OPTIMAL SUBSTITUTION OF RENEWABLE AND NONRENEWABLE NATURAL RESOURCES IN PRODUCTION*

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A B S T R A C T

A theoretical model is presented in order to study the optimal combination of natural resources, used as inputs, taking into account their natural growth ability and the technical possibilities of input substitution. The model enables us to consider renewable resources, nonrenewable, or both. The relative use of resources evolves through time according to the difference between both resources' natural growth and technological flexibility, as measured by the elasticity of substitution of the production function. Output evolves according to a version of the traditional Keynes-Ramsey rule, where the marginal productivity of capital is substituted by the "marginal productivity of natural capital", that is a combination of both resources' marginal growth weighted by each resource return in production.

Keywords: Renewable Resources, Nonrenewable Resources, Production, Optimal Control.

1 Introduction

This paper focuses on the optimal combination of renewable or nonrenewable natural resources, used as inputs, taking into account their natural growth and technological substitution possibilities.

Since Hotelling (1931) first researched on the optimal use of exhaustible resources[1] and the 70's oil crisis showed the importance of this matter, many economic research articles have addressed questions related to natural resources. The optimal extraction patterns of an exhaustible resource have been studied by Dasgupta and Heal (1974) and Weinstein and Zeckhauser (1975), among others. The article by Stiglitz and Dasgupta (1982) pays attention to the effect of the market structure on the extraction rate of a nonrenewable resource. Herfindahl (1967) studies the optimal depletion on several deposits of a nonrenewable resource without extraction costs, while the same problem is solved by Weitzman (1976) with different extraction costs and Hartwick, Kemp and Long (1986) with set-up costs. Pyndick (1978) analyzes a joint problem of optimal extraction and investment on exploration to find new resources. Pyndick (1980) and Pyndick (1981) study the effect of different types of uncertainty on resource management. Dasgupta and Heal (1979) present a broad discussion on the basic aspects concerning the influence of exhaustible resources on economic theory. For a more recent survey see, for example, the chapter 7 of Hanley et.al. (1997).

As shown in Beckman (1974, 1975) and Hartwick (1978a, 1978b, 1990), productive processes do not usually depend on a single natural resource, but it is possible to choose among several resources or combinations of resources. So, apart from the whole quantity of resources employed, it is also interesting to determine the optimal substitution among them. Hartwick (1978a) obtains some results regarding substitution among nonrenewable resources. The growing awareness and interest about renewable resources suggest the need for study the substitution among renewable and nonrenewable resources.

Given that renewable and nonrenewable resources give rise to qualitatively different economic matters, the research efforts related to both kinds of resources have largely evolved as two separate branches in the economic literature. As a consequence, most economic articles about natural resource economics focus on just renewable or just nonrenewable resources[2], depending on the specific research purposes of each paper. In order to the address the matter of substitution among renewable and nonrenewable resources, we need to model explicitly the possibility of different combinations of renewable (RR) and nonrenewable (NR) resources to manage situations such as the following:

NR: Energy generation from oil or coal. Manufacturing cars from different metal combinations.

RR: Energy generation from different renewable (such as hydraulic or solar) sources. Making furniture from different types of wood. Different fishing species.

[1] Devarajan and Fisher (1981) remark that «there are only a few fields in economics whose antecedents can be traced to a single, seminal article. One such field is natural resource economics, which is currently experiencing an explosive revival of interest; its origin is widely recognized as Harold Hotelling's 1931 paper».

[2] Of course, there are several economic articles which include several types of resources in order to discuss some specific issue. For example, Swallow (1990) studies the joint exploitation of a renewable and a nonrenewable resource when the interaction between both resources happens through the natural growth rate of the renewable resource. To the best of our knowledge, there is not any economic paper addressing the matter of combining renewable and nonrenewable resources as inputs for production in an economically optimal way.

NR and RR: renewable versus nonrenewable energy sources. Combining some metals (NR) with wood (RR). Making packages from paper (made from wood, RR) and plastic (obtained from oil, NR).

The results show that, when both resources are nonrenewable, it is optimal to use them in a constant proportion to each other, depending on their scarcity and their weight in production, while output decreases along time. When production depends on a renewable and a nonrenewable resource, the renewable resource tends to be more and more intensively used through time with respect to the nonrenewable resource and output is more sustainable when production rests more intensively on the renewable resource. In the two-renewable-resources case, it is possible to obtain a sustainable solution, represented by an interior steady state, whose uniqueness and saddlepoint stability are proved.

The remainder has the following structure: section 2 presents a theoretical model in which production depends on two natural resources, including the possibility of employing renewable or nonrenewable resources. In section 3, the solution is discussed stressing the time properties of the resource substitution, the output path and the existence of an interior steady state. The particular cases (NR,NR); (NR, RR) and (RR, RR) possess specific economic features which are presented in subsections 3.1, 3.2 and 3.3 respectively. In order to obtain some further insight about the results and to compare the solution corresponding to different combinations of resources, section 4 presents and analyzes in detail an example with a Cobb-Douglas production function. Section 5 shows the main conclusions and section 6 is a mathematical appendix.

2 Model, solution and economic interpretation

From a general equilibrium viewpoint, suppose an economy with a single consumption good, whose quantity is denoted by $Y \geq 0$, obtained from two natural resources used as inputs in quantities $X_1 \geq 0$ and $X_2 \geq 0$, according to the production function $Y = F(X_1, X_2)$, which is assumed to be of class $C^{(2)}$, homogeneous of degree 1, and verify[3] F_1, $F_2 > 0$, F_{11}, $F_{22} < 0$, $F_{11}F_{22} - F_{12}^2 > 0$. In order to focus the attention on natural resources, we take as exogenously given the quantities of any other input, such as capital and labor. Furthermore, a model with two resources is rich enough to address the questions raised in this paper. The solution provides simple and economically meaningful results which can be useful to manage any arbitrary number of resources.

$S_i(t)$ measures the stock of resource i $(i = 1, 2)$ at instant t and the time evolution of S_i is given by the following differential equation:

$$\dot{S}_i(t) \equiv \frac{dS_i(t)}{dt} = g_i(S_i(t)) - X_i(t), \tag{1}$$

where $g_i(S_i)$ is the natural growth function of resource i, which is concave, of class $C^{(2)}$ and verifies $g_i(0) = 0$. As noted for example in Smith (1968), the nonrenewable case is a particular one with $g_i(S_i) = 0 \; \forall \, S_i$. $X_i(t)$ is the instantaneous extraction rate of resource i at instant t.

The whole output Y is consumed by a single consumer in the economy, whose preferences are represented by the utility function $U(Y)$, which we assume is of class $C^{(2)}$ and verifies $U' > 0$, $U'' < 0$. A social planner has the objective of maximizing the consumer total discounted utility, so that he solves the problem

$$\begin{aligned}
&\underset{\{X_1, X_2\}}{Max} \int_0^\infty U(Y)e^{-\delta t}dt \\
&s.t.: \\
&\quad Y = F(X_1, X_2), \\
&\quad \left.\begin{aligned} &\dot{S}_i = g_i(S_i) - X_i, \\ &S_i(0) = S_i^0, \\ &0 \leq X_i \leq S_i, \end{aligned}\right\} \quad i = 1, 2,
\end{aligned} \tag{P}$$

δ being the time discount rate and S_i^0 the initial stock of resource i, which is exogenously given. To simplify the notation, the time variable t is omitted when there is no ambiguity.

(P) is an infinite horizon, continuous time, optimal control problem with two state variables and two control variables. If there is a time $T \in [0, \infty)$ in which both resource stocks are depleted under the optimal solution[4], then from T on, we necessarily have $X_1 = X_2 = 0$, in such a way that the objective functional of problem (P) can be written as

$$\int_0^\infty U(Y)e^{-\delta t}dt \equiv \int_0^T U(Y)e^{-\delta t}dt + \int_T^\infty e^{-\delta t}U_0 dt \equiv \int_0^T U(Y)e^{-\delta t}dt + \frac{1}{\delta}e^{-\delta T}U_0,$$

where $U_0 = U(F(0, 0))$ is a constant representing the utility obtained without resource extraction. Because T is not given a priori, but is a decision variable, (P) becomes a free time horizon problem.

[3] F_i denotes the partial derivative $\dfrac{\partial F}{\partial X_i}$ and F_{ij} denotes $\dfrac{\partial^2 F}{\partial X_i \partial X_j}$.

[4] The question of resource stock depletion is usually addressed in nonrenewable resource rather than renewable resource literature. Nevertheless, renewable resources are in fact subject to the possibility of depletion and, in many cases, this is an important concern in practice.

Note that problem (P) resembles a neoclassical optimal economic growth model with two activity sectors, each one exploiting a different natural resource, where the stocks of both resources play the role of productive capital stocks and the natural growth functions g_i play the role of two sector production functions[5].

Substituting the production function in the objective functional of problem (P), the current value Hamiltonian and the current value Lagrangian are defined as

$$\begin{aligned}\mathcal{H}(S_1, S_2, X_1, X_2, \lambda_1, \lambda_2) &= U[F(X_1, X_2)] + \sum_{i=1}^{2} \{\lambda_i [g_i(S_i) - X_i]\}, \qquad \text{and} \\ \mathcal{L}(S_1, S_2, X_1, X_2, \lambda_1, \lambda_2) &= \mathcal{H}(S_1, S_2, X_1, X_2, \lambda_1, \lambda_2) + \sum_{i=1}^{2} \{\mu_i X_i + \Psi_i (S_i - X_i)\},\end{aligned}$$

where λ_i is the costate variable related to resource i, which can be interpreted as the social valuation of a further unit of stock of the resource i or, equivalently, the social cost of extracting a unit of such a resource, and μ_i and Ψ_i are the multipliers related to constraints $X_i \geq 0$ and $X_i \leq S_i$.

