IMPLEMENTATION, ELIMINATION OF WEAKLY DOMINATED STRATEGIES AND EVOLUTIONARY DYNAMICS*

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Abstract

This paper studies convergence and stability properties of Sjöström's (1994) mechanism, under the assumption that boundedly rational players find their way to equilibrium using monotonic learning dynamics and best-reply dynamics. This mechanism implements most social choice functions in economic environments using as a solution concept one round of deletion of weakly dominated strategies and one round of deletion of strictly dominated strategies. However, there are other sets of Nash equilibria, whose payoffs may be very different from those desired by the social choice function. With monotonic dynamics, all these sets of equilibria contain limit points of the learning dynamics. Furthermore, even if the dynamics converge to the "right" set of equilibria (i.e. the one which contains the solution of the mechanism), it may converge to an equilibrium which is worse in welfare terms. In contrast with this result, any interior solution of the best-reply dynamics converges to the equilibrium whose outcome the planner desires.

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1 Introduction

The theory of implementation studies the problem of designing decentralized institutions ("mechanisms") through which certain socially desirable objectives can be achieved. These social arrangements should be able to operate in a wide variety of environments, without extensive knowledge by the planner about the agents' preferences. Once it is ensured that agents respect the rules of the mechanism, these rules are designed so that it is in the best interest of agents to take those actions that lead to the socially desirable outcome. More precisely, a *social choice rule* is implemented by a (*game-form*) mechanism if, for every possible environment (preference profile), the solution (set of equilibrium outcomes) of the mechanism coincides with the set of outcomes of the social choice rule.

This definition implicitly assumes that agents are always able to play equilibrium strategies. However, there is substantial empirical and experimental evidence against this theoretical presumption. What we learn from experiments is that subjects usually fail to play an equilibrium, unless they are given the chance to acquire enough experience through repeated play. Furthermore, for some games, players may still fail to play an equilibrium, even with experience, specially if the equilibrium notion is fairly refined.[1]

In spite of this evidence, research in implementation theory has paid little attention to the problem of how equilibrium is achieved.[2] One of the reasons is that, in describing the mechanism to the agents, the planner has always the option of explaining the reasons why the required actions correspond to "the obvious way to play" the mechanism, (that, is, why it is in the best interest of agents to follow the social rule dictat). However, since the planner should be concerned with the performance of the mechanism when some (if not all of the) agents are not as "rational" as expected, it is useful to test the mechanism's performance in the presence of some form of bounded rationality.

A more fundamental approach to these issues would require the planner to take bounded rationality into account, when designing the game agents play. This necessarily leads to an alternative definition of implementation which includes, among the variables which specify the "environment", the learning protocols agents use, as well as initial conditions of the learning process. In this respect, we propose the following definition: a social choice rule is *dynamically implemented* by a mechanism if, for every possible environment, the limiting set of outcomes, when the game is played repeatedly, coincides with the set of outcomes of the social choice rule.

[1] See Cooper *et al.* (1991) for the prisoner's dilemma, a strictly dominance solvable game; McKelvey and Palfrey (1991) for the centipede game, a game with a unique Nash equilibrium; and Güth *et al.* (1982) for the ultimatum game, which has a unique subgame perfect equilibrium.

[2] Noticeable exceptions are the papers of Muench and Walker (1984), Walker (1984), Jordan (1986), Vega-Redondo (1989), De Trenqualye (1988,1989) and Cabrales (1997).

There is a caveat here. Why should we focus only on limiting outcomes? The planner may also care about what happens on the way to equilibrium, as the learning path may include outcomes significantly different than what the choice rule prescribes. This, in turn, would require to fully characterize the planner's preferences, rather than specify the most preferred outcome, for any given state of the environment. This is something the implementation literature traditionally leaves unspecified, as it has focused on implementing "exactly", that is, designing games that produce the most desired outcomes in all states of the world. The main advantage of this approach is that it avoids the thorny problem of having "ad-hoc" preferences for the planner. Moreover, if the planner does not discount the future and the game is played infinitely often, then it is legitimate to look at limiting outcomes. This would be the case, for example, if we consider the planner as the writer of a constitution which is concerned about the welfare of many generations of users, each of those being equally important.

In this paper we study the *dynamic* implementation of Sjöström's (1994) mechanism[3]. First, we study the performance of the mechanism under monotonic dynamics (Samuelson and Zhang 1992, Weibull 1995), which essentially imply higher growth rates for those strategies which perform better.[4] We also study the mechanism under best-reply dynamics (Matsui, 1992), a limiting case of monotonic dynamic by which only strategies that are a best response to the current mixed strategy profile grow. This choice of dynamics allows us to understand the effects of increasing levels of responsiveness to past payoffs of the players (which could be interpreted as a proxy for "sophistication") on the performance of the mechanism.

We concentrate on Sjöström's mechanism for several reasons. First, the conditions for implementation are quite weak. Although the environments that are permitted are not universal, they are rich enough for most economic purposes. Furthermore, this reduction in the domain allows the author to implement the social choice rule with a "bounded" game, that is, a game which does not exploit equilibrium nonexistence to rule out undesirable outcomes.[5] Finally, the game can be solved by one round of deletion of weakly dominated strategies, and then another round of deletion of strictly dominated strategies. This feature of the mechanism makes it particularly attractive since, under some assumptions of

[3]Sjöström's (1994) mechanism and the one proposed by Jackson *et al.* (1994) for separable environments are very similar. Most of our results would generalize easily for that mechanism as well.

[4]One particularly well known member of the family of monotonic dynamics is the so-called *replicator dynamics* of evolutionary game theory (Taylor and Jonker, 1978). These dynamics have been given a learning theoretic foundation by Börgers and Sarin (1997), and they can also be interpreted as a model of imitation (Schlag, 1994).

[5]For example, in the canonical mechanism for Nash implementation (Repullo, 1987), if agents disagree widely on the announced preferences, they have to play a game in which the agent announcing the highest integer wins a prize. Jackson (1992) provides a good treatment of this issue.

imperfect knowledge of agents,[6] the appropriate solution concept implies one round of deletion of weakly dominated strategies, and then the iterated deletion of strictly dominated strategies.

In Sjöström's (1994) mechanism agents are required to simultaneously announce their own preferences, together with the preferences of their two closest neighbors. The mechanism is designed in such a way that the truthful report of one's own preferences is weakly dominant, as it does not affect one's payoff, except for a set of (so-called) *totally inconsistent* states, where it is (strictly) preferable to report preferences truthfully. Since, for this mechanism, it is always advantageous to report the same preferences about your neighbors as they report about themselves, the only equilibrium that survives the first round of deletion of weakly dominated strategies is the truth-telling one.

However, there are many other Nash equilibria. In particular, for every preference profile R, there is a component (i.e. a closed and connected set) of equilibria in which all agents report the preferences for their neighbors indicated in R, and report the preferences about themselves indicated in R with high enough (this need not be very high) probability. This is because it is important for the mechanism that all agents match their neighbors' announcements about themselves, but the report about oneself is only important in some unlikely (totally inconsistent) state.

As for monotonic dynamics, we show (Proposition 4) that many equilibria in all these latter components are limit points of trajectories of the learning dynamics that have completely mixed initial conditions (that is, initial conditions that give strictly positive weights to all possible messages). Even when the dynamics converge to the "right" component of equilibria (i.e. the one which contain the solution of the mechanism), they need not go to the "right" equilibrium. This implies a welfare loss, since Nash equilibria in the same component are not outcome equivalent. We also show by example (Proposition 2) that the initial conditions that lead to these equilibria need not be close to the limiting point. Similar considerations apply when we look at the *structural stability* properties of the various equilibrium components, that is, when we study how the dynamic structure react to the introduction of (arbitrarily small) perturbations in the vectorfield. In this respect, we use the example to show (Proposition 6) that, although there is a unique structurally stable component (namely, the component which contains the solution of the mechanism), the untruthful component is stable for a non-negligible set of admissible perturbations.

As can be seen from figure 4, the less responsive the dynamics are to payoffs (the further the initial conditions from the "right" equilibrium), the more difficult it is to converge to the desired solution. Only in the extreme case of best-reply dynamics (in which the response to arbitrarily small payoff differences is infinite), we show (Propositions 7-8) that any interior trajectory converges to

[6]Either because of payoff uncertainty, as in Dekel and Fudenberg (1990), or through lack of common knowledge of rationality, as in Börgers (1994).

the pure strategy equilibrium in which players reveal their true preferences and the outcome desired by the planner is achieved.

The fact that evolution need not eliminate weakly dominated strategies has been known since, at least, Nachbar (1990). However, we are far from possessing a sound theory on the evolutionary properties of weakly dominance solvable games, as we have examples in which a single round of deletion is not allowed if we want to characterize the limiting set of the evolutionary dynamics[7], as well as games in which only strategies which survive an (arbitrarily large) number of rounds of deletion can be in the support of the limiting play.[8] Since the theory has not proposed, so far, a suitable framework to explain these differences, it is important to test the evolutionary properties of (game-form) mechanisms in which the iterated deletion of dominated strategies plays such a crucial role.

The remainder of the paper is arranged as follows. In section 2 we introduce some notation, we describe the mechanism and we make the assumptions about the dynamics. In section 3 we fully characterize (for all interior initial conditions) the set of limit points of any monotonic dynamic for the game in Figure 1, Sjöström (1994), to be considered as a simplified version of the mechanism. In section 4 we give local results on the convergence and stability properties of the Nash equilibrium components of the general game. In section 5 we describe the structural stability properties of the equilibria of the simplified mechanism. Section 6 explores the dynamic properties of best-reply dynamics for this game. Finally, section 7 concludes, together with an appendix containing the proofs of the relevant propositions.

2 The model and the dynamics

We introduce few changes to Sjöström's (1994) model for analytical convenience. First, we employ a Von Neumann-Morgenstern utility function instead of a preference relation. The reason is that we need to specify the payoff functions for mixed strategies, as the dynamics are defined on the mixed strategy space. We also assume that the set of possible preference parameters is finite. This is because the dimension of the pure strategy space is related with the set of preferences. If we had an infinite dimensional pure strategy space, the dynamics, which account for the relative frequency with which each pure strategy is being used, would have to describe the evolution of a measure over an infinite space. This seems an unnecessary complication for our purposes.

There is a set $I \equiv \{1, ..., n\}, n \geq 3$, of agents and a set $A \subseteq \Re^m_+$ of feasible consumption plans. The preferences of agent $i \in I$ are represented with a (Von Neumann-Morgenstern) utility function $v_i : A \times \Phi_i \to \Re$, where Φ_i specifies a finite set of possible preference parameters. An element R_i of Φ_i represents the

[7]See, for example, Samuelson (1993) and Gale *et al.* (1995).

[8]See, for example, the finitely repeated prisoners' dilemma (Cressman, 1996), or the centipede game (Ponti, 1997).

preferences of agent i over A. A *preference profile* is a vector $R=(R_1,...,R_n)$, which is assumed to be common knowledge among the agents. The following assumptions refine the sets of feasible consumption plans and preferences profiles.

Assumption p.1. Free disposal. If $a \in A$ and $0 \leq a' \leq a$, then $a' \in A$.

Assumption p.2. The set of feasible consumption plans A is convex. For all $a, a' \in A$ and for all $\lambda \in [0,1]$ then $\lambda a + (1-\lambda)a' \in A$.

Assumption p.3. The preferences represented by $R_i \in \Phi_i$are strictly convex. For any $a, a' \in R_+^m$ and for all $\lambda \in (0,1)$, if $a \neq a'$ and $v_i(a, R_i) \geq v_i(a', R_i)$, then

$v_i(\lambda a + (1-\lambda)a', R_i) > v_i(a', R_i)$.

Assumption p.4. For any $R_i \in \Phi_i$ if $a \geq 0$ and $a \neq 0$ then $v_i(a, R_i) > v_i(0, R_i)$.

Assumption p.5. Preference reversal. For any $R_i, R_i' \in \Phi_i$ if $R_i \neq R_i'$ then there are $a, \tilde{a} \in A$ such that $v_i(a, R_i) > v_i(\tilde{a}, R_i)$ and $v_i(\tilde{a}, R_i') > v_i(a, R_i')$.

For any set $B \subseteq \Re_+^m$ and any $R_i \in \Phi_i$ a *choice correspondence* is defined as follows: $c(B, R_i) \equiv \{a \in B|$ for all $b \in B, v_i(a, R_i) \geq v_i(b, R_i)\}$.

