi}_{l-1}$ are actually invariant with respect to changes in the values $\hat{y}_{l-1} \in \hat{\pi}$ that are associated to reserve prices out of the support of G^J. This is clear from the definition of $\tilde{z}^J$, see Eq.(4).

Finally, consider two consecutive reserve prices in the support of G^J such that there are other reserve prices between them in the increasing sequence $\{\hat{r}_l\}_{l=1}^R$ that describes Π. Then, we can substitute recursively conditions (II) and (III) for the reserve prices that are in between and get some new conditions that relate directly the two reserve prices in the support of G^J and their corresponding cut-offs. These conditions are essentially the same as conditions (ii) and (iii), respectively. This completes the proof that Conditions (I), (II), and (III) with an appropriate extension of the definition of $\tilde{\Psi}_{l-1}$ define the equilibrium cut-offs associated to an arbitrary distribution function $G^J \in \mathcal{G}^J$.

The final step is to prove convergence of conditions (I), (II) and (III) in this extended version to the limit conditions (i'), (ii'), and (iii'), when J tends to infinity. This proof is quite similar to the proof that we provide for sequences of distribution functions G^J with support Π. Again, this proof assures convergence of the values $\{\hat{y}_l^J\}_{l=1}^R$, and that their limit is in fact the solution of conditions (i'), (ii'), and (iii'). ■

Proof of Lemma 13.

We start the proof with the following result.

Lemma 19. *Consider a given sequence of distributions of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ ($G^J \in \mathcal{G}^J$) that converges to $G \in \mathcal{G}$, and a type $x \in [\hat{y}_1^*, 1]$, then:*

(a) For all $x \notin \hat{\pi}^*$:

$$\tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ} \xrightarrow{J\to\infty} \bar{z}(x;\hat{\pi}^*, G).$$

(b) For all $x \notin \hat{\pi}^*$, and $x > \hat{y}_j^*$:

$$\Psi_j^J(x;\hat{\pi}^J, G^J) \xrightarrow{J\to\infty} \bar{\Psi}_j(x;\hat{\pi}^*, G).$$

(c) For all $j \in \{1, 2, ..., R\}$:

$$\tilde{z}^J(\hat{y}_j^J;\hat{\pi}^J, G^J)^{kJ} \xrightarrow{J\to\infty} \bar{z}(\hat{y}_j^*;\hat{\pi}^*, G).$$

(d) For all $x < \underline{r}(G)$:

$$kJ\,\tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ-1} \xrightarrow{J\to\infty} 0.$$

Remember that for $x < \underline{r}(G)$, $\bar{z}(x;\hat{\pi}^*, G) = 0$.

Proof. The first result (a) is direct for $x = 1$. Consider now, $x \in (\underline{r}(G), 1)$. It can be deduced from conditions (i'), and (ii') that $\hat{y}_l^* < \hat{y}_{l+1}^*$ for all $\hat{y}_l^* < 1$. Hence,[13] if $x \in (\hat{y}_l^*, \hat{y}_{l+1}^*)$, for an $l \in \{1, 2, ..., R\}$, remember that $\hat{y}_{R+1}^* = 1$:

$$\lim_{J\to\infty} \tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ} =$$

$$\lim_{J\to\infty}\left(1 - \frac{F(\hat{y}_{l+1}^J) - F(x)}{JG^J(\hat{r}_l)} - \sum_{q=l+1}^{R}\frac{F(\hat{y}_{q+1}^J) - F(\hat{y}_q^J)}{JG^J(\hat{r}_q)}\right)^{kJ} =$$

$$\lim_{J\to\infty}\left(1 - \frac{\frac{F(\hat{y}_{l+1}^J)-F(x)}{G^J(\hat{r}_l)} + \sum_{q=l+1}^{R}\frac{F(\hat{y}_{q+1}^J)-F(\hat{y}_q^J)}{G^J(\hat{r}_q)}}{J}\right)^{kJ} =$$

$$e^{-k\left[\frac{F(\hat{y}_{l+1}^*)-F(x)}{G(\hat{r}_l)} + \sum_{q=l+1}^{R}\frac{F(\hat{y}_{q+1}^*)-F(\hat{y}_q^*)}{G(\hat{r}_q)}\right]} =$$

$$\bar{z}(x, \hat{\pi}^*, G).$$

[13] See Footnote 12 for the computation of the limit.