From the optimal conditions for problem (P), which are discussed in section 6.1 of the appendix, we obtain that, if at a certain time t, $U^{'}F_i < \lambda_i$ holds, that is, the marginal utility of using the resource i (defined as its marginal productivity F_i times the marginal utility of consumption) is smaller than the social valuation of maintaining such a resource for its future use (measured by its shadow price), then $X_i = 0$, so that it is optimal not to employ any amount of resource i at all. If the resource i is *essential* for production[6], then the output Y has also a value of zero at instant t. Conversely, if $U^{'}F_i > \lambda_i$, then under the optimal solution $X_i = S_i$. The marginal utility of employing resource i being larger than the social valuation of keeping that resource unextracted, it is optimal to extract the whole available quantity of such a resource. This paper mainly focuses on the third case, that of interior solutions, with $0 < X_i < S_i$. In such a case, from the first order conditions, we obtain $U^{'}F_i = \lambda_i$, which is the usual equality between marginal utility and marginal cost for each resource, and $\frac{\dot{\lambda}_i}{\lambda_i} = \delta - g_i'(S_i)$, $i = 1, 2$, which is a usual condition in renewable resource models. For nonrenewable resources, it becomes the Hotelling rule $\dot{\lambda}_i/\lambda_i = \delta$, according to which, the shadow price of resource i grows at a constant rate equal to δ. The second order sufficient conditions for the maximization of the Hamiltonian are $U'F_{ii} + U''F_i^2 \leq 0$, $i = 1, 2$ and

$$D = \begin{vmatrix} U'F_{11} + U''F_1^2 & U'F_{12} + U''F_1F_2 \\ U'F_{12} + U''F_1F_2 & U'F_{22} + U''F_2^2 \end{vmatrix} \geq 0, \tag{2}$$

$|\cdot|$ denoting the determinant of the matrix. The assumptions on U and F guarantee $U'F_{ii} + U''F_i^2 \leq 0$, henceforth the second order conditions reduce to (2).

Let us define the relative use of resources as the ratio $x = \frac{X_1}{X_2}$. The main results of this paper, concerning the optimal path of relative resource employ (as measured by x) and output production (as measured by Y), are discussed in sections 2.1 and 2.2.

[5] There is a wide literature related to economic growth with renewable (Tahvonen and Kuuluvainen (1991, 1993)) and nonrenewable resources (Dasgupta and Heal (1974), Stiglitz (1974a, 1974b, 1976)).

[6] In chapter 8 of Dasgupta and Heal (1979) the concepts of *essential* and *nonessential resource* are discussed. In the present article, as in Hartwick (1978a), resource i is said to be essential for production if $X_i = 0 \Rightarrow F(X_1, X_2) = 0\ \forall X_j,\quad j \neq i$, i.e., if it is not possible to obtain any output with no resource i. This happens, for example, with a Cobb-Douglas production function $F(X_1, X_2) = X_1^{\alpha_1} X_2^{\alpha_2}$ or a Leontieff function $F(X_1, X_2) = \min\{a_1X_1, a_2X_2\}$. It does not happen with a linear function $F(X_1, X_2) = a_1X_1 + a_2X_2$.

2.1 Relative use of resources

Proposition 1 *In an interior solution of problem (P), x evolves according to the following differential equation:*

$$\frac{\dot{x}}{x} = \sigma \left[g_1'(S_1) - g_2'(S_2) \right], \tag{3}$$

where $\sigma = \dfrac{d\,(X_1/X_2)}{d\;MRTS} \dfrac{MRTS}{(X_1/X_2)}$ *represents the elasticity of substitution of the function F and* $MRTS = \dfrac{F_2}{F_1}$ *the Marginal Rate of Technical Substitution between both resources.*

Proof: see section 6.2∎

Proposition 1 has the following economic interpretation: throughout the optimal (interior) solution of problem (P), the evolution of x is determined by an environmental component -the difference between the marginal growth of both resources- and a technological component -the elasticity of substitution of the production function-. Given that $\sigma \geq 0$, (3) states that x increases (decreases), or equivalently, that X_1 (X_2) grows faster than X_2 (X_1)[7], if the marginal growth of resource 1 is larger (smaller) than that of resource 2. If we draw an analogy between a natural resource stock and a physical capital stock, the resource with a higher marginal productivity always tends to be more intensively employed.

In addition, the higher the elasticity of substitution, the faster the response of x to a difference between g_1' and g_2'. Remember that the elasticity of substitution is a measure of the technical *flexibility* to substitute inputs while keeping the output unchanged. As an extreme case, if $F(X_1, X_2) = \min\{\alpha_1 X_1, \alpha_2 X_2\}$, so that both resources are perfect complements and $\sigma = 0$[8], then the production technology is so rigid that x remains at a constant value given by the technological component, whatever the natural growth of resources are. For a linear production function $F(X_1, X_2) = a_1 X_1 + a_2 X_2$, with $\sigma = \infty$, if $g_1'(S_1) \neq g_2'(S_2)$ a corner solution is obtained with a full instantaneous adjustment towards the use of one of both resources. If $g_1'(S_1) = g_2'(S_2)$ an indeterminacy occurs. In the intermediate case of a Cobb-Douglas production function, with $\sigma = 1$, which is usual in the economic literature, equation (3) simplifies to $\dfrac{\dot{x}}{x} = g_1'(S_1) - g_2'(S_2)$.The Cobb-Douglas case is studied in section 4. See chapter 2 of André (2000) for an in-depth analysis of the extreme cases.

2.2 Optimal output path and extraction rates

Proposition 2 *In an interior solution for problem (P), the optimal output path is given by the following differential equation:*

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)} \left[\xi_1 g_1'(S_1) + \xi_2 g_2'(S_2) - \delta \right], \tag{4}$$

where ξ_1 and ξ_2 are the returns to the i-th input, given by[9]

$$\xi_i = \frac{X_i F_i}{F(X_1, X_2)} \geq 0, \qquad i = 1, 2, \tag{5}$$

[7] Note that $\dfrac{\dot{x}}{x} = \dfrac{\dot{X}_1}{X_1} - \dfrac{\dot{X}_2}{X_2}$ and so $\dfrac{\dot{x}}{x} > 0 \Leftrightarrow \dfrac{\dot{X}_1}{X_1} > \dfrac{\dot{X}_2}{X_2}$.

[8] Note that this function is not differentiable so that the results shown do not directly apply. Nevertheless, the perfect complements case can be regarded as an extreme case of the general result, taking limits when σ tends to 0 in a production function of the CES type $F(X_1, X_2) = \left[\alpha_1 X_1^{\frac{\sigma-1}{\sigma}} + \alpha_2 X_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$, with $0 < \alpha_1, \alpha_2 < 1$.

[9] See, for example, Nadiri (1982).

and $\eta(Y)$ *is the intertemporal substitution elasticity of the utility function* U*, given by* $\eta(Y) = \dfrac{-U''(Y)\,Y}{U'(Y)} \geq 0$.

Proof: see section 6.3∎

The equation (4) is a generalization of a single renewable resource extraction rule, given by

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)}\left[g'(S) - \delta\right]. \tag{6}$$

Condition (6) can be interpreted as a version of the Keynes-Ramsey rule of a standard neoclassical optimal economic growth model, where the marginal growth of the natural resource plays the role of the marginal capital productivity. Using this analogy, the stock of a natural resource S can be called *natural capital* and its marginal growth $g'(S)$ can be called the marginal productivity of natural capital. Using the same analogy, (4) can also be interpreted as a version of the Keynes-Ramsey rule, where the stock of natural capital is given by the linear convex combination $\xi_1 S_1 + \xi_2 S_2$ and the marginal productivity of natural capital is given by $\xi_1 g_1'(S_1) + \xi_2 g_2'(S_2)$, where each resource is weighted by its return ξ_i. If, at any instant of the solution, both marginal growths coincide, $g_1' = g_2' = g'$, (4) simplifies to (6), given that $\xi_1 + \xi_2 = 1$.

Given the strict concavity assumption on U, $\eta(Y)$ is positive for any positive value of Y and equations (4) and (6) state that, throughout the optimal solution, the output path grows with the difference between the marginal productivity of natural capital and the discount rate δ. This growth is smoother when U is more concave, as measured by $\eta(Y)$.

Deriving $F(X_1, X_2)$ with respect to t and dividing by $F(X_1, X_2)$, we obtain $\frac{\dot{Y}}{Y} = \xi_1 \frac{\dot{X}_1}{X_1} + \xi_2 \frac{\dot{X}_2}{X_2}$. Rearranging this equation and using $\frac{\dot{x}}{x} = \frac{\dot{X}_1}{X_1} - \frac{\dot{X}_2}{X_2}$, we have the identities $\frac{\dot{X}_1}{X_1} = \frac{\dot{Y}}{Y} + \xi_2 \frac{\dot{x}}{x}$ and $\frac{\dot{X}_2}{X_2} = \frac{\dot{Y}}{Y} - \xi_1 \frac{\dot{x}}{x}$ which, after using (3) and (4) and rearranging, become

$$\frac{\dot{X}_1}{X_1} = \frac{1}{\eta(Y)}\left[\gamma_{11} g_1'(S_1) + \gamma_{12} g_2'(S_2) - \delta\right], \tag{7}$$

$$\frac{\dot{X}_2}{X_2} = \frac{1}{\eta(Y)}\left[\gamma_{21} g_1'(S_1) + \gamma_{22} g_2'(S_2) - \delta\right], \tag{8}$$

where

$$\gamma_{11} = \xi_1 + \xi_2 \eta(Y)\sigma, \qquad \gamma_{12} = \xi_2\left(1 - \eta(Y)\sigma\right),$$

$$\gamma_{21} = \xi_1\left(1 - \eta(Y)\sigma\right), \qquad \gamma_{22} = \xi_2 + \xi_1 \eta(Y)\sigma$$

and $\gamma_{11} + \gamma_{12} = \gamma_{21} + \gamma_{22} = 1$. Note that (7) and (8) are similar to (4), in such a way that X_1 and X_2 evolve depending on the difference between a linear convex combination of both resources growth and the discount rate, and their evolution is smoother when the intertemporal substitution elasticity is higher.