For any $i \in I$, a *social choice function* for player i is a mapping $f_i : \Phi \to A$, where $f(R) \equiv (f_1(R), \ldots, f_n(R))$.

Assumption p.6. Individual rationality. For all i and R, $f_i(R) \neq (0, 0, \ldots, 0)$.

A *mechanism* is a pair $\Gamma \equiv (M, \alpha)$, where $M \equiv \times_{i\in I} M_i$ and $\alpha(m) \equiv (\alpha_1(m), \ldots, \alpha_n(m)) \in A$. M_i is the *message* space of agent i and α is the *outcome function.* A *mechanism* and a *preference profile* define a game.

Let $M_{-i} \equiv M_1 \times ... \times M_{i-1} \times M_{i+1} \times ... \times M_n$. Given a *mechanism* Γ and a *preference profile* R, we say that m_i is *weakly dominated* for some set of messages $F \equiv \times_{i\in I} F_i \subseteq M$ if there exists a message $m_i' \in F_i$ such that $v_i(\alpha_i(m_i', m_{-i}), R_i) \geq v_i(\alpha_i(m_i, m_{-i}), R_i)$ for all $m_{-i} \in F_{-i}$ and there is some $m_{-i}^* \in F_{-i}$ such that $v_i(\alpha_i(m_i', m_{-i}^*), R_i) > v_i(\alpha_i(m_i, m_{-i}^*), R_i)$. Define the set $U_i(F : (\Gamma, R)) \equiv \{m_i \in F_i | m_i$ is not weakly dominated in F for the game $(\Gamma, R)\}$. The message m_i is a best response for player i, to $m_{-i} \in M_{-i}$ if

$$v_i(\alpha_i(m_i, m_{-i}), R_i) \geq v_i(\alpha_i(m_i', m_{-i}), R_i) \forall m_i' \in M_i.$$

A message profile m is a *Nash equilibrium* (NE) if m_i is a best response to m_{-i} for all $i \in I$. A message profile $m \in M$ is an *undominated Nash equilibrium (UNE) for the game* (Γ, R) if it is a Nash equilibrium and $m_i \in U_i(M : (\Gamma, R))$. Let $UNE(\Gamma, R) \equiv \{\alpha(m) \in A | m$ is an UNE for the game $(\Gamma, R)\}$.

We say that a mechanism Γ *implements* a social choice function f in *undominated Nash equilibrium* if for all $R \in \Phi$, $f(R) = UNE(\Gamma, R)$.

For the *iterated deletion of weakly dominated strategies* let $U_i^1(\Gamma, R) = U_i(M : (\Gamma, R))$, and if $U_i^k(\Gamma, R)$ has been defined for $k \geq 1$.

Let $U_i^{k+1}(\Gamma, R) \equiv U_i(\times_{j\in I} U_j^k(\Gamma, R) : (\Gamma, R))$. Let $U_i^\infty(\Gamma, R) \equiv \cap_{k=1}^\infty U_i^k(\Gamma, R)$. Let $IWD(\Gamma, R) \equiv \{\alpha(m) \in A | m_i \in U_i^\infty(\Gamma, R)$ for all $i\}$.

We say that a mechanism Γ *implements* a social choice function f with *iterated deletion of weakly dominated strategies* if for all $R \in \Phi$, $f(R) = IWD(\Gamma, R)$.

We now construct a mechanism.

Let $M_i = \Phi_{i-1} \times \Phi_i \times \Phi_{i+1}$, so that each individual announces the preferences of her two neighbors, and let members of M_i and M be denoted m_i and m respectively. A generic strategy is therefore $m_i = (R^i_{i-1}, R^i_i, R^i_{i+1})$. A K-tuple of messages $\{m_{j_1}, \ldots, m_{j_K}\}$ is *totally consistent* if whenever agents $i, k \in \{j_1, \ldots, j_K\}$ both announce the preference of player $j \in I$, then $R^i_j = R^k_j$. On the other hand, a K-tuple of messages $\{m_{j_1}, \ldots, m_{j_K}\}$ is *totally inconsistent* if whenever agents $i, k \in \{j_1, \ldots, j_K\}$ both announce the preference of player $j \in I$, then $R^i_j \neq R^k_j$.

Consider $R_i, R'_i \in \Phi_i$, where $R_i \neq R'_i$. By assumption p.5 there are $a, \tilde{a} \in A$ such that $v_i(a, R_i) > v_i(\tilde{a}, R_i)$ and $v_i(\tilde{a}, R'_i) > v_i(a, R'_i)$. We can choose a and $\tilde{a}$ so that $v_i(a, R_i) > v_i(a', R_i)$ for all a' in the line segment between a and $\tilde{a}$. Given this pair $(a, \tilde{a})$ let $\beta_i(R_i, R'_i) \equiv \{b \in \Re^m_+ | b = \lambda a + (1-\lambda)\tilde{a}, \text{ for } \lambda \in [0,1]\}$. By construction, for all $R_i, R'_i \in \Phi_i$, $c(\beta_i(R_i, R'_i), R_i) \neq c(\beta_i(R_i, R'_i), R'_i)$. Let $\phi(i, m) \equiv (R^n_1, R^1_2, \ldots, R^{i-1}_i, R^i_{i+1}, R^{i+1}_{i+2}, \ldots, R^{n-1}_n)$ and for every i and m_{-i}, define

$$B_i(m_{-i}) = \begin{cases} f_i(\phi(i,m)) & \text{if } \mathrm{m_{-i} is totally consistent} \\ \beta_i(R^{i-1}_i, R^{i+1}_i) & \text{if } \mathrm{m_{-i} is totally inconsistent} \\ \frac{1}{n} f_i(\phi(i,m)) & \mathit{Otherwise} \end{cases}$$

Now we can define α :

$$\alpha_i(m) = \begin{cases} c(B_i(m_{-i}), R^i_i) & \text{if } \mathrm{R^i_{i-1} = R^{i-1}_{i-1} and R^i_{i+1} = R^{i+1}_{i+1}} \\ 0 & \mathit{otherwise} \end{cases}$$

Let $\hat{R}$ be the true preference profile and R^* an arbitrary preference profile. To understand how the mechanism works, notice that the only time when the choice of an announcement R^i_i has any effect on i's payoffs is when m_{-i} is totally inconsistent. In this case, the outcome is the optimal choice within the set $\beta_i(R^{i-1}_i, R^{i+1}_i))$ according to the announced R^i_i. This is the reason why, for player i, announcing her true preference $\hat{R}_i$ can never hurt. Furthermore, for every alternative announcement $R^i_i = R^*_i$, there is some totally inconsistent m_{-i} with $R^{i-1}_i = \hat{R}_i$ and $R^{i+1}_i = R^*_i$ and the set $\beta_i(.,.)$ is constructed in such a way that $c(\beta_i(\hat{R}_i, R^*_i), \hat{R}_i)$ is strictly preferred to $c(\beta_i(\hat{R}_i, R^*_i), R^*_i)$. Therefore, a message $m_i = (R^i_{i-1}, R^*_i, R^i_{i+1})$ is weakly dominated by a message $m_i = (R^i_{i-1}, \hat{R}_i, R^i_{i+1})$, i.e. untruthful announcements about oneself are weakly dominated.

Once these weakly dominated strategies are eliminated and all agents announce the true preferences about themselves, $R^i_i = \hat{R}_i$, it is strictly dominated to announce untruthful preferences about your neighbors, $R^i_{i+1} \neq \hat{R}_{i+1} = R^{i+1}_{i+1}$ or $R^i_{i-1} \neq \hat{R}_{i-1} = R^{i-1}_{i-1}$, since disagreeing with your neighbors is punished with the 0 consumption bundle.

These two facts establish the main theorem in Sjöström (1994).

Proposition 0. Let f be an arbitrary social choice function. The mechanism described above implements f in UNE and in IWD.

It is important to notice, for the discussion we undertake below, that the set of states in which not announcing the true preferences about oneself is weakly dominated are themselves states that typically produce very bad outcomes for other opponents (at least one of them will have 0 consumption and probably many). If agents learn fast to avoid these (totally inconsistent) states, there is no incentive to tell the truth about oneself. The mechanism we have just described focuses on consensus announcements, since disagreement is punished with 0 consumption; truth-telling is only rewarded in a set of states which need not be very prominent in the minds of the players. This is precisely the reason why, if agents are boundedly rational in the way we describe, convergence to the social choice outcome function may fail to occur.

We now move on to the characterization of the evolutionary dynamics we analyze.

Fix a given mechanism Γ and a given preference profile $R \in \Phi$. Let $x_i^{m_i}$ be the probability assigned by agent i to message m_i, and $x_i \in \Delta_i$ be a mixed strategy for agent i (where Δ_i denotes the $|M_i - 1|$-dimensional simplex which describes player i's mixed strategy space). Let also $x_{-i} \in \Delta_{-i} \equiv \times_{j \neq i} \Delta_j$ be a mixed strategy profile for agents other than i, with $x \equiv (x_i, x_{-i}) \in \Delta \equiv \times_{i \in I} \Delta_i$. Finally, let $u_i(x_i, x_{-i}) = \Sigma_{m \in M} v_i(\alpha_i(m_i, m_{-i}), R_i) \Pi_{j \in I} x_j^{m_j}$.

We formalize player i's behavior in terms of the mixed strategy $x_i(t)$ she adopts at each point in time. The vector $x(t)$ will then describe the *state of the system* at time t, defined over the state space Δ, with Δ^0 denoting its relative interior, i.e. the set of completely mixed strategy profiles.

Assumption d.1 The evolution of $x(t)$ is given by a system of continuous-time differential equations:

$$\dot{x}_i^{m_i} = D_i^{m_i}(x(t)) \tag{1}$$

We require that the autonomous system (1) satisfies the standard regularity condition, i.e., D must be *i)* Lipschitz continuous with *ii)* $\sum_{m_i \in M_i} D_i^{m_i}(x(t)) = 0$.[9] Furthermore, D must also satisfy the following requirements:

Assumption d.2. D is a regular (payoff) *monotonic* selection dynamic. More explicitly, let $g_i(m_i, x_{-i}(t)) \equiv \frac{\dot{x}_i^{m_i}(t)}{x_i^{m_i}(t)}$ denote the growth rate of strategy m_i. Then for all $m_i, m_i' \in M_i$ and all $x_{-i} \in \Delta_{-i}$ it must be that

$$sign\left[g_i(m_i, x_{-i}(t)) - g_i(m_i', x_{-i}(t))\right] = sign\left[u_i(m_i, x_{-i}(t)) - u_i(m_i', x_{-i}(t))\right].$$

Assumption d.2 is commonly used in the literature to capture the essence of

[9] A useful implication of this regularity assumption is that the solution of the dynamical system leaves Δ, as well as Δ^0, invariant (and, a fortiori, forward invariant): any solution path starting from Δ (Δ^0) does not leave Δ (Δ^0). This property will prove to be useful to obtain some of the results of the paper.

a *selective* evolutionary process.[10] Given the mixed strategy profile played at each point in time, strategies with higher expected payoff grow faster than poorly performing ones.

Assumption d.3. $x(0) \in \Delta.^0$

Assumption d.3 is also standard in the evolutionary literature. It excludes the possibility that the selection dynamic acts only on a subset of the strategy space. This possibility arises because the system is forward invariant, and therefore a strategy that has zero weight at time zero would also have zero weight at all subsequent times. If Assumption d.3 did not hold, the selection dynamics would then operate on a different game.

3 An example.

We prefix the dynamic analysis of the mechanism with the following example, taken from Sjöström (1994), p. 504, which is intended to convey the essence of our results. There is one unit of a single divisible private good, which has to be divided among three players: 1, 2 and 3. Preferences of players 1 and 2 are increasing in the amount of the good they consume, and are common knowledge for all players and the planner. There are two possible types for player 3's preferences, which are indexed by 0 and 1. Preferences of type 0 peak at consumption 1/3; preferences of type 1 peak at consumption 1/2. Player 3's type is common knowledge among the players, but the planner does not know it.

