LONG-LIVED ASSETS, INCOMPLETE MARKETS, AND OPTIMALITY*

Subir Chattopadhyay

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Correspondence: Universidad de Alicante, Departamento de Fundamentos de Análisis Económico, Campus San Vicente del Raspeig, s/n, 03071 Alicante (Spain). Tel.: +34 96 590 3400 x 3224 / Fax: +34 96 590 3898 / E-mail: subir@merlin.fae.ua.es.

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A B S T R A C T

We consider general OLG economies under uncertainty, with dividend paying assets of infinite maturity and money, and in which one good is available for consumption. We study the optimality properties of equilibria when asset markets are allowed to be sequentially incomplete. We show that if equilibrium in asset markets has to be restored once an intervention has been made, then all non-monetary competitive equilibria are locally constrained optimal. We proceed to a notion of optimality which allows asset markets to not clear and provide a complete characterization of those equilibria that are optimal in terms of the prices and dividends of assets of infinite maturity and feasible portfolio reassignments. Results for various special cases follow; in particular, we show that if dividends are non-negative and assets are freely disposable then every non-monetary equilibrium allocation is optimal. Other results shed light on the role played by money vis-a-vis other assets of infinite maturity in determining the optimality properties of equilibria when markets are sequentially complete/incomplete and free disposal of assets is or is not allowed.

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1. INTRODUCTION

We consider a general class of pure exchange overlapping generations (OLG) economies under uncertainty in which one good is available for consumption in each period and intertemporal transfers are carried out by trading in assets. Our objective is to analyze the role of long-lived assets (dividend paying and money), which can have negative payoffs and need not be freely disposable, in ensuring optimality of competitive allocations in economies in which asset markets are, potentially, not even sequentially complete, i.e., when agents are unable to insure against all sources of uncertainty affecting them after their birth.

The motivation for our exercise comes from many sources. The fact that competitive OLG economies need not allocate resources efficiently is well known and is aptly summarized in the "chocolate" parable where more can be consumed today by making every young agent give one chocolate to a contemporaneously old agent. This inefficiency occurs even in the most basic deterministic model and has been characterized in OLG economies under certainty and also some particular instances of uncertainty.[1] That market incompleteness has important consequences for the welfare properties of equilibria is also well known. In two-period models with one consumption good and an incomplete set of assets, equilibrium allocations are known to be constrained optimal in the sense that there is no alternative allocation which is Pareto improving and can be induced using the same asset structure. The situation changes dramatically when one considers two or more consumption goods since, with missing markets, the market solution fails to use even the existing markets in an efficient manner in a strong sense.[2] It seems natural to ask how, if at all, the chocolate problem interacts with incompleteness in an OLG economy under uncertainty. As to the potential role of long-lived dividend paying assets, Wilson (1981) showed that in an economy with contingent commodities, the presence of a non-negligible individual, who can be interpreted as an individual who owns a dividend paying asset of infinite maturity, forces the value of the aggregate endowment to be finite which in turn forces the equilibrium to be Pareto optimal; Scheinkman (1980) made a similar point. Santos and Woodford (1997) extended the result to a multi-good model with sequentially complete asset markets when dividends are non-negative and assets are freely disposable.[3] Our interest in looking at OLG models with long-lived assets and sequentially incomplete markets stems from these earlier contributions. At a more applied level, the model that we consider is the canonical framework for the analysis of intertemporal risk sharing and

[1] See Geanakoplos and Polemarchakis (1991) for the case in which markets are complete, and Chattopadhyay and Gottardi (1999) for the case in which markets are sequentially complete. Cass (1972) provided a characterization of efficient competitive production paths in sequence economies and is the seminal work in the area.

[2] Geanakoplos and Polemarchakis (1986) showed that in two-period general equilibrium models with incomplete markets, equilibrium allocations are generically constrained suboptimal provided that two or more goods are traded. The notion of optimality used is that of Pareto comparisons between allocations that can be induced as equilibria with trade in spot markets but with portfolios that are assigned by the planner.

[3] Allen and Gale (1997) studied a model with one good and one agent in every period, so that any equilibrium allocation obtained with a sequentially complete market can be induced with just one asset, and showed that the existence of a dividend paying asset implies that the resulting stationary equilibrium allocation is optimal.

the results obtained provide insights in a wide range of applications; an evident one is the problem of identifying desirable features of any reform of social security systems.

Our model is similar to that of Santos and Woodford (1997), though less general in that agents live for two periods and that there is only one consumption good, and more general in that assets are unrestricted in terms of the sign of the dividends and they need not be freely disposable.[4] We consider the one-good model since one expects that in a multi-good model the results will be very different as market incompleteness will have a dominating effect. If one thinks of the assets as reduced forms of the right to use a technology, where buying an asset today requires a commitment to investing a certain amount in every event tomorrow, then the gross return is random and could be low enough to induce a negative net dividend tomorrow and this justifies the consideration of a general dividend process; Brock's example reminds us that free disposability of assets can be an essential restriction. The set of assets available is allowed to be sparse so that agents might not be able to overcome the fact that they face a sequence of budget constraints, i.e., markets are allowed to be sequentially incomplete. Fiat money and sequentially complete markets (and hence a model without uncertainty) are special cases of our model. In order to reduce notation we assume that the same set of assets is available at different nodes in a date-event tree.

The model covers three broad categories of long-lived assets: the case where the long-lived asset can be freely disposed and pays a non-negative dividend which is sometimes positive, land being the standard example; the case where the long-lived asset never pays a dividend, i.e., fiat money; and cases where free disposal fails and/or the dividend is not required to be non-negative. Our results vary according to the case we consider. Also, we assume that the endowment of each of the assets is independent of time and the realization of uncertainty, i.e., a constant. For dividend paying assets it is reasonable to assume that the endowment never changes (one can redo the analysis allowing for endowment growth). However, when dealing with money it is not always reasonable to assume that the endowment is constant but, to economize on notation and inessential details related to the specification of how newly created money enters into the budget set of each agent, we analyze a model with a constant money endowment; hence, when dealing with situations in which it not reasonable to assume a constant money endowment, we restrict attention to non-monetary equilibria.

We proceed by specifying a notion of optimality. The criterion called *conditional Pareto optimality* (CPO), proposed by Muench (1977), is one in which agents' welfare is evaluated by conditioning their utility on the event at the date of their birth; agents are thus distinguished not only according to their type and their date of birth but also according to the event at that date. CPO is particularly suitable when markets are sequentially complete. However, it is known that when markets fail to be sequentially complete, the equilibrium allocation is typically not CPO since different agents will have different marginal valuations of income in different states. So we need to weaken the notion further. We do so by requiring that a dominating allocation be obtainable via existing markets in the sense that consumption when young is induced by using transfers (in a

[4]Santos and Woodford (1997) noted the importance of imposing free disposal of assets and non-negativity of dividends in order to derive their results on the non-existence of bubbles; the restriction is essential since an example due to Brock (1990) showed how a standard economy can have a stationary equilibrium in which the price of the asset is negative.

one good framework this is equivalent to allocating consumption when young directly) while consumption when old can only be allocated indirectly by specifying a portfolio reassignment which then induces income and hence consumption according to the returns of the various assets. As for the specification of the returns of the assets, we consider two possibilities: in the first, in the spirit of Geanakoplos and Polemarchakis (1986), asset prices are allowed to adjust to post-intervention equilibrium levels and asset markets are required to clear, while in the second the asset reassignment is allowed to allocate less of the asset in aggregate than the asset endowment (but non-negative amounts) and asset prices are kept fixed.[5] Of course, if markets happen to be sequentially complete, the criteria proposed reduce to CPO.

Our first result, Theorem 1, is that when equilibrium needs to be restored after intervention, all non-monetary equilibria are locally constrained optimal. This result is quite striking and does not require the use of arguments which refer to the infinite horizon; it follows from the fact that any feasible portfolio reassignment induces an average change in utility which is zero upto first order. It makes essential use of the fact that the endowment of the assets is assumed to never change and that asset markets are required to clear with equality.[6] So when the notion of optimality requires asset prices to adjust, the most basic feature of the OLG model, the chocolates problem, disappears in non-monetary economies and does not discriminate between economies with positive dividends and free disposal of assets against the rest. This makes it natural to consider alternative notions in which intertemporal inefficiency can occur. Essentially, one must allow the net transfer between generations to be non-zero and then check to see whether a CPO improving allocation can be constructed. That is precisely what our second criterion permits by not requiring asset markets to clear exactly, and keeping asset prices fixed. Our main results, Theorems 2 and 3, provide necessary and sufficient conditions for a competitive allocation to be constrained optimal according to the second criterion. The results say that a competitive allocation is not optimal if and only if there is a set of histories, and a change in the asset allocation, such that, on every history in the set, the discounted value of the change in the asset reallocation is positive and uniformly bounded. The discount factors used are event dependent (random discounting) and defined in terms of the returns at each event to the portfolio which is defined by the asset reallocation.

The result on sufficiency, Theorem 2, can be modified and generalized to incorporate a notion of constrained optimality which combines the two notions that we propose by al-

[5]When the asset in question is money, the asset reassignment is also allowed to allocate more of the asset than the endowment. This corresponds to being able to create money since it can be done costlessly (by definition money never pays a dividend) and allows us to incorporate monetary policy.

[6]The formal argument allows money to be one of the assets but, as we noted earlier, it is not reasonable to assume that the endowment of money is constant and here such an assumption plays a crucial role in generating the result; hence, we restrict attention to non-monetary equilibria.

Cass, Green, and Spear (1992) showed that there are no locally improving stationary asset redistributions which improve over the stationary monetary equilibria of a one-good stochastic OLG economy with incomplete asset markets, no dividend paying long-lived asset, and freely disposable money, when the price of money is allowed to adjust.

Demange (2000) showed that constrained optimality obtains when asset prices are allowed to adjust without the restriction to local price changes but assuming free disposal of assets and dividends which are uniformly bounded away from zero; portfolio reassignments are, however, restricted to be local.

lowing the asset market not to clear exactly while at the same time allowing asset prices to undergo local changes; the precise statement and the argument are simple reformulations of the statement and proof of Theorem 2 and are, therefore, not included.

We provide results for various special cases with only one long-lived asset including the case in which the equilibrium is stationary; these results follow easily from the characterization and shed light on the range of situations covered by the result. An important implication of the main result is that, in a non-monetary economy, an equilibrium allocation with free disposal of assets and non-negative dividends, is necessarily constrained optimal.[7] Another implication is that if money is the only long-lived asset then all stationary monetary equilibria are constrained optimal.[8]

In terms of generalizations, one conjectures that in an OLG model with more than one good and sequentially incomplete markets, the equilibrium allocation will generically fail to be constrained optimal with the first criterion of optimality; this should follow by embedding the Geanakoplos and Polemarchakis (1986) economy in a multi-good OLG model but is not trivial as one needs to follow the price effects along different paths. So there is every reason to believe that our Theorem 1, the local constrained optimality result, is an artifact of the one-good structure. As for our characterization result with the second criterion, it seems clear that it can be extended to the case of many goods.

The rest of the paper is structured as follows. Section 2 presents the model and notation. In Section 3 we present a definition of constrained optimality where asset prices are allowed to readjust and state the local constrained optimality result. In Section 4a we present our second definition of optimality where the asset market clearing condition takes a weaker form. In Section 4b we state and discuss our main result. Various special cases of interest are discussed in Section 4c which can be read after glancing through Section 4b. The proofs of the three theorems are relegated to Section 5.

[7]When we apply the generalized version of Theorem 2 to non-monetary economies with non-negative dividends and free disposal of assets, we obtain a generalization of the result in Demange (2000) without the need to assume a positive uniform lower bound on dividends.

[8]Gottardi (1996) considers stationary redistributions, starting from a stationary monetary equilibrium of a one-good stochastic OLG economy with incomplete asset markets and money. He ignores the welfare of the initial old. His results are very partial.

2. THE MODEL

We consider a general pure exchange overlapping generations (OLG) economy under uncertainty where only one consumption good is traded and agents live for two periods. The economy evolves in discrete time with uncertainty in the environment described by an abstract date-event tree as in Chapter 7 of Debreu (1959). We turn to a formal description of the model and the notation used.[9]

Time is discrete and dates are denoted $t = 1, 2, 3, \cdots$.

Let $\mathcal{S} = \{1, 2, \cdots, S\}$ be a *state space*, the set from which a state is chosen at each date; so $\#\mathcal{S} = S$. The structure of the date-event *tree* induced by all possible realizations of states from an initial date $t = 0$ is as follows. The *root* of the tree is $\sigma_0 \in \mathcal{S}$; Σ_t is the set of *nodes at date* t where we set $\Sigma_1 := \{\sigma_0\} \times \mathcal{S}$, and iteratively set $\Sigma_t := \Sigma_{t-1} \times \mathcal{S}$ for $t = 2, 3, \cdots$. Define $\Sigma := \cup_{t \geq 1} \Sigma_t$ and $\Gamma := \{\sigma_0\} \cup \Sigma$. Elements of Γ are called *nodes* (to be thought of as the "date-events" or simply "events"), and a generic node is denoted σ. Given a node $\sigma \in \Sigma$, $t(\sigma)$ denotes the value of t at which $\sigma \in \Sigma_t$. Clearly, a node $\sigma \in \Sigma_t$ is nothing but a string of states $(\sigma_0, s_1, s_2, \cdots, s_t)$ where $s_\tau \in \mathcal{S}$ denotes the state realized at date τ, $\tau = 1, \cdots, t$ (σ_0 is the realization at the initial date). It follows that the *predecessor* of a node $\sigma \in \Sigma_t$ is uniquely defined and it will be denoted by σ_{-1}, an element of Σ_{t-1}; the set of immediate successor nodes of a node σ is denoted σ^+. A *path* is defined by an infinite sequence of nodes $\{\sigma_t\}_{t \geq 1}$ such that, for all $t \geq 1$, σ_t is the predecessor of σ_{t+1}; σ^∞ will denote a path.