3 Production with renewable and/or nonrenewable resources

The general solution takes different particular forms depending on the grow ability of both resources. The following table summarizes the main differences and commonalities among the different situations: two nonrenewable resources (NR) a nonrenewable and a renewable resource (NR and RR) and two renewable resources (RR)

case	Resource Intensity	Output	Steady State
NR	$\dot{x} = 0$ Constant throughout the solution depending on technical return and resource scarcity	$\dot{Y} < 0$ Decreasing throughout the solution	No interior steady state exists
NR and RR	$\dot{x} < 0$ renewable resource substitutes renewable resource	$\dot{Y} \lesseqgtr 0$ May increase or decrease depending on the properties of the renewable resource	No interior steady state exists
RR	$\dot{x} \lesseqgtr 0$	$\dot{Y} \lesseqgtr 0$	Interior ss. may exist. If $g"_1$, $g"_2 < 0$, then it is unique and saddlepoint stable

Some different economic conclusions can be drawn for each situation. In the following subsections, each case is studied with further detail.

3.1 Production with two nonrenewable resources

The case with two nonrenewable resources has already been addressed in the literature (see Beckman (1974, 1975) and Hartwick (1978a)) from slightly different perspectives[10]. Given that our purpose is to offer a unified framework to study all the possible renewable and nonrenewable resources, let us consider this case as a comparison pattern for more complex models with one or two renewable resources. Equation (3) shows that, if both resources are nonrenewable, $\dot{x} = 0$ so that x remains constant throughout the solution, or equivalently, the use of both resources increases (or decreases) at the same rate. The specific value of x is given by lemma 3 and proposition 4.

Lemma 3 *Let Λ be defined as the ratio of the shadow price of both resources, $\Lambda = \frac{\lambda_2}{\lambda_1}$. If both resources are nonrenewable, in an interior solution for problem (P), this ratio remains constant throughout the solution, so that, $\Lambda = \frac{\lambda_2(0)}{\lambda_1(0)}$. Furthermore, $\frac{F_2}{F_1} = \Lambda$ holds throughout the solution.*

Proof: Readily obtained from the Maximum Principle first order conditions (see section 6.1) with $g_1(S_1) = g_2(S_2) = 0$■

[10] Beckman studies a particular case with Cobb-Douglas production function and logarithmic utility function, instead of generic functions F and U. Hartwick focuses on efficient, not necessarily optimal paths.

The ratio Λ can be interpreted as a non-dimensional measure of both resources relative valuation and, given that economic valuation is linked to scarcity, Λ can also be interpreted as a measure of both resources relative scarcity. Lemma 3 states that, for two nonrenewable resources, this measure remains constant throughout the solution. Furthermore, X_1 and X_2 are used in such a way that the Marginal Rate of Technical Substitution is also constant and equals Λ.

Proposition 4 *In an interior solution for problem (P), if both resources are nonrenewable, the relative use of resources is given by*

$$x = \frac{\xi_1}{\xi_2}\Lambda, \tag{9}$$

where ξ_1 and ξ_2 were defined in (5) and Λ was defined in lemma 3.

Proof: see section 6.4■

Given that F is homogeneous of degree 1, we know that $\xi_1 + \xi_2 = 1$. The homogeneity assumption also implies that, if X_1 and X_2 grow at the same rate, then ξ_1 and ξ_2 $\left(\text{and hence } \frac{\xi_1}{\xi_2}\right)$ remain constant[11].

According to equation (9), the ratio x is given by the product of $\frac{\xi_1}{\xi_2}$, which is a measure of resource's relative technical weight, and Λ, which is a measure of relative scarcity. The greater the weight of resource 1 in production with respect to 2, and the scarcer resource 2 with respect to 1, the higher the optimal value of x.

When $g_1(S_1) = g_2(S_2) = 0$, from (4), (7) and (8) we know

$$\frac{\dot{Y}}{Y} = \frac{\dot{X}_1}{X_1} = \frac{\dot{X}_2}{X_2} = \frac{-\delta}{\eta(Y)} < 0, \tag{10}$$

so that the output and both resources extraction rate decrease at the same rate. Such reduction is faster the higher the discount rate and the smaller the elasticity of temporal substitution. For high values of δ, present and near future weigh very strongly with respect to the distant future in the objective function, so that it is optimal to extract resources very intensively at the beginning of the time horizon. For high values of $\eta(Y)$, the utility function is very concave and smooth consumption paths are preferred.

3.2 Production with renewable and nonrenewable resources

Assume that resource 2 is renewable and resource 1 is nonrenewable[12], so that $g_1(S_1) = 0$. Given that the natural growth is the only intrinsic property of resources we are interested in, from a macroeconomic point of view, we can interpret resource 1(2) as an aggregate of all the nonrenewable (renewable) resources existing in the economy which are relevant for production. Then equation (3) becomes

$$\frac{\dot{x}}{x} = -\sigma g_2'(S_2), \tag{11}$$

[11] Assume that X_1 and X_2 are multiplied by a positive constant α. For homogeneity of F we know that $F(\alpha X_1, \alpha X_2) = \alpha F(X_1, X_2)$. The first derivatives of a degree one homogeneous function are degree zero homogeneous, so that $F_i(\alpha X_1, \alpha X_2) = F_i(X_1, X_2)$ $i = 1, 2$ and ξ_i becomes

$$\xi_i(\alpha X_1, \alpha X_2) = \frac{\alpha X_i F_i(X_1, X_2)}{\alpha F(X_1, X_2)} = \xi_i(X_1, X_2).$$

[12] Given the symmetry of the model, this distinction is arbitrary.

and the sign of the time evolution of x is given just by the marginal growth of resource 2. The speed of such evolution is also affected by the elasticity of substitution of the technology. Provided that $g_2'(S_2) > 0$, x decreases, so it is optimal for the renewable resource to be more and more intensively used with respect to the nonrenewable resource.

If resource 1 is nonrenewable and resource 2 is renewable, equation (4) becomes

$$\frac{\dot{Y}}{Y} = \frac{1}{\eta(Y)} \left[\xi_2 g_2' \left(S_2\right) - \delta\right]. \tag{12}$$

In this case, the output path may be time increasing or decreasing, depending on the sign of the difference $\xi_2 g_2' \left(S_2\right) - \delta$. Such a path would be more increasing (or less decreasing) when the marginal growth of the renewable resource is larger and such a resource has a greater weight on the production technology. Note the economic meaning of this result. The existence of a renewable resource makes it possible to maintain the production through time; when the marginal growth of resource 2 and its weight on technology is smaller this case becomes more similar to the one with two nonrenewable resources.

Equations (11) and (12) express, in a mathematical way, the interest (and, in the long run, the need) to promote the research and use of renewable energy sources, such as solar, hydraulic or wind energy to substitute nonrenewable energies, such as oil, coal and atomic energy. A production process is *more sustainable* the more it depends on renewable resources instead of nonrenewable resources[13].

As for resource extraction rates, equations (7) and (8), take the form

$$\frac{\dot{X}_1}{X_1} = \frac{1}{\eta\left(Y\right)} [\gamma_{12} g_2' \left(S_2\right) - \delta], \qquad \frac{\dot{X}_2}{X_2} = \frac{1}{\eta\left(Y\right)} \left[\gamma_{22} g_2' \left(S_2\right) - \delta\right],$$

being $\gamma_{12} = \xi_2 \left(1 - \eta\left(Y\right)\sigma\right)$ and $\gamma_{22} = \xi_2 + \xi_1 \eta\left(Y\right)\sigma$.

3.3 Two renewable resources: steady state analysis

The most interesting issue concerning the two renewable resource cases is the possibility of obtaining an interior steady state (that is, one in which control and state variables have a strictly positive value). A steady state of problem (P) is defined as a set of sustainable values for state, control and costate variables $(\bar{\lambda}_1, \bar{\lambda}_2)$ such that, if those values are simultaneously reached at a certain point of the optimal solution, they keep indefinitely constant, that is, $\dot{X}_1 = \dot{X}_2 = \dot{Y} = \dot{S}_1 = \dot{S}_2 = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$.

Given the structure of problem (P), note that the only possibility of obtaining an interior steady state requires that both resources are renewable, given that if a resource i is nonrenewable, $X_i > 0$ is not compatible with $\dot{S}_i = 0$.