Consider next the case $x \in (y_1^*, \underline{r}(G))$. In this case, $\tilde{z}_J(x;\hat{\pi}^J, G^J)$ is bounded away from 1. Then, there exists an $\eta > 0$, such that, $\lim_{J\to\infty} \tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ} \leq \lim_{J\to\infty}(1-\eta)^{kJ} = 0$.

We can proof (b) following similar steps to those followed in the proof of Lemma 18. The proof of (c) can be done in a similar way to the proof of (a). In order to prove (d) we note again that for all $x \in (y_1^*, \underline{r}(G))$, $\tilde{z}_J(x;\hat{\pi}^J, G^J)$ is bounded away from 1. Then, there exists an $\eta > 0$, such that,

$$\lim_{J\to\infty} kJ(1-\tilde{z}^J(x;\hat{\pi}^J, G^J))\tilde{z}^J(x;\hat{\pi}^J, G^J)^{kJ-1} \leq \lim_{J\to\infty} kJ(1-\eta)^{kJ-1} = 0,$$

for an $\eta > 0$. The last step follows from [20, Theorem 3.20 (d), p. 57]. ■

For $\hat{r}_j \geq \underline{r}(G)$, and using Lemma 19 (a) and (b), and the Lebesgue bounded convergence theorem (see [19, Theorem 16, p. 91]) in the third step below:

$$\lim_{J\to\infty} \tilde{\Phi}^J(\hat{r}_j, G^J|w_j) =$$

$$\lim_{J\to\infty}\left\{\int_{\hat{y}_j^J}^{1}\left(\tilde{\Psi}_j^J(x;\hat{y}_j^J,\hat{y}_{j+1}^J,...,\hat{y}_R^J,G^J) - w_j\right) d\tilde{z}^J(x;\hat{\pi}^J,G^J)\right\} =$$

$$\lim_{J\to\infty}\left\{\sum_{l=j}^{R}\int_{\hat{y}_l^J}^{\hat{y}_{l+1}^J}\left(\tilde{\Psi}_j^J(x;\hat{y}_j^J,\hat{y}_{j+1}^J,...,\hat{y}_R^J,G^J) - w_j\right) \tilde{z}^J(x;\hat{\pi}^J,G^J)^{kJ-1}k\frac{f(x)}{G^J(r_l)}dx\right\} =$$

$$\sum_{l=j}^{R}\int_{\hat{y}_l^*}^{\hat{y}_{l+1}^*}\left(\bar{\Psi}_j(x;\hat{y}_j^*,\hat{y}_{j+1}^*,...,\hat{y}_R^*,G) - w_j\right) \bar{z}(x;\hat{\pi}^*,G)k\frac{f(x)}{G(r_l)}dx =$$

$$\int_{\hat{y}_j^*}^{1}\left(\bar{\Psi}_j(x;\hat{y}_j^*,\hat{y}_{j+1}^*,...,\hat{y}_R^*,G) - w_j\right) d\bar{z}(x;\hat{\pi}^*,G) =$$

$$\bar{\Phi}(\hat{r}_j, G|w_j).$$

Consider now the case $r_j < \underline{r}(G)$, then we split the integral in the following two halves:

$$\tilde{\Phi}^J(\hat{r}_j, G^J|w_j) = \int_{\hat{y}_j^J}^{\hat{y}_l^J}\left(\tilde{\Psi}_j^J(x;\hat{y}_j^J,\hat{y}_{j+1}^J,...,\hat{y}_R^J,G^J) - w_j\right) d\tilde{z}^J(x;\hat{\pi}^J,G^J) +$$
$$\int_{\hat{y}_l^J}^{1}\left(\tilde{\Psi}_j^J(x;\hat{y}_j^J,\hat{y}_{j+1}^J,...,\hat{y}_R^J,G^J) - w_j\right) d\tilde{z}^J(x;\hat{\pi}^J,G^J),$$

where $\hat{y}_l^J$ is the equilibrium cut-off associated to an announcement $\underline{r}(G)$ in the game with J auctioneers.