One commodity is available for consumption at each node $\sigma \in \Sigma$.

At each node $\sigma \in \Sigma$, a generation of agents, denoted $\mathcal{H}$, is born, where $H := \#\mathcal{H}$. A member of generation σ of type $h \in \mathcal{H}$ is denoted (σ, h). In addition, there is a set, $\mathcal{H}$, of H agents who enter the economy at each node $\sigma \in \Sigma_1$ at date 1; they constitute the generation of the "initial old", and are denoted $(\sigma; o, h)$, where $\sigma \in \Sigma_1$. The set of agents is denoted $\mathcal{I}$ where $\mathcal{I} := (\Sigma_1 \times \{o\} \times \mathcal{H}) \cup (\Sigma \times \mathcal{H})$. $\sigma(i)$ identifies the node at which i was born; so $\sigma(i) \in \Sigma$ with $\sigma(i) \in \Sigma_1$ for the initial old.

For $i \in \mathcal{I}$, Σ_i denotes the set of nodes at which she is *alive*. We assume that the initial old live only at the node at which they enter the economy, $\Sigma_i = \{\sigma\}$ if $i = (\sigma; o, h)$, so $\#\Sigma_i = 1$, while every other agent is alive at the node of birth and at every node which is an immediate successor of the node at which she was born, $\Sigma_i = \{\sigma(i), \sigma^+\}$ if $i \in \Sigma \times \mathcal{H}$, so $\#\Sigma_i = 1 + S$. The set of agents alive at a node σ is denoted $\mathcal{I}(\sigma)$ where $\mathcal{I}(\sigma) := \{i \in \mathcal{I} : \sigma \in \Sigma_i\}$.

Each agent $i \in \mathcal{I}$ is described by a consumption set, $X_i \subset R^{\#\Sigma_i}$, an endowment vector, $\omega_i = ((\omega_i(\sigma))_{\sigma \in \Sigma_i}) \in X_i$, and a utility function, $u_i : X_i \to R$.

There is a set $\mathcal{J} = \{1, 2, \cdots, J\}$ of one-period lived *short maturity assets*, with payoffs (per unit) in the commodity described by the function $s : \Sigma \to R^J$. There is also a set of *dividend paying assets* of infinite maturity, denoted $\mathcal{K} = \{1, 2, \cdots, K\}$, with payoffs (per unit) in the commodity specified by the function $d : \Sigma \to R^K$. Finally, we consider an asset of infinite maturity called *fiat money*, denoted m. Fiat money is characterized by the fact that it *never* pays a dividend. The set of assets available is $\mathcal{A} := \mathcal{J} \cup \mathcal{K} \cup \{m\}$.

Only the initial old are endowed with these assets and their endowments are denoted $\omega^a(i)$, $a \in \mathcal{A}$, $i \in \Sigma_1 \times \{o\} \times \mathcal{H}$. A negative endowment of an asset indicates a pre-existing

[9]We extend the notation developed in Chattopadhyay and Gottardi (1999).

debt.

We assume that, for every asset, the total endowment is independent of the node at date $t = 1$. Define $\omega^a := \sum_{i \in \Sigma_1 \times \{o\} \times \mathcal{H}} \omega^a(i)$ for $a \in \mathcal{A}$, the total endowment of each asset.

We introduce a notational convention. For $z \in R^{J+K+1}$ we write $z = (z_{\rm s}, z_{\rm d}, z_{\rm m})$ where $z_{\rm s} := (z^1, z^2, \cdots, z^J)$, $z_{\rm d} := (z^1, z^2, \cdots, z^K)$, and $z_{\rm m} := z^m$. So $\omega = (\omega_{\rm s}, \omega_{\rm d}, \omega_{\rm m})$, $\omega \in R^{J+K+1}$, gives the total endowment of each asset.

An asset is an *inside asset* if its total endowment is zero, i.e., it is in zero net supply.

Denoting $\omega(\sigma)$ the total endowment of commodities at node σ, we have:

$\omega(\sigma) := \sum_{i \in \mathcal{I}(\sigma)} \omega_i(\sigma) + \omega_{\rm s} \cdot s(\sigma) + \omega_{\rm d} \cdot d(\sigma) \quad$ for $\sigma \in \Sigma$.

We assume

ASSUMPTION 1:

(i) $1 \leq H < \infty, \; 1 \leq S < \infty, \; 0 \leq \#\mathcal{A} < \infty$.

(iia) For all $i \in \Sigma_1 \times \{o\} \times \mathcal{H}$, $\omega^a(i) \in R$ for all $a \in \mathcal{A}$, with $\omega_{\rm s} = \underline{0}$ and with $\omega_{\rm d} \geq \underline{0}$.[10]

(iib) For all $i \in \Sigma_1 \times \{o\} \times \mathcal{H}$, $\Sigma_i = \{\sigma(i)\}$, so $\#\Sigma_i = 1$,
for all $i \in \Sigma \times \mathcal{H}$, $\Sigma_i = \{\sigma(i), \sigma(i)^+\}$, so $\#\Sigma_i = 1 + S$.

(iic) For all $i \in \mathcal{I}$, $X_i = R_+^{\#\Sigma_i}$, $\omega_i(\sigma(i)) \in R_+/\{\underline{0}\}$ and $(\, (\omega_i(\sigma'))_{\sigma' \in \Sigma_i/\sigma(i)}) \in R_+^{(\#\Sigma_i - 1)}/\{\underline{0}\}$, $u_i : X_i \to R$ is C^2, strictly monotone, and differentiably strictly quasi-concave.

(iii) For all $\sigma \in \Sigma$, $\omega(\sigma) \in R_{++}$.

We have imposed monotonicity and a differentiable form of strict quasi-concavity of utility functions, and the condition that every commodity is available in a strictly positive quantity. We have also imposed the condition that the short maturity assets are inside assets, a natural restriction, and that the dividend paying assets are available in non-negative quantities; we have not imposed any restriction on the sign of the total amount of money in the economy.

A consumption plan for agent i is denoted $x_i = ((x_i(\sigma))_{\sigma \in \Sigma_i}) \in X_i$.

The next definition is standard and specifies the set of feasible allocations.

DEFINITION 1: A *feasible allocation* x is given by an array $((x_i)_{i \in \mathcal{I}})$ such that
$x_i \in X_i$ for all $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}(\sigma)} x_i(\sigma) \leq \omega(\sigma)$ for all $\sigma \in \Sigma$.

We now introduce the notion of equilibrium. Given the nature of the problem, it is easy to see that the price of the commodity can be normalized to 1 at every node. Asset prices are denoted $q = (q_{\rm s}, q_{\rm d}, q_{\rm m}) \in R^{J+K+1}$. The vector valued function specifying all asset prices is denoted $q : \Sigma \to R^{J+K+1}$.

Given q, the vector of *asset returns* $r(\sigma) := (s(\sigma), q_{\rm d}(\sigma) + d(\sigma), q_{\rm m}(\sigma))$ can be specified. This induces $r : \Sigma \to R^{J+K+1}$, the vector valued function specifying all asset returns. Evidently, r is a function of q, a fact which we supress in the notation so as not to clutter it.

Agents choose their asset holdings and consumption levels at a node using their endowments at that node and returns on assets carried over from the previous node. For $i \in \mathcal{I}$, let $\theta(i) \in R^{J+K+1}$, denote an agent's portfolio. Let $\theta = (\, (\theta(i))_{i \in \mathcal{I}})$ denote the array which specifies asset holdings for all agents.

[10]The notation $\underline{0}$ denotes the vector $(0, 0, \cdots, 0)$ in a space of conformal dimension.

We can now define a competitive equilibrium. It requires that the allocation of commodities is feasible, that the allocation of assets clears asset markets *exactly*, and that agents optimize subject to a multiplicity of budget constraints.

DEFINITION 2 (CE-S): $(x^*, \theta^*, q^*, r^*)$ is a *competitive equilibrium with a sequence of markets (CE-S)* if:
(i) x^* is a feasible allocation;
(ii) for all $\sigma \in \Sigma$, $\sum_{h \in \mathcal{H}} \theta^*(\sigma, h) = \omega$;
(iii) for all $\sigma \in \Sigma$, $r^*(\sigma) := (s(\sigma), q^*_{\mathrm{d}}(\sigma) + d(\sigma), q^*_{\mathrm{m}}(\sigma))$;
(iv) for all $i \in \Sigma_1 \times \{o\} \times \mathcal{H}$, $\theta^*(i) = \omega(i)$ and $x^*_i = \omega_i + \theta^*(i) \cdot r(\sigma(i))$;
(v) for all $i \in \Sigma \times \mathcal{H}$,
(a) $x^*_i(\sigma(i)) + \theta^*(i) \cdot q^*(\sigma(i)) \leq \omega_i(\sigma(i))$, $x^*_i(\sigma) \leq \omega_i(\sigma) + \theta^*(i) \cdot r^*(\sigma)$ for all $\sigma \in \sigma(i)^+$;
(b) if $u_i(x) > u_i(x^*_i)$ for $x \in X_i$, then
$x(\sigma(i)) + \theta^*(i) \cdot q^*(\sigma(i)) > \omega_i(\sigma(i))$ or $x(\sigma) > \omega_i(\sigma) + \theta^*(i) \cdot r^*(\sigma)$ for some $\sigma \in \sigma(i)^+$.

REMARK 1: We have imposed the condition that all asset markets must clear exactly, i.e., we have not allowed for *free disposal* of assets. We will also treat equilibria with free disposal of assets in which the obvious changes are made to (ii) in Definition 2 (market clearing with a weak inequality, non-negativity of asset prices, and complementary slackness) and non-negativity of the dividends is imposed as an additional condition. Also, the definition of equilibrium applies even when markets are *sequentially complete*, that is, if at every node the returns from the $J + K + 1$ assets span R^S.

REMARK 2: The model developed above appears to be special to the extent that (i) the number of agents born at each node, (ii) the number of nodes that succeed any given node, and (iii) the number of assets of each type and their total endowment, are all taken to be independent of the node. This is without loss of generality as all our results go through with a more general specification (except Theorem 1 which requires a constant asset endowment) but at the cost of more notation. Furthermore, the notation and definitions extend in a straighforward manner to the case in which L consumption goods are traded at each node and asset payoffs are denominated in the first good, and the case with a more general demographic structure.

REMARK 3: The optimization problem solved by an agent i can be written as

$\max_{x,\theta} u_i(x, (\omega_i(\sigma) + \theta \cdot r^*(\sigma))_{\sigma \in \sigma(i)^+})$

subject to : $x + \theta \cdot q^*(\sigma) \leq \omega_i(\sigma(i))$.

So each agent, effectively, solves an optimization problem with a single budget constraint and will meet the constraint with equality. This property leads to the constrained optimality of all equilibria in two-period economies in which one consumption good is traded in each state.

We close the section by introducing a definition of optimality which is used in the case where markets are sequentially complete. Applying the notion of Pareto efficiency to the economy described above, where agents are distinguished by the event at their birth, yields the criterion of conditional Pareto optimality, first proposed by Muench (1977):

DEFINITION 3 (CPO): Let x be a feasible allocation. x is *conditionally Pareto optimal (CPO)* if there does not exist another feasible allocation $\widehat{x}$ such that
(i) for all $i \in \mathcal{I}$, $u_i(\widehat{x}_i) \geq u_i(x_i)$,
(ii) for some $i' \in \mathcal{I}$, $u_{i'}(\widehat{x}_{i'}) > u_{i'}(x_{i'})$.

A CPO allocation requires that the risk borne in the second period of the agents' lives be allocated optimally.[11]

This completes the description of the model. Stationary equilibria of stationary economies constitute a special case which will be briefly developed in Section 4c.

3. A LOCAL CONSTRAINED OPTIMALITY RESULT

In this section we gauge the efficiency properties of competitive equilibria, when intertemporal transfers are carried out via trades in asset markets, by carrying out CPO comparisons between a given equilibrium allocation and alternative allocations with the proviso that the alternatives must be induceable as equilibria. We show that there are no local interventions which can CPO improve under a reasonable definition of post-intervention equilibria.

Let the planner be constrained to using the existing assets in reallocating resources. Let the planner directly assign an asset portfolio and a transfer when young after which agents are allowed to trade in the market for assets where an equilibrium is established; this allows the determination of consumption when young and old. Since only one consumption good is traded, consumption when old is completely determined by the post-intervention equilibrium portfolio and asset prices (and the endowment); similarly, consumption when young is determined by the transfer and the post-intervention equilibrium portfolio. The portfolio chosen will depend on the finer aspects of the intervention, i.e., the restrictions that are placed on the set of markets in which the agent can trade; for example, the agent could have unrestricted access to all asset markets or she could be prevented from trading in the markets for those assets in which the planner specifies an

[11] For a complete characterization of those competitive equilibrium allocations that are CPO when markets are sequentially complete and trade takes place in contingent commodities with many goods at each date, see Chattopadhyay and Gottardi (1999, Theorems 1 and 2).

assignment for them.[12] However, we do not need to specify the details of the intervention since we can prove our result with a reduced form of interventions in which the following ingredients are specified in the post-intervention equilibrium: consumption when young, $\widehat{x}_i(\sigma(i))$, the portfolio, $\widehat{\theta}(i)$, and asset prices, $\widehat{q}(\sigma)$, for all agents and all nodes, with the proviso that asset markets clear and the allocation of commodities is feasible. Formally

DEFINITION 4: $(\widehat{x}, \widehat{\theta}, \widehat{q}, \widehat{r})$ is *compatible with a post-intervention competitive equilibrium with a sequence of markets (PI-CE-S)* if:
(i) $\widehat{x}$ is a feasible allocation;
(iia) for all $\sigma \in \Sigma_1$, $\sum_{h\in\mathcal{H}} \widehat{\theta}(\sigma; o, h) = \omega$;
(iib) for all $\sigma \in \Sigma$, $\sum_{h\in\mathcal{H}} \widehat{\theta}(\sigma, h) = \omega$;
(iii) for all $\sigma \in \Sigma$, $\widehat{r}(\sigma) := (s(\sigma), \widehat{q}_{\mathrm{d}}(\sigma) + d(\sigma), \widehat{q}_{\mathrm{m}}(\sigma))$;
(iva) for all $i \in \Sigma_1 \times \{o\} \times \mathcal{H}$, $\widehat{x}_i = \omega_i + \widehat{\theta}(i) \cdot \widehat{r}(\sigma(i))$;
(ivb) for all $i \in \Sigma \times \mathcal{H}$, $\widehat{x}_i(\sigma) = \omega_i(\sigma) + \widehat{\theta}(i) \cdot \widehat{r}(\sigma) \quad$ for all $\sigma \in \sigma(i)^+$.

