OPTIMAL SHARING OF SURGICAL COSTS IN THE PRESENCE OF QUEUES*

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WP-AD 2001-27

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Editor: Instituto Valenciano de Investigaciones Económicas, S.A.
First Edition October 2001.
Depósito Legal: V-4063-2001

* We would like to thank Rebeca Jiménez, Nicolas Porteiro and an anonymous referee for their helpful comments. Financial support from the IVIE is gratefully acknowledged. All remaining errors are our own responsibility.

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A B S T R A C T

We deal with a cost allocation problem arising from sharing a medical service in the presence of queues. We use a standard queuing theory model in a context with several medical procedures, a certain demand of treatment and a maximum average waiting time guarantee set by the government. We show that sharing the use of an operating theatre to treat the patients of the di¤erent procedures, leads to a cost reduction. Then, we compute an optimal fee per procedure for the use of the operating theatre, based on the Shapley value. Afterwards, considering the post-operative time, we characterize the conditions under which this cooperation among treatments has a positive impact on the average post-operative costs. Finally, we provide a numerical example constructed on the basis of real data, to highlight the main features of our model.

KEYWORDS: Surgical Waiting Lists; Queueing Theory; Cost-Sharing Game.

1 INTRODUCTION

The widespread access to Public Health Care in western European countries is placing the system at a point in which optimal allocation of resources becomes a major management problem. On the one hand, and since health services are among the critical aspects to control the quality of public services, the regularity and adequacy of hospital services turn out to be crucial for the support of a certain government. On the other hand, management mistakes could have a tremendous impact on the Health Administration budget.

Citizens are particularly sensitive to some phenomena related to health services. One of those phenomena is the persistency of waiting lists for surgical treatment. The popular discomfort under this phenomenon forces the government to perform some especial programs to temporarily alleviate the problem. Notwithstanding, temporary programs cannot solve the problem, and may be extremely costly.

In the existing literature on waiting lists for surgical treatment there seems to be two separate traditions: the queueing theory tradition, which considers that the arrivals and service times are stochastic events, and the welfare economics literature, where queues are understood as a system for the distribution and allocation of resources.

Queueing theory studies this kind of problems from a statistical or operational research point of view.[1] Any system in which arrivals place demands upon a …nite-capacity resource may be termed a queueing system. In particular, if the arrival times of the demands are unpredictable, then con‡icts for the use of the resource will arise and queues of waiting customers will form. The main idea to predict the behavior of the system, is, nonetheless, extremely simple: the length of the queue depends upon the average rate of arrivals, and on the statistical ‡uctuations of this rate. Certainly, if the average rate of arrivals exceeds the capacity, then the system breaks down, and unbounded queues will form. However, when the average rate is less than the system capacity, then here too, we have the formation of queues due to the statistical ‡uctuations and spurts of arrivals that may occur.

The formation of waiting lists to get elective surgery, can be framed as a queueing system. Queueing theory predicts several characteristics of the waiting lists such as the average waiting time of the agents or the average length of the queue. Assuming that the agents are served respecting their arrival order, the only control variable is the capacity to install. Consequently, the theory can help us to take decisions concerning that capacity, taking into account that the higher the capacity the higher the associated costs, but the shorter the expected queues.

The queueing system arising in surgical treatment has some speci…c characteristics: (1)

[1] For a general view of this topic see Gross and Harris (1997), Hillier and Lieberman (1995), Kleinrock (1975) and Prabhu (1997).

There are two sources for the formation of waiting lists. On the one hand, the capacity of the operation theatre, and, on the other hand, the bed capacity of the hospital; (2) Several medical procedures share both servers, namely, customers from di¤erent treatments need to use both the operation theatre and the beds; (3) Each of those procedures have their own rate of arrival; (4) Not all medical procedures are considered as equally urgent, in the sense that the average waiting time politically considered as adequate di¤ers among procedures.

In the managing of such a situation, a cost allocation problem arises: Since di¤erent procedures share both the operation theatre and the hospital beds, we have to design a cost allocation rule in order to share the joint costs. This is the main purpose of this paper. In order to construct a cost allocation rule, we use a game theoretical perspective, designing a cost allocation game. In the …rst part of the paper we concentrate ourselves on the costs associated to the operating theatre. Then, we construct a game by confronting two situations: one in which each medical procedure has its own operating theatre, and another one in which there is a unique theatre that serves all the diseases. We show that sharing the use of the operating theatre to treat the patients of the di¤erent medical procedures, leads to a cost reduction. Then, we construct a cost-sharing game and, given the characteristics of the game, we suggest a cost-sharing rule that recommends the Shapley value allocation of the cost-sharing game. Thus, our optimal tari¤ has all the nice properties of the Shapley value.[2] The fact that this cooperative solution can be computed easily, is certainly an important property in a practical environment.