Using the definition of steady state in (1) and (29), an interior steady state is given by

$$g_i\left(S_i\right) = X_i, \quad i = 1, 2, \tag{13}$$

$$g_1' = g_2' = \delta. \tag{14}$$

The existence of an interior steady state is guaranteed if there exist a pair of positive values $\bar{S}_1 > 0$ and $\bar{S}_2 > 0$ such that $\bar{S}_1 > g_1\left(\bar{S}_1\right) > 0$, $\bar{S}_2 > g_2\left(\bar{S}_2\right) > 0$ and $g_1'\left(\bar{S}_1\right) = g_2'\left(\bar{S}_2\right) = \delta$. Those values,

[13] Of course, a further reason for this substitution is the fact that nonrenewable energy sources are, in general, more intensively polluting than renewable sources. The extreme case is that of atomic power, whose potential consequences for health and life are known.

if they exist, can be obtained from equations (14) and, using them in (13), they allow us to obtain the steady state controls $\bar{X}_1$ and $\bar{X}_2$. By substitution in the production function we have $\bar{Y}$, and from (28), we obtain $\bar{\lambda}_1$ and $\bar{\lambda}_2$. If a steady state exists, a sufficient condition for it to be unique is g_1'', $g_2'' < 0$.

In a steady state of (P), global stability is discarded[14] and the only possibilities are instability and saddlepoint stability. The following proposition states necessary and sufficient conditions for local saddlepoint stability:

Proposition 5 *If second order conditions (2) hold with strict inequality, an interior steady state for problem (P) is saddlepoint stable if and only if* $g_1''\left(\bar{S}_1\right)$, $g_2''\left(\bar{S}_2\right) < 0$.

Proof: see section 6.5■

Next, a sensitivity analysis exercise is developed with respect to the only parameter of problem (P): the discount rate δ. Deriving (13) and (14) with respect to δ, we obtain

$$g_i'(\bar{S}_i)\frac{\partial S_i}{\partial \delta} = \frac{\partial X_i}{\partial \delta}, \qquad i = 1, 2, \tag{15}$$

$$g_i''\frac{\partial S_i}{\partial \delta} = 1, \qquad i = 1, 2. \tag{16}$$

Rearranging (15) and (16), using (14) and assuming $g_1''\left(\bar{S}_1\right)$, $g_2''\left(\bar{S}_2\right) < 0$, we obtain

$$\left.\begin{array}{l} \dfrac{dS_i}{d\delta} = \dfrac{1}{g_i''(\bar{S}_i)} < 0, \\ \dfrac{dX_i}{d\delta} = \dfrac{g_i'(\bar{S}_i)}{g_i''(\bar{S}_i)} = \dfrac{\delta}{g_i''(\bar{S}_i)} < 0, \end{array}\right\} \qquad i = 1, 2,$$

Observe that this result is economically meaningful: the higher the value of δ, the higher the weight of the present moment in the objective function with respect to future. So, for high values of δ, it becomes optimal to extract both resources intensively in the early stage of the solution, reducing the available long run stock, and hence the steady state stock. To make this stock sustainable, the resource extraction rates must also be smaller.

[14] Kurz (1968).

4 Example: Cobb-Douglas production function

For a further study of the solution, a particular case with Cobb-Douglas production function is presented[15]. Assume that the production function is $F(X_1, X_2) = X_1^{\alpha_1} X_2^{\alpha_2}$, with $\alpha_1 + \alpha_2 = 1$. Assume also that the utility function is of the constant temporal substitution elasticity type $U(Y) = \dfrac{Y^{1-\eta}}{1-\eta}$ with $0 < \eta < 1$.

4.1 Two nonrenewable resources

As obtained in section 6.6 of the appendix, the optimal extraction rate of each resource is given by

$$X_i = A_i e^{\frac{-\delta t}{\eta}} \quad i = 1,2, \qquad \text{where} \quad A_i = \frac{\delta S_i^0}{\eta} > 0 \quad i = 1,2. \tag{17}$$

As a consequence, Y and x take the expressions

$$Y = A_1^{\alpha_1} A_2^{1-\alpha_1} e^{\frac{-\delta t}{\eta}} = \frac{\delta S_2^0}{\eta} \left(\frac{S_1^0}{S_2^0} \right)^{\alpha_1} e^{\dfrac{-\delta t}{\eta}}, \tag{18}$$

$$x = \frac{X_1}{X_2} = \frac{S_1^0}{S_2^0}. \tag{19}$$

Note that x only depends on S_1^0 and S_2^0. We obtain that $\dfrac{\partial Y}{\partial S_i^0} > 0$, $i = 1,2$, so that increasing the initial stock S_i^0 of any resource leads to augmenting that resource extraction rate and hence to shift upwards the whole output path. The more intense the weight of such resource in production (as measured by α_i), the larger the shift. The remaining parameters affect X_1 and X_2 in the same direction and intensity, so that it is enough to study their aggregate influence on Y. We obtain that $\dfrac{\partial Y}{\partial \alpha_1} \geq 0$ if and only if $S_1^0 \geq S_2^0$, implying that an increment in the weight of the most abundant (scarce) resource, causes the output to increase (decrease) throughout the whole optimal path, implying that technologies intensively depending on abundant resources allow us to obtain more output than technologies mainly depending on scarce resources. We also obtain that $\dfrac{\partial Y}{\partial \delta} \geq 0$ and $\dfrac{\partial Y}{\partial \eta} \leq 0$ if and only if $t \leq \dfrac{\eta}{\delta}$, so that, an increment of the discount rate leads to increase the resource extraction and output for *low* values of t and to reduce them for the distant future, whereas increasing parameter η causes the utility function to be more concave so that, under the optimal solution, it is preferable to smooth the output path, increasing long term with respect to short term consumption.

4.2 A renewable and a nonrenewable resource

Assume that resource 1 is nonrenewable ($g_1(S_1) = 0$) and resource 2 is renewable with a constant growth rate, so that, $g_2(S_2) = \gamma_2 S_2$. Make also the technical assumption $\delta > \alpha_2\gamma_2$ to ensure solution existence.

As shown in section 6.7 the solution for X_1 and X_2 is given by

$$X_i = K_1 S_i^0 e^{-K_i t} > 0, \tag{20}$$

where $K_1 = \dfrac{\delta - \alpha_2\gamma_2(1-\eta)}{\eta} > 0, \qquad K_2 = \dfrac{\delta - \gamma_2[1-\alpha_1(1-\eta)]}{\eta} \lesseqgtr 0.$

[15] In the chapter 2 of André (2000) a similar analysis is developed with other technologies: perfect complements and perfect substitutes.

From (20) we obtain

$$Y = \left(S_1^0\right)^{\alpha_1} \left(S_2^0\right)^{1-\alpha_1} \frac{\left[\delta - (1-\alpha_1)\gamma_2(1-\eta)\right]}{\eta} e^{\frac{(1-\alpha_1)\gamma_2 - \delta}{\eta} t} > 0, \qquad x = \frac{X_1}{X_2} = \frac{S_1^0}{S_2^0} e^{-\gamma_2 t},$$

Therefore X_1 tends asymptotically to zero, whereas X_2 may increase or decrease depending on the sign of K_2. We can also conclude that $\frac{\dot{Y}}{Y} = \frac{1}{\eta}\left[\alpha_2\gamma_2 - \delta\right] < 0$, showing that the output decreases more slowly than in the two nonrenewable resources case. The output reduces faster for lower values of the growth rate γ_2, the renewable resource returns α_2, the temporal substitution elasticity η, and for higher values of the discount rate δ. In a similar way $\frac{\dot{x}}{x} = -\gamma_2$, showing that, throughout the solution, x decreases at a constant rate γ_2.

The effect of the parameters S_1^0, S_2^0, δ and η on the solution are similar to those shown in section 4.1. The effect of the parameter α_1 (measuring the nonrenewable resource elasticity) is given by

$$\frac{\partial Y}{\partial \alpha_1} \geq 0 \Longleftrightarrow t \leq t^*, \qquad \text{where } t^* = \frac{\eta}{\gamma_2} \log\left(\frac{S_1^0}{S_2^0}\right) + \frac{\eta(1-\eta)}{\delta - \alpha_2\gamma_2(1-\eta)}, \tag{21}$$

so that, short term output increases with α_1 if resource 1 is initially abundant enough with respect to 2[16]. The larger the proportion $\frac{S_1^0}{S_2^0}$, the more intensive and the longer term the positive effect of α_1 on output. Nevertheless, $\frac{S_1^0}{S_2^0}$, and hence t^*, is always finite, in such a way that, in the long run, increasing α_1 reduces output.

The following table shows the effect of γ_2 on variables X_1, X_2 and Y:

	$0 \leq t < t_{X_2}$	$t_{X_2} < t < t_Y$	$t_Y < t < t_{X_1}$	$t_{X_1} < t < \infty$
X_1	$-$	$-$	$-$	$+$
X_2	$-$	$+$	$+$	$+$
Y	$-$	$-$	$+$	$+$

$t_{X_2} < t_Y < t_{X_1}$ being given by

$$\begin{cases} t_{X_2} = \dfrac{\alpha_2\eta(1-\eta)}{\left[1-\alpha_1(1-\eta)\right]\left[\delta - \alpha_2\gamma_2(1-\eta)\right]} > 0, \\ \\ t_Y = \dfrac{\eta(1-\eta)}{\delta - \alpha_2\gamma_2(1-\eta)} > 0, \qquad t_{X_1} = \dfrac{\eta}{\delta - \alpha_2\gamma_2(1-\eta)} > 0. \end{cases}$$

Increasing γ_2 leads to reducing short term, and to increasing long term resource extraction and output, in order to take long term advantage from the larger natural growth ability. Nevertheless, the effect of γ_2 has a different timing on each variable. The extraction rate of resource 2 begins to increase at instant t_{X_2}, whereas resource 1 extraction begins to increase later, at instant t_{X_1}. In the interval (t_{X_2}, t_Y) the X_2 increment does not suffice to compensate the X_1 drop, causing output to diminish. From t_Y to t_{X_1}, although X_1 is still lower than initially, the compensation by larger value of X_2 allows output to increase. From t_{X_1}on, both resources extraction rate (and, of course, output) is larger.