DEFINITION 5: Let $(x^*, \theta^*, q^*, r^*)$ be a *CE-S*. x^* is *constrained CPO* if there is no $\widehat{x}$ which is a CPO improvement over x^* and there exist $\widehat{\theta}$, $\widehat{q}$ and $\widehat{r}$ such that $(\widehat{x}, \widehat{\theta}, \widehat{q}, \widehat{r})$ is PI-CE-S.

We can now state our result which shows that if attention is restricted to interventions that induce equilibria that are nearby, then *every* non-monetary equilibrium allocation is constrained CPO, i.e., a form of optimality which takes into account the incompleteness of the market reigns even though asset prices are allowed to adjust to clear markets. The result is obtained by using the agents' first order conditions for choosing the optimal portfolio to write the change in utility induced by a move from one equilibrium to a nearby one in terms of the marginal utility at the initial equilibrium and the prices of the assets. The restriction to nearby equilibria allows us to ignore changes in the derivative of the utility function of an agent when the allocation changes.[13]

THEOREM 1: *Let $(x^*, \theta^*, q^*, r^*)$ be a non-monetary competitive equilibrium with a sequence of markets (CE-S) and suppose Assumption 1 holds. Assume that $x_i^* \in R_{++}^{1+S}$, for all $i \in \Sigma \times \mathcal{H}$. There exists an $\epsilon > 0$ such that x^* is constrained CPO in an ϵ-neigbourhood of $(x^*, \theta^*, q^*, r^*)$ in the sup norm.*

For the result to go through it is essential that asset markets clear exactly and that the aggregate endowment of assets not change across nodes. The latter restriction is reasonable for non-monetary assets but not for fiat money whose endowment can be changed costlessly. The same result holds in the uninteresting case of monetary equilibria with a fixed endowment of money; hence, we state the result only for non-monetary equilibria.

4a. A DIFFERENT DEFINITION OF CONSTRAINED OPTIMALITY

[12] For the two period model, Geanakoplos and Polemarchakis (1986) consider the case in which all the asset markets are closed for all agents.

[13] To our knowledge, most analyses of constrained optimality are local; an example due to Hart (1975) is an exception.

The result in the previous section is quite disturbing since it applies even when markets are sequentially complete. It appears to indicate that the Samuelson problem of passing chocolates from young to old vanishes when one considers the more realistic structure in which intertemporal transfers are carried out via trades in non-monetary assets instead of trade in an ethereal market for contingent commodities. Alternatively, it indicates that the appropriate definition of constrained optimality is a different one. We propose a possible alternative definition which shows that the Samuelson problem continues to affect the non-monetary equilibria of the model.

The result in Theorem 1 depends crucially on the fact that all asset markets clear exactly. Such a restriction makes sense when the asset in question is in zero net supply; not imposing the restriction on assets in positive net supply is a possibility. So consider an intervention in which the planner is allowed to “confiscate” a part of the asset endowment. A reassignment of consumption when young and of the assets is carried out and post-intervention trade in assets is not permitted; as a consequence, asset prices are not allowed to change. This induces an allocation of commodities which is required to be feasible.

More specifically, feasibility of the asset reassignment is imposed in the following form: the aggregate holding of every non-monetary asset is required to be between zero and the endowment of the asset in question; in particular, the aggregate holding of every inside asset is required to be zero. The aggregate holding of money is allowed to exceed or fall short of the endowment in absolute value to accomodate monetary policy, though only in a rudimentary manner. So the planner is not allowed to create or destroy the endowment of assets (except money) but can decide to not allocate the entire stock of those assets which are in positive net supply; this requirement seems to be reasonable.

The change in consumption when young is induced by using taxes and subsidies while the change in consumption when old is induced by the change in the portfolio of existing assets. The allocation of commodities must be feasible; so, even though the planner can decide to assign less of some dividend paying asset than its endowment, this possibility does not affect the aggregate feasibility constraint. This is particularly important in the case of an asset whose total return is negative at a node, so that it reduces the endowment of the economy, as in such a case the planner would want to “freely dispose” of the asset thereby preventing the loss of resources; as we shall see, it is this feature which drives the characterization results that we obtain.

DEFINITION 6: $\widehat{x}$ is *q-constrained feasible*, for $q : \Sigma \to R^{J+K+1}$ a specification of asset prices, if there exists $((\widehat{\theta}(i))_{i\in\mathcal{I}})$ such that:
(i) $\widehat{x}$ is a feasible allocation;
(iia) for all $\sigma \in \Sigma_1$,
$\underline{0} \leq \sum_{h\in\mathcal{H}} \widehat{\theta}^a(\sigma; o, h) \leq \omega^a$, $a \in \mathcal{J} \cup \mathcal{K}$, $|\sum_{h\in\mathcal{H}} \widehat{\theta}_{\mathrm{m}}(\sigma; o, h) - \omega_{\mathrm{m}}| \leq H \cdot \Delta M$, $\Delta M \geq 0$;
(iib) for all $\sigma \in \Sigma$,
$\underline{0} \leq \sum_{h\in\mathcal{H}} \widehat{\theta}^a(\sigma, h) \leq \omega^a$, $a \in \mathcal{J} \cup \mathcal{K}$, $|\sum_{h\in\mathcal{H}} \widehat{\theta}_{\mathrm{m}}(\sigma, h) - \omega_{\mathrm{m}}| \leq H \cdot \Delta M$, $\Delta M \geq 0$;
(iii) for all $\sigma \in \Sigma$, $\widehat{r}(\sigma) := (s(\sigma), \widehat{q}_{\mathrm{d}}(\sigma) + d(\sigma), \widehat{q}_{\mathrm{m}}(\sigma))$;
(iva) for all $i \in \Sigma_1 \times \{o\} \times \mathcal{H}$, $\widehat{x}_i = \omega_i + \widehat{\theta}(i) \cdot \widehat{r}(\sigma(i))$;
(ivb) for all $i \in \Sigma \times \mathcal{H}$, $\widehat{x}_i(\sigma) = \omega_i(\sigma) + \widehat{\theta}(i) \cdot \widehat{r}(\sigma)$ for all $\sigma \in \sigma(i)^+$.

DEFINITION 7: An allocation x is *q-constrained CPO* if there is no $\widehat{x}$ which is q-constrained feasible and a CPO improvement over x.

The fact that asset prices are held fixed in Definition 6 may seem like a major drawback. However, a minor change in the definition allows us to incorporate the possibility of allowing local price changes in response to the changed situation in asset demand and supply.

This completes the discussion of the notion of constrained optimality that we shall use; its implications are the subject of the next two sub-sections.

4b. A COMPLETE CHARACTERIZATION RESULT

In this sub-section we state our main result; we provide a complete characterization, in terms of asset prices and dividends, of those competitive equilibrium allocations that are optimal under a criterion that takes into account the fact that markets are incomplete and that assets are not freely disposable (Definition 7). The result also applies when markets are sequentially complete and/or assets are freely disposable. It uses in an essential way the fact that the asset reallocations need not meet the feasibility restriction with equality. The characterization result that we present becomes all the more interesting in the light of Theorem 1, i.e., the fact that with a notion of optimality in which asset markets clear exactly and asset prices adjust (Definition 5), non-monetary equilibria are necessarily locally constrained CPO. For notational convenience, we state the result for the case where asset prices are held fixed, as in Definitions 6 and 7, even though the result goes through when local changes to asset prices are permitted.

Our result is a criterion of the type first obtained by Cass (1972) and it is well known that a pair of curvature conditions are an essential ingredient in generating the result; for sufficiency one needs to impose a uniformity condition that the curvature of every agent's upper contour set at the competitive allocation exceeds a strictly positive number (the condition is slightly stronger than differentiable strict quasi-concavity of the utility function of every agent) while for necessity one needs to impose a finite upper bound on the curvature of every agent's upper contour set at the competitive allocation. We use $\underline{\rho}_i$ and $\bar{\rho}_i$ to denote the greatest lower bound and the least upper bound, respectively, on the curvature of an agent's upper contour set at the competitive allocation.[14]

[14]Precise statements of the definitions of the bounds on the curvature of an agent's indifference surface, and sufficient conditions under which they are well defined, are notationally heavy; for the case in which

We introduce a bit of notation. Given an equilibrium $(x^*, \theta^*, q^*, r^*)$ and a function $f : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{J+K+1}$, induce the functions $\mathcal{P}_f : \Sigma \to R$ and $\mathcal{C}_f : \Sigma \to R$

$$\mathcal{P}_f(\sigma) := f(\sigma, o) \cdot r^*(\sigma) \quad \text{for all } \sigma \in \Sigma_1 \qquad \mathcal{P}_f(\sigma) := f(\sigma_{-1}) \cdot r^*(\sigma) \quad \text{for all } \sigma \in \cup_{t \geq 2} \Sigma_t$$

$$\mathcal{C}_f(\sigma) := f(\ \sigma) \cdot q^*(\sigma) \quad \text{for all } \sigma \in \Sigma.$$

The function f should be thought of as specifying a feasible variation of the equilbrium portfolio where the initial old are explicitly taken into account; $\mathcal{P}_f$ identifies the payoff from the portfolio at different nodes, while $\mathcal{C}_f$ identifies the cost of the portfolio. Of course, there is no way to set the cost of the variation to the initial old.

We need two additional concepts. Given $\tilde{\sigma} \in \Gamma$, we define a *sub-tree* (of the tree Γ) with root $\tilde{\sigma}$, denoted $\Gamma_{\tilde{\sigma}}$, as a collection of nodes such that $\Gamma_{\tilde{\sigma}}$ itself is a tree with $\tilde{\sigma}$ as its root. A *path in the sub-tree* $\Gamma_{\tilde{\sigma}}$, denoted $\sigma^\infty(\Gamma_{\tilde{\sigma}})$, is a collection of nodes which are ordered by precedence and for $t \geq t(\tilde{\sigma})$ all the nodes are elements of the sub-tree, i.e., $\sigma^\infty(\Gamma_{\tilde{\sigma}}) \subset \{\sigma_1^\infty, \sigma_2^\infty, \cdots, \sigma_{t(\tilde{\sigma})-1}^\infty\} \cup \Gamma_{\tilde{\sigma}}$, where σ_t^∞ denotes the tth coordinate of the path.

We can now state our characterization result.[15]

THEOREM 2 (Sufficiency): *Let $(x^*, \theta^*, q^*, r^*)$ be a competitive equilibrium with a sequence of markets (CE-S) and suppose Assumption 1 holds. Assume that $x_i^* \in R_{++}^{1+S}$, for all $i \in \Sigma \times \mathcal{H}$, and that there are real numbers $\Omega > 0$ and $\underline{\rho} > 0$ such that*
(i) $\omega(\sigma) \leq \Omega$ for all nodes $\sigma \in \Sigma$, and (ii) $\underline{\rho} \leq \underline{\rho}_i$ for all $i \in \Sigma \times \mathcal{H}$.
If the equilibrium allocation is not q^*-constrained CPO then there exists a sub-tree $\Gamma_{\tilde{\sigma}}$, with $\tilde{t} := t(\tilde{\sigma}) \geq 1$, a function $\bar{\Delta}\theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{J+K+1}$, and real numbers $\bar{P} > 0$ and $B > 0$, such that
(a) $\bar{\Delta}\theta_{\rm s}(\sigma) = \underline{0}$, $-(1/H)\omega_{\rm d} \leq \bar{\Delta}\theta_{\rm d}(\sigma) \leq \underline{0}$, and $|\bar{\Delta}\theta_{\rm m}(\sigma) - (1/H)\omega_{\rm m}| \leq \Delta M$,
(b) $0 < \mathcal{P}_{\bar{\Delta}}(\sigma) \leq \bar{P}$ and $0 < \mathcal{C}_{\bar{\Delta}}(\sigma)$ if $\sigma \in \Gamma_{\tilde{\sigma}}$,
$-\bar{P} \leq \mathcal{P}_{\bar{\Delta}}(\sigma) \leq 0$ for every σ such that $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ and $\sigma \notin \Gamma_{\tilde{\sigma}}$,
(c) for every path $\sigma^\infty(\Gamma_{\tilde{\sigma}})$ in the sub-tree

$$\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma_\tau)}\right] \leq 1 \ \ \textit{for all } t \geq \tilde{t}, \quad \textit{and} \quad \lim_{T\to\infty} \sum_{t=\tilde{t}}^{T} \left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma_\tau)}\right]\right\}\mathcal{C}_{\bar{\Delta}}(\sigma_t) \leq B.$$

THEOREM 3 (Necessity): *Let $(x^*, \theta^*, q^*, r^*)$ be a competitive equilibrium with a sequence of markets (CE-S) and suppose Assumption 1 holds. Assume that $x_i^* \in R_{++}^{1+S}$, for all $i \in \Sigma \times \mathcal{H}$, and that there are real numbers $\varepsilon > 0$ and $\bar{\rho} > 0$, and a sub-tree $\Gamma_{\tilde{\sigma}}$, with $\tilde{t} := t(\tilde{\sigma}) \geq 1$, such that for all nodes $\sigma \in \Gamma_{\tilde{\sigma}}$ there exists an agent $h_\sigma \in \mathcal{H}$ for whom (i) $\bar{\rho} \geq \bar{\rho}_{\sigma,h_\sigma}$, (ii) $x_{\sigma,h_\sigma}^* \geq \varepsilon \cdot 1_{(1+S)\times 1}$.*
If there exist a function $\Delta\theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{J+K+1}$, and real numbers $\bar{P} > 0$ and $B > 0$, such that
(a) $\Delta\theta_{\rm s}(\sigma) = \underline{0}$, $-(1/H)\omega_{\rm d} \leq \Delta\theta_{\rm d}(\sigma) \leq \underline{0}$, and $|\Delta\theta_{\rm m}(\sigma) - (1/H)\omega_{\rm m}| \leq \Delta M$,

markets are assumed to be sequentially complete, a detailed description can be found in Chattopadhyay and Gottardi (1999 Definitions 4 and 5 and Lemma 1). Given our assumption of differentiable strict quasi-concavity and smoothness of the utility function, a parallel development can be carried out for the case of asset markets which is why we do not formalize the concepts.