Note that we can compute the limit of the second part of the above integral following exactly the same steps as in the case $r_j \geq \underline{r}(G)$. For the second part note the following algebraic transformations:[14]

$$\int_{\hat{y}_j^J}^{\hat{y}_l^J} \left(\tilde{\Psi}_j^J(x;\hat{y}_j^J,\hat{y}_{j+1}^J,...,\hat{y}_R^J,G^J) - w_j\right) d\tilde{z}^J(x)^{kJ} =$$

$$\int_{\hat{y}_j^J}^{\hat{y}_l^J} \left(\int_{\hat{y}_j^J}^{x} \tilde{x} d\frac{\tilde{z}^J(\tilde{x})^{kJ-1}}{\tilde{z}^J(x)^{kJ-1}} + \hat{r}_j \frac{\tilde{z}^J(\hat{y}_j^J)^{kJ-1}}{\tilde{z}^J(x)^{kJ-1}} - w_j\right) d\tilde{z}^J(x)^{kJ} =$$

$$\int_{\hat{y}_j^J}^{\hat{y}_l^J} \left[x - w_j - (\hat{y}_j^J - \hat{r}_j)\frac{\tilde{z}^J(\hat{y}_j^J)^{kJ-1}}{\tilde{z}^J(x)^{kJ-1}} - \int_{\hat{y}_j^J}^{x} \frac{\tilde{z}^J(\tilde{x})^{kJ-1}}{\tilde{z}^J(x)^{kJ-1}} d\tilde{x}\right] d\tilde{z}^J(x)^{kJ} =$$

$$\int_{\hat{y}_j^J}^{\hat{y}_l^J} (x - w_j)\, d\tilde{z}^J(x)^{kJ}$$

$$- \int_{\hat{y}_j^J}^{\hat{y}_l^J} kJ \left[(\hat{y}_j^J - \hat{r}_j)\tilde{z}^J(\hat{y}_j^J)^{kJ-1} + \int_{\hat{y}_j^J}^{x} \tilde{z}^J(\tilde{x})^{kJ-1}\, d\tilde{x}\right] d\tilde{z}^J(x) =$$

$$(\hat{y}_l^J - w_j)\tilde{z}(\hat{y}_l^J)^{kJ} - (\hat{y}_j^J - w_j)\tilde{z}(\hat{y}_j^J)^{kJ} - \int_{\hat{y}_j^J}^{\hat{y}_l^J} \tilde{z}^J(x)^{kJ}\, dx$$

$$- (\hat{y}_j^J - \hat{r}_j)\, kJ\, \tilde{z}^J(\hat{y}_j^J)^{kJ-1} \left[\tilde{z}^J(\hat{y}_l^J) - \tilde{z}^J(\hat{y}_j^J)\right]$$

$$- kJ \int_{\hat{y}_j^J}^{\hat{y}_l^J} \int_{x}^{\hat{y}_l^J} d\tilde{z}^J(\tilde{x})\, \tilde{z}^J(x)^{kJ-1} dx =$$

$$(\hat{y}_l^J - w_j)\tilde{z}(\hat{y}_l^J)^{kJ} - (\hat{y}_j^J - w_j)\tilde{z}(\hat{y}_j^J)^{kJ} - \int_{\hat{y}_j^J}^{\hat{y}_l^J} \tilde{z}^J(x)^{kJ}\, dx$$

$$- (\hat{y}_j^J - \hat{r}_j)\, kJ\, \tilde{z}^J(\hat{y}_j^J)^{kJ-1} \left[\tilde{z}^J(\hat{y}_l^J) - \tilde{z}^J(\hat{y}_j^J)\right]$$

$$- \int_{\hat{y}_j^J}^{\hat{y}_l^J} kJ\, \tilde{z}^J(x)^{kJ-1} \left[\tilde{z}^J(\hat{y}_l^J) - \tilde{z}^J(x)\right] dx.$$

Hence, we can apply results (a), (c), and (d) in Lemma 19 to prove using the Lebesgue bounded convergence theorem (see [19, Theorem 16, p. 91]):

$$\lim_{J\to\infty} \int_{\hat{y}_j^J}^{\hat{y}_l^J} \left(\tilde{\Psi}_j^J(x;\hat{y}_j^J,\hat{y}_{j+1}^J,...,\hat{y}_R^J,G^J) - w_j\right) d\tilde{z}^J(x;\hat{\pi}^J,G^J)^{kJ} =$$

$$(\underline{r}(G) - w_j)\, \bar{z}(\underline{r}(G);\hat{\pi}^*,G).$$

This last result completes the proof of the Lemma. ■

Proof of Lemma 15.