[16] If $\frac{S_1^0}{S_2^0} \leq e^{\frac{-\gamma_2(1-\eta)}{\delta - \alpha_2\gamma_2(1-\eta)}}$, then $t^* \leq 0$, meaning that an increment on α_1 reduces the value of Y throughout the whole solution.

4.3 Two renewable resources

Assume that the stocks of both resources grow according to the usual logistic function

$$g_i(S_i) = \theta_i \left(S_i - \frac{S_i^2}{K_i} \right) \qquad i = 1, 2,$$

θ_i and K_i being two positive parameters known as intrinsic growth rate and carrying capacity. According to equations (13) and (14), the steady state is given by

$$\theta_i \left(S_i - \frac{S_i^2}{K_i} \right) = X_i, \qquad \theta_i \left(1 - \frac{2S_i}{K_i} \right) = \delta, \qquad i = 1, 2. \tag{22}$$

As for the problem of steady state existence, the condition for equations (22) having an interior solution is θ_1, $\theta_2 > \delta$. If this condition holds, we can solve (22) for X_i and S_i to obtain

$$\bar{S}_i = \frac{K_i}{2} \left(1 - \frac{\delta}{\theta_i} \right), \qquad \bar{X}_i = K_i \frac{\theta_i^2 - \delta^2}{4\theta_i}, \qquad i = 1, 2. \tag{23}$$

Note that $\bar{S}_i$ and $\bar{X}_i$ do not depend on the specific utility and production functions, but the transition to steady state depends on such functions according to equations (3) and (4). The utility and production functions also have influence on the steady state value for costate variables λ_1 and λ_2, according to equations (28). Given $F(X_1, X_2) = X_1^{\alpha_1} X_2^{\alpha_2}$ and $U(Y) = \dfrac{Y^{1-\eta}}{1-\eta}$, such values are given by

$$\bar{\lambda}_i = \alpha_i \bar{X}_i^{\alpha_i(1-\eta)-1} \bar{X}_j^{\alpha_j(1-\eta)} = \alpha_i \left(K_i \frac{\theta_i^2 - \delta^2}{4\theta_i} \right)^{\alpha_i(1-\eta)-1} \left(K_j \frac{\theta_j^2 - \delta^2}{4\theta_j} \right)^{\alpha_j(1-\eta)} \qquad \begin{array}{c} i, j = 1, 2, \\ i \neq j. \end{array}$$

Provided that $g_i''(S_i) = -\dfrac{2\theta_i}{K_i} < 0$, $i = 1, 2$, both growth functions are strictly concave and, as a consequence, if a θ_1, $\theta_2 > \delta > 0$, then a unique steady state exists and i tis local saddlepoint stabile. As shown in section 6.8, we have the following staeady state sensitivity analysis results:

$$\frac{\partial \bar{S}_i}{\partial \delta}, \frac{\partial \bar{X}_i}{\partial \delta} < 0, \qquad \frac{\partial \bar{S}_i}{\partial K_j}, \frac{\partial \bar{S}_i}{\partial \theta_j}, \frac{\partial \bar{X}_i}{\partial K_j}, \frac{\partial \bar{X}_i}{\partial \theta_j} \begin{cases} > 0 & \text{if } i = j \\ = 0 & \text{if } i \neq j \end{cases} \qquad i, j = 1, 2,$$

As discussed in section 3.3 for the general case, the larger δ, the smaller the steady state value for both resources stock and extraction rate. Parameters θ_i and K_i do not affect the steady state value for resource j when $i \neq j$, because no natural interaction exists between both resources. Note, however, that according to equations (7) and (8) both growth functions are relevant for determining both resources optimal extraction path. Increasing resource i's intrinsic growth rate or carrying capacity makes such resource steady state stock and extraction rate increase.

5 Conclusions and extensions

A theoretical model has been presented in order to study the optimal combination of natural resources used as inputs, taking into account their natural growth ability and the technical substitution possibilities. The model allows us to include either renewable, nonrenewable resources or both, connecting renewable and nonrenewable resources economics.

The relative use of resources evolves according to two factors: the difference between both resources' natural growth and technical flexibility, as measured by the production elasticity of substitution. Resource 1 proportion with respect to 2 increases (decreases) when resource 1's marginal growth is greater (smaller) than that of resource 2. This adjustment is faster when the production elasticity of substitution is higher.

The optimal output path obtained from two natural resources follows an equation similar to the classical Keynes-Ramsey rule of economic growth models, where the role of physical capital productivity is played by the marginal productivity of natural capital, which is formed as a linear convex combination of both resources' marginal growth, using returns to i-th input as weight in the combination. Output grows (drops) through time when natural capital marginal productivity is greater (smaller) than the temporal discount rate. This adjustment is slower when the temporal substitution elasticity of the utility function is higher.

When both resources are nonrenewable, they are used at a constant proportion determined as the product of a relative production weight measure and a relative scarcity measure. Output and both resources extraction rate decrease at a rate that is greater in absolute value when the temporal discount rate is larger and the temporal substitution elasticity is smaller.

When production depends on a renewable and a nonrenewable resource, the natural capital marginal productivity is formed by the renewable resource marginal growth times such resource return. If the marginal growth is positive, the renewable resource is more and more intensively used through time with respect to the nonrenewable resource. Output is more sustainable when the return to the renewable resource and its marginal growth are higher.

The case with two renewable resources is the only one in which a sustainable solution, represented by an interior steady state can exist. If such a steady state exists, its uniqueness and saddlepoint stability are guaranteed if both resources natural growth functions are strictly concave.

Some plausible ways to extend the results obtained in this paper are the following: first, take into account some further features of the natural resources which are relevant for their optimal use, such as their recycling ability and their impact on environmental quality. Second, include some additional elements in the theoretical model, such as physical capital accumulation and technical change. From a market equilibrium point of view, it would be relevant to model the interaction among different economic agents with decision capacity on resources management.

6 Appendix : Mathematical results

6.1 Solution to problem (P)

Together with the state equations ($\dot{S}_i = g_i(S_i) - X_i$, with $S_i(0) = S_i^0$), The Pontryagin Maximum Principle conditions for problem (P) are

$$\dot{\lambda}_i = \lambda_i \left(\delta - g_i'\left(S_i\right)\right) - \Psi_i, \qquad i = 1, 2. \tag{24}$$

$$\begin{aligned} U^{'} F_i - \lambda_i + \mu_i - \Psi_i &= 0, \qquad i = 1, 2, && (25) \\ \mu_i X_i &= 0, \qquad i = 1, 2, && (26) \\ \Psi_i \left[S_i - X_i\right] &= 0, \qquad i = 1, 2, && (27) \\ X_i, \mu_i, \Psi_i, S_i - X_i &\geq 0, \qquad i = 1, 2. \end{aligned}$$

The transversality conditions for the terminal value of λ_i and T are

$$\begin{aligned} \lim_{t\to\infty} e^{-\delta t}\lambda_i &\geq 0 \qquad \text{with} \qquad \lim_{t\to\infty} e^{-\delta t}\left(\lambda_i S_i\right) = 0, \qquad i = 1, 2, \\ e^{-\delta T}\mathcal{H}\left(T\right) &= -\frac{\partial}{\partial T}\left(\frac{1}{\delta}e^{-\delta T}U_0\right) = e^{-\delta T}U_0 \Leftrightarrow \mathcal{H}\left(T\right) = U_0, \end{aligned}$$

$\mathcal{H}(T)$ being the current value Hamiltonian at T.

Mangasarian global maximum sufficient conditions hold[17], given that U, F and g_i are assumed to be concave functions and $\lambda_i \geq 0$ throughout the solution. To prove the latter statement, note that (25) can be expressed as $\lambda_i = U^{'} F_i + \mu_i - \Psi_i$ and, by (27), we obtained that, if $S_i > X_i$[18], then $\Psi_i = 0$ holds and λ_i is nonnegative because U', F_i and μ_i are nonnegative.

Whenever $U^{'} F_i < \lambda_i$ holds, from (25) we know that $\mu_i > 0$, and because of (26), $X_i = 0$. If $U^{'} F_i > \lambda_i$, then because of (25), $\Psi_i > 0$ and, from (27), we know that $X_i = S_i$. As for interior solutions, from (26) and (27), we have $\mu_i = \Psi_i = 0$, so that (25) and (29) become

$$\begin{aligned} \lambda_i &= U^{'} F_i\,, \qquad \text{and} && (28) \\ \frac{\dot{\lambda}_i}{\lambda_i} &= \delta - g_i'(S_i), \qquad i = 1, 2. && (29) \end{aligned}$$

6.2 Proof of proposition 1

Deriving (28) with respect to time and dividing by (28) we obtain

$$\dot{\lambda}_i = U' \frac{dF_i}{dt} + \frac{dU'}{dt} F_i \quad \Rightarrow \frac{\dot{\lambda}_i}{\lambda_i} = \frac{1}{F_i}\frac{dF_i}{dt} + \frac{1}{U'}\frac{dU'}{dt} \qquad i = 1, 2. \tag{30}$$

Using (29) to substitute $\frac{\dot{\lambda}_i}{\lambda_i}$, and rearranging,

$$\frac{1}{F_1}\frac{dF_1}{dt} - \frac{1}{F_2}\frac{dF_2}{dt} = g_2'(S_2) - g_1'(S_1). \tag{31}$$

[17] Mangasarian (1966).