[15] For a vector $x \in R^N$, $\| x \| := \left(\sum_{i=1}^N x_i^2\right)^{1/2}$, the usual Euclidean norm.

(b) $0 < \mathcal{P}_\Delta(\sigma) \leq \bar{P}$ *and* $0 < \mathcal{C}_\Delta(\sigma)$ *if* $\sigma \in \Gamma_{\tilde{\sigma}}$,
$-\bar{P} \leq \mathcal{P}_\Delta(\sigma) \leq 0$ *for every* σ *such that* $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ *but* $\sigma \notin \Gamma_{\tilde{\sigma}}$,
(c) for every path $\sigma^\infty(\Gamma_{\tilde{\sigma}})$ *in the sub-tree:*

$$\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_\Delta(\ \sigma_\tau)}{\mathcal{C}_\Delta(\ \sigma_\tau)}\right] \leq 1 \textit{ for all } t \geq \tilde{t}, \quad \textit{and} \quad \lim_{T\to\infty}\sum_{t=\tilde{t}}^{T}\left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_\Delta(\ \sigma_\tau)}{\mathcal{C}_\Delta(\ \sigma_\tau)}\right]\right\}\mathcal{C}_\Delta(\ \sigma_t) \leq B,$$

then the equilibrium allocation is not q^**-constrained CPO.*

REMARK 4: Assets in zero net supply do not play any role in the criterion that we obtain (see the first part of the condition labelled (a) in each result) since they do not play any role in determining the possibilities for intertemporal reassignments; if there are no assets of infinite maturity in non-zero net supply, then all equilibria are necessarily q^*-constrained CPO. However, every non-redundant asset is important in that it contributes towards determining the equilibrium; in particular, enough inside assets can lead to a sequentially complete market (a case in which our result applies) where the only source of inefficiency is the infinite horizon. But given an equilibrium, the existence of intertemporal improvements is determined without reference to the assets in zero net supply.

From here onwards by the *usual assumptions* we will mean that Assumption 1 holds, the aggregate endowment is uniformly bounded above across nodes, the allocation is uniformly interior for all agents in all coordinates, and the curvature of every agent's upper contour set lies in a compact subset of the strictly positive real numbers. These assumptions are easy to state and can be expected to hold in applications even though they are much stronger than the ones under which Theorem 2 or Theorem 3 holds.

The characterization result can be paraphrased as follows: Under the usual assumptions, a q^*-constrained improvement over an equilibrium allocation exists if and only if there is (a1) a set of nodes which form a sub-tree, denoted $\Gamma_{\tilde{\sigma}}$, and (a2) a portfolio, denoted $\bar{\Delta}\theta$ or $\Delta\theta$, which is a feasible variation on the equilibrium asset allocation as per (ii) in Definition 6, with the property that (b) for every node in $\Gamma_{\tilde{\sigma}}$, a node which is its immediate successor is also included in $\Gamma_{\tilde{\sigma}}$ if and only if the portfolio has a positive payoff at that successor node (in particular, there is at least one such successor node) and the payoff from the portfolio is uniformly bounded across σ such that $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$, and (c1) the *random discount factors*, determined by the product along nodes of the return on the portfolio at a node (where the return at a node is specified by dividing the payoff from the portfolio carried from the previous node by the value of the portfolio at the node, denoted $\mathcal{P}_{\bar{\Delta}}/\mathcal{C}_{\bar{\Delta}}$), are strictly positive and uniformly bounded across nodes in $\Gamma_{\tilde{\sigma}}$ at which they are evaluated, and (c2) the *value* of the portfolio, calculated by considering the sum of the price at which the portfolio can be bought discounted by the random discount factors mentioned above, is strictly positive, converges along the path, and is uniformly bounded across paths in $\Gamma_{\tilde{\sigma}}$.

The essence of the proof consists in showing that an improvement exists if and only if there exists a set of nodes such that the per capita value of the net transfer to the agents of a generation increases at a quadratic rate as we move to successive generations; this happens since preferences are assumed to be strictly convex and monotone. Since we work with asset markets, and the improvement is restricted to be one that can be obtained via a reallocation of existing assets, we are able to use the no arbitrage property of asset prices to write the values of the net transfers in terms of asset prices and the portfolio $\bar{\Delta}\theta$, or $\Delta\theta$, a feasible variation on the equilibrium asset allocation. A complication arises due to the fact that asset payoffs are unrestricted in sign, so that for a given portfolio a node could have successors at which the payoff is positive and other successors at which the payoff is negative; we need to separate these nodes and do so by showing that for the portfolio $\bar{\Delta}\theta$, or $\Delta\theta$, the ones with positive payoff have the structure of a sub-tree. In addition, because asset payoffs are unrestriced in sign, in Theorem 3 we need to work harder to guarantee that the allocation that is constructed leaves every agent with a vector in his consumption set which is why we require that the equilibrium consumption vector be uniformly interior in every coordinate. Finally, one uses the fact that the payoff of the portfolio $\bar{\Delta}\theta$, or $\Delta\theta$, is uniformly bounded to obtain the boundedness of the discount factors and the convergence of the family of sums stated as condition (c) in the theorems as a necessary and sufficient condition for the existence of an improvement.

In the skeleton of the argument one finds the seminal proof in Cass (1972).[16] One expects some extension of the Cass result to an environment with uncertainty to hold. Since we work with asset payoffs that are unrestricted in sign and a constrained notion of an improvement, the extension is neither obvious nor simply a matter of cranking through in a mechanical manner. When we compare our results with earlier results on the Cass criterion we see that a very important difference is that our conditions for necessity are stronger since we require that the payoff of the portfolio be uniformly bounded (which can be ensured by assuming that the aggregate endowment is uniformly bounded) and that consumption is uniformly interior in *every* coordinate; we need these conditions since asset payoffs are unrestricted in sign. Also, the no arbitrage property of asset prices plays a very important role. The proofs that we give are self-explanatory because of which we prefer not to discuss them in the main text.

REMARK 5: Since the portfolio $\bar{\Delta}\theta$, or $\Delta\theta$, is uniformly bounded, one can also consider an alternative formulation of the result in which one uses normalized versions of the functions $\mathcal{P}_{\bar{\Delta}}$ and $\mathcal{C}_{\bar{\Delta}}$, obtained by dividing them by the norm of the portfolio, and one restricts their domain of definition to the set of nodes on which the norm is positive, i.e., to the sub-tree that is identified; this manouvre only changes the values of the constants which give the bounds on the series that appear in Theorems 2 and 3. The reformulation is particularly useful when one deals with the case in which there is only one long-lived asset since the normalized portfolio appears with values in the set $\{-1, 1\}$.

4c. SOME SPECIAL CASES

In this sub-section we turn to see what the result has to say in cases of particular interest. Doing so gives us additional insight into the nature of the characterization.

One Asset

[16]Chattopadhyay and Gottardi (1999) provide a discussion and a detailed bibliography of the many refinements to Cass' proof.

Consider first the case in which there is only one dividend paying asset and no money.

COROLLARY 1: *Let* $(x^*, \theta^*, q^*, r^*)$ *be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that* $K = 1$ *and that fiat money is not available as an asset. The equilibrium allocation is not* q^**-constrained CPO if and only if there exists a sub-tree,* $\Gamma_{\tilde{\sigma}}$ *with* $\tilde{t} := t(\tilde{\sigma})$, *and real numbers* $\bar{P} > 0$, $A > 0$, *and* $C > 0$, *such that*
$-\bar{P} \leq r^*_{\mathrm{d}}(\sigma) < 0$ *and* $q^*_{\mathrm{d}}(\sigma) < 0$ *if* $\sigma \in \Gamma_{\tilde{\sigma}}$,
$0 \leq r^*_{\mathrm{d}}(\sigma) \leq \bar{P}$ *for every* σ *such that* $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ *and* $\sigma \notin \Gamma_{\tilde{\sigma}}$,
and on every path in the sub-tree

$$0 < \Pi^t_{\tau=\tilde{t}} \left[\frac{r^*_{\mathrm{d}}(\sigma_\tau)}{q^*_{\mathrm{d}}(\sigma_\tau)} \right] \leq A \quad for\, all\; t \geq \tilde{t}, \quad and$$

$$0 < \lim_{T\to\infty} \sum_{t=\tilde{t}}^{T} \Pi^t_{\tau=\tilde{t}} \left[\frac{r^*_{\mathrm{d}}(\sigma_\tau)}{q^*_{\mathrm{d}}(\sigma_\tau)} \right] [-q^*_{\mathrm{d}}(\sigma_t)] \leq C.$$

The proof follows from the reformulation of the result noted in Remark 5. Since the only asset is a long-lived dividend paying one, the feasibility condition on asset reallocations implies that $\bar{\Delta}\theta(\sigma) \leq 0$ so that $(\bar{\Delta}\theta(\sigma)/ \parallel \bar{\Delta}\theta(\sigma) \parallel) = -1$ for all nodes σ in the sub-tree; it suffices to substitute these values into the criterion obtained in the reformulation after writing out explicitly the functions $\mathcal{P}_{\bar{\Delta}}$ and $\mathcal{C}_{\bar{\Delta}}$.

If the only asset of infinite maturity is fiat money then we have

COROLLARY 2: *Let* $(x^*, \theta^*, q^*, r^*)$ *be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that* $K = 0$ *but that fiat money is available as an asset. The equilibrium allocation is not* q^**-constrained CPO if and only if there exists a sub-tree,* $\Gamma_{\tilde{\sigma}}$ *with* $\tilde{t} := t(\tilde{\sigma})$, *a function* $\bar{\Delta}\theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R$, *and a real number* $B > 0$, *such that*
$\mathrm{sign}\, q^*_{\mathrm{m}}(\sigma) = \mathrm{sign}\, q^*_{\mathrm{m}}(\tilde{\sigma})$ *if* $\sigma \in \Gamma_{\tilde{\sigma}}$,
$\mathrm{sign}\, q^*_{\mathrm{m}}(\sigma) \neq \mathrm{sign}\, q^*_{\mathrm{m}}(\tilde{\sigma})$ *for every* σ *such that* $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ *but* $\sigma \notin \Gamma_{\tilde{\sigma}}$,

$$\mathrm{sign}\bar{\Delta}\theta(\sigma) = \mathrm{sign}\bar{\Delta}\theta(\tilde{\sigma}_{-1})\; for\; all\; \sigma \in \Gamma_{\tilde{\sigma}},$$

and on every path

$$0 < \lim_{T\to\infty} \bar{\Delta}\theta(\tilde{\sigma}_{-1}) \sum_{t=\tilde{t}}^{T} q^*_{\mathrm{m}}(\;\sigma_t) \leq B.$$

The proof follows since, using the fact that the only long-lived asset is money,

$$\Pi^t_{\tau=\tilde{t}} \left[\frac{\mathcal{P}_{\bar{\Delta}}(\sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\sigma_\tau)} \right] = \Pi^t_{\tau=\tilde{t}} \left[\frac{\bar{\Delta}\theta(\;\sigma_{\tau-1}) \cdot q^*_{\mathrm{m}}(\sigma_\tau)}{\bar{\Delta}\theta(\sigma_\tau) \cdot q^*_{\mathrm{m}}(\sigma_\tau)} \right] = \frac{\bar{\Delta}\theta(\tilde{\sigma}_{-1})}{\bar{\Delta}\theta(\sigma_t)}$$

showing that the portfolio variation must have the same sign at every node in the sub-tree since the random discount factors are positive. Further, the condition on the series in Theorems 2 and 3 becomes

$$\lim_{T\to\infty}\sum_{t=\tilde{t}}^{T}\left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\sigma_\tau)}\right]\right\}\mathcal{C}_{\bar{\Delta}}(\ \sigma_t)=\lim_{T\to\infty}\sum_{t=\tilde{t}}^{T}\left\{\frac{\bar{\Delta}\theta(\tilde{\sigma}_{-1})}{\bar{\Delta}\theta(\sigma_t)}\right\}\bar{\Delta}\theta(\ \sigma_t)\cdot q_{\mathrm{m}}^*(\sigma_t)$$

$$=\lim_{T\to\infty}\bar{\Delta}\theta(\tilde{\sigma}_{-1})\sum_{t=\tilde{t}}^{T}q_{\mathrm{m}}^*(\ \sigma_t)\leq B.$$

Finally, the conditions on the functions $\mathcal{P}_{\bar{\Delta}}$ and $\mathcal{C}_{\bar{\Delta}}$ take the form

$0<\bar{\Delta}\theta(\sigma_{-1})\cdot q_{\mathrm{m}}^*(\sigma)\leq\bar{P}$ and $0<\bar{\Delta}\theta(\sigma)\cdot q_{\mathrm{m}}^*(\sigma)$ if $\sigma\in\Gamma_{\tilde{\sigma}}$,

$-\bar{P}\leq\bar{\Delta}\theta(\sigma_{-1})\cdot q_{\mathrm{m}}^*(\sigma)\leq 0$ for every σ such that $\sigma_{-1}\in\Gamma_{\tilde{\sigma}}$ but $\sigma\notin\Gamma_{\tilde{\sigma}}$.