For preferences of type 0, the social choice function recommends the outcome $f(0) = (1/4, 1/4, 1/2)$; for preferences of type 1, $f(1) = (1/3, 1/3, 1/3)$. Notice that the social choice function is such that type 3 would prefer the outcome $f(1)$ when she is of type 0, and the outcome $f(0)$ when she is of type 1. This provides her an incentive to conceal her type, and therefore the planner needs a nontrivial mechanism to elicit her true preferences.

The mechanism proposed by Sjöström requires the three players to make a simultaneous statement about the preferences of player 3. Let $m_i^1(m_i^0), i \in I$ represent the message in which preferences of type 1 (type 0) for player 3 are announced by player i. Figure 1 illustrates the outcome function of the mechanism. As for its dynamic analysis, we shall focus on the case in which true preferences of player 3 are of type 1, and assume that Figure 1 also represents the game's payoffs when player 3's preferences are of type 1. We denote this game by G. Player 1 picks a row, player 2 a column, and player 3 picks a matrix. We first note that the mechanism leads to a game which is *weakly dominance solvable*, as it can be reduced to a single outcome (the *solution)* by the iterated deletion of weakly dominated strategies. Unlike in other weakly dominance solvable games, the same outcome is selected independently on the order by which strategies are

[10] See, for example, Samuelson and Zhang, (1992) and Weibull (1995).

deleted. We start by deleting the weakly dominated strategy m_3^0 for player 3 (the other agents have no dominated strategies at this stage). The reason is that, like in the mechanism described in section 2, truthtelling about your own preferences never hurts, and is strictly optimal when the opponents disagree on your own type. Once m_3^0 has been removed, strategies m_1^0 and m_2^0 become strictly dominated. The reason is that, like in the mechanism described in section 2, if all the players tell the truth about their own preference, lying about a neighbor is punished with 0 consumption.

	m_2^0	m_2^1
m_1^0	$\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	$\frac{1}{3}, 0, \frac{1}{3}$
m_1^1	$0, \frac{1}{3}, \frac{1}{3}$	$0, 0, \frac{1}{3}$

m_3^0

	m_2^0	m_2^1
m_1^0	$0, 0, \frac{1}{2}$	$0, \frac{1}{3}, \frac{1}{2}$
m_1^1	$\frac{1}{3}, 0, \frac{1}{2}$	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

m_3^1

Figure 1
Sjöström's Example: game G.

The unique strategy profile selected is then (m_1^1, m_2^1, m_3^1), that is, the pure strategy profile in which the true preferences are consistently revealed (i.e. the *solution).*

We begin by fully characterizing the set of Nash equilibria of game G. Since each player has only two pure strategies in her support, we abuse our notation setting $x_i \equiv x_i^{m_i^1}$.[11]

Proposition 1 *The set NE of Nash equilibria of G is the union of precisely two disjoint components NE^0 and NE^1, where:*

$$NE^0 \equiv \{x \in \Delta | x_1 = x_2 = 0, x_3 \leq \tfrac{3}{7}\},$$
$$NE^1 \equiv \{x \in \Delta | x_1 = x_2 = 1, x_3 \geq \tfrac{1}{2}\}.$$

Proof. See the Appendix. ■

We now move on to dynamics. Denote by $RE(G)$ the set of restpoints of G under any monotonic dynamic. It is straightforward to show that $RE(G)$ contains (together with all the pure strategy profiles) only the following components:

$$RE^0 \equiv \{x \in \Delta | x_1 = x_2 = 0, x_3 \in [0,1]\}$$

and

$$RE^1 \equiv \{x \in \Delta | x_1 = x_2 = 1, x_3 \in [0,1]\}.$$

Our task is to study the asymptotics of a monotonic selection dynamic whose initial state lies in the relative interior of the state space.

[11] The fact that each player has only two available options also allows us to express the dynamics in terms of the payoff difference between player i's truthful and untruthful strategy, which we call $\Delta\Pi_i(x_{-i}(t))$ (i.e. $\Delta\Pi_i(x_{-i}(t)) \equiv u_i\left(m_i^1, x_{-i}(t)\right) - u_i\left(m_i^0, x_{-i}(t)\right)$).

Proposition 2 *Any interior solution $x(t, x(0))$ of a monotonic selection dynamics $\dot{x} = D(x)$ converges to NE.*

Proof. See the Appendix. ■

If initial conditions are completely mixed, we then know from Proposition 2 that the evolutionary dynamics will eventually converge to a Nash equilibrium of the game. In the next section we show that this result generalize locally also in the case of Sjöström's (1994) mechanism, as described in Section 2.

4 Local results for the general game

In the following Proposition 3 we characterize some components of Nash equilibria for the game induced by the mechanism. In particular, we show that any message profile in which the agents are unanimous in the (arbitrary) preference profile they announce, R^* (more precisely, the preferences they announce about their neighbors and themselves are taken from R^*), is an equilibrium. Furthermore, any mixed strategy profile in which agents mix between messages consistent with R^* and other messages that only differ in the announcements agents make about their own preferences, is also an equilibrium, provided that messages in R^* are given a high enough weight. The equilibria in each of these components are not payoff equivalent, since disagreeing with a neighbor (event with nonzero probability in these mixed equilibria) results in a punishment. Nevertheless, Proposition 4 shows that this punishment is not high enough to prevent these equilibria to be the limit points of some interior path of any monotonic selection dynamic.

Let $m_i^* = (R_{i-1}^*, R_i^*, R_{i+1}^*)$ be a consensus announcement by agent i; $U_i = \max_R v_i(f_i(R), \hat{R}_i)$ be the utility associated to the most preferred outcome from the social choice function for agent i with true preferences $\hat{R}_i$, and $U_{in} = \max_R v_i\left(\frac{1}{n} f_i(R), \hat{R}_i\right)$ be the utility associated to the most preferred consumption bundle among those that result from dividing the bundles assigned by the social choice function by n. Let also S_i denote the set of all pure strategies in which announcements about the neighbors agree with R^*, i.e.

$$S_i = \{m_i \in M_i | R_{i-1}^i = R_{i-1}^*, R_{i+1}^i = R_{i+1}^*\} \tag{2}$$

where $\bar{S}_i = \{m_i \in M_i | m_i \notin S_i\}$ denoting the complement of S_i with respect to M_i and $S_{-i} \equiv \times_{j \neq i} S_j$ ($\bar{S}_{-i} \equiv \times_{j \neq i} \bar{S}_j$). Finally, denote by $S_i^{k_i}$ the following

$$S_i^{k_i} = \{x_i | x_i^{m_i} = 0, \text{ for all } \mathrm{m_i} \notin \mathrm{S_i} \text{ and } \mathrm{x_i^{m_i^*}} > \mathrm{k_i}\}, \tag{3}$$

where we assume

$$(k_i)^n \geq \frac{U_{in} - v_i(0, \hat{R}_i)}{v_i(f_i(\phi(i, R^*)), \hat{R}_i) - v_i(0, \hat{R}_i) + U_{in} - v_i(0, \hat{R}_i)} \tag{4}$$

for all i and all $j \neq i$. The set $S_i^{k_i}$ is the set of all mixed strategies in which i's announcements about her neighbors agrees with R^*, and the probability of announcing R_i^* is higher than k_i

Proposition 3 *For all $\hat{R}, R^* \in \Re$ and $x_i \in S_i^{k_i}$, x is a Nash equilibrium of $(\Gamma, \hat{R})$.*

Proof. See the Appendix. ■

To understand the role of (4) in the proof of Proposition 3, notice that, against any $x_{-i} \in \times_{j \neq i} S_j^{k_j}$, the payoff for agent i using strategy $m_i \in S_i$ satisfies the following condition:

$$u(m_i, x_{-i}) \geq (\min_{j \neq i} k_j)^{n-1} v_i(f_i(\phi(i, R^*)), \hat{R}_i) + (1 - (\min_{j \neq i} k_j)^{n-1}) v_i(0, \hat{R}_i). \quad (5)$$

The reason is that, for all $j \neq i$, $x_j^{m_j^*} \geq k_j$, which in turn implies a lower bound (*i.e.* $(\min_{j \neq i} k_j)^{n-1}$) on the probability with which m_{-i} is *totally consistent* with $m_i \in S_i$ and, therefore, the payoff $v_i(f_i(\phi(i, R^*)), \hat{R}_i)$ is achieved. With the remaining probability $1 - (\min_{j \neq i} k_j)^{n-1}$, the worse that can happen to player i is that her message does not match the announcements of her neighbors about themselves, in which case her payoff is $v_i(0, \hat{R}_i)$. By the same token, against any $x_{-i} \in \times_{j \neq i} S_j^{k_j}$, the payoff for agent i announcing a message $m_i' \in \overline{S_i}$ is at most

$$u(m_i', x_{-i}) \leq (\min_{j \neq i} k_j)^{n-1} v_i(0, \hat{R}_i) + (1 - (\min_{j \neq i} k_j)^{n-1}) U_{in} \quad (6)$$

because with probability at least $(\min_{j \neq i} k_j)$ each of the opponents will select strategy m_j^*, which in turn implies a lower bound on the probability with which the payoff is $v_i(0, \hat{R}_i)$. With the remaining probability $1 - (\min_{j \neq i} k_j)^{n-1}$, agent i gets at most U_{in}. The reason is that it is not possible that m_i forms a *totally consistent* announcement with m_{-i}, since there must exist some $j \neq i$, such that $R_j^j \neq R_j^*$ and $R_j^{j+1} = R_j^*$. From equations (5) and (6), it follows that

$$u_i(m_i, x_{-i}) - u_i(m_i', x_{-i}) \geq$$

$$v_i(0, \hat{R}_i) - U_{in} + (\min_{j \neq i} k_j)^{n-1} \left(v_i(f_i(\phi(i, R^*)), \hat{R}_i) + U_{in} - 2 v_i(0, \hat{R}_i) \right), \quad (7)$$

which implies $u_i(m_i, x_{-i}) - u_i(m_i', x_{-i}) \geq 0$, provided that (4) is satisfied.

Also note that, for all $x_{-i} \in \times_{j \neq i} S_j^{k_j}$, if $m_i, m_i' \in S_i$, then $u_i(m_i, x_{-i}) - u_i(m_i', x_{-i}) = 0$. The reason is that, in playing any strategy in S_i, agent i rules out the possibility that totally inconsistent states occur (at least the announcements about i have to coincide). These are the only states in which i's announcement about her own preferences makes a difference to her own payoff.

We shall now prove that elements in all the Nash equilibria components characterized by Proposition 3 are *reachable,* i.e. are limit points for some interior solution. By Lipschitz continuity, there exists a constant $K > 0$ such that for all m_i, x_{-i} and x'_{-i} , we have that

$$|g_i(m_i, x_{-i}(t)) - g_i(m_i, x'_{-i}(t))| \leq K|x_{-i} - x'_{-i}|,$$

where the $| \, . \, |$ denotes the norm of a vector. This in turn implies that, for all $h_v > 0$ with $u_i(m_i, x_{-i}(t)) - u_i(m'_i, x_{-i}(t)) \leq -h_v$, there exists some $h_g > 0$, such that $g_i(m_i, x_{-i}(t)) - g_i(m'_i, x_{-i}(t)) \leq -h_g$. By analogy with (7), for any $m_i \in \overline{S}_i$, it also must be

$$u_i(m_i, x_{-i}) - u_i(m'_i, x_{-i}) < U_i - v_i(0, \hat{R}_i) - \Pi_{j \neq i} x_j^{m_j^*}(t) \left(v_i(f_i(\phi(i, R^*)), \hat{R}_i) + U_i - 2v_j(0, \hat{R}_i) \right).$$

Therefore, if h_v is a constant such that $0 \leq h_v < \min_{i,R} v_i(f_i(\phi(i, R^*)), \hat{R}_i) - v_i(0, \hat{R}_i)$, then there exists another constant $H \in [0, 1)$, with

$$H = \max_i \left\{ \left(\frac{U_i - v_i(0, \hat{R}_i) + h_v}{v_i(f_i(\phi(i, R^*)), \hat{R}_i) + U_i - 2v_i(0, \hat{R}_i)} \right)^{\frac{1}{n-1}} \right\},$$

such that, if $x_j^{m_j^*}(t) > H$ for all j and t, then strategies not in S_i are decreasing at a rate not higher than $-h_g$.