[18] If, at a certain instant $\bar{t}$, we have $X_i = S_i$, the whole quantity of resource i is extracted. Given that $g_i\left(0\right) = 0$, from $\bar{t}$ on, resource i ceases to be available for production.

Using the definition of x and the homogeneity assumption for F, $f(x)$ is defined as

$$\frac{F(X_1,X_2)}{X_2}=F\left(\frac{X_1}{X_2},1\right)=F\left(x,1\right)=f(x) \tag{32}$$

and we know that

$$F_1=f'(x), \qquad F_2=f(x)-x\,f'(x), \qquad \frac{dF_1}{dt}=f''(x)\dot{x}, \qquad \frac{dF_2}{dt}=-x\,f''(x)\dot{x}. \tag{33}$$

Using (33) to substitute F_1, F_2, $\frac{dF_2}{dt}$ and $\frac{dF_2}{dt}$ in (31), and rearranging,

$$\frac{\dot{x}}{x}\frac{x\,f(x)\;f''(x)}{f'(x)\,[f(x)-xf'(x)]}=g_2'(S_2)-g_1'(S_1). \tag{34}$$

F being homogeneous of degree one, its elasticity of substitution may be expressed as[19]

$$\sigma=\frac{-f'(x)\,[f\,(x)-x\,f'\,(x)]}{x\,f(x)\,f''(x)}>0. \tag{35}$$

Using (35) in (34) and rearranging, we obtain (3)■

6.3 Proof of proposition 2

Adding the equations for $\dot{\lambda}_1/\lambda_1$ and $\dot{\lambda}_2/\lambda_2$ in (30), we have $\frac{1}{U'}\frac{dU'}{dt}=\frac{1}{2}\left[\frac{\dot{\lambda}_1}{\lambda_1}+\frac{\dot{\lambda}_2}{\lambda_2}-\frac{1}{F_1}\frac{dF_1}{dt}-\frac{1}{F_2}\frac{dF_2}{dt}\right]$ and, using (29), becomes

$$\frac{1}{U'}\frac{dU'}{dt}=\frac{1}{2}\left[2\,\delta-g_1'(S_1)-g_2'(S_2)-\left(\frac{1}{F_1}\frac{dF_1}{dt}+\frac{1}{F_2}\frac{dF_2}{dt}\right)\right]. \tag{36}$$

Using (33) and (3), we know that

$$\frac{1}{F_1}\frac{dF_1}{dt}+\frac{1}{F_2}\frac{dF_2}{dt}=\frac{f''(x)\,[f(x)-2\,x\,f'(x)]}{f'(x)\,[f(x)-xf'(x)]}\dot{x}=\frac{f''(x)\,[f(x)-2\,x\,f'(x)]\,x}{f'(x)\,[f(x)-xf'(x)]}\sigma\,[g_1'(S_1)-g_2'(S_2)]\,. \tag{37}$$

Substituting in (37) the value of σ given by (35), rearranging and using (33) again,

$$\frac{1}{F_1}\frac{dF_1}{dt}+\frac{1}{F_2}\frac{dF_2}{dt}=-\left[g_1'(S_1)-g_2'(S_2)\right]\frac{f(x)-2xf'(x)}{f(x)}=-\left[g_1'(S_1)-g_2'(S_2)\right]\frac{F(X_1,X_2)-2X_1F_1}{F(X_1,X_2)}. \tag{38}$$

Substituting (38) in (36), using the definition of ξ_i given in (5) and the Euler theorem for homogeneous functions, according to which $X_1F_1+X_2F_2=F\,(X_1,X_2)$, we have

$$\frac{1}{U'}\frac{dU'}{dt}=\delta-g_1'(S_1)\frac{X_1F_1}{F\,(X_1,X_2)}-g_2'(S_2)\frac{F\,(X_1,X_2)-\,X_1F_1}{F\,(X_1,X_2)}=\delta-\xi_1g_1'(S_1)-\xi_2g_2'(S_2),$$

that becomes (4) after using $\frac{1}{U'}\frac{dU'}{dt}=\frac{1}{U'}U''\dot{Y}=-\eta(Y)\frac{\dot{Y}}{Y}$■

6.4 Proof of proposition 4

Using lemma 3 and (33), we have $\Lambda=\frac{F_2}{F_1}=\frac{f\,(x)-xf'\,(x)}{f'\,(x)}=\frac{f\,(x)}{f'\,(x)}-x$, and using (32) and (33),

$$x=\frac{f\,(x)}{f'\,(x)}-\Lambda=\frac{F\,(X_1,X_2)}{X_2F_1\,(X_1,X_2)}-\Lambda=x\frac{F\,(X_1,X_2)}{X_1F_1\,(X_1,X_2)}-\Lambda=\frac{x}{\xi_1}-\Lambda.$$

Rearranging, we obtain $x=\frac{\xi_1}{1-\xi_1}\Lambda$, and using $\xi_1+\xi_2=1$, (9) is obtained■

[19] See, for example, Dasgupta and Heal (1974).

6.5 Proof of proposition 5

First, we need to express the optimal value of the control variables as a function of the state and costate variables. Taking the total differential in the two-equations system (28) we obtain

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} d\lambda_1 \\ d\lambda_2 \end{bmatrix}, \tag{39}$$

where $A = U^{'}F_{11} + U^{''}F_1^2 < 0$, $B = U^{'}F_{12} + U^{''}F_1F_2 \lesseqgtr 0$, $C = U^{'}F_{22} + U^{''}F_2^2 < 0$. If (2) holds with strict inequality, then $D = AC - B^2 > 0$ and the Implicit Function Theorem guarantees the local existence of the $C^{(2)}$ functions

$$X_i = \hat{X}_i(\lambda_1, \lambda_2), \qquad i = 1, 2, \tag{40}$$

implicitly defined in (28). By Cramer's rule, we obtain

$$\frac{\partial X_1}{\partial \lambda_1} = \frac{C}{D} < 0, \qquad \frac{\partial X_2}{\partial \lambda_2} = \frac{A}{D} < 0, \qquad \frac{\partial X_1}{\partial \lambda_2} = \frac{\partial X_2}{\partial \lambda_1} = \frac{-B}{D} \lesseqgtr 0. \tag{41}$$

Substituting (40) in (1) we have

$$\dot{S}_i = g_i(S_i) - \hat{X}_i(\lambda_1, \lambda_2), \qquad i = 1, 2 \tag{42}$$

which, together with (29), form the *canonical or modified Hamiltonian dynamical system.* Following Dockner (1985), we make a first order approximation at a steady state. Deriving (29) and (42) with respect to S_i and λ_2, the Jacobian matrix of the canonical system, evaluated at a steady state, is defined as

$$J = \begin{bmatrix} \frac{\partial \dot{S}_1}{\partial S_1} & \frac{\partial \dot{S}_1}{\partial S_2} & \frac{\partial \dot{S}_1}{\partial \lambda_1} & \frac{\partial \dot{S}_1}{\partial \lambda_2} \\ \frac{\partial \dot{S}_2}{\partial S_1} & \frac{\partial \dot{S}_2}{\partial S_2} & \frac{\partial \dot{S}_2}{\partial \lambda_1} & \frac{\partial \dot{S}_2}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_1}{\partial S_1} & \frac{\partial \dot{\lambda}_1}{\partial S_2} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_2}{\partial S_1} & \frac{\partial \dot{\lambda}_2}{\partial S_2} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_1} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} \end{bmatrix}_{\bar{S}_i, \bar{\lambda}_i} = \begin{bmatrix} \delta & 0 & -\frac{\partial X_1}{\partial \lambda_1} & -\frac{\partial X_1}{\partial \lambda_2} \\ 0 & \delta & -\frac{\partial X_2}{\partial \lambda_1} & -\frac{\partial X_2}{\partial \lambda_2} \\ -\bar{\lambda}_1 g_1''(\bar{S}_1) & 0 & 0 & 0 \\ 0 & -\bar{\lambda}_2 g_2''(\bar{S}_2) & 0 & 0 \end{bmatrix}. \tag{43}$$

Following Dockner (1985) and Tahvonen (1989) the necessary and sufficient conditions for local saddle-point stability are $|J| > 0$ and $K < 0$, being $|J|$ the determinant of J and K defined as

$$\begin{aligned} K &= \left[\begin{vmatrix} \frac{\partial \dot{S}_1}{\partial S_1} & \frac{\partial \dot{S}_1}{\partial \lambda_1} \\ \frac{\partial \dot{\lambda}_1}{\partial S_1} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_1} \end{vmatrix} + \begin{vmatrix} \frac{\partial \dot{S}_2}{\partial S_2} & \frac{\partial \dot{S}_2}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_2}{\partial S_2} & \frac{\partial \dot{\lambda}_2}{\partial \lambda_2} \end{vmatrix} + 2 \begin{vmatrix} \frac{\partial \dot{S}_1}{\partial S_2} & \frac{\partial \dot{S}_1}{\partial \lambda_2} \\ \frac{\partial \dot{\lambda}_1}{\partial S_2} & \frac{\partial \dot{\lambda}_1}{\partial \lambda_2} \end{vmatrix} \right]_{\bar{S}_i, \bar{\lambda}_i} \\ &= \begin{vmatrix} \delta & -\frac{\partial X_1}{\partial \lambda_1} \\ -\lambda_1 g_1'' & 0 \end{vmatrix} + \begin{vmatrix} \delta & -\frac{\partial X_2}{\partial \lambda_2} \\ -\lambda_2 g_2'' & 0 \end{vmatrix} + 2 \begin{vmatrix} 0 & -\frac{\partial X_1}{\partial \lambda_2} \\ 0 & 0 \end{vmatrix}. \end{aligned}$$

After some operations, we obtain $|J| = \frac{\lambda_1 g_1'' \lambda_2 g_2''}{D}$ and $K = -\left[\lambda_1 g_1'' \frac{\partial X_1}{\partial \lambda_1} + \lambda_2 g_2'' \frac{\partial X_2}{\partial \lambda_2}\right]$, where D is defined in (2) and has a positive value if the second order conditions hold with strict inequality.