Since we have shown that $\mathrm{sign}\Delta\theta(\sigma)=\mathrm{sign}\Delta\theta(\tilde{\sigma}_{-1})$ for all $\sigma\in\Gamma_{\tilde{\sigma}}$, we are led to conclude that

$\mathrm{sign}\, q_{\mathrm{m}}^*(\sigma)=\mathrm{sign}\, q_{\mathrm{m}}^*(\tilde{\sigma})$ if $\sigma\in\Gamma_{\tilde{\sigma}}$,

$\mathrm{sign}\, q_{\mathrm{m}}^*(\sigma)\neq\mathrm{sign}\, q_{\mathrm{m}}^*(\tilde{\sigma})$ for every σ such that $\sigma_{-1}\in\Gamma_{\tilde{\sigma}}$ but $\sigma\notin\Gamma_{\tilde{\sigma}}$.

As a special case, under certainty and free disposal of money we have:

COROLLARY 3: *Let (x^*,θ^*,q^*,r^*) be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that $K=0$ but fiat money is available as an asset. Suppose further that there is no uncertainty, $S=1$, and that money is freely disposable. The equilibrium allocation is not q^*-constrained CPO if and only if there exists a real number $A>0$, such that*

$$0<\lim_{T\to\infty}\sum_{t=\tilde{t}}^{T}q_t^*\leq A:=\frac{B}{\bar{\Delta}\theta(\tilde{\sigma}_{-1})}.$$

The proof follows since the price of money is non-negative by free disposal. So, $\bar{\Delta}\theta(\tilde{\sigma}_{-1})>0$ by the condition in Corollary 2 stating that the series is positive. Now the result is a restatement of Corollary 2.

In Corollary 3 we recover the classical Cass criterion for one-good deterministic OLG economies since the discounted price of the commodity can be identified with the reciprocal of the price of money. Furthermore, it shows that our characterization result applies to economies with sequentially complete markets and that in such economies all equilibrium allocations need not always be q^*-constrained CPO.

Free Disposal of Assets

An extremely important special case is the one in which assets are freely disposable and dividends are non-negative. Corollary 3 has shown that one cannot rule out the possibility of inefficiency even with free disposability of assets if fiat money is one of the assets; hence, we assume that fiat money is not available.

PROPOSITION 1:[17] *Let (x^*,θ^*,q^*,r^*) be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that fiat money is not available as*

[17]For the special case of stationary equilibria with a single dividend paying asset, Chattopadhyay and Jimenez (2000) provide a simple proof of this result.

an asset and that dividends are always non-negative and the assets are freely disposable, $d : \Sigma \to R^K_+$ and $q^* : \Sigma \to R^K_+$. Then the equilibrium allocation is q^*-constrained CPO.

Proposition 1 is proved by noting that free disposal of the asset together with the fact that dividends are non-negative, $q^*(\sigma) \geq 0$ and $d(\sigma) \geq 0$, and the fact that there is no money in the economy, so $\bar{\Delta}\theta(\sigma) \leq 0$, implies that the value of the change in any possible portfolio reassignment is necessarily non-positive at all nodes, $\mathcal{P}_{\bar{\Delta}}(\sigma) \leq 0$, and so is the cost of the possible portfolio reassignment, $\mathcal{C}_{\bar{\Delta}}(\sigma) \leq 0$. But then the series in Theorem 2 cannot be positive contradicting an implication of the existence of an improving allocation.

Allowing money to be one of the long-lived assets in Proposition 1 breaks the result. The reason is that we have allowed the quantity of money to be increased and decreased so that even if its price is positive, its net payoff can be positive if $\Delta\theta(\sigma) > 0$ at some nodes, i.e., inflationary policies might lead to an inefficiency which free disposal is unable to cure.

Sequentially Complete Markets

Evidently, our characterization result also applies when markets are sequentially complete so that agents can insure against all risks that arise after their birth. We now show that our result covers a wider range of situations relative to existing results on optimality with sequentially complete markets.

Consider the result obtained by applying Proposition 1 to an economy with sequentially complete markets. Adapting Santos and Woodford (1997) to our framework, one shows that if dividends are *non-negligible* then the allocation in *every* free disposal equilibrium with sequentially complete markets is CPO; their proof extends an argument due to Wilson (1981). Our proof does not require the additional condition of non-negligibility since it is based on Theorem 2 which takes into account the second order effects on utility induced by the reallocation. More generally, the characterization result in terms of contingent claims prices obtained by Chattopadhyay and Gottardi (1999) can be used to determine whether an equilibrium allocation is CPO when markets are sequentially complete; this can be done by constructing the contingent claims prices from asset prices.[18] However, there is an important caveat. There could exist an equilibrium with a sequence of markets which cannot be represented as an equilibrium in contingent commodity markets. This happens when the dividend paying asset is not freely disposable. Brock (1990) gives a robust deterministic example of such a phenomenon; extensions to stochastic enviroments are easy to obtain. Since Chattopadhyay and Gottardi restrict attention to allocations that can be obtained via trade in ex-ante contingent commodities and their agents have endowments only during their lifetimes, their results cannot be applied; however, our result allows us to determine the optimality properties of the allocation in such cases (in the specific case of Brock's example the allocation is inefficient).

Stationary Equilibrium

We turn to the case in which the equilibrium is stationary which is of particular interest. We will provide results which are analogues of the results in Corollaries 1 and 2.

[18]The contingent claims prices are unique upto normalization, when markets are sequentially complete (see, e.g., Santos and Woodford (1997)).

We assume that the economy is stationary, i.e., that the characteristics (endowments and utility functions) of each agent only depend on the realizations of the state during her lifetime, not on time nor on past realizations. Elements of $\mathcal{S}$ can now be interpretated as the realizations of a time homogeneous Markov process. Given a node σ let $s(\sigma)$ denote the realization of the Markov state. Stationarity of the environment requires that the characteristics of the agents born at a node σ and the payoffs of the assets at a node σ depend only on $s(\sigma) \in \mathcal{S}$. This lets us convert the general model of Section 2 into a stationary one.

DEFINITION 8: An economy is *stationary* if for all $(\sigma, \widehat{\sigma}) \in \Sigma \times \Sigma$, $s(\sigma) = s(\widehat{\sigma})$ implies that $X_{\sigma,h} = X_{\widehat{\sigma},h} := X_{s(\sigma),h}$, $\omega_{\sigma,h} = \omega_{\widehat{\sigma},h} := \omega_{s(\sigma),h}$, $u_{\sigma,h} = u_{\widehat{\sigma},h} := u_{s(\sigma),h}$, $s(\sigma) = s(\widehat{\sigma}) := s_{s(\sigma)}$ and $d(\sigma) = d(\widehat{\sigma}) := d_{s(\sigma)}$.

Under stationarity, asset returns will be denoted $((d_s)_{s \in \mathcal{S}}) \in R^S$ since we will have either one or no long-lived dividend paying asset. Stationary prices of the assets will be denoted $q_s := (q_{\mathrm{s},s}, q_{\mathrm{d},s}, q_{\mathrm{m},s})$, $s \in \mathcal{S}$.

Stationarity of the equilibrium requires that, given stationary prices, $x_{\sigma,h} = x_{s(\sigma),h}$ for all $(\sigma, h) \in \Sigma \times \mathcal{H}$ (i.e., the consumption allocation of each agent only depends on the state at the date of his birth and not on the past). Stationary asset demands will be denoted by $\theta(s, h)$ where $(s, h) \in \mathcal{S} \times \mathcal{H}$.[19] We can now define a stationary competitive equilibrium.

DEFINITION 9 (SCE-S): $(x^*, \theta^*, (q_1^*, \cdots, q_S^*), (r_1^*, \cdots, r_S^*))$ is a *stationary competitive equilibrium with a sequence of markets (SCE-S)* if it is a CE-S such that

for all $(\sigma, \widehat{\sigma}) \in \Sigma \times \Sigma$, if $s(\sigma) = s(\widehat{\sigma})$ then $x_{\sigma,h} = x_{\widehat{\sigma},h}$ and $\theta(\sigma, h) = \theta(\widehat{\sigma}, h)$.

REMARK 6: We make no claims regarding existence; results on existence are available in certain special cases, e.g., when the only long-lived asset is money.

PROPOSITION 2:[20] *Let $(x^*, \theta^*, (q_1^*, \cdots, q_S^*), (r_1^*, \cdots, r_S^*))$ be a stationary competitive equilibrium with a sequence of markets (SCE-S) and suppose Assumption 1 holds. Suppose that $K = 1$ and that fiat money is not available as an asset. An interior equilibrium allocation is not q^*-constrained CPO if and only if the set $\bar{\mathcal{S}} := \{s \in \mathcal{S} : r_{\mathrm{d},s}^* < 0\}$ is non-empty, $\bar{\mathcal{S}} \neq \emptyset$, and such that $0 < r_{\mathrm{d},s}^*/q_{\mathrm{d},s}^* < 1$ for all $s \in \bar{\mathcal{S}}$ in which case there exists a stationary improvement.*

To prove Proposition 2, we invoke Corollary 1. If the allocation is not q^*-constrained CPO then there exists a sub-tree on which the discount factors are uniformly bounded, a family of sums converges, $r_{\mathrm{d},s(\sigma)}^* < 0$ and $q_{\mathrm{d},s(\sigma)}^* < 0$ if $\sigma \in \Gamma_{\tilde{\sigma}}$, and $0 \leq r_{\mathrm{d},s(\sigma)}^*$ for every σ such that $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ but $\sigma \notin \Gamma_{\tilde{\sigma}}$; it follows that there are Markov states for which $r_{\mathrm{d},s}^* < 0$, and for every Markov state $\bar{s}$ for which $r_{\mathrm{d},\bar{s}}^* < 0$, there is a node $\bar{\sigma} \in \Gamma_{\tilde{\sigma}}$ with $r_{\mathrm{d},s(\bar{\sigma})}^* < 0$.

[19]We also need to specify the characteristics and behaviour of the initial old. The way in which this is done is important for questions which deal with existence of equilibrium. Since our interest is in optimality, we shall be sloppy and forego a full specification of the model.

[20]In Propositions 2 and 3 we no longer need to add "under the usual assumptions" since stationarity of the equilibrium together with Assumption 1 and interior consumption guarantee that the usual assumptions hold.

In other words the Markov states that appear in the sub-tree $\Gamma_{\tilde{\sigma}}$ are exactly the ones in $\bar{\mathcal{S}}$. Also, $q^*_{\mathrm{d},s} < 0$ if $s \in \bar{\mathcal{S}}$. If it were to be the case that $1 \leq r^*_{\mathrm{d},\tilde{s}}/q^*_{\mathrm{d},\tilde{s}}$, for some $\tilde{s} \in \bar{\mathcal{S}}$, then, along the path in the sub-tree $\Gamma_{\tilde{\sigma}}$ in which the realization is always $\tilde{s}$, convergence of the series would fail since the sum would exceed the expression $T \cdot (-q^*_{\mathrm{d},\tilde{s}})$ which diverges since $q^*_{\mathrm{d},\tilde{s}} < 0$ as $\tilde{s} \in \bar{\mathcal{S}}$. This proves Proposition 2 in one direction.

Going in the other direction, note first that $0 < r^*_{\mathrm{d},\tilde{s}}/q^*_{\mathrm{d},\tilde{s}} < 1$ for all $s \in \bar{\mathcal{S}}$ implies that $q^*_{\mathrm{d},\tilde{s}} < 0$ for all $s \in \bar{\mathcal{S}}$ since $r^*_{\mathrm{d},\tilde{s}} < 0$ for all $s \in \bar{\mathcal{S}}$. Now construct the sub-tree generated by elements of the set $\bar{\mathcal{S}}$. Given the conditions on asset returns that are satisfied in the states in the set $\bar{\mathcal{S}}$, the family of sums in Corollary 1 converges, $r^*_{\mathrm{d}}(\sigma) < 0$ and $q^*_{\mathrm{d}}(\sigma) < 0$ if $\sigma \in \Gamma_{\tilde{\sigma}}$, and $0 \leq r^*_{\mathrm{d}}(\sigma)$ for every σ such that $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ but $\sigma \notin \Gamma_{\tilde{\sigma}}$, on the induced sub-tree, indicating the existence of an improvement by an application of Corollary 1.

We forego the proof of the existence of a stationary improvement since it is available in Chattopadhyay and Jimenez (2000).

Consider replacing the dividend paying asset in Proposition 2 with money. The condition for obtaining a stationary reallocation which also improves can no longer be satisfied ($\bar{\mathcal{S}}$ may be non-empty but since money does not pay a dividend, the condition on the return being less than one is not satisfied). It is easy to show that now the first order effect of a stationary reallocation of money is zero. So we find that all stationary monetary equilbrium allocations are optimal, i.e., Corollary 3 on monetary equilibria does not extend to the case of stationary equilibrium.

PROPOSITION 3: *Let* $(x^*, \theta^*, (q_1^*, \cdots, q_S^*), (r_1^*, \cdots, r_S^*))$ *be a stationary competitive equilibrium with a sequence of markets (SCE-S) and suppose Assumption 1 holds. Suppose that* $K = 0$ *but fiat money is available as an asset. If the allocation is interior it is* q^**-constrained CPO.*

The proposition is proved by invoking Corollary 2 applied to a stationary equilibrium. Suppose that the allocation is not optimal. Since $\operatorname{sign} q^*_{\mathrm{m}}(\sigma) = \operatorname{sign} q^*_{\mathrm{m}}(\tilde{\sigma})$ if $\sigma \in \Gamma_{\tilde{\sigma}}$, and the number of Markov states is finite, stationarity of prices implies that the series cannot converge contradicting an implication of the hypothesis that the allocation is not optimal.[21]

Of course, Proposition 1 continues to apply and lets us prove the optimality of interior stationary equilibrium allocations when assets are freely disposable.