The cost-sharing game emerging among the treatments is the sum of an additive game plus an "airport game",[3] where the di¤erent landing track capacities are translated in our model to the capacity required by the operating theatre in order to satisfy its demand, according to the maximum average waiting time guarantee. A similar idea has been applied in Fragnelli et al. (2000) to the construction of a railway path, as a proposal for the reorganization of the railway sector in Europe. In their case, as we do here, the proposed solution to the cost-sharing game is the Shapley value.

Up to this point, only the direct costs derived from surgical interventions were considered. However, we have to take into account that an operation generates also other costs, more precisely the costs incurred during the patients' hospitalization time for recovering. Then, we introduce in the model the post-operative costs and we study how they are a¤ected by the cooperation among medical procedures. Treating the beds as servers, we may model the hos-

[2] See Shapley (1953), Tijs and Driesen (1986), Young (1994) and Moulin and Shenker (1996).

[3] It is a game-theoretic approach to a cost- allocation problem arising at airports. Di¤erent types of airplanes need di¤erent runway lengths, but the largest runway is su¢ cient for all of them. Then, the problem lies in allocating the capital costs of constructing the largest track among the set of users. See Littlechild and Owen (1973) and Littlechild and Thompson (1977).

pitalization stage also as a queueing system. Then, the number of servers (beds) required to guarantee the service, can be computed in different scenarios. Nonetheless, there is no possibility of arriving at general results, due to the lack of analytical solvability of the model.

In spite of that, something general can be said about the average number of beds. By so doing, we show that sharing the use of the operating theatre has an ambiguous effect on average post-operative costs. If the medical procedure with the highest priority level, has a higher recovering time than the average hospitalization time of the rest of the pathologies, we can ensure that in average terms cooperation leads to post-operative cost savings.

A numerical example with real data is analyzed then. In this example, we compute the distribution of surgical costs, applying the theoretical results obtained previously. As for the number of beds required, we also compute them, under different scenarios. Also, we estimate the distribution of bed costs among the procedures, provided that an upper bound of $.1$ is set on the probability of waiting after the intervention.

Most of the contributions to the literature of hospital waiting lists have focused on the demand side. Culyer and Cullis (1976) and Cullis and Jones (1985) highlight demand factors as the ones affecting the waiting list problem.

However, some papers address the problem from the supply side. For instance, Iversen (1993), shows that the non-cooperative character of resource allocation in Public Health Services may contribute to excessive waiting lists. Our work fits in this supply side branch of the literature, since we study the costs derived from increasing the capacity of the operating theatre in order to decrease the time spent by the patients on the waiting list.

Recent papers mainly focus on the effects of such waiting lists on patients' welfare and on the purchase of private health insurance. Johannesson (1998) develops a model of the benefits and costs of being on a waiting list. Since changes in the length of the waiting time causes complex shifts in utility streams, shorter waiting time need not necessarily be preferred to a longer one. Besley (1999) shows that longer waiting lists for public treatment, are associated with greater purchases of private health insurance. In our analysis, neither patients' welfare nor private provision is considered.

The problem of the hospital bed supply has been also treated in the literature. Joskow (1980) and Worthington (1987) use a queueing model to analyze the characteristics of the hospital bed supply. They both consider that the beds are the servers of the system. Hence, the waiting lists are determined by the interaction between two facts: on the one hand, the arrival of new patients and their lengths of stay and, on the other hand, the amount of available beds. Instead, we consider that the queue is formed in the previous stage. Then, when a patient leaves the queue and enters the operating theatre, we put a small upper bound to the probability that there is not a bed prepared for him.

The rest of the paper is organized as follows: Section 2 presents the model with its basic assumptions. Section 3 studies the operating theatre costs. Section 4 computes the optimal cost sharing. Section 5 introduces in the model the post-operative time. Section 6 provides a numerical example. Finally, Section 7 gives some concluding remarks.

2 THE MODEL

We consider the basic queueing process: customers requiring a service are generated over time by an input source. These customers arrive to the system and join a queue. In different moments, one of the customers is selected to receive the service by means of a queue discipline. Then, the mechanism of service provides the service and the customer leaves the system.

In our problem, the customers are patients that require surgical treatment and the mechanism of service is a hospital. Actually, there are two different sort of servers in our model: (1) the operating theatre, and (2) the hospital beds. Any individual entering the system should go first throughout the operating theatre, and once he/she is out of this server, a bed should be waiting for him/her. The patient only leaves the system once he is released from the hospital.