In an interior solution, because of (28), λ_1, $\lambda_2 > 0$ holds and, as shown in (41), $\frac{\partial X_1}{\partial \lambda_1}, \frac{\partial X_2}{\partial \lambda_2} < 0$. $|J| > 0$ requires that g_1'' and g_2'' are different from zero and have the same sign, whereas $K < 0$ requires such a sign to be negative. Hence, a steady state is saddlepoint stable if and only if g_1'', $g_2'' < 0$■

6.6 Example with two nonrenewable resources (section 4.1)

The problem to solve is

$$\underset{\{X_1,X_2\}}{Max}\int_0^{\infty}\tfrac{1}{1-\eta}\left(X_1^{\alpha_1(1-\eta)}X_2^{\alpha_2(1-\eta)}\right)e^{-\delta t}dt$$
$$s.t.:\quad \left.\begin{array}{l}\dot{S}_i=-X_i,\\ S_i(0)=S_i^0,\\ 0\le X_i\le S_i,\end{array}\right\} i=1,2.$$

We will find an interior solution, and then check that the constraints $0 \le X_i \le S_i$ are nonbinding. The current-value Hamiltonian is $\mathcal{H}=\dfrac{1}{1-\eta}\left(X_1^{\alpha_1(1-\eta)}X_2^{\alpha_2(1-\eta)}\right)-\lambda_1X_1-\lambda_2X_2$. Apart from the state equations and the initial conditions, the Pontryagin Maximum Principle conditions are

$$\frac{\partial\mathcal{H}}{\partial X_i} = \alpha_i X_i^{\alpha_i(1-\eta)-1}X_j^{\alpha_j(1-\eta)}-\lambda_i=0, \qquad i,j=1,2 \quad i\neq j, \tag{44}$$
$$\frac{\dot{\lambda}_i}{\lambda_i} = \delta \rightarrow \lambda_i=\lambda_i(0)\,e^{\delta t}, \qquad i=1,2. \tag{45}$$

We check that the second order sufficient conditions for the maximization of $\mathcal{H}$ hold:

$$\frac{\partial^2\mathcal{H}}{\partial X_i^2}=\alpha_i\left[\alpha_i(1-\eta)-1\right]X_i^{\alpha_i(1-\eta)-2}X_j^{\alpha_j(1-\eta)}<0 \qquad i,j=1,2,$$
$$|Hes(\mathcal{H})|=\alpha_1\alpha_2\left[1-(\alpha_1+\alpha_2)(1-\eta)\right]X_1^{2\alpha_1(1-\eta)-2}X_2^{2\alpha_2(1-\eta)-2}>0.$$

Solving (44) for λ_1 and λ_2, deriving with respect to t and dividing the result by (44) we obtain

$$\frac{\dot{\lambda}_i}{\lambda_i}=\left[\alpha_i(1-\eta)-1\right]\frac{\dot{X}_i}{X_i}+\alpha_j(1-\eta)\frac{\dot{X}_j}{X_j}, \qquad i,j=1,2. \tag{46}$$

Equating (46) and (45), using $\alpha_1+\alpha_2=1$ and rearranging, we obtain $\dfrac{\dot{X}_1}{X_1}=\dfrac{\dot{X}_2}{X_2}=\dfrac{-\delta}{\eta}$ whose solution is $X_i=X_i(0)\,e^{\frac{-\delta}{\eta}t}$. Substituting in the state equation for S_i and solving we obtain $S_i=S_i^0+\dfrac{X_i^0}{K}\left(1-e^{\frac{-\delta}{\eta}t}\right)$. Using the solution for λ_i, the transversality conditions become

$$\lim_{t\to\infty}\left[\lambda_i e^{-\delta t}\right]=\lambda_i(0)\ge 0 \quad \left(\text{with "=" if } \lim_{t\to\infty}S_i>0\right), \qquad i=1,2.$$

From (28) we know that $\lambda_1(0),\ \lambda_2(0)>0$, so that $\lim_{t\to\infty}S_i=0$ and both resources get exhausted. A simple argument shows that they get exhausted at the same time, $T\le\infty$[20]. Using the terminal conditions $S_i(T)=0$ in the solution for S_i, we obtain $X_i(0)=\dfrac{\delta S_i^0}{\eta\left(1-e^{\frac{-\delta}{\eta}T}\right)}$. Using (44) to substitute λ_i and employing the solution for X_i, the transversality condition for T becomes $\mathcal{H}(T)=J\cdot e^{\frac{-\delta(1-\eta)}{\eta}T}=0$, where $J=\dfrac{\eta}{1-\eta}\left(S_1^0\right)^{\alpha_1(1-\eta)}\left(S_2^0\right)^{\alpha_2(1-\eta)}\left[\dfrac{\delta}{\eta\left(1-e^{\frac{-\delta}{\eta}T}\right)}\right]^{1-\eta}>0$, so that the condition reduces to

[20] Assume that $S_i(T_i)=0$, $S_j(T_i)>0$ for $j\neq i$. Given that both resources are essential for production, it is clear that $Y=0$ from T_i on. Then, it is possible to increase the extraction of resource j in some subinterval $[t_1,t_2]\subset[0,T_i]$ so that $S_j(T_j)=0$, and keeping unchanged the rest of the solution. In the interval $[t_1,t_2]$ a strictly larger output is obtained, and so, the production function has a larger value.

$e^{\frac{-\delta(1-\eta)}{\eta}T} = 0$, which holds when $T = \infty$ and both resources exhaust asymptotically. Using this result in the expression for X_i, we obtain (17).

To do comparative static exercises, derive (18) with respect to α_1, δ, S_1^0 and S_2^0, obtaining

$$\begin{aligned}
\frac{\partial Y}{\partial \alpha_1} &= \frac{\delta S_2^0}{\eta}\left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1} e^{\frac{-\delta t}{\eta}} \log\left(\frac{S_1^0}{S_2^0}\right) \geq 0 \Longleftrightarrow \frac{S_1^0}{S_2^0} \geq 1 \Longleftrightarrow S_1^0 \geq S_2^0, \\
\frac{\partial Y}{\partial \delta} &= \left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1} \frac{S_2^0}{\eta}\left[1 - \frac{\delta t}{\eta}\right] e^{\frac{-\delta t}{\eta}} \geq 0 \Longleftrightarrow 1 - \frac{\delta t}{\eta} \geq 0 \Longleftrightarrow t \leq \frac{\eta}{\delta}, \\
\frac{\partial Y}{\partial \eta} &= \left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1} \frac{\delta S_2^0}{\eta^2}\left[\frac{\delta t}{\eta} - 1\right] e^{\frac{-\delta t}{\eta}} \geq 0 \Longleftrightarrow \frac{\delta t}{\eta} - 1 \geq 0 \Longleftrightarrow t \geq \frac{\eta}{\delta},
\end{aligned}$$

$$\frac{\partial Y}{\partial S_1^0} = \frac{\delta \alpha_1}{\eta} e^{\frac{-\delta t}{\eta}} \left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1 - 1} > 0, \qquad \frac{\partial Y}{\partial S_2^0} = \frac{\delta(1-\alpha_1)}{\eta}\left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1} e^{\frac{-\delta t}{\eta}} > 0.$$

6.7 A renewable and a nonrenewable resource (section 4.2)

As in section 6.6, the problem

$$\begin{aligned}
&\underset{\{X_1, X_2\}}{Max} \quad \int_0^\infty \tfrac{1}{1-\eta}\left(X_1^{\alpha_1(1-\eta)} X_2^{\alpha_2(1-\eta)}\right) e^{-\delta t} dt \\
&s.t.: \\
&\quad \dot{S}_1 = -X_1, \\
&\quad \dot{S}_2 = \gamma_2 S_2 - X_2, \\
&\quad \left.\begin{aligned} S_i(0) &= S_i^0, \\ 0 \leq X_i &\leq S_i, \end{aligned}\right\} \quad i = 1, 2
\end{aligned}$$

is solved assuming that an interior solution exists. Together with state equations and initial conditions, the Maximum Principle conditions are

$$\frac{\partial \mathcal{H}}{\partial X_i} = \alpha_i X_i^{\alpha_i(1-\eta)-1} X_j^{\alpha_j(1-\eta)} - \lambda_i = 0, \quad i, j = 1, 2, \tag{47}$$

$$\frac{\dot{\lambda}_1}{\lambda_1} = \delta, \qquad \frac{\dot{\lambda}_2}{\lambda_2} = \delta - \gamma_2, \tag{48}$$

and solving the equations for λ_1 and λ_2, we obtain $\lambda_1 = \lambda_1(0) e^{\delta t}$, $\lambda_2 = \lambda_2(0) e^{(\delta - \gamma_2)t}$.