We make a final comment. It is known that when markets are sequentially complete, i.e., $J + K + 1 \geq S$ and the assets span R^S, a stationary equilibrium at which the agents' common matrix of marginal rates of intertemporal substitution has a Perron root which is less than or equal to one is CPO.[22] One wonders about the existence of a similar relationship when markets fail to be sequentially complete. We refer the reader to Chattopadhyay and Jimenez (2000) who show that it is indeed possible to obtain such a *unit root* type result even with incompleteness if we assume that dividends are non-negative.

5. PROOFS

We introduce some notational conventions and concepts that will be used throughout.

For each proof, we will consider an equilibrium tuple $(x^*, \theta^*, q^*, r^*)$ which will remain fixed. We will also use an *alternative* tuple, denoted $(\widehat{x}, \widehat{\theta}, \widehat{q}, \widehat{r})$, which will be a local variation of $(x^*, \theta^*, q^*, r^*)$ and, in addition, either (i) compatible in the sense of Definition 4 or (ii) such that $\widehat{\theta}$ makes $\widehat{x}$ q^*-constrained feasible, i.e., according to Definition 6.

For an agent i, let

$$\Delta x_i := \widehat{x}_i - x_i^* \qquad \text{and} \qquad \Delta\theta(i) := \widehat{\theta}(i) - \theta^*(i).$$

Also, let

$$\Delta q(\sigma) := \widehat{q}(\sigma) - q^*(\sigma) \qquad \text{and} \qquad \Delta r(\sigma) := \widehat{r}(\sigma) - r^*(\sigma).$$

From the budget constraints and monotonicity of preferences we have, for agent i

$$x_i^*(\sigma) = \omega_i(\sigma) + \theta^*(i) \cdot r^*(\sigma) \qquad \text{for all } \sigma \in \sigma(i)^+.$$

Also, for the alternative triple, using Definition 4 (iv) or Definition 6 (iv), we have

$$\widehat{x}_i(\sigma) = \omega_i(\sigma) + \widehat{\theta}(i) \cdot \widehat{r}(\sigma) \qquad \text{for all } \sigma \in \sigma(i)^+.$$

Consequently, for agent i and for all $\sigma \in \sigma(i)^+$,

$$\Delta x_i(\sigma) = \Delta\theta(i) \cdot \widehat{r}(\sigma) + \widehat{\theta}(i) \cdot \Delta r(\sigma). \tag{1}$$

[21] The claim in Gottardi (1996) regarding the suboptimality of stationary monetary equilibria with money prices changing signs is not clear to us.

[22] Very different proofs of the same result can be found in Chattopadhyay and Gottardi (1999), Demange and Laroque (1999), Chattopadhyay (2000).

Let us define the aggregate quantities

$$\bar{\Delta}x(\sigma) := (1/H) \sum_{h\in\mathcal{H}} \Delta x_{\sigma,h}(\sigma),$$

$$\bar{\Delta}\theta^a(\sigma) := (1/H) \sum_{h\in\mathcal{H}} \Delta\theta^a(\sigma,h) \qquad \text{for} \quad a \in \mathcal{A},$$

$$\bar{\Delta}\theta(\sigma) := ((\bar{\Delta}\theta^a(\sigma))_{a\in\mathcal{A}}).$$

Aggregate feasibility of x^* and of $\widehat{x}$, Definition 1, and the fact that, given monotonicity of preferences, the aggregate feasibility constraint holds with equality in an equilibrium, implies that

$$\bar{\Delta}x(\sigma) + (1/H) \sum_{h\in\mathcal{H}} \Delta x_{\sigma_{-1},h}(\sigma) \leq 0$$

which, upon substituting (1), leads to

$$\bar{\Delta}x(\sigma) + \bar{\Delta}\theta(\sigma_{-1}) \cdot \widehat{r}(\sigma) + \Big\{(1/H) \sum_{h\in\mathcal{H}} \widehat{\theta}(\sigma_{-1},h)\Big\} \cdot \Delta r(\sigma) \leq 0 \tag{2}$$

where $\Delta r(\sigma) = (\underline{0}, \Delta q_{\mathrm{d}}(\sigma), \Delta q_{\mathrm{m}}(\sigma))$ since dividends don't vary.

We recall the notation for the payoff from a portfolio and its cost, $\mathcal{P}_f$ and $\mathcal{C}_f$. If the portfolio being considered is the change in the average portfolio relative to its value in the initial equilibrium, $\bar{\Delta}\theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{J+K+1}$, the functions $\mathcal{P}_{\bar{\Delta}} : \Sigma \to R$ and $\mathcal{C}_{\bar{\Delta}} : \Sigma \to R$ are given by

$$\mathcal{P}_{\bar{\Delta}}(\sigma) := \bar{\Delta}\theta(\sigma,o)\cdot r^*(\sigma) \quad \text{for all } \sigma \in \Sigma_1 \qquad \mathcal{P}_{\bar{\Delta}}(\sigma) := \bar{\Delta}\theta(\sigma_{-1})\cdot r^*(\sigma) \qquad \text{for all } \sigma \in \cup_{t\geq 2}\Sigma_t$$

$$\mathcal{C}_{\bar{\Delta}}(\sigma) := \bar{\Delta}\theta(\ \sigma) \cdot q^*(\sigma) \qquad \text{for all } \sigma \in \Sigma.$$

So the feasibility condition (2) takes the form

$$\bar{\Delta}x(\sigma) + \mathcal{P}_{\bar{\Delta}}(\sigma) + \Big\{(1/H) \sum_{h\in\mathcal{H}} \widehat{\theta}(\sigma_{-1},h)\Big\} \cdot \Delta r(\sigma) \leq 0. \tag{3}$$

We turn to notation for marginal utility comparisons.

For $f : R^N_{++} \to R$, $\frac{\partial f(\bar{x})}{\partial x_i}$ denotes the partial derivative of the function f with respect to its i-th coordinate evaluated at the point $\bar{x}$.

All derivatives will be evaluated at the chosen equilibrium tuple $(x^*, \theta^*, q^*, r^*)$; hence, notation for the allocation being considered will be supressed. For an agent i, $\frac{\partial u_i}{\partial x_\sigma}$ will denote marginal utility from consumption at the node $\sigma \in \Sigma_i$. $\lambda_i(\sigma)$ denotes the Lagrange multiplier on the budget constraint faced by agent i at the node $\sigma \in \Sigma_i$.

The first order necessary and sufficient conditions for optimization on the part of agent i at an interior equilibrium allocation are given by:

$$\frac{\partial u_i}{\partial x_\sigma} = \lambda_i(\sigma) \qquad\qquad \text{for all } \sigma \in \Sigma_i, \tag{4}$$

$$\lambda_i(\sigma(i)) \cdot q^{a^*}(\sigma(i)) = \sum_{\sigma\in\sigma(i)^+} \lambda_i(\sigma) \cdot r^{a^*}(\sigma) \qquad\qquad \text{for all } a \in \mathcal{A}. \tag{5}$$

In particular, equilibrium asset prices satisfy the no arbitrage property of asset prices so that at any given node $\sigma \in \Gamma$, there exists a vector $a_\sigma \in R^S_{++}$ such that

$$q^*(\sigma) = \sum_{\sigma' \in \sigma^+} a_\sigma(\sigma') \cdot r^*(\sigma'). \tag{6}$$

With these preliminaries in place, we proceed to the proofs of the various results.

PROOF OF THEOREM 1: To economize on notation, we do the proof allowing money to be traded. This allows us to see clearly that a constant endowment of money has drastic consequences. Setting the price of money, and its variation, to zero at every node, we obtain the proof for a non-monetary equilibrium.

Suppose that the allocation is not locally constrained CPO. So, a constrained improvement must exist; denote it $(\widehat{x}, \widehat{\theta}, \widehat{q}, \widehat{r})$, a local variation of $(x^*, \theta^*, q^*, r^*)$.

Consider an agent (σ, h). The change in her utility, up to first order, because of the change in the allocation, is given by

$$du_i = \frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \Delta x_i(\sigma(i)) + \sum_{\sigma \in \sigma(i)^+} \frac{\partial u_i}{\partial x_\sigma} \cdot \Delta x_i(\sigma)$$

which, upon using (4) and (1), can be written as

$$du_i = \lambda_i(\sigma(i)) \cdot \Delta x_i(\sigma(i)) + \sum_{\sigma \in \sigma(i)^+} \lambda_i(\sigma) \cdot \left\{ \Delta\theta(i) \cdot \widehat{r}(\sigma) + \widehat{\theta}(i) \cdot \Delta r(\sigma) \right\}. \tag{7}$$

Since we have assumed that the alternative is a CPO improvement, the term in (7) must be non-negative for every agent and strictly positive for some agent who could be the initial old.

By using (5), the first order condition for optimal portfolio choice, we obtain

$$\sum_{\sigma \in \sigma(i)^+} \lambda_i(\sigma) \cdot \Delta\theta(i) \cdot \widehat{r}(\sigma) = \lambda_i(\sigma(i)) \cdot \Delta\theta(i) \cdot \widehat{q}(\sigma(i)). \tag{8}$$

By a similar argument

$$\sum_{\sigma \in \sigma(i)^+} \lambda_i(\sigma) \cdot \widehat{\theta}(i) \cdot \Delta r(\sigma) = \lambda_i(\sigma(i)) \cdot \widehat{\theta}(i) \cdot \Delta q(\sigma(i)). \tag{9}$$

Using (8) and (9) in (7) and averaging over the set of agents we obtain

$$(1/H) \sum_{h \in \mathcal{H}} (du_{\sigma,h}/\lambda_{\sigma,h}(\sigma)) = \bar{\Delta}x(\sigma) + \bar{\Delta}\theta(\sigma) \cdot \widehat{q}(\sigma) + (1/H) \sum_{h \in \mathcal{H}} (\widehat{\theta}(\sigma, h)) \cdot \Delta q(\sigma). \tag{10}$$

Using the facts that, at every node σ,

$$\bar{\Delta}\theta(\sigma) = \underline{0} \qquad \sum_{h \in \mathcal{H}} \widehat{\theta}(\sigma, h) = \omega$$

since asset markets must clear exactly, in (10), we obtain

$$(1/H) \sum_{h \in \mathcal{H}} (du_{\sigma,h}/\lambda_{\sigma,h}(\sigma)) = \bar{\Delta}x(\sigma) + (1/H)\omega \cdot \Delta q(\sigma)$$

which, by the feasibility condition, (2), and the fact that $\Delta r(\sigma) = (\underline{0}, \Delta q_{\mathrm{d}}(\sigma), \Delta q_{\mathrm{m}}(\sigma))$ implies that

$$(1/H) \sum_{h \in \mathcal{H}} (du_{\sigma,h}/\lambda_{\sigma,h}(\sigma)) \leq 0.$$

So the first order effect is always zero. Hence, the young will never be willing to effect a transfer to the old since the first order effect on utility is zero while the second order effect is necessarily negative because of strict quasi-concavity of the utility functions.

This completes the proof of the non-existence of a constrained improving allocation which is a local variation and is also an equilibrium. ■

PROOF OF THEOREM 2: Since the allocation is not q^*-constrained CPO, a constrained improvement must exist; denote it $(\widehat{x}, \widehat{\theta})$. Since both x^* and $\widehat{x}$ are q^*-feasible, so is any convex combination of the two, so that without loss of generality we can assume that $\widehat{x}$ is a local variation of x^*.

As in the proof of Theorem 1, (7), rewritten below, gives an evaluation of the change in the utility, up to first order, of an agent i because of the reallocation:

$$\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \left\{ \Delta x_i(\sigma(i)) + \Delta\theta(i) \cdot q^*(\sigma(i)) + \widehat{\theta}(i) \cdot \Delta q(\sigma(i)) \right\}.$$

However, $\Delta q^a(\sigma) = 0$ for all $a \in \mathcal{A}$ by the nature of the improvement being considered since asset prices do not change. This leads to

$$\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \left\{ \Delta x_i(\sigma(i)) + \Delta\theta(i) \cdot q^*(\sigma(i)) \right\}.$$

Strict convexity of preferences implies that we need to take into account the second order effect in order to ensure that we have a weak improvement. So it is not sufficient that the expression above be non-negative for every agent; it must exceed a quadratic term given by[23]

$$\underline{\rho}_i \cdot \frac{\left[\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \Delta x_i(\sigma(i)) \right]^2}{\frac{\partial u_i}{\partial x_{\sigma(i)}}},$$

where $\underline{\rho}_i$ is the greatest lower bound on the curvature of the upper contour set of agent i at the competitive allocation. So in order to have an improvement, the following inequality must hold for every agent and must be strict for some agent:

$$\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \left[\Delta x_i(\sigma(i)) + \Delta\theta(i) \cdot q^*(\sigma(i)) \right] \geq \underline{\rho}_i \cdot \frac{\left[\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \Delta x_i(\sigma(i)) \right]^2}{\frac{\partial u_i}{\partial x_{\sigma(i)}}}. \tag{11}$$

By averaging the inequality in (11) across agents born at the same node, and using Jensen's Inequality applied to a quadratic function, we obtain

$$\bar{\Delta} x(\sigma) + \mathcal{C}_{\bar{\Delta}}(\sigma) \geq \underline{\rho} \cdot [\bar{\Delta} x(\sigma)]^2 \tag{12}$$

[23]See Definition 4 and Lemma 1 in Chattopadhyay and Gottardi (1999) for an explicit derivation of the required quadratic term in a related context. A similar argument can be used here.