We also need to establish a link between the weight with which messages $m_{-i} \in \overline{S}_{-i}$ are played and the relative performance of strategies $m_i \in S_i$. This is done by means of the following function

$$X_i(t) = \sum_{j \neq i} \left(\left(\sum_{m_j \in \overline{S}_j} x_j^{m_j}(t) \right)^2 + \sum_{m_j \in \overline{S}_j} (x_j^{m_j}(t))^2 \right).$$

with $X(t) = \max_i [X_i(t)]$ .The function $X_i(t)$ accounts for the relative weight of messages $m_{-i} \in \overline{S}_{-i}$ in x_{-i}, since only against these messages strategies in S_i yield different payoffs for player i. Therefore, the maximum difference in payoffs between strategies in S_i, and therefore in growth rates by monotonicity, is connected to $X_i(t)$, as shown in Lemma 3.[12] Finally, let

$$L = \min_i \left\{ \exp \left(\frac{-KX(0)}{h_g} \frac{H^2}{(x_i^{m_i^*}(0))^2} \right) \right\}.$$

The constant L appears because we want to show that $x_i^{m_i^*}(t)$ need not go to one in the limit, even if there is convergence to the equilibrium component to which m^* belongs. For any $m_i \in S_i$, the ratio

$$\frac{x_i^{m_i^*}(t)}{x_i^{m_i}(t)} \frac{x_i^{m_i}(0)}{x_i^{m_i^*}(0)}$$

[12] In the appendix.

is the integral of the differences in growth rates (thus connected to the difference in payoffs by monotonicity) between m_i and m_i^*. This integral depends on $X(t)$, as we show in Lemma 3. But $X(t)$ depends on $X(0)$ also, as well as on the growth rates of strategies of i's opponents in $\overline{S}_{-i}$. As shown in the following Proposition 4, also the weight of this latter strategies has an upper bound which depends on h_g, K and H. Thus, the constant L can be used to set an upper bound for the integral of the difference in growth rates between strategies m_i and m_i^*.

Also notice that $X(0)$ can be made arbitrarily close to zero (and, therefore, L arbitrarily close to 1) by selecting an initial condition in which the aggregate weight of strategies in $\overline{S}_{-i}$ is arbitrarily small.

Proposition 4 *Assume that, for all* $i \in I$, $x_i^{m_i^*}(0)L > H$. *Then*

a) *For all* $m_i \in \bar{S}_i$, $\frac{x_i^{m_i}(t)}{x_i^{m_i}(0)} < \exp[-h_g t]\frac{H}{x_i^{m_i^*}(0)}$ *for all* t *and all* i;

b) $x_i^{m_i^*}(t) > H$ *for all* t;

c) $\frac{x_i^{m_i^*}(t)}{x_i^{m_i}(t)} < \frac{x_i^{m_i^*}(0)}{x_i^{m_i}(0)}\frac{1}{L}$ *for all* t *and all* $m_i \in S_i$.

Proof. See the Appendix. ■

By Proposition 4*a)*, for any $i \in I$, the weight of any strategy in $\bar{S}_i$ decreases over time at a rate higher than h_g. This is important because the strategies against which not telling the truth about oneself is strictly suboptimal for player i are all in $\overline{S}_{-i}$; if the weight of these strategies decreases over time, the payoff advantage of the dominating strategy vanishes, making it possible for a dominated strategy to be in the support of the limiting play.[13]

By Proposition 4*b)*, the weight of m_i^* is always high enough, which in turn implies that messages in $\overline{S}_{-i}$ yield lower payoffs than messages in S_{-i}, since an announcement about your neighbor that does not match her announcement about herself is always punished.

In fact, Proposition 4*a)* and 4*b)* reinforce each other. While m_i^* keeps a high enough weight, the weight of strategies in $\overline{S}_{-i}$ decreases. If strategies in $\overline{S}_{-i}$ decrease fast enough, the weight of m_i^* will stay bounded away from zero, provided that $x^{m_i^*}(0)$ was high enough. (as the following Figure 2 shows, $x^{m_i^*}(0)$ need not be very high).

Notice that Propositions 4*a)* and 4*b)* guarantee that equilibria in the "wrong" components are attractors of interior paths. By Proposition 4*c)*, the limiting weight of m_i^* is less than 1 (provided L is sufficiently close to 1), and therefore some mixed strategy equilibria are attractors as well, if the initial conditions give sufficiently little weight to strategies in $\overline{S}_{-i}$. This guarantees that, even if there is convergence to the "right" component, it need not be to the pure strategy equilibrium (remember that the equilibria are not payoff equivalent, as the mixed strategy equilibria have lower expected payoff because agents are punished for announcing discordant preferences).

[13]See Ponti (1997), Proposition 4.1.

Convergence to mixed equilibria may occur because payoffs to all strategies in S_i are "close", if the weight of strategies in $\bar{S}_{-i}$ is small. We know, by Proposition 4*a)*, that the weight of strategies in $\bar{S}_{-i}$ is indeed decreasing. So, even though m_i^* has a payoff advantage, this advantage vanishes, and assumption d.2 (plus Lipschitz continuity) guarantees that it does not accumulate fast enough.

5 Stability with/out drift.

In the previous section, we have extended the convergence result of Proposition 2 to the general mechanism, showing that the limit points of the dynamics for interior initial conditions are generally different from the outcomes intended by the planner. We now go back to our example to test the stability properties of the various equilibrium components.

Definition 1 *Let $x(t, x(0))$ be the solution of (1) on state space Δ given initial conditions $x(0)$. Let also C be a closed set of restpoints in Δ of the same differential equation. Then:*

(i) C is (interior) stable *if, for every neighborhood O of C, there is another neighborhood U of C, with $U \subset O$, such that, for any $x(0) \in U \cap \Delta$ $\left(U \cap \Delta^0\right)$ we have $x(t, x(0)) \in O$;*

(ii) C is (interior) attracting *if is contained in an open set O such that for any $x(0) \in O \cap \Delta$ $\left(O \cap \Delta^0\right)$ we have $\lim_{t \to \infty} x(t, x(0)) \in O$;*

(iii) C is globally *(interior) attracting if for any $x(0) \in \Delta \left(\Delta^0\right)$ we have $\lim_{t \to \infty} x(t, x(0)) \in O$;*

(iii) C is called (interior) asymptotically stable *if it is (interior) attracting and (interior) stable.*

To simplify the analysis, we set additional conditions on the dynamics, which is the purpose of the following assumption, (which replaces Assumptions d.1-3):
Assumption d.5. The evolution of $x(t)$ is given by the following system of continuous-time differential equations:

$$\dot{x}_i \equiv \tilde{D}_i(x(t), \lambda) = x_i(t)\,(1 - x_i(t))\,\Delta\Pi_i(.) + \lambda\,(\beta_i - x_i(t)) \tag{8}$$

with $\lambda \geq 0$, $\beta_1 = \beta_2 = \frac{1}{2}$ and $\beta_3 = \beta \in (0, 1)$.

In words: the evolutionary dynamic is now composed of two additive terms. The first represents the standard replicator dynamic, while the second term ensures that, at each point in time, each strategy is played with positive probability, no matter how it performs against the current opponents' mixed strategy profile (i.e. it points the dynamic *toward the relative interior* of the state space Δ). Following Binmore and Samuelson (1996), this latter term is called *drift:* it opens the model to the possibility of a heterogeneity of behaviors. Gale *et al.* (1995), derive an analogous system in the following way. At each point in time, a fixed proportion of players (of measure $\frac{\lambda}{1+\lambda}$) is replaced by new individuals whose aggregate behavior is represented by a generic, constant, completely mixed strategy (i.e. β_i), while the rest of the population aggregate behavior follows the replicator dynamics. The relative importance of the drift is measured by λ,which we refer to as the *drift level.* We assume λ to be "very small", reflecting the fact that all the major forces which govern the dynamics should be captured by the evolutionary dynamic defined by D, which here takes the form of the replicator dynamics.

We check how the model reacts to the introduction of such a perturbation. The stability analysis of the replicator dynamics with drift will give us information about the effects of small changes in the vector field on the equilibria of the system defined by the replicator dynamic (in other words, it will test the *structural stability* of such equilibria). To simplify the exposition, β_1 and β_2 have been chosen to be 1/2, since only the value of β_3 turns out to be genuinely significant.

We start by looking at the case of the replicator dynamic without drift (i.e. when $\lambda = 0$). We know from Proposition 2, that NE is globally interior attracting, since it attracts every interior path under any monotonic selection dynamic (of which the replicator dynamic is a special case). We now take a closer look at the stability properties of each component of Nash equilibria separately (i.e. NE^0 and NE^1)) in figure 2.

Figure 2 shows a phase diagram describing trajectories of the replicator dynamic starting from some interior initial conditions. The Nash equilibrium component NE^0 $\left(NE^1\right)$ is represented by a bold segment in the bottom-left (top-right) corner of the state space Δ. First notice that, as we know from Proposition 2, all trajectories converge to a Nash equilibrium of the game. Moreover, the diagram shows (consistently with Proposition 4) that there are some trajectories of the replicator dynamic which converge to NE^0, the Nash equilibrium component in which both players 1 and 2 deliver the false message with probability 1. However, this latter component is not asymptotically stable, as can be easily spotted from the diagram. Trajectories starting arbitrarily close to NE^0, provided $x_3 > \frac{3}{7}$, will eventually converge to the truth-telling component.

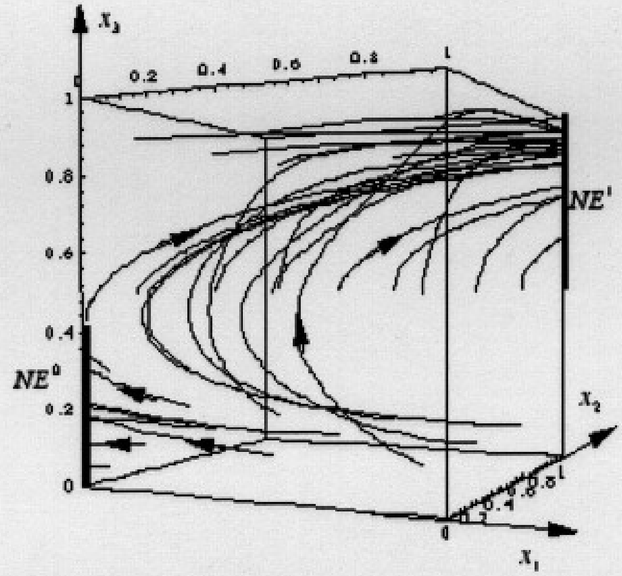

Figure 2
The replicator dynamic and game G

We summarize the key properties of these trajectories in the following proposition:

Proposition 5 *Under the replicator dynamic* (i) NE^1 *is interior asymptotically stable, whereas* (ii) NE^0 *is not.*

Proof. See the Appendix. ■

We now move to the analysis of the replicator dynamics with drift. of the replicator dynamic with drift. Let $\beta \in (0,1)$ be a generic element of the space of the feasible perturbations. Figure 3 shows trajectories of the replicator dynamic with drift under two different specifications of β. Figure 3*b)* represents a situation in which, in the proximity of NE^0, the drift against m_i^0 is uniform across players, where in Figure 3*a)* the drift against m_3^0 is lower. As the diagrams show, there is a local attractor close to NE^1 in both cases. Moreover, none of the elements of NE^0 is a restpoint of the dynamic with drift in figure 3*b)*. In contrast, in figure 3*a)* there is an additional local attractor which belongs to NE^0 : trajectories starting close to NE^0 converge to it, as it happens in the case of the replicator dynamics without drift.