Suppose that we have a family $\mathcal{S}$ of sequences of distribution of reserve prices $\{G^J\}_{J\in\mathbb{N}^*}$ that converge to G. The lemma follows if we show that the expected utility

[14] To simplify the notation we denote $\tilde{z}^J(x;\hat{\pi}^J,G^J)$ simply by $\tilde{z}^J(x)$.

of a bidder i with type x_i evaluated at the unique symmetric equilibrium of the entry game converges to the same value for each sequence in the family $\mathcal{S}$. Again for the sake of simplicity assume that each distribution of reserve prices has support G^J. Then, Lemma 3 implies that the expected utility of bidder i is simply the integral of the probability of winning for all types $\tilde{x} \in (\hat{y}_1^J, x_i)$ (i.e. the integral $\int_{\hat{y}_1^J}^{x_i} \tilde{z}^J(\tilde{x}, \hat{\pi}^J, G^J)^{kJ-1} d\tilde{x}$). We can show as in Lemma 19 that $\tilde{z}^J(\tilde{x}, \hat{\pi}^J, G^J)^{kJ-1}$ converges to $\bar{z}(\tilde{x}, \hat{\pi}^*, G)$ for all $\tilde{x} \notin \hat{\pi}^*$ and all sequences in $\mathcal{S}$ when J tends to infinity. Hence, we can apply the Lebesgue bounded convergence theorem (see [19, Theorem 16, p. 91]) to show that the expected utility of bidder i converges to the same value for all sequences in $\mathcal{S}$. ■

Proof of Proposition 3.

The basic idea of the proof is to use the strict payoff comparisons of the limit auctioneer's payoffs to rule out strategies from the equilibrium set in the finite game, for J large enough. Lemma 13 computes the limit of the auctioneer's expected profit for convergence sequences of games in which each of the other auctioneers announces a reserve price with probability one. This result is, however, insufficient for our purpose. The reason is that typically, in equilibrium the other auctioneers randomise among a set of reserve prices instead of announcing a reserve price with probability one. Hence, if we want to approximate the auctioneer's payoffs for J large we must use the limit when the other auctioneers are allowed to randomise.

Consider an infinite sequence of reduced games of competition among auctioneers defined by the sequence of payoff functions $\{\tilde{\Phi}^J(\hat{r}_j, G^J|w_j)\}_{J\in\mathbb{N}^*}$, each of which corresponds to a reduced game with J auctioneers. Let also $\mu^J \equiv \{\mu_1^J, \mu_2^J, ..., \mu_J^J\}$ be some distributional strategies for each of the auctioneers. Recall that j's distributional strategy is a probability measure μ_j^J on the set $\Pi_H \times \Pi$, such that the marginal distribution on Π_H is the distribution of the auctioneers' types H. The empirical distribution of reserve prices generated by these distributional strategies in any play of the game is a random variable given by $\tilde{\mu}^J$ with expectation $\frac{1}{J}\sum_{j=1}^J \bar{\mu}_j^J$, where $\bar{\mu}_j^J$ is the marginal distribution of μ_j^J on Π. Let ξ^J be the probability measure that this induces on $\mathcal{G}^J$ in the game consisting of J auctioneers. Then, if one generic auctioneer j with production cost w_j announces a reserve price $\hat{r}_j$ (without loss of generality) with probability one, then his expected payoffs equal: $\int_{G^J\in\mathcal{G}^J} \tilde{\Phi}^J(\hat{r}_j, G^J|w_j) d\xi^J(G^J)$.