The second order sufficient conditions for the maximization of $\mathcal{H}$, which are identical to that of example 4.1 (see solution in section 6.6), hold. Solving (47) for λ_1 and λ_2, deriving with respect to t and dividing by λ_1 and λ_2 we have $\frac{\dot{\lambda}_i}{\lambda_i} = [\alpha_i(1-\eta) - 1]\frac{\dot{X}_i}{X_i} + \alpha_j(1-\eta)\frac{\dot{X}_j}{X_j}$, $\quad i, j = 1, 2$. Equating to (48) and rearranging, we obtain $\frac{\dot{X}_1}{X_1} = -K_1$, $\frac{\dot{X}_2}{X_2} = -K_2$, with

$$K_1 = \frac{\delta - \alpha_2 \gamma_2 (1-\eta)}{\eta} > 0, \qquad K_2 = \frac{\delta - \gamma_2[1 - \alpha_1(1-\eta)]}{\eta} \lesseqgtr 0,$$

from which $X_1 = X_1(0) e^{-K_1 t}$, $X_2 = X_2(0) e^{-K_2 t}$. Substituting in the equations for $\dot{S}_1$ and $\dot{S}_2$ and solving with initial conditions $S_i(0) = S_i^0$ we have

$$S_1 = S_1^0 + \frac{X_1(0)}{K_1}\left(e^{-K_1 t} - 1\right), \qquad S_2 = \frac{X_2(0)}{\gamma_2 + K_2} e^{-K_2 t} + \left[S_2^0 - \frac{X_2(0)}{\gamma_2 + K_2}\right] e^{\gamma_2 t}.$$

Using the solutions for λ_1 and λ_2, the transversality conditions become

$$\begin{aligned}\lim_{t\to\infty}\left[\lambda_1 e^{-\delta t}\right] &= \lambda_1(0)\geq 0 \quad \left(\text{with "=" if } \lim_{t\to\infty} S_1>0\right),\\ \lim_{t\to\infty}\left[\lambda_2 e^{-\delta t}\right] &= \lim_{t\to\infty}\lambda_2(0)\,e^{-\gamma_2 t}\geq 0 \quad \left(\text{with "=" if } \lim_{t\to\infty} S_2>0\right).\end{aligned}$$

From (28) we know that $\lambda_1(0)$, $\lambda_2(0)>0$. Transversality conditions guarantee that the stock of the resource 1 depletes: $\lim_{t\to\infty} S_1=0$. A simple argument allows us to assure that the resource 2 exhausts too, and furthermore, both resources exhaust simultaneously[21]. Using the terminal conditions $S_i(T)=0$ and $K_2+\gamma_2=K_1$[22], we obtain the initial values for X_1 and X_2:

$$X_1(0)=\frac{K_1 S_1^0}{1-e^{-K_1 T}},\qquad X_2(0)=\frac{(K_2+\gamma_2)\,S_2^0}{1-e^{-(K_2+\gamma_2)T}}=\frac{K_1 S_2^0}{1-e^{-K_1 T}},$$

and the solution for X_i becomes $X_i=\dfrac{K_1 S_i^0}{(1-e^{-K_1 T})}e^{-K_i t}$. Using (47) to substitute λ_i, employing the solution for X_i and the condition $S_2(T)=0$, the transversality condition for T becomes $\mathcal{H}(T)=J\cdot e^{-(\alpha_1 K_1+\alpha_2 K_2)T}=0$, where $J=\dfrac{\eta}{1-\eta}\left(S_1^0\right)^{\alpha_1(1-\eta)}\left(S_2^0\right)^{\alpha_2(1-\eta)}K_1^{1-\eta}\left[\dfrac{1}{1-e^{-K_1 T}}\right]^{1-\eta}>0$, so that $\mathcal{H}=0$ collapses to

$$e^{-(1-\eta)(\alpha_1 K_1+\alpha_2 K_2)T}=e^{(1-\eta)\frac{\alpha_2\gamma_2-\delta}{\eta}T}=0$$

that, under $\alpha_2\gamma_2<\delta$, implies $T=\infty$ and the solution for X_1 and X_2 becomes (20).

Substituting in the production function, we obtain the expression for Y,

$$Y=K_1\left(S_1^0\right)^{\alpha_1}\left(S_2^0\right)^{a_2}e^{-(\alpha_1 K_1+\alpha_2 K_2)t}=\left(S_1^0\right)^{\alpha_1}\left(S_2^0\right)^{\alpha_2}\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}e^{\frac{\alpha_2\gamma_2-\delta}{\eta}t}. \tag{49}$$

Deriving (49) with respect to S_1^0, S_2^0, δ, η and α_1 we obtain

$$\begin{aligned}\frac{\partial Y}{\partial S_i^0} &= \alpha_i\left(S_i^0\right)^{\alpha_i-1}\left(S_j^0\right)^{\alpha_j}\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}e^{\frac{\alpha_2\gamma_2-\delta}{\eta}t}>0 \qquad i=1,2,\\ \frac{\partial Y}{\partial \delta} &= \left(S_1^0\right)^{\alpha_1}\left(S_2^0\right)^{\alpha_2}\frac{1}{\eta}\left[1-\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}t\right]e^{\frac{\alpha_2\gamma_2-\delta}{\eta}t}\geq 0\Longleftrightarrow t\leq\frac{\eta}{\delta-\alpha_2\gamma_2(1-\eta)},\\ \frac{\partial Y}{\partial \eta} &= \frac{\alpha_2\gamma_2-\delta}{\eta^2}\left(S_1^0\right)^{\alpha_1}\left(S_2^0\right)^{\alpha_2}\left[1-\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}t\right]e^{\frac{\alpha_2\gamma_2-\delta}{\eta}t}\geq 0\Longleftrightarrow t\geq\frac{\eta}{\delta-\alpha_2\gamma_2(1-\eta)},\\ \frac{\partial Y}{\partial \alpha_1} &= \left(\frac{S_1^0}{S_2^0}\right)^{\alpha_1}S_2^0 e^{\frac{\alpha_2\gamma_2-\delta}{\eta}t}\left\{\frac{\gamma_2}{\eta}\left(1-\eta-\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}t\right)+\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}\log\left(\frac{S_1^0}{S_2^0}\right)\right\},\end{aligned}$$

so that

$$\frac{\partial Y}{\partial \alpha_1}\geq 0\Longleftrightarrow t\leq\frac{\eta(1-\eta)}{\delta-\alpha_2\gamma_2(1-\eta)}+\frac{\eta}{\gamma_2}\log\left(\frac{S_1^0}{S_2^0}\right).$$

Finally, deriving (20) and (49) with respect to γ_2,

$$\begin{aligned}\frac{\partial X_1}{\partial \gamma_2} &= \frac{\alpha_2(1-\eta)}{\eta}S_1^0\left[\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}t-1\right]e^{\frac{\alpha_2\gamma_2(1-\eta)-\delta}{\eta}t},\\ \frac{\partial X_2}{\partial \gamma_2} &= \frac{S_2^0}{\eta}\left[\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}\left[1-\alpha_1(1-\eta)\right]t-\alpha_2(1-\eta)\right]e^{\frac{\gamma_2[1-\alpha_1(1-\eta)]-\delta}{\eta}t},\\ \frac{\partial Y}{\partial \gamma_2} &= \left(S_1^0\right)^{\alpha_1}\left(S_2^0\right)^{\alpha_2}\frac{\alpha_2}{\eta}\left[\frac{\delta-\alpha_2\gamma_2(1-\eta)}{\eta}t-(1-\eta)\right]e^{\frac{\alpha_2\gamma_2-\delta}{\eta}t},\end{aligned}$$

[21] See footnote 20.

[22] Substituting the value for K_2 and operating we have $K_2+\gamma_2=\dfrac{\delta-(1-\alpha_1)\gamma_2(1-\eta)}{\eta}$ and, using the assumption $\alpha_1+\alpha_2=1$, we obtain $K_2+\gamma_2=K_1$.

so that

$$
\begin{aligned}
\frac{\partial X_1}{\partial \gamma_2} &\geq 0 \Longleftrightarrow t \geq \frac{\eta}{\delta - \alpha_2 \gamma_2 (1-\eta)}, \\
\frac{\partial X_2}{\partial \gamma_2} &\geq 0 \Longleftrightarrow t \geq \frac{\alpha_2 (1-\eta) \eta}{[1-\alpha_1 (1-\eta)] [\delta - \alpha_2 \gamma_2 (1-\eta)]}, \\
\frac{\partial Y}{\partial \gamma_2} &\geq 0 \Longleftrightarrow t \geq \frac{\eta (1-\eta)}{\delta - \alpha_2 \gamma_2 (1-\eta)}.
\end{aligned}
$$

6.8 Two renewable resources (section 4.3)

Deriving (23) we obtain the following comparative static results:

$$
\left.
\begin{array}{ll}
\dfrac{\partial \bar{S}_i}{\partial \delta} = \dfrac{-K_i}{2\theta_i} < 0, & \dfrac{\partial \bar{X}_i}{\partial \delta} = \dfrac{-K_i \delta}{2\theta_i} < 0, \\
\dfrac{\partial \bar{S}_i}{\partial K_i} = \dfrac{\theta_i - \delta}{2\theta_i} > 0, & \dfrac{\partial \bar{X}_i}{\partial K_i} = \dfrac{\theta_i^2 - \delta^2}{4\theta_j} > 0, \\
\dfrac{\partial \bar{S}_i}{\partial \theta_i} = \dfrac{\delta K_i}{2\theta_i^2} > 0, & \dfrac{\partial \bar{X}_i}{\partial \theta_i} = K_i \dfrac{\theta_i^2 + \delta^2}{4\theta_i^2} > 0
\end{array}
\right\} \qquad i = 1, 2,
$$

$$
\frac{\partial \bar{S}_i}{\partial K_j} = \frac{\partial \bar{S}_i}{\partial \theta_j} = \frac{\partial \bar{X}_i}{\partial K_j} = \frac{\partial \bar{X}_i}{\partial \theta_j} = 0, \qquad i, j = 1, 2 \quad i \neq j.
$$

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