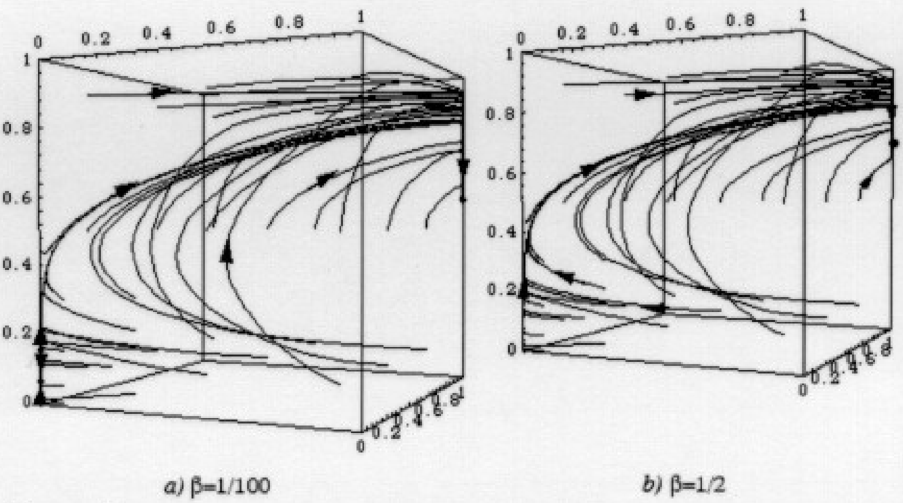

Figure 3
The dynamic with drift and game G

We are interested in the convergence and stability properties of (8) when $\lambda \to 0$, considering two different configurations of the drift parameter β :

$$CASEA : \beta \in \left(0, \frac{23-4\sqrt{30}}{49}\right)$$
$$CASEB : \beta \in \left(\frac{23-4\sqrt{30}}{49}, 1\right)$$

Given $\frac{23-4\sqrt{30}}{49} \approx .0222673$, CASE A depicts a situation in which, for small values of x_i, the drift against the untruth-telling strategy is substantially lower for player 3 than for her opponents.

In the following proposition we characterize the set of restpoints of the dynamic with drift, together with their stability properties:

Proposition 6 *Let $\hat{R}E(\beta)$ be the set of restpoints of (8) for λ sufficiently close to 0. The following properties hold:*

a)$\forall\beta \in (0,1)$, *$\hat{R}E(\beta)$ contains an element of NE^1, which is also asymptotically stable.*

b) *under CASE A $\hat{R}E(\beta)$ contains also two additional restpoints, both belonging to NE^0, one of which is asymptotically stable.*

Proof. See the Appendix. ■

There is a striking similarity between the content of Proposition 6 and the findings of Gale *et al.* (1995). They analyze the classic *Chain Store* game, in one of whose equilibrium components a player selects a weakly dominated strategy with positive probability. This component is reachable, as it attracts a (non-zero measure) set of interior trajectories. Moreover, like our NE^0, it fails to be interior asymptotically stable, but for certain parameter values it may be asymptotically stable when the system is slightly perturbed. Given the failure of asymptotic stability without perturbations, one would expect any perturbation to move the system away from the unstable component and the weakly dominated strategy to become extinct. Proposition 6 tells us that evolutionary game theory does not provide a ground for such a claim. The intuition here is similar to the one in Gale *et al.* (1995). When there is drift, strategies against which the weakly dominated strategy does poorly will have positive weight at all times and, therefore, the part of the dynamics that depends on payoffs pushes against the dominated strategy. On the other hand, drift may provide a direct push in favor of the dominated strategy (and more crucially, in favor of those strategies of the other players which do well against the dominated strategy). When the balance between these two forces is right, one gets a stable equilibrium with non-negligible weight for the dominated strategy.

6 Best-Reply Dynamics and Sjöström's Mechanism

In this section, we consider an alternative scenario. Suppose that $x(t)$ evolves according to the following dynamics:

$$\dot{x} = BR(x) - x \tag{9}$$

with $BR(x)$ denoting the *mixed strategy best-reply correspondence* $BR : \Delta \mapsto \Delta$. This alternative dynamic defines a (continuous-time) version of the classic *best-reply* dynamics, often proposed as an alternative learning model to the evolutionary dynamics studied hereto. We can give two interpretations to (9). Following Matsui (1992), we can use (9) to approximate the evolution of an infinite population of players who occasionally update their strategy, selecting a best reply to the current population state $x(t)$.[14] Alternatively, (9) can be regarded as the continuous-time limit (up to a reparametrization of time) of the well known *fictitious play* dynamic.[15] This dynamic accounts for the evolution of players' *beliefs*, when these beliefs follow the empirical frequencies with which each pure strategy profile has been played (and perfectly observed) in the past, and agents select, at each point in time, a pure strategy among those which maximize their expected payoff, given their current beliefs.

[14] See also Gilboa and Matsui (1991).

[15] Firstly introduced by Brown (1951) as an algorithm to compute Nash equilibria, fictitious play has been recently re-interpreted as a learning model in the works of Fudenberg and Kreps (1993). Milgrom and Roberts (1990) extend some of the properties of fictitious play to the more general class of adaptive learning dynamics. We prefer here the non-standard version in continuous-time to be consistent with the rest of the paper. Nevertheless, in an earlier version of this paper we prove that the same results still hold if the dynamics are defined in discrete-time.

Notice that, for some $x \in \Delta$, $BR(x)$ can take infinitely many values. Thus, uniqueness of the solution of (9) is not guaranteed. However, since $BR(x)$ is upper-hemicontinuous with closed and convex values, it can be shown[16] that the differential inclusion $\dot{x} \in BR(x) - x$ has at least one (interior) solution $x(t, x(0))$, which is Lipschitz continuous and defined, for any $t \geq 0$.

We begin by characterizing the asymptotics of (9) in the case of Sjöström's example, that is, game G.

Proposition 7 *Any interior solution* $x(t, x(0))$ *of (9) converges to* $(1, 1, 1)$.

Proof. Since m_3^0 is weakly dominated by m_3^1, we have $\dot{x}_3 = 1 - x_3$, for any interior solution $x(t, x(0))$ of (9). This in turn implies $\lim_{t\to\infty} x_3(t) = 1$. Moreover, for any $0 < \varepsilon < 1/2$, there exists some $T(\varepsilon)$ such that $x_3(t) \geq \frac{1}{2} + \varepsilon$, for any $t \geq T(\varepsilon)$. We evaluate $T(\varepsilon)$ explicitly:

$$T(\varepsilon) = \left\{ 0 if x_3(0) > \frac{1}{2} + \varepsilon \log \left[\frac{1 - x_3(0)}{1/2 - \varepsilon} \right] if x_3(0) \leq \frac{1}{2} + \varepsilon \right. \tag{10}$$

By virtue of (10), $T(\varepsilon) < \infty$, for ε sufficiently small. Since $BR(x(t)) = (1, 1, 1)$ for any $t \geq T(\varepsilon)$, any interior solution of (9) is characterized by the following system of differential equations:

$$\dot{x}_i(t) = 1 - x_i(t), i = 1, 2, 3; \tag{11}$$

for t sufficiently large. This, in turn, implies $\lim_{t\to\infty} x_i(t) = 1, i = 1, 2, 3.$■

Similar considerations hold for the general mechanism. By analogy with (2-3), let $\widehat{S}_i = \{m_i \in M_i \left| R_i^i = \widehat{R}_i \right.\}$, with $\widehat{s}_i = \{m_i \in \widehat{S}_i \left| R_{i-1}^i = \widehat{R}_{i-1}, R_{i+1}^i = \widehat{R}_{i+1} \right.\}$ denoting the pure Nash equilibrium in which all agents consistently reveal their true preferences (i.e. the "solution" of Γ given the true preference profile $\widehat{R}$). For any given arbitrary preference profiles $R \in \Phi$, with $R \neq \widehat{R}$, $m_i = \{m_i \notin \widehat{S}_i \left| R_{i-1}^i = R_{i-1}, R_{i+1}^i = R_{i+1} \right.\}$ is weakly dominated by $\widehat{m}_i = \{m_i \in \widehat{S}_i \left| R_{i-1}^i = R_{i-1}, R_{i+1}^i = R_{i+1} \right.\}$, which in turn implies that, for any interior solution $x(t, x(0))$ of (9)

$$\lim_{t\to\infty} x_i^{m_i'}(t) = 0. \tag{12}$$

for any $m_i \notin \widehat{S}_i$. Let $\widehat{\Delta}$ denote the face of Δ spanned by the restricted game $(\Gamma, \widehat{R}) \Big|_{\times \widehat{S}_i}$. An implication of (12) is that $\widehat{\Delta}$ is globally interior attracting for the best-reply dynamics (9), as it contains the set of undominated mixed strategies. Consider now the following system of differential equations:

[16]See Aubin and Cellina (1984), Chapter 2. On the stability properties of (9) see Hofbauer (1997).

$$
\begin{array}{c}
\dot{x}_i^{m_i}(t) = 1 - x_i^{m_i}(t), m_i = \widehat{s}_i \\
\dot{x}_i^{m_i}(t) = -x_i^{m_i}(t), m_i \neq \widehat{s}_i
\end{array} \tag{13}
$$

Since, for all i, $\widehat{s}_i$ is the unique best reply for player i in the restricted game $(\Gamma, \widehat{R})\Big|_{\times \widehat{S}_i}$, (13) defines the unique solution of (9) for (not necessarily interior) trajectories starting from $\widehat{\Delta}$ and, therefore, for any interior trajectory (9) starting from Δ, for t sufficiently large. We have just proved

Proposition 8 *Any interior solution of (9) converges to* $\widehat{s}_i$.

For best-reply dynamics we have shown that every interior solution converges to the unique equilibrium whose outcome is the one the planner wants to implement. This is so because completely mixed initial beliefs make the weakly dominated strategies in which agents lie about their own type suboptimal. Furthermore, since initial beliefs are completely mixed, they will always be completely mixed, so these weakly dominated strategies will always remain suboptimal, will never be played and their weight in beliefs will eventually vanish. This implies that nonequilibrium strategies by which agents misrepresent their neighbors' preferences become also suboptimal, and agents will learn not to use them.

The results obtained here are so different from those we derived in the previous sections essentially because the difference in growth rates between two pure strategies, in the case of the best-reply dynamics (9), need not satisfy Lipschitz continuity. The only strategies with a positive growth rate are best responses; this implies that there is an infinite response of growth rates to changes in the sign of the differences in payoffs, which is precisely what Lipschitz continuity rules out.

To understand the effects of increasing levels of responsiveness to payoffs on the performance of the mechanism, consider the following (monotonic) dynamics:

$$
\dot{x}_i^{m_i} = x_i^{m_i} \left(\frac{\exp[\sigma u_i(m_i, x_{-i})]}{\sum\limits_{m_k \in M_i} x_i^{m_k} \exp[\sigma u_i(m_k, x_{-i})]} - 1 \right) \tag{14}
$$

The dynamic (14) has been proposed by Björnerstedt and Weibull (1996) to approximate the evolution of a population of agents who revise their pure strategy imitating at random other agents in the same player position; the more successful is strategy m_i given the current population state (i.e. the higher is $u_i(m_i, x_{-i})$), the higher is the probability of m_i being imitated (i.e. the higher is $\dot{x}_i^{m_i}$).[17] Given the functional form (14), we can interpret σ as a "responsiveness"

[17]On the evolutionary properties of (14), see also Weibull (1995) and Hofbauer and Weibull (1996).

parameter. For small σ, (14) approaches (up to a reparametrization of time) the standard replicator dynamics, whose dynamic properties are described in Figure 2; for large σ, (14) approaches, at least for interior solutions, the best-reply dynamics (9).

The phase diagrams of figure 4 trace the interior solutions of (14) in the case of game G, starting from the same initial conditions, under two different realizations of σ :

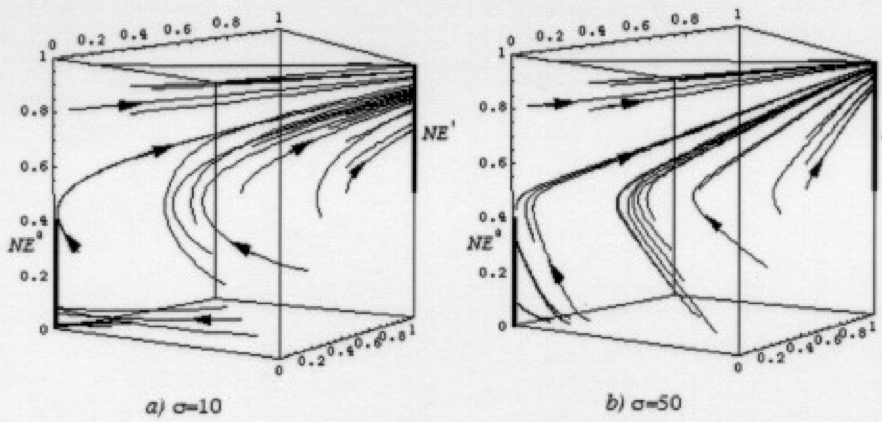

Figure 4
Increasing responsivness to payoffs.