Lemma 20. *Let $\{\mu^J\}$ be any sequence of distributional strategies having the property that $\frac{1}{J}\sum_{j=1}^J \bar{\mu}_j^J$ converges to some probability distribution $G \in \mathcal{G}$. Then:*

- *the probability measure ξ^J converges weakly to a measure that assigns point mass one to the distribution G.*
- *if there is one auctioneer j that plays a distributional strategy μ_j^J which marginal distribution on Π puts probability mass one in $\hat{r}_j \in \Pi$ for all $J \in \mathbb{N}^*$, then:*

$$\lim_{J\to\infty} \int_{G^J\in\mathcal{G}^J} \tilde{\Phi}^J(\hat{r}_j, G^J|w_j) d\xi^J(G^J) = \bar{\Phi}(\hat{r}_j, G|w_j).$$

Proof. The reserve prices offered by the auctioneers form a triangular system of row-wise independent random variables. Thus $\sup\left|\tilde{\mu}^J(x) - \sum_{j=1}^{J} \bar{\mu}_j^J(x)\right|$ converges almost surely to zero when J goes to infinity by an extension of the Glivenko-Cantelli theorem, see [22, Theorem 1, page 105]. Almost surely convergence implies that the probability measure ξ^J converges weakly to a measure that assigns point mass one to the distribution G.

>From Lemma 13, $\tilde{\Phi}^J(\hat{r}_j, G^J|w_j) \xrightarrow{J\to\infty} \bar{\Phi}(\hat{r}_j, G|w_j)$ for any sequence $\{G^J\}_{J\in\mathbb{N}^*}$ such that $G^J \xrightarrow{J\to\infty} G$. Moreover, the distribution ξ^J converges weakly to a degenerate distribution with mass point one in the distribution G. Thus,

$$\lim_{J\to\infty} \int_{G^J\in\mathcal{G}^J} \tilde{\Phi}^J(\hat{r}_j, G^J|w_j) d\xi^J(G^J) = \bar{\Phi}(\hat{r}_j, G|w_j),$$

by [3, Theorem 5.5, p. 34].

■

Assume next that we have a convergent sequence of reduced games of competition among auctioneers with increasing numbers of auctioneers and bidders. Each of these games must have at least one Nash equilibrium.[15] Hence, we can always take a subsequence of equilibrium distributional strategies with convergence mean in the sense of Lemma 20. We shall call the limit distribution function of this mean $G \in \mathcal{G}$.

Lemma 20 shows that we can approximate the auctioneer's payoffs when J is large enough with the limit of the auctioneer's payoffs evaluated at the limit distribution of reserve prices G. We can thus use the strict payoff comparisons in Lemma 14 to rule out certain strategies from the auctioneer' strategy set that will not be played in equilibrium for J large enough. We proceed in three steps.

Step 1: *In equilibrium, if $w < \overline{r}(G)$ and for all $w \in \Pi_H$, no auctioneer announces with positive probability a reserve price above w conditional on having a production cost w for J large enough.*

Lemma 14 says that in the limit when J goes to infinity if an auctioneer has a production cost $w < \overline{r}(G)$, he gets strictly higher expected utility with a reserve price w against the limit distribution of reserve prices G than with any other reserve price strictly above w and weakly above $\underline{r}(G)$. Strictness implies that this should also be true for J large enough. Hence, in equilibrium no auctioneer with production cost w announces a reserve price weakly above $\underline{r}(G)$ and strictly above w for J large enough. We next show that this actually implies the above statement.

The strong law of large numbers implies that when J goes to infinity, with probability one the fraction of auctioneers with production cost w equals the probability measure that a given auctioneer has a production cost w (see [4, Theorem 6.1, p.

[15] Proposition 2 shows that all these games have a Nash equilibrium in distributional strategies. Although this proof was done under the assumption that the strategy space is continuous some obvious modifications show that we can apply it also to a discrete strategy space. Nevertheless, given the discrete nature of the auctioneers' strategy space and the auctioneers' private types we can more naturally apply Nash's (1950) existence theorem.

85]). Hence, the finiteness assumption of the support of the distribution of production costs H implies that $\lim_{J\to\infty} P_w^J \geq \underline{r}(G)$, where P_w^J is the maximum reserve price that is announced with positive probability in equilibrium by an auctioneer conditional on a production cost w. According to the paragraph above this implies that $\lim_{J\to\infty} P_w^J \leq w$. This completes the proof of Step 1.