(using the function $\mathcal{C}_{\bar{\Delta}}$ that we introduced earlier) since $\underline{\rho}_i \geq \underline{\rho} > 0$ by hypothesis (ii) in Theorem 2 on the existence of a positive lower bound on the curvature of the upper contour sets. If an improvement exists then (12) must hold at every node $\sigma \in \Sigma$ with a strict inequality at some node.

By the feasibility condition on the reallocation of assets $\bar{\Delta}\theta_{\mathrm{s}}(\sigma) = \underline{0}$, since the short maturity assets are in zero net supply, $-(1/H)\omega_{\mathrm{d}} \leq \bar{\Delta}\theta_{\mathrm{d}}(\sigma) \leq \underline{0}$, and $|\bar{\Delta}\theta_{\mathrm{m}}(\sigma) - (1/H)\omega_{\mathrm{m}}| \leq \Delta M$. This lets us induce a function, denoted $\bar{\Delta}\theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{J+K+1}$, which has all the properties stated in the theorem.

Aggregate feasibility of x^* and of $\widehat{x}$, taking into account the fact that asset prices do not change, implies that (3) takes the form

$$\bar{\Delta}x(\sigma) + \mathcal{P}_{\bar{\Delta}}(\sigma) \leq 0. \tag{13}$$

To be able to construct the sub-tree that interests us, we will need an implication of the no arbitrage property of asset prices. Given a node $\sigma \in \Sigma$, let $A(\sigma)$ denote the set of immediate successors at which the payoff from the portfolio is positive, i.e.,

$$A(\sigma) := \{\sigma' \in \sigma^+ : \mathcal{P}_{\bar{\Delta}}(\sigma') > 0\}.$$

Using the no arbitrage property of asset prices, (6), we have that for every node $\sigma \in \Sigma$ there is $a_\sigma \in R^S_{++}$ such that

$$\mathcal{C}_{\bar{\Delta}}(\sigma) = a_\sigma \cdot ((\mathcal{P}_{\bar{\Delta}}(\sigma'))_{\sigma' \in \sigma^+}). \tag{14}$$

A direct implication of (14) is that if $\mathcal{C}_{\bar{\Delta}}(\sigma) > 0$ then $A(\sigma) \neq \emptyset$.

We now identify the root of the sub-tree that interests us. First we show that $\mathcal{P}_{\bar{\Delta}}(\sigma) = 0$ for all $\sigma \in \Sigma$ cannot hold. If $\mathcal{P}_{\bar{\Delta}}(\sigma) = 0$ for all $\sigma \in \Sigma$ then, by (13), $\bar{\Delta}x(\sigma) \leq 0$ for all $\sigma \in \Sigma$, while, by (14), $\mathcal{C}_{\bar{\Delta}}(\sigma) = 0$ for all $\sigma \in \Sigma$, so that (12) can never hold with a strict inequality contradicting the existence of an improvement.

Let $\bar{\sigma}$ be such that $\mathcal{P}_{\bar{\Delta}}(\bar{\sigma}) \neq 0$ and $\mathcal{P}_{\bar{\Delta}}(\sigma) = 0$ for all σ such that $t(\sigma) < t(\bar{\sigma})$. In the partial order of dates, $\bar{\sigma}$ is the "first node" (it need not be the unique node with this property) with non-zero aggregate transfer to the old. We proceed to verify the existence of a node $\tilde{\sigma} \in \bar{\sigma}^+_{-1}$ with the additional property that $\mathcal{P}_{\bar{\Delta}}(\tilde{\sigma}) > 0$. By the definition of $\bar{\sigma}$, $\mathcal{P}_{\bar{\Delta}}(\bar{\sigma}_{-1}) = 0$ so that, by (13), $\bar{\Delta}x(\bar{\sigma}_{-1}) \leq 0$; but then, by (12), $\mathcal{C}_{\bar{\Delta}}(\bar{\sigma}_{-1}) \geq 0$ necessarily. In case $\mathcal{P}_{\bar{\Delta}}(\sigma) \leq 0$ for all $\sigma \in \bar{\sigma}^+_{-1}$, with a strict inequality at some node, i.e., the aggregate transfer to the old born at $\bar{\sigma}_{-1}$ is never positive and is negative at some node, then, by (14), $\mathcal{C}_{\bar{\Delta}}(\bar{\sigma}_{-1}) < 0$ contradicting $\mathcal{C}_{\bar{\Delta}}(\bar{\sigma}_{-1}) \geq 0$. So, in order to have an improvement $\mathcal{P}_{\bar{\Delta}}(\sigma) > 0$ for some $\sigma \in \bar{\sigma}^+_{-1}$; denote $\tilde{\sigma}$ one of the nodes at which $\mathcal{P}_{\bar{\Delta}}(\sigma) > 0$ for $\sigma \in \bar{\sigma}^+_{-1}$.[24]

$\tilde{\sigma}$, with date $\tilde{t} := t(\tilde{\sigma})$, is the root of the sub-tree that interests us. Since $\tilde{\sigma} \in \bar{\sigma}^+_{-1}$, $\tilde{\sigma}_{-1} = \bar{\sigma}_{-1}$; so $\mathcal{C}_{\bar{\Delta}}(\tilde{\sigma}_{-1}) \geq 0$. Since $\mathcal{P}_{\bar{\Delta}}(\tilde{\sigma}) > 0$, $\bar{\Delta}\theta(\tilde{\sigma}_{-1}) \neq \underline{0}$. $\bar{\Delta}\theta(\sigma) = \underline{0}$ for all σ such that $t(\sigma) < t(\tilde{\sigma}_{-1}) = t(\tilde{\sigma}) - 1$. Of course, $\tilde{t} \geq 1$.[25]

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[24] Clearly, there might exist nodes which are successors to the predecessor of $\bar{\sigma}$, i.e., for $\sigma \in \bar{\sigma}^+_{-1}$, such that $\mathcal{P}_{\bar{\Delta}}(\sigma) < 0$.

[25] The argument needs to be changed slightly when $\tilde{t} = 1$. In this case $\mathcal{C}_{\bar{\Delta}}(\tilde{\sigma}_{-1})$ is not defined so one cannot say that $\mathcal{C}_{\bar{\Delta}}(\tilde{\sigma}_{-1}) \geq 0$. However, monotonicity of the preferences of the initial old guarantees that $\mathcal{P}_{\bar{\Delta}}(\sigma) \geq 0$ for all $\sigma \in \Sigma_1$ so that $\mathcal{P}_{\bar{\Delta}}(\sigma) \neq 0$ for all $\sigma \in \Sigma_1$ directly implies that $\mathcal{P}_{\bar{\Delta}}(\tilde{\sigma}) > 0$ for some $\tilde{\sigma} \in \Sigma_1$.

We can now define the sub-tree that interests us.

$$\Gamma_{\tilde{\sigma}} := \{\tilde{\sigma}\} \cup A(\tilde{\sigma}) \cup_{\sigma \in A(\tilde{\sigma})} A(\sigma) \cup_{\sigma' \in \cup_{\sigma \in A(\tilde{\sigma})} A(\sigma)} A(\sigma') \cdots .$$

The definition is recursive and starts by including $\tilde{\sigma}$ and the set $A(\tilde{\sigma})$.

We have shown that $\mathcal{P}_{\bar{\Delta}}(\tilde{\sigma}) > 0$ so, by (13), $\bar{\Delta}x(\tilde{\sigma}) < 0$; so (12) at the node $\tilde{\sigma}$ implies that $\mathcal{C}_{\bar{\Delta}}(\tilde{\sigma}) > 0$. Therefore, $A(\tilde{\sigma}) \neq \emptyset$. So $\mathcal{P}_{\bar{\Delta}}(\sigma) > 0$ for $\sigma \in A(\tilde{\sigma})$ and (13) implies that

$$0 < \mathcal{P}_{\bar{\Delta}}(\sigma) \leq -\bar{\Delta}x(\sigma), \qquad \text{for all } \sigma \in A(\tilde{\sigma}), \tag{15}$$

which, when substituted into (12), leads to

$$-\mathcal{P}_{\bar{\Delta}}(\sigma) + \mathcal{C}_{\bar{\Delta}}(\sigma) \geq \underline{\rho} \cdot \left[\mathcal{P}_{\bar{\Delta}}(\sigma)\right]^2 \qquad \text{for all } \sigma \in A(\tilde{\sigma}). \tag{16}$$

By repeating the argument, we see that the sub-tree $\Gamma_{\tilde{\sigma}}$ has the following properties:

$$-\bar{\Delta}x(\sigma) \geq \mathcal{P}_{\bar{\Delta}}(\sigma) > 0 \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \tag{17}$$

by (17) and (12)

$$\mathcal{C}_{\bar{\Delta}}(\sigma) \geq \mathcal{P}_{\bar{\Delta}}(\sigma) + \underline{\rho} \cdot \left[\mathcal{P}_{\bar{\Delta}}(\sigma)\right]^2 \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \tag{18}$$

by (17) and (18)

$$\mathcal{C}_{\bar{\Delta}}(\sigma) > 0 \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \tag{19}$$

by (19) and (14)

$$A(\sigma) \neq \emptyset \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}},$$

which, with (13), implies that (17) holds.

By aggregate feasibility of consumption and non-negativity of consumption when young, the net intergenerational transfer in equilibrium, given by the return on the aggregate asset endowment, satisfies

$$-\omega(\sigma) \leq \omega \cdot r^*(\sigma) \leq \omega(\sigma).$$

Clearly, the inequality above implies that the payoff from the aggregate feasible portfolio $\bar{\Delta}\theta$ is bounded,

$$|\mathcal{P}_{\bar{\Delta}}(\sigma)| \leq \omega(\sigma) \leq \Omega \qquad \text{for all } \sigma \in \Gamma \tag{20}$$

using hypothesis (i) in Theorem 2, i.e., that the aggregate endowment of the consumption good is uniformly bounded across nodes. Let $\bar{P} := \Omega$. So, by (17) and (20), the function $\mathcal{P}_{\bar{\Delta}}$ satisfies $0 < \mathcal{P}_{\bar{\Delta}}(\sigma) \leq \bar{P}$ on the set of nodes such that $\sigma \in \Gamma_{\tilde{\sigma}}$. Furthermore, it is clear that if a node σ satisfies $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ and $\sigma \notin \Gamma_{\tilde{\sigma}}$, then $\sigma \notin A(\sigma_{-1})$ so that $-\bar{P} \leq \mathcal{P}_{\bar{\Delta}}(\sigma) \leq 0$, as stated in (b) of Theorem 2.

By (19), $0 < \mathcal{C}_{\bar{\Delta}}(\sigma)$ on the set $\Gamma_{\tilde{\sigma}}$ as stated in (b) of Theorem 2.

Since, by (17), $\mathcal{P}_{\bar{\Delta}}(\sigma) > 0$, for any node in the sub-tree $\Gamma_{\tilde{\sigma}}$ (18) can be rewritten as

$$\frac{1}{\mathcal{C}_{\bar{\Delta}}(\sigma)} \leq \frac{1}{\mathcal{P}_{\bar{\Delta}}(\sigma)} - \frac{\underline{\rho}}{1 + \underline{\rho}\mathcal{P}_{\bar{\Delta}}(\sigma)} \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \tag{21}$$

while $\underline{\rho} > 0$ and (20) imply that

$$\frac{\underline{\rho}}{1+\underline{\rho}\Omega} \leq \frac{\underline{\rho}}{1+\underline{\rho}\mathcal{P}_{\bar{\Delta}}(\ \sigma)} \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}.$$

Using the above in (21) we obtain the condition

$$\frac{1}{\mathcal{C}_{\bar{\Delta}}(\ \sigma)} + \frac{\underline{\rho}}{1+\underline{\rho}\Omega} \leq \frac{1}{\mathcal{P}_{\bar{\Delta}}(\ \sigma)} \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}},$$

$$\Leftrightarrow \qquad \frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma)} + \frac{\underline{\rho}}{1+\underline{\rho}\Omega}\mathcal{P}_{\bar{\Delta}}(\ \sigma) \leq 1 \qquad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}. \tag{22}$$

By iterating on the inequality (22) along paths in the sub-tree $\Gamma_{\tilde{\sigma}}$ we obtain

$$\Pi_{t=\tilde{t}}^{T-1}\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_t)}{\mathcal{C}_{\bar{\Delta}}(\sigma_t)} + \frac{\underline{\rho}}{1+\underline{\rho}\Omega}\sum_{t=\tilde{t}}^{T-1}\left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\sigma_\tau)}\right]\right\}\mathcal{C}_{\bar{\Delta}}(\ \sigma_t) \leq 1$$

$$\Rightarrow \qquad \Pi_{t=\tilde{t}}^{T-1}\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_t)}{\mathcal{C}_{\bar{\Delta}}(\sigma_t)}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_T)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma_T)} + \frac{\underline{\rho}}{1+\underline{\rho}\Omega}\mathcal{P}_{\bar{\Delta}}(\ \sigma_T)\right] + \frac{\underline{\rho}}{1+\underline{\rho}\Omega}\sum_{t=\tilde{t}}^{T-1}\left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\sigma_\tau)}\right]\right\}\mathcal{C}_{\bar{\Delta}}(\ \sigma_t) \leq 1$$

$$\Rightarrow \qquad \Pi_{t=\tilde{t}}^{T}\frac{\mathcal{P}_{\bar{\Delta}}(\sigma_t)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma_t)} + \frac{\underline{\rho}}{1+\underline{\rho}\Omega}\sum_{t=\tilde{t}}^{T}\left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma_\tau)}\right]\right\}\mathcal{C}_{\bar{\Delta}}(\ \sigma_t) \leq 1.$$

Since all the terms in the inequality above, including each term in the series, are positive, the series, being bounded and increasing, converges. It follows that

$$\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\ \sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\ \sigma_\tau)}\right] \leq 1 \qquad \lim_{T\to\infty}\sum_{t=\tilde{t}}^{T}\left\{\Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_{\bar{\Delta}}(\sigma_\tau)}{\mathcal{C}_{\bar{\Delta}}(\sigma_\tau)}\right]\right\}\mathcal{C}_{\bar{\Delta}}(\ \sigma_t) \leq B := \frac{1}{\underline{\rho}} + \Omega$$

along the path. The proof is completed by noting that the argument applies to every path in the sub-tree. ■

PROOF OF THEOREM 3: We need to construct a feasible allocation, obtainable via a reallocation of assets, which improves over the equilibrium allocation. We do so by first proposing a candidate asset reallocation, then verifying that the induced consumption allocation is aggregate feasible and gives consumption vectors in the consumption set of each agent, and finally verifying that the proposed transfers satisfy a curvature condition which guarantees that we have an improvement.