As can be spotted by the diagrams, the higher is σ, the smaller is the basin of attraction of the untruthful component NE^0. Furthermore, for any given initial condition, the higher is σ, the closer is the limit point of the corresponding trajectory to (1,1,1). However, no matter how high is σ, it still remains true that *i)* NE^0 attracts a non-negligible set of interior trajectories and *ii)* no interior trajectory converges to (1,1,1).

Figure 4 helps us in understanding the limitations of the theoretical findings of this section. For example, it is crucial that only exact best responses grow. If small differences in payoffs did not lead to such a large effect on difference in growth rates, then the results about monotonic dynamics would still hold.

7 Conclusions

We have argued that there is room for doubt about the practicability of one the of the leading examples of implementation with iterated deletion of weakly dominated strategies when agents are boundedly rational. As we said in the introduction, there are only few papers that study implementation with boundedly rational players, so a deeper theoretical study with evolutionary tools of other mechanisms studied in the literature would enhance our understanding of the performance of these mechanisms with this type of agents, a necessary step before mechanisms are used in real life.

Further empirical study is at least as necessary. It would, for example, help to answer the question about which of the dynamics assumptions is more appropriate. In this sense, there is already some evidence on mechanism design and learning algorithms. Chen and Tang (1996) have done experiments with the Basic Quadratic mechanism by Groves and Ledyard (1977) and the Paired-Difference mechanism by Walker (1981). They estimate different learning models using experimental data, showing that variants of stimulus-response learning algorithms (whose expected law of motion is the replicator dynamics) outperform the generalized fictitious play model. This is also consistent with the good performance that Roth and Erev (1995) show for stimulus-response learning algorithms in mimicking the behavior of a range of experimental data, which includes other weakly dominance solvable games, like the ultimatum game.[18]

But even more importantly, the empirical and experimental work would help to design games with good convergence properties to the preferred social outcome, by revealing how people adjust their play in games like that studied in this paper, as well as in other mechanisms proposed by the literature. We have already begun to do such experimental studies.

[18]In their paper, Roth and Erev (1995) show that these dynamics explain the data significantly better, according to quadratic deviation measures and others, than a generalized fictitious play model which can accommodate behaviors ranging from fictitious play to best response dynamics by the estimation of a "forgetfulness parameter" which weights past information. For the experimental evidence on learning rules, see also Tang (1996), Chen *et al.*, forthcoming, and Mookherjee and Sopher (1996).

8 Appendix

Proof of Proposition 1. We already noticed that agent 3 has a weakly dominated strategy (namely, m_3^0). In particular, m_3^1 (truth-telling) makes agent 3 (strictly) better off than m_3^0 (lying), unless agents 1 and 2 coordinate their actions completely, that is, unless they play m_i^0 $i = 1, 2$ with probability 1 or they play m_i^1 $i = 1, 2$ with probability 1, (in which case, 3 is completely indifferent). This leads to the following lemma:

Lemma 1 *No strategy profile in which $x_3 \in (0, 1)$ can be a Nash equilibrium unless $x_1 = x_2 = 1$ or $x_1 = x_2 = 0$, that is, unless agents 1 and 2 play the same strategy with probability 1.*

With this in mind, we construct the proof as follows: we fix the mixed strategy of player 3 and check which mixed strategies for player 1 and 2 can sustain a Nash equilibrium. Noting that

$$\Delta\Pi_1 \equiv u_1(m_1^1, x) - u_1(m_1^0, x) = \frac{1}{12}(x_2(x_3 - 1) + 7x_3 - 3) \tag{15}$$

$$\Delta\Pi_2 \equiv u_2(m_2^1, x) - u_2(m_2^0, x) = \frac{1}{12}(x_1(x_3 - 1) + 7x_3 - 3) \tag{16}$$

we can make the following observations:

a) When $x_3 < \frac{3}{7}$, m_i^0 (lying) yields a strictly higher payoff than m_i^1 for both 1 and 2, independently of what the other player does. Therefore, strategy profiles in NE^0 (and only those) are Nash equilibria.

b) When $x_3 = \frac{3}{7}$, m_1^0 yields a strictly higher payoff than m_1^1 unless $x_2 = 0$, and $x_2 = 0$ makes player 1 indifferent between m_1^0 and m_1^1 (a symmetric argument holds for player 2). This excludes the possibility of $\left(1, 1, \frac{3}{7}\right)$ being a Nash equilibrium of the game, leaving $\left(0, 0, \frac{3}{7}\right) \in NE^0$ as the unique Nash equilibrium when $x_3 = \frac{3}{7}$.

c) When $x_3 \in \left(\frac{3}{7}, \frac{1}{2}\right)$ there are no Nash equilibria. The reason is that, in this case, if $x_1 = 1$, the best response of player 2 is $x_2 = 0$; if $x_1 = 0$, the best response for player 2 is $x_2 = 1$. However, neither $(0, 1, x_3)$ nor $(1, 0, x_3)$ can be Nash equilibria when $x_3 \in \left(\frac{3}{7}, \frac{1}{2}\right)$ by Lemma 1.

d) $x_3 = \frac{1}{2}$. By analogy with the case where $x_3 = \frac{3}{7}$, it is an implication of Lemma 1 that $\left(1, 1, \frac{1}{2}\right) \in NE^1$ is the unique Nash equilibrium when $x_3 = \frac{1}{2}$.

e) When $x_3 > \frac{1}{2}$ announcing m_i^1 (truth-telling) is optimal for $i = 1$ and 2, independently of what the other player does. Thus, strategy profiles in NE^1 (and only those) will be Nash equilibria.

Since this exhausts all cases the result follows. ■

Proof of Proposition 2. To prove the proposition, it is enough to show that any interior trajectory converges. The reason is that, once convergence has been proved, we can apply the standard result "convergence implies Nash under any monotonic selection dynamics" (see, e.g. Weibull, 1995, Theorem 5.2 *(iii)*).

We start by observing that the dynamic is forward invariant. This implies that $x_i(t)$ is always defined and positive, for any nonnegative t. By monotonicity, $x_3(t)$ is also a positive, increasing function of t and bounded above by 1 (since m_3^1 is a weakly dominant strategy). Therefore, $x_3(t)$ must converge (this already implies convergence of player 3's mixed strategy). Let $x_i^* \equiv \lim_{t\to\infty} x_i(t)$, when such a limit exists. Three alternative cases have to be discussed:

a) $x_3^* = 0$. If $x_3^* = 0$ there must be a time t' such that $x_3(t) < \frac{3}{7}$ for $t > t'$. This implies that there is a $k > 0$ such that for all $t' > t$, $\Delta\Pi_i(x(t)) < -k$ for $i = 1, 2$. This implies, by monotonicity, $\lim_{t\to\infty} x_i(t) = 0$ for $i = 1, 2$, thus $x^* = (0, 0, 0)$.

b) $x_3^* = 1$. By a similar argument, monotonicity implies $x^* = (1, 1, 1)$.

c) $x_3^* \in (0, 1)$. We want to prove that x_3^* cannot converge to a value within this range unless the system converges to a Nash equilibrium. To do so (given the special features of our example) it is enough to show that, if $x_3^* \in (0, 1)$, then both players 1 and 2 select, in the limit, the same pure strategy. Given that this implies convergence of the full mixed strategy profile, the result follows. More formally, what we need to prove is contained in the following lemma:

Lemma 2 *If $x_3^* \in (0, 1)$ then:*

$$
\begin{gathered}
either \\
x_i^* = 0, i = 1, 2 (CASE\ 0 hereafter) \\
or \\
x_i^* = 1, i = 1, 2. (CASE\ 1)
\end{gathered}
$$

Proof. Assume, for the purpose of contradiction, that neither of the above statements is true. In this case, there must exist a sequence $\{t_k\}_{k=1}^{\infty}$ and a positive constant $\epsilon > 0$ such that either $x_i(t_k) > \epsilon, i = 1, 2$ or $x_i(t_k) < 1 - \epsilon, i = 1, 2$ for all k (in other words, assume that the system stays infinitely often an ϵ away from the faces of Δ in which player 1 and 2 play the same pure strategy). We already noticed that these are the only faces of Δ in which both pure strategies for player 3 yield the same payoff. If the system stays away from these faces infinitely often along the solution path, then the integral of the payoff difference $\Delta\Pi_3(x(t))$ goes to infinity as t goes to infinity.

To show this, notice that $\Delta\Pi_i(x(t))$ is a continuous function of $x(t)$ defined over a compact set (Δ). In the case of player 3, such function takes the following form:

$$
\Delta\Pi_3(x(t)) \equiv \frac{(x_1(t) - x_2(t))^2 + x_1(t)(1 - x_1(t)) + x_2(t)(1 - x_2(t))}{6} \tag{17}
$$

Take $g_M \equiv \max_{i \in I, x_{-i} \in \Delta_{-i}} [|g_i(m_i, x_{-i}(t))|]$, i.e. the highest possible growth rate (in absolute value) over all strategies and players (we know a max exists, since also $g_i(.)$ is continuous in Δ). Then define τ_1, τ_2, τ_3 and τ_4 as follows:

τ_1 solves $\epsilon \exp[-g_M \tau_1] = \frac{\epsilon}{2}$ (i.e. $\tau_1 = \frac{\ln[2]}{g_M}$);

τ_2 solves $(1-\epsilon)\exp\left[-g_M\tau_2\right]=\frac{\epsilon}{2}$ (i.e. $\tau_2=\frac{\ln\left[-2+\frac{2}{\epsilon}\right]}{g_M}$);
τ_3 solves $\epsilon\exp\left[g_M\tau_3\right]=1-\frac{\epsilon}{2}$ (i.e. $\tau_3=\frac{\ln\left[-\frac{1}{2}+\frac{1}{\epsilon}\right]}{g_M}$);
τ_4 solves $(1-\epsilon)\exp\left[g_M\tau_4\right]=1-\frac{\epsilon}{2}$ (i.e. $\tau_4=\frac{\ln\left[\frac{2-\epsilon}{2-2\epsilon}\right]}{g_M}$).

Let $\partial\tau\equiv\min[\tau_1,\tau_2,\tau_3,\tau_4]$ be the lower bound for the time interval in which, after each t_k, $\frac{\epsilon}{2}<x_i<1-\frac{\epsilon}{2}, i=1,2$ and therefore $\Delta\Pi_3(x(t))$ still remains bounded away from 0 (i.e. $\Delta\Pi_3(x(t))>\frac{\epsilon(1-\frac{\epsilon}{2})}{3}>0, \forall t\in[t_k,t_k+\partial\tau]$). Denote by $G_\epsilon=\left\{x\in\Delta|\Delta\Pi_3(x)\geq\frac{\epsilon(1-\frac{\epsilon}{2})}{3}\right\}$. Now define $\gamma_i(x(t))$ the time derivative of the log of the ratio between the probabilities with which each of player i's pure strategies are played, which can be expressed in terms of the difference in the growth rates:

$$\gamma_i(x(t))\equiv\frac{\partial}{\partial t}\ln\left(\frac{x_i(t)}{1-x_i(t)}\right)=\frac{\dot{x}_i(t)}{x_i(t)}-\frac{(1-\dot{x}_i(t))}{1-x_i(t)}=\frac{\dot{x}_i(t)}{x_i(t)-(x_i(t))^2}$$

Also $\gamma_3(x(t))$ is a positive number bounded away from 0 infinitely often since, by assumption d.1, is a continuous function of $x(t)$ defined on a compact set, which preserves the same sign of $\Delta\Pi_3(x(t))$. This implies that we can always define a constant $g_\epsilon=\min_{x\in G_\epsilon}\gamma_3(x(t))$, with $g_\epsilon>0$ by assumption d.2. Also by assumption d.2, $\gamma_3(x(t))>g_\epsilon\Longleftrightarrow\Delta\Pi_3(x(t))>\frac{\epsilon(1-\frac{\epsilon}{2})}{3}$. If we integrate the value of $\gamma_3(x(t))$ over time we then obtain:

$$\lim_{t\to\infty}\int_0^t\gamma_3(x(t))dt\geq\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\partial\tau}\gamma_3(x(t))dt>g_\epsilon\sum_{k=1}^{\infty}\int_{t_k}^{t_k+\partial\tau}dt=\infty$$

which implies that $x_3^*=1$, which leads to a contradiction.■

To summarize, Lemma 2 shows that, if $x_3^*\in(0,1)$, $x_1(t)$ and $x_2(t)$ must converge (and therefore $x(t)$ must converge to a Nash equilibrium). Since this exhausts all cases the result follows. ■

Proof of Proposition 3. We begin by noting that, against any $m_{-i}\in M_{-i}$, all strategies $m_i\in S_i$ yield the same payoff, as they only differ in i's announcement about herself. Since supp$[x_{-i}]\subseteq S_{-i}$, totally inconsistent states (the only states where announcements about i'excluded.