Step 2: *Let $\underline{w}$ be the minimum production cost in the support of the distribution of production costs H. Then, for all w in Π_H such that $\underline{w} < w < \overline{r}(G)$, all auctioneers conditional on having a production cost w announce a reserve price w in equilibrium and for J large enough.*[16]

Lemma 14 says that in the limit when J goes to infinity if $\underline{r}(G) < w < \overline{r}(G)$, then an auctioneer with a production cost w strictly prefers a reserve price w against the limit distribution of reserve prices G. Strictness implies that this is also true for J large enough. This means that in equilibrium, and for J large enough, all auctioneers with a production cost w such that $\underline{r}(G) < w < \overline{r}(G)$ announce a reserve price equal to w. Hence, we only need to show that $w > \underline{w}$ implies that $w > \underline{r}(G)$.

Since $\underline{w} < \overline{r}(G)$, Step 1 implies that the auctioneers with production cost $\underline{w}$ announce a reserve price smaller than or equal to w in equilibrium and for J large enough. Due to the finiteness of the support of the distribution of production costs, the strong law of large numbers implies that when J goes to infinity, the fraction of auctioneers with production cost $\underline{w}$ is strictly positive with probability one, see [4, Theorem 6.1, page 85]. This means that in the limit when J goes to infinity $w > \underline{r}(G)$ for all $w > \underline{w}$. This completes the proof of Step 2.

In step 2, we rule out some strategies that involve reserve prices below the production cost mainly when the production cost is strictly above $\underline{w}$. The impossibility to compare payoffs for J large enough with the limit payoffs when J goes to infinity for reserve prices below or equal to $\underline{r}(G)$ precludes to extend Step 2 to production costs $\underline{w}$. In the next step, we produce a weaker statement for the production cost $w = \underline{w}$ than Step 2 for $w \neq \underline{w}$. This weaker statement is, nonetheless, sufficient for the Proposition.

Step 3: *For all $\epsilon > 0$, the fraction of auctioneers that announce in equilibrium a reserve price different to his production cost with probability greater than ϵ and conditional on a production cost $\underline{w}$ goes to zero as J tends to infinity if $\underline{w} < \overline{r}(G)$.*

Similarly to Step 2, we only need to show that if the conditions in the statement of Step 3 are not met then $\underline{w} > \underline{r}(G)$. Step 1 says that no auctioneer with production cost $\underline{w} < \overline{r}(G)$ announce a reserve price above $\underline{w}$ in equilibrium and for J large enough. Suppose next that there exists an $\epsilon > 0$ such that the fraction of auctioneers that announce in equilibrium a reserve price different to his production cost, i.e. strictly below w, with probability greater than ϵ and conditional on a production cost $\underline{w}$ goes to $\delta > 0$ as J tends to infinity. The strong law of large numbers (see [4, Theorem 6.1, p. 85]) says that the limit of the fraction of auctioneers that announce a reserve price strictly less than $\underline{w}$ is at least $\epsilon\,\delta > 0$ with probability one. This means that $\underline{w} > \underline{r}(G)$.

[16] Note that in this step and in the first step we prove more than required by the Proposition.

This completes the proof of Step 3.

In order to complete our proof, we only need to show that Step 1, Step 2, and Step 3 imply the Proposition. Step 1, Step 2, and Step 3 imply that for all $\epsilon > 0$ the fraction of auctioneers that announce a production cost different to their production cost conditional on having a production cost below the reserve price $\overline{r}(G)$ tends to zero as J goes to infinity. Lemma 14 says that auctioneers with production costs weakly above $\overline{r}(G)$ strictly prefer to announce a reserve price weakly above $\overline{r}(G)$ for J large enough. Hence, we can use Lemma 20 to show that the distribution of reserve prices strictly below $\overline{r}(G)$ that are observed in equilibrium when J tends to infinity converges weakly to the distribution of production costs strictly below $\overline{r}(G)$. Note that production costs weakly above $\overline{r}(G)$ never trade in the market and hence, they do not actually affect to the level of $\overline{r}(G)$. This implies that the $\overline{r}(G)$ associated to the limit of the equilibrium strategies must be actually equal to the $\overline{r}(H)$, where H is the distribution of the auctioneers' production costs. This completes the proof of Proposition 3. ■

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