$\Gamma_{\tilde{\sigma}}$, a sub-tree, denotes the set of nodes that we will work with. From here onwards $\sigma \in \Gamma_{\tilde{\sigma}}$ unless otherwise noted. $\Delta\theta$, $\mathcal{C}_\Delta$, $\mathcal{P}_\Delta$, etc., refer to the functions whose existence is assumed in the statement of Theorem 3.

From now on consider a fixed path in the sub-tree, $\sigma^\infty(\Gamma_{\tilde{\sigma}})$.

Let

$$\mathcal{R}_\Delta(\sigma_t) := \Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_\Delta(\ \sigma_\tau)}{\mathcal{C}_\Delta(\sigma_\tau)}\right].$$

By hypothesis (c) in Theorem 3, $0 < \mathcal{R}_\Delta(\sigma_t) \leq 1$ and

$$0 < \lim_{T\to\infty} \sum_{t=\tilde{t}}^{T} \mathcal{R}_\Delta(\ \sigma_t)\mathcal{C}_\Delta(\sigma_t) \leq B. \tag{23}$$

Define

$$a(\ \sigma_t) := \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{1}{\max\{1,B^2\}} \cdot \mathcal{R}_\Delta(\sigma_t)\Big[\sum_{s=\tilde{t}}^{t} \mathcal{R}_\Delta(\ \sigma_s)\mathcal{C}_\Delta(\ \sigma_s)\Big] \qquad \text{for } \ \sigma_t \in \Gamma_{\tilde{\sigma}},$$

$$a(\sigma) := 0 \qquad \text{otherwise},$$

where $\kappa := (1/2)\frac{\varepsilon}{\bar{P}}$ and $\bar{\rho} > 0$, $\varepsilon > 0$, and $\bar{P} > 0$ are specified in hypotheses (i), (ii) and (c) of Theorem 3. Clearly,

$$\frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]} < 1 \tag{24}$$

$$\frac{B}{\max\{1,B^2\}} \leq 1. \tag{25}$$

So, using $\mathcal{R}_\Delta(\sigma_t) \leq 1$, (23), (24), and (25), we have

$$0 \leq a(\sigma) \leq \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{1}{\max\{1,B^2\}} 1 \cdot B = \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{B}{\max\{1,B^2\}} < 1. \tag{26}$$

The reallocation can now be defined. For assets

$$\widehat{\theta}(\sigma,h) := \theta^*(\sigma,h) + a(\sigma)\Delta\theta(\sigma) \qquad \text{for } h = h_\sigma,$$

$$\widehat{\theta}(\sigma,h) := \theta^*(\sigma,h) \qquad \text{for } h \neq h_\sigma,$$

where h_σ is specified in Theorem 3. Clearly, aggregate feasibility of the asset allocation is maintained since the portfolio reallocation $\Delta\theta$ was assumed to be feasible, and, by (26), $0 \leq a(\sigma) < 1$.

Consumption when young is reassigned as follows

$$\widehat{x}_{\sigma,h}(\sigma) := x^*_{\sigma,h}(\sigma) + (-1)a(\sigma_{-1}) \cdot \mathcal{P}_\Delta(\sigma) \qquad \text{for } \sigma \in \Sigma,$$

while consumption when old is induced according to

$$\widehat{x}_{\sigma,h}(\sigma') := x^*_{\sigma,h}(\sigma') + a(\sigma) \cdot \mathcal{P}_\Delta(\sigma') \qquad \text{for } \sigma' \in \sigma^+, \quad \text{for } \sigma \in \Sigma,$$

and similarly for the initial old.

Evidently, at each node the change in the consumption allocation of an old agent is offset by an identical change of opposite sign in the consumption allocation of a young agent; thus aggregate feasibility is always maintained by construction. We now show that every individual obtains a consumption vector in his consumption set. For a young agent born at σ, where $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$,

$$\Delta\widehat{x}_{\sigma,h}(\sigma) := \widehat{x}_{\sigma,h}(\sigma) - x^*_{\sigma,h}(\sigma) = (-1)a(\sigma_{-1}) \cdot \mathcal{P}_{\bar{\Delta}}(\sigma)$$

$$\Rightarrow \qquad \| \Delta\widehat{x}_{\sigma,h}(\sigma) \| \leq |a(\sigma)| \cdot \bar{P}$$

since, by hypothesis (b) of Theorem 3, $|\mathcal{P}_{\bar{\Delta}}(\sigma)| \leq \bar{P}$ for every σ such that $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$,

$$\Rightarrow \qquad \| \Delta\widehat{x}_{\sigma,h}(\sigma) \| \leq \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{B}{\max\{1,B^2\}}\bar{P}$$

$$= (1/2)\varepsilon \cdot \frac{1}{1+(1/2)(\epsilon/\bar{P})[\max\{1,\bar{\rho}\}]}\frac{B}{\max\{1,B^2\}} < \varepsilon$$

using (25) and the definition of κ. By hypothesis (ii) of Theorem 3, $\varepsilon > 0$ is a uniform lower bound on every coordinate of the equilibrium consumption vector, i.e., when young and in all states faced when old. So in states in which the young transfer the good to the old, the post-transfer consumption of the young is strictly positive (and the old consume a positive amount since they receive the transfer); the states in which the young receive the good are the states in which the old are forced to pay up but the bound above is valid for all states and that implies that the old never surrender more than ε which leaves them with strictly positive post-transfer consumption. So every agent gets a vector in the interior of her consumption set.

We have shown that the proposed reallocation is feasible. We proceed to show that we have an improvement.

Consider a node which is a successor to a node in the sub-tree but which is not an element of the sub-tree, $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$ but $\sigma \notin \Gamma_{\tilde{\sigma}}$. In such a case the old agent delivers the commodity to a young agent, so $\mathcal{P}_{\Delta}(\sigma) < 0$ and $\widehat{x}_{\sigma,h}(\sigma) > x^*_{\sigma,h}(\sigma)$, and the young agent's asset holding is not perturbed so that, by monotonicity of preferences, such a young agent is improved.

For an agent whose asset holding is perturbed a sufficient condition for a local change from the equilibrium allocation to be weakly improving is that the inequality below holds[26]

$$\frac{\partial u_{\sigma,h}}{\partial x_\sigma} \cdot \Big[\Delta x_{\sigma,h}(\sigma) + \Delta\theta(\sigma,h)\cdot q^*(\sigma)\Big] \geq \bar{\rho}_{\sigma,h} \cdot \frac{\Big[\frac{\partial u_{\sigma,h}}{\partial x_\sigma}\cdot \Delta x_{\sigma,h}(\sigma)\Big]^2}{\frac{\partial u_{\sigma,h}}{\partial x_\sigma}}$$

where $\Delta\widehat{\theta}(\sigma,h) := \widehat{\theta}(\sigma,h) - \theta^*(\sigma,h)$. By replacing the values of $\Delta x_{\sigma,h}(\sigma)$ and $\Delta\theta(\sigma,h)$ by the proposed reassignment we obtain the inequality

$$-a(\sigma_{t-1})\mathcal{P}_{\Delta}(\sigma_t) + a(\sigma_t)\mathcal{C}_{\Delta}(\sigma_t) \geq \bar{\rho}_{\sigma,h}\cdot\Big[a(\sigma_{t-1})\mathcal{P}_{\Delta}(\sigma_t)\Big]^2. \tag{27}$$

We proceed to check that the proposed reallocation does indeed satisfy (27).

$$a(\sigma_t) - \frac{\mathcal{P}_{\Delta}(\sigma_t)}{\mathcal{C}_{\Delta}(\sigma_t)}a(\sigma_{t-1}) = \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{1}{\max\{1,B^2\}}\Bigg[\mathcal{R}_{\Delta}(\sigma_t)\mathcal{R}_{\Delta}(\sigma_t)\mathcal{C}_{\Delta}(\sigma_t)$$

$$+\mathcal{R}_{\Delta}(\sigma_t)\sum_{s=\tilde{t}}^{t-1}\mathcal{R}_{\Delta}(\sigma_s)\mathcal{C}_{\Delta}(\sigma_s) - \frac{\mathcal{P}_{\Delta}(\sigma_t)}{\mathcal{C}_{\Delta}(\sigma_t)}\mathcal{R}_{\Delta}(\sigma_{t-1})\sum_{s=\tilde{t}}^{t-1}\mathcal{R}_{\Delta}(\sigma_s)\mathcal{C}_{\Delta}(\sigma_s)\Bigg]$$

[26]See Definition 5 and Lemma 1 in Chattopadhyay and Gottardi (1999) for an explicit derivation of the required quadratic term in a related context. A similar argument can be used here.

$$= \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{1}{\max\{1,B^2\}}\Big[\mathcal{R}_\Delta(\ \sigma_t)\mathcal{R}_\Delta(\ \sigma_t)\mathcal{C}_\Delta(\ \sigma_t)\Big]$$

where we substitute for $a(\sigma)$ and use the fact that

$$\mathcal{R}_\Delta(\ \sigma_t) = \Pi_{\tau=\tilde{t}}^{t}\left[\frac{\mathcal{P}_\Delta(\ \sigma_\tau)}{\mathcal{C}_\Delta(\ \sigma_\tau)}\right] = \frac{\mathcal{P}_\Delta(\ \sigma_t)}{\mathcal{C}_\Delta(\ \sigma_t)}\cdot \Pi_{\tau=\tilde{t}}^{t-1}\left[\frac{\mathcal{P}_\Delta(\ \sigma_\tau)}{\mathcal{C}_\Delta(\ \sigma_\tau)}\right] = \frac{\mathcal{P}_\Delta(\ \sigma_t)}{\mathcal{C}_\Delta(\ \sigma_t)}\cdot\mathcal{R}_\Delta(\ \sigma_{t-1}). \quad (28)$$

So

$$a(\sigma_t) - \frac{\mathcal{P}_\Delta(\ \sigma_t)}{\mathcal{C}_\Delta(\sigma_t)}a(\sigma_{t-1}) = \frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{1}{\max\{1,B^2\}}\mathcal{C}_\Delta(\ \sigma_t)\ \Big[\mathcal{R}_\Delta(\ \sigma_t)\Big]^2. \quad (29)$$

Using (23) we have

$$1 \geq \left[\frac{\sum_{s=\tilde{t}}^{t-1}\mathcal{R}_\Delta(\ \sigma_s)\mathcal{C}_\Delta(\ \sigma_s)}{\max\{1,B\}}\right]. \quad (30)$$

Using (24) and (30) we obtain

$$1 \geq \frac{\kappa[\max\{1,\bar{\rho}\}]}{1+\kappa[\max\{1,\bar{\rho}\}]}\left[\frac{\sum_{s=\tilde{t}}^{t-1}\mathcal{R}_\Delta(\ \sigma_s)\mathcal{C}_\Delta(\ \sigma_s)}{\max\{1,B\}}\right]^2. \quad (31)$$

So (29) and (31) imply that

$$\begin{aligned} a(\sigma_t) - \frac{\mathcal{P}_\Delta(\ \sigma_t)}{\mathcal{C}_\Delta(\sigma_t)}a(\sigma_{t-1}) &\geq [\max\{1,\bar{\rho}\}]\mathcal{C}_\Delta(\sigma_t) \\ &\cdot\left[\frac{\kappa}{1+\kappa[\max\{1,\bar{\rho}\}]}\frac{1}{\max\{1,B\}}\mathcal{R}_\Delta(\ \sigma_t)\frac{\sum_{s=\tilde{t}}^{t-1}\mathcal{R}_\Delta(\ \sigma_s)\mathcal{C}_\Delta(\ \sigma_s)}{\max\{1,B\}}\right]^2 \\ &= [\max\{1,\bar{\rho}\}]\mathcal{C}_\Delta(\ \sigma_t)\left[\frac{\mathcal{P}_\Delta(\ \sigma_t)}{\mathcal{C}_\Delta(\ \sigma_t)}a(\ \sigma_{t-1})\right]^2 \\ &\geq \bar{\rho}_{\sigma,h}\mathcal{C}_\Delta(\ \sigma_t)\left[\frac{\mathcal{P}_\Delta(\ \sigma_t)}{\mathcal{C}_\Delta(\ \sigma_t)}a(\sigma_{t-1})\right]^2 \end{aligned}$$

where we use (28) and the facts that (i) $[\max\{1,B\}]^2 = \max\{1,B^2\}$ as $B > 0$, (ii) $\max\{1,\bar{\rho}\} \geq \bar{\rho} \geq \bar{\rho}_{\sigma,h}$ for all $\sigma \in \Gamma_{\tilde{\sigma}}$, and $h = h_\sigma$. This shows that the proposed reallocation does indeed satisfy (27) so that an agent whose asset holding is perturbed is also at least weakly improved.

Finally, the first agent to receive an asset transfer is one who is born at the node $\tilde{\sigma}$. She does not transfer consumption to those who are old when she is young since, by construction, $a(\tilde{\sigma}_{-1}) = 0$; however, $a(\tilde{\sigma}) > 0$ by construction and $\mathcal{C}_\Delta(\tilde{\sigma}) > 0$ by hypothesis (b) of Theorem 3, so for her (27) is satisfied with a strict inequality ensuring that overall we have a strict improvement.

We have verified that the proposed reallocation is an improvement thus completing the proof of the theorem. ■

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