For all $\hat{x}_i$, such that $\hat{x}_i^{m_i}>0$ only if $m_i\in S_i$ we have,

$$u_i(\hat{x}_i,x_{-i})\geq\Pi_{j\neq i}x_j^{m_j^*}v_i(f_i(\phi(i,R^*)),\hat{R}_i)+(1-\Pi_{j\neq i}x_j^{m_j^*})v_i(0,\hat{R}_i).$$

For all $\bar{x}_i\neq\hat{x}_i$,

$$u_i(\bar{x}_i,x_{-i})\leq\sum_{m_i\in S_i}\bar{x}_i^{m_i}u_i(\hat{x}_i,x_{-i})+\left(1-\sum_{m_i\in S_i}\bar{x}_i^{m_i}\right)\left[\Pi_{j\neq i}x_j^{m_j^*}v_i(0,\hat{R}_i)+(1-\Pi_{j\neq i}x_j^{m_j^*})U_{in}\right].$$

Then

$$u_i(\hat{x}_i, x_{-i}) - u_i(\bar{x}_i, x_{-i}) \geq \left(1 - \sum_{m_i \in S_i} \bar{x}_i^{m_i}\right) \Big[\Pi_{j\neq i} x_j^{m_j^*}(v_i(f_i(\phi(i,R^*)), \hat{R}_i) - v_i(0,\hat{R}_i)) \\ +(1-\Pi_{j\neq i} x_j^{m_j^*})(v_i(0,\hat{R}_i) - U_{in})\Big],$$

which is greater than zero since, by (4),

$$\Pi_{j\neq i} x_j^{m_j^*} \geq \Pi_{j\neq i} k_j \geq \frac{U_{in} - v_i(0,\hat{R}_i)}{v_i(f_i(\phi(i,R^*)), \hat{R}_i) - v_i(0,\hat{R}_i) + U_{in} - v_i(0,\hat{R}_i)}.$$

The following lemma will be useful in the proof of Proposition 4.

Lemma 3 . *Let any $m_i, m_i' \in S_i$ and x_i. Then*

$$g_i(m_i, x_{-i}) - g_i(m_i', x_{-i}) \geq -2KX_i.$$

Proof.

Let $\hat{x}_{-i}$ such that $\hat{x}_j^{m_j} = x_j^{m_j}$ for all $m_j \in S_j \backslash m_j^*$; $\hat{x}_j^{m_j} = 0$ for all $m_j \in \overline{S}_j$; and $\hat{x}_j^{m_j^*} = x_j^{m_j^*} + \sum_{m_j \in \overline{S}_j} x_j^{m_j}$.

Since $u_i(m_i, x_{-i}) = u_i(m_i', x_{-i})$ for all $x_{-i} \in S^{-i}$, then $g_i(m_i, \hat{x}_{-i}) = g_i(m_i', \hat{x}_{-i})$.

By Lipschitz continuity we have that,

$$g_i(m_i, x_{-i}) - g_i(m_i, \hat{x}_{-i}) \geq -K|x_{-i} - \hat{x}_{-i}| \tag{18}$$

$$g_i(m_i', \hat{x}_{-i}) - g_i(m_i', x_{-i}) \geq -K|x_{-i} - \hat{x}_{-i}| \tag{19}$$

Since $g_i(m_i, \hat{x}_{-i}) = g_i(m_i', \hat{x}_{-i})$ and $|x_{-i} - \hat{x}_{-i}| = X_i$, the result follows by adding up inequalities (18) and (19). ■

Proof of Proposition 4. By contradiction.

Suppose that *a)* is the statement that stops being true earliest, that it does so for agent i and strategy $m_i \in \bar{S}_i$ and that the boundary time is t'. Then it must be

$$\frac{x_i^{m_i}(t')}{x_i^{m_i}(0)} = \exp[-h_g t'] \frac{H}{x_i^{m_i^*}(0)}.$$

Notice that, for all t,

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$$u_i(m_i, x_{-i}(t)) - u_i(m_i^*, x_{-i}(t)) \leq v_i(0,\hat{R}_i)\Pi_{j\neq i} x_j^{m_j^*}(t) + U_i(1 - \Pi_{j\neq i} x_j^{m_j^*}(t)) \\ - \Big(v_i(f_i(\phi(i,R^*)), \hat{R}_i)\Pi_{j\neq i} x_j^{m_j^*}(t) + v_i(0,\hat{R}_i)(1-\Pi_{j\neq i} x_j^{m_j^*}(t))\Big) \\ = U_i - v_i(0,\hat{R}_i) - \Pi_{j\neq i} x_j^{m_j^*}(t)\Big(v_i(f_i(\phi(i,R^*)), \hat{R}_i) + U_i - 2v_i(0,\hat{R}_i)\Big)$$

Since *b)* is true for $t < t'$,

$$u_i(m_i, x_{-i}(t)) - u_i(m_i^*, x_{-i}(t)) < U_i - v_i(0, \hat{R}_i) - H^{n-1}\left(v_i(f_i(\phi(i, R^*)), \hat{R}_i) + U_i - 2v_i(0, \hat{R}_i)\right).$$

Thus,

$$u_i(m_i, x_{-i}(t)) - u_i(m_i^*, x_{-i}(t)) < -h_v,$$

which, by assumption d.2 and the definition of h_v and h_g, implies that

$$g_i(m_i, x_{-i}(t)) - g_i(m_i^*, x_{-i}(t)) < -h_g.$$

Given $x_i^{m_i^*}(t') \leq H$, if we integrate $g_i(m_i, x_{-i}(t)) - g_i(m_i^*, x_{-i}(t))$ from 0 to t', we obtain the following:

$$\frac{x_i^{m_i}(t')}{x_i^{m_i}(0)} < \exp[-h_g t'] \frac{H}{x_i^{m_i^*}(0)}.$$

This is a contradiction.

Suppose that *b)* is the statement that stops being true earliest, that it does so for agent i, and that the boundary time is t'. Then, it must be true that $x_i^{m_i^*}(t') = H$.

Notice that Lemma 3 implies that, for all $m_i \in S_i \backslash \{m_i^*\}$,

$$g_i(m_i^*, x_{-i}(t)) - g_i(m_i, x_{-i}(t)) \geq -2KX_i(t). \tag{20}$$

Since *a)* holds for $t < t'$, (20) implies that

$$\begin{aligned} g_i(m_i^*, x_{-i}(t)) - g_i(m_i, x_{-i}(t)) &> -2K\left(\exp[-2h_g t] \tfrac{H^2}{(x_i^{m_i^*}(0))^2} X_i(0)\right) \\ &\geq -2K\left(\exp[-2h_g t] \tfrac{H^2}{(x_i^{m_i^*}(0))^2} X(0)\right). \end{aligned}$$

By integration,

$$\frac{x_i^{m_i^*}(t')}{x_i^{m_i}(t')} \frac{x_i^{m_i}(0)}{x_i^{m_i^*}(0)} > \exp\left[\frac{-2KX(0)}{2h_g} \frac{H^2}{(x_i^{m_i^*}(0))^2}\right] \geq L$$

Adding over all strategies in S_i,

$$\frac{x_i^{m_i^*}(t')}{x_i^{m_i^*}(0)} > \frac{x_i^{S_i}(t')}{x_i^{S_i}(0)} L = \frac{1 - x_i^{\bar{S}_i}(t')}{1 - x_i^{\bar{S}_i}(0)} L \geq L$$

This implies $x_i^{m_i^*}(t') > H$ (using the assumption $x_i^{m_i^*}(0)\ L > H$), which is a contradiction.

Suppose that *c)* is the statement that stops being true earliest, that it does so for agent i, and that the boundary time is t'. Then it must be $\frac{x_i^{m_i^*}(t')}{x_i^{m_i}(t')} = \frac{x_i^{m_i^*}(0)}{x_i^{m_i}(0)} \frac{1}{L}$.

By Lemma 3, for all $m_i \in S_i \backslash \{m_i^*\}$,

$$g_i(m_i, x_{-i}(t)) - g_i(m_i^*, x_{-i}(t)) \geq -2KX_i(t). \tag{21}$$

Since *a)* holds for $t < t'$, (21) implies that

$$\begin{aligned} g_i(m_i, x_{-i}(t)) - g_i(m_i^*, x_{-i}(t)) &> -2K \left(\exp[-2h_g t] \frac{H^2 X_i(0)}{(x_i^{m_i^*}(0))^2} \right) \\ &\geq -2K \left(\exp[-2h_g t] \frac{H^2}{(x_i^{m_i^*}(0))^2} X(0) \right). \end{aligned}$$

By integration,

$$\frac{x_i^{m_i}(t')}{x_i^{m_i^*}(t')} \frac{x_i^{m_i^*}(0)}{x_i^{m_i}(0)} > \exp\left[\frac{-2KX_0}{2h_g} \frac{H^2}{(x_i^{m_i^*}(0))^2} \right] \geq L.$$

which implies that

$$\frac{x_i^{m_i^*}(t')}{x_i^{m_i}(t')} < \frac{x_i^{m_i^*}(0)}{x_i^{m_i}(0)} \frac{1}{L}$$

which is a contradiction. Since this exhausts all cases the result follows. ■

Proof of Proposition 5. (i) We know, from Proposition 2, that $\dot{x}_3 > 0$ in any interior point. This implies that, if there is a time t such that $) > \frac{1}{2}$ for all $t' \geq t$. From equations (15-16) we have that, whenever $x_3(t) > \frac{1}{2}$, $\Delta\Pi_i(x) > 0$ for players 1 and 2. This implies that, if there is a time t such that $x_3(t) > \frac{1}{2} \geq t$ for $i = 1, 2$ and, therefore, $x(t)$ must converge. Since convergence must be to a Nash equilibrium and x_1 and x_2 have been increasing, x must converge to NE^1. To show the stability of NE^1 it suffices to show that there is a neighborhood of NE^1 such that, for all $x(0)$ in this neighborhood, there is a time t such that $x_3(t) > \frac{1}{2}$. Let

$$x_3(0) = \frac{1}{2} \exp\left[-\frac{(1-\epsilon_1)(\frac{\epsilon_1^2}{2})}{12} \right] < \frac{1}{2}$$

. From (15-16) we also have that $-1 < \Delta\Pi(x) < 1$ for $i = 1, 2$, thus

$$\exp[-t](1-\epsilon_i) < x_i(t) < \exp[t](1-\epsilon_i). \tag{22}$$

Since $\Delta\Pi_3(x) \leq \frac{x_1(1-x_1)}{6}$, (22) implies

$$\frac{\dot{x}_3(t)}{x_3(t)} > \frac{(1-\epsilon_1)(\exp[-t](1-\exp[t](1-\epsilon_1))}{6}.$$

Thus,

$$\frac{\dot{x}_3(t)}{x_3(t)} > \frac{(1-\epsilon_1)(\exp[-t]-(1-\epsilon_1))}{6} > \frac{(1-\epsilon_1)(-t+\epsilon_1)}{6}.$$

This implies that

$$x_3(t) > \exp\left[\frac{(1-\epsilon_1)(-\frac{t^2}{2}+\epsilon_1 t)}{6}\right] x_3(0).$$

Note that, for $t=\epsilon_1$,

$$x_3(\epsilon_1) > \exp\left[\frac{(1-\epsilon_1)(\frac{\epsilon_1^2}{2})}{6}\right] \exp\left[-\frac{(1-\epsilon_1)(\frac{\epsilon_1^2}{2})}{12}\right]\frac{1}{2} = \exp\left[\frac{(1-\epsilon_1)(\frac{\epsilon_1^2}{2})}{12}\right]\frac{1}{2} > \frac{1}{2}.$$

Which is what we wanted to show.

(ii). Assume $x_3(0) > \frac{3}{7}$. Since $\dot{x}_3(t) > 0$ for all t, $x_3(t)$ must converge. Furthermore, since $x_3(0)$ is larger than $\frac{3}{7}$, $x_3(t)$ must converge to a number larger than $\frac{3}{7}$. We know that $x(t)$ converges to a Nash equilibrium by Proposition 2. Since there is no equilibrium in NE^0 with $x_3 > \frac{3}{7}$, $x(t)$ cannot converge to a point in NE^0. Since $x_3(0)$ can be arbitrarily close to $\frac{3}{7}$ and therefore to the set NE^0, this set must be unstable. ■

Proof of Proposition 6. The proof is constructed as follows. We first characterize the limit of the set of rest points $\hat{R}E(\beta)$, and then analyze the stability properties of each of its elements.

We start by observing that, given $\beta \in (0,1)$, any rest point must be completely mixed, and it also must be $x_3 > \beta$, as $\Delta\Pi_3(.)$ is always positive in the interior of the state space Δ (because m_3^0 is a weakly dominated strategy).We also know, by continuity of the vectorfield with respect to λ, that every limiting rest point of the dynamic, as λ goes to zero, must lie in the set of restpoints of the unperturbed dynamic $RE\,(G)$.

First, we analyze the limit set of rest points under CASE 0. In this case, both players 1 and 2 play their strategy m_i^0 with probability 1, that is $x_i = 0$, for $i = 1, 2$. Setting $\dot{x}_1 = 0$ yields the following equation:

$$\frac{x_1}{\lambda} = \frac{12\left(\frac{1}{2}-x_1\right)}{(1-x_1)\left(3+x_1-x_3(7-x_2)\right)} \tag{23}$$

and an analogous expression can be obtained for $\frac{x_2}{\lambda}$. Denote by x_3^0 a limiting value in a rest point, if a limit exists, for x_3. When the limiting values for x_1 and x_2 are zero we have:

$$\lim_{x_i\to 0\lambda\to 0}\frac{x_i}{\lambda} = \frac{6}{(3-7x_3^0)} \tag{24}$$

Notice that, in this case, if a rest point exists, it must be $x_3^0 < \frac{3}{7}$, since $\frac{x_i}{\lambda} > 0$. We set $\frac{\dot{x}_3}{\lambda} = 0$,substitute $\frac{x_i}{\lambda}$ with the expression in (24), solve for x_3, and substitute $x_i, i = 1, 2$ and λ by their limiting value of zero. The solutions for x_3^0 take the following form:

$$\hat{x}_3^0 = \frac{1 + 7\beta + \sqrt{1 - \beta(46 - 49\beta)}}{10} and \check{x}_3^0 = \frac{1 + 7\beta - \sqrt{1 - \beta(46 - 49\beta)}}{10}$$

Remember that x_3^0 must be a real, positive number, with $\beta < x_3^0 < \frac{3}{7}$. For the expression under the square root at the numerator to be nonnegative, it must be that $\beta \in \left[0, \frac{23-4\sqrt{30}}{49} \approx .0222673\right]$, which determines the feasible range for both roots. Within this interval of values for β, $\hat{x}_3^0$ $\left(\check{x}_3^0\right)$ is a strictly decreasing (increasing) function of β, which has a minimum and a maximum, whose values are $\frac{15-2\sqrt{30}}{35}$ (0) and $\frac{2}{10}\left(\frac{15-2\sqrt{30}}{35}\right)$ respectively. As $\beta \to \frac{23-4\sqrt{30}}{49}$, both solutions converge to $\frac{15-2\sqrt{30}}{35}$.

We now deal with the subset of limiting rest points under CASE 1, i.e. with limiting values for $x_i = 1$ for $i = 1, 2$. The equations corresponding to (23-24) are now the following:

$$\frac{(1 - x_1)}{\lambda} = \frac{12\left(x_1 - \frac{1}{2}\right)}{x_1\left(7x_3 + x_2(1 - x_3) - 3\right)} \tag{25}$$

$$\lim_{x_i \to 1 \lambda \to 0} \frac{(1 - x_i)}{\lambda} = \frac{3}{2(2x_3^1 - 1)} \tag{26}$$

where x_3^1 denotes a limiting value for x_3 (if a limit exists). By analogy with CASE 0, we know from (26) that, if a rest point exists, it must be $x_3^1 > \frac{1}{2}$. There is a unique feasible solution for $x_3^1, \forall \beta \in (0, 1)$ which has the following form:

$$\hat{x}_3^1 = \frac{3 + 4\beta + \sqrt{9 - 16\beta(1 - \beta)}}{10}.$$

Following the same procedure for the remaining rest points of the unperturbed dynamics (i.e. the pure strategy profiles which belong to $RE\,(G)$ and do not satisfy either CASE 0 or CASE 1) does not add any element to the limiting set of rest points of the perturbed dynamics. This should not be surprising, as any other rest point of the unperturbed replicator dynamics is unstable with respect to the interior. Since this exhausts all cases, the result follows.

We now move to establish the stability properties of each limiting restpoint separately. The Jacobian matrix for the dynamic system is as follows:

$$J(x, \lambda) = \begin{matrix} (1 - 2x_1)\Delta\Pi_1 - \lambda & \frac{-(1-x_1)x1(1-x_3)}{12} & \frac{(1-x1)x1(7+x_2)}{12} \\ \frac{-(1-x_2)x_2(1-x_3)}{12} & (1 - 2x_2)\Delta\Pi_2 - \lambda & \frac{(1-x_2)x_2(7+x_1)}{12} \\ \frac{(1-2x_2)(1-x_3)x_3}{6} & \frac{(1-2x_1)(1-x_3)x_3}{6} & (1 - 2x_3)\Delta\Pi_3 - \lambda \end{matrix}$$

We analyze CASE 0 first. We know that, in this case, we have two restpoints, which we call $\hat{x}^0 \equiv (0, 0, \hat{x}_3^0)$ and $\check{x}^0 \equiv (0, 0, \check{x}_3^0)$. We evaluate the Jacobian when x_1, x_2 and λ are equal to their limiting value (i.e. 0). The corresponding eigenvalues are $\left\{0, \frac{-3+7x_3^0}{12}, \frac{-3+7x_3^0}{12}\right\}$. There are then two (identical) negative eigenvalues (since any limiting $x_3^0 < \frac{3}{7}$ for CASE 0), while the third eigenvalue is equal to zero. To determine the stability properties of the perturbed system, the sign of the eigenvalue whose limit is zero becomes crucial given that continuity of $J(.)$ ensures that the other two will be negative, for any λ sufficiently small. We now linearize the rest points (as a function of λ) around NE^0. We set $\tilde{x}(\lambda, \delta) \equiv \left(\delta_1\lambda, \delta_2\lambda, x_3^0 + \delta_3\lambda\right)$, where $\delta \equiv (\delta_1, \delta_2, \delta_3)$ denotes the vector collecting the coefficients of the linearized system. We then evaluate the following expression:

$$\phi^0(x_3^0, \delta) \equiv \lim_{\lambda \to 0} \frac{\partial \det \left(J(x, \lambda) \left|_{\tilde{x}(\lambda,\delta)}\right.\right)}{\partial \lambda}$$

We do so because $\det\left(J(x, \lambda)\right)$, which is equal to zero $\forall x \in NE^0$, will preserve the sign of the third eigenvalue, given that the sign of the other two will stay constant (and negative) when x is sufficiently close to NE^0 and λ is sufficiently small. For CASE 0 we get the following result:

$$\phi^0(x_3^0, \delta) = \frac{-54 + x_3^0(252 + 294x_3^0) + (\delta_1+\delta_2)\left(9 - 39x_3^0 + 63\left(x_3^0\right)^2 - 49\left(x_3^0\right)^3\right)}{864} \tag{27}$$

We first notice that (27) does *not* depend on δ_3. To evaluate $sign\left(\phi^0(x_3^0, \delta)\right)$ we only need to get estimates of δ_1 and δ_2, the linear coefficients which measure the responsiveness of the equilibrium values of $x_i, i = 1, 2$ to small changes in λ. We do so by setting $\lim_{\lambda \to 0} \frac{d}{d\lambda} \tilde{D}(x, \lambda)\Big|_{\tilde{x}(\lambda,\delta)} = 0$, and solving for $\left\{\delta_1, \delta_2, x_3^0\right\}$. There are two alternative set of solutions, each of them corresponds to each of the restpoints. In particular:

$$\check{\delta}_1^0 = \check{\delta}_2^0 = \frac{23 - 49\beta - 7\sqrt{1 - \beta(46 - 49\beta)}}{8}$$

$$\hat{\delta}_1^0 = \hat{\delta}_2^0 = \frac{23 - 49\beta + 7\sqrt{1 - \beta(46 - 49\beta)}}{8}$$

We evaluate the numerator of (27) for both sets of solutions, obtaining the following expressions:

$$\check{\phi}(\beta) = \frac{3\left(-7 + 322\,\beta - 343\,\beta^2 + (49\,\beta - 23)\sqrt{1 - 46\,\beta + 49\,\beta^2}\right)}{10} \tag{28}$$

$$\hat{\phi}(\beta) = \frac{2863 - 147476\,\beta + 882882\,\beta^2 - 1546244\,\beta^3 + 823543\,\beta^4 + k\sqrt{146\,\beta + 49\,\beta^2}}{1000} \tag{29}$$

with $k = \left(3887 - 60123\beta + 165669\,\beta^2 - 117649\,\beta^3\right)$.

Both $\check{\phi}^0(\beta)$ and $\hat{\phi}^0(\beta)$ are plotted in Figure 5. As the diagram shows, $\check{\phi}^0(\beta)$ is always negative in the domain $\left[0, \frac{23-4\sqrt{30}}{49}\right]$, whereas $\hat{\phi}^0(\beta)$ is not. In consequence, $\check{x}^0$ is asymptotically stable whereas $\hat{x}^0$ is not.

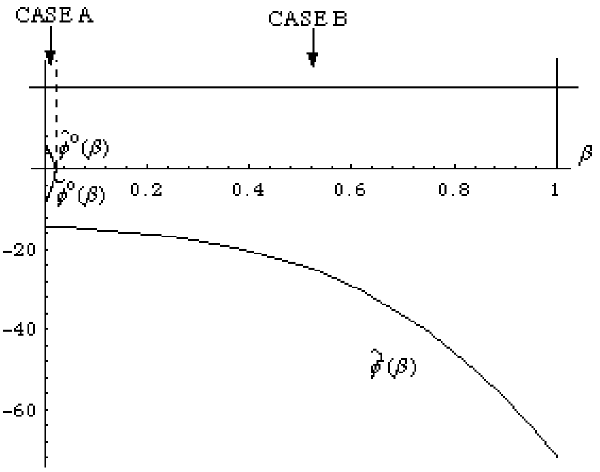

Figure 5
Asymptotic stability of the dynamic with drift

We now move on to CASE 1. Here we have a unique rest point, which we call $\hat{x}^1 \equiv \left(1, 1, \hat{x}_3^1\right)$.The eigenvalues of the unperturbed dynamics are as follows:$\left\{0, \frac{1-2x_3}{3}, \frac{1-2x_3}{3}\right\}$. As in CASE 0, there are two (identical) negative eigenvalues (given that $x_3 > \frac{1}{2}$), and the remaining eigenvalue equal to zero. By analogy with CASE 0, we define $\tilde{x}(\lambda) \equiv \left(1 - \delta_1\lambda, 1 - \delta_2\lambda, x_3^0 + \delta_3\lambda\right)$and solve $\lim_{\lambda\to 0} \frac{d}{d\lambda} \tilde{D}(x, \lambda)\Big|_{\tilde{x}(\lambda,\delta)} = 0$ to get estimates of δ. The unique feasible solution (corresponding to the unique limiting equilibrium), takes the following form:

$$\hat{\delta}_1^1 = \hat{\delta}_2^1 = \frac{3\left(2 - 4\,\beta_3 + \sqrt{9 - 16\,\beta + 16\,\beta^2}\right)}{2}$$

The function corresponding to (28-29) takes now the following form:

$$\hat{\phi}^1(\beta) = \frac{24\ (-\alpha + (2 - 4\beta)\sqrt{\alpha})}{5}$$

with $\alpha = 9 - 16\beta$. The function $\hat{\phi}^1(\beta)$ is also plotted in Figure 5. As the diagram shows, $\hat{\phi}^1(\beta)$ stays negative $\forall\beta \in (0, 1)$. Thus, $\hat{x}^1$ is asymptotically stable under any drift configuration. ■

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