Let us consider a situation in which we have n kind of diseases or medical procedures and a certain number of patients requiring a service from the different procedures. Let $N = \{1, 2, ..., n\}$ denote the set of fields of treatment.

We assume that the number of potential patients is infinite. This is a standard assumption in queueing theory, which simply makes the model analytically more tractable. The main implication of this assumption is that the number of individuals in the queue does not affect the amount of potential entrants. It seems reasonable in our framework, since the probability of needing some medical treatment is, in principle, independent from the amount of people requiring it.[4] We put no restriction on the length that the queue can reach, which is also standard in the literature even when dealing with situations where actually a finite upper bound exists, but it is large enough.

We assume that the patients' arrivals to the medical system follow a Poisson process. This means that every period of certain length, has the same probability of receiving a patient. We can define $\lambda_i \in R_{++}$ as the average number of arrivals per unit of time, from the i_{th} medical procedure. This is equivalent to say that the time between arrivals of patients of the same type is given by an exponential distribution with mean $\frac{1}{\lambda_i}$.

[4] In our work, we completely abstract from physicians' strategic behavior. If we consider this possibility, this assumption would be difficult to sustain, since the lenght of the waiting lists could affect the incentives of the General Practitioners to send more or less patients to an elective surgery treatment.

The work an arriving patient brings into the operating theatre, equals the time of service he requires. We consider this service time to follow a random process. Even if in principle the length of a surgical intervention from a given medical procedure is fixed, an operation can unexpectedly become more complicated, and hence require some extra time. Therefore, it is necessary to consider some randomness in the service process.

We measure the service time not in absolute terms (length of the operation) but in relative ones, using as reference the total time that the operating theatre is opened per day. For example, if an intervention lasts 2 hours (in expected terms) and the service is opened 8 hours per day, the expected service time of a patient would be $\frac{1}{4}$. The interpretation would be that each patient covers one-fourth of the total time the server is working in a day.

With this construction, the service time of the i_{th} medical procedure would follow an exponential distribution with average $\frac{1}{\mu_i}$, and $\mu_i \in \mathbb{R}_{++}$, where $\frac{1}{\mu_i}$ captures the fraction of the total working time of the server employed in one patient.

Analogously, μ_i would stand for the maximum expected rate (capacity) at which the system can perform its work (different among treatments), i.e., the potential average rate of type i patients' departures per unit of time.

We require that $\lambda_i < \mu_i \; \forall i \in \{1, 2, ..., n\}$, otherwise the queue would "explode" and the system will break down.

We also assume that the queue discipline is "first come, first served", i.e., the patients will be chosen to receive the service by their order of arrival.[5]

As in many queueing theory models, we assume that the arrivals and departures from the system behave as a "birth and death process". Hence, we impose that in any given instant, only one "birth" (arrival of a patient to the queue) and one "death" (departure of a patient from the operating theatre) can occur.

Finally, we perform our analysis considering that we are in a situation in which the steady state of the system has been reached.

3 OPERATING THEATRE COSTS

In order to characterize the costs associated with giving the operating theatre enough capacity to provide the service in the legal maximum time, we consider two alternative scenarios. Recall that in this Section, the system is simply the operating theatre.

In the first scenario, each surgical procedure has its own operating theatre to treat its pa-

[5] Another possibility would be to consider the case in which the queue discipline is based in some priority rule. However, the mathematical analysis becomes more complicated and only limited results are available.

tients. Let us denote by W_i the average waiting time of an individual of type i in the system. Notice that this time includes not only waiting in the queue, but also the time spent in the operating theatre.[6] If we considered only the time in the queue, what in principle could seem more reasonable, the qualitative results would not change and the model would become less tractable. Moreover, the time spent in the operating theatre is negligible with respect to the total time in the system. Under all previous assumptions, it is well-known that the average time of an individual in the system is given by:

$$W_i = \frac{1}{\mu_i - \lambda_i}. \qquad (1)$$

In the second scenario the different medical procedures share the use of a unique operating theatre. Proceeding analogously, and denoting by W the average time of an individual in the system, independently from his type, we have:

$$W = \frac{1}{\mu - \sum_{i=1}^{n} \lambda_i}; \qquad (2)$$

where μ is the average number of patients that leave the operating theatre per unit of time (coming from any of the treatments) and $\sum_{i=1}^{n} \lambda_i$ is the total average number of arrivals per unit of time. Since the arrivals of patients are independent events across specialities, the total number of arrivals follows also a Poisson process and its average is computed as the sum of the average of the arrivals of the patients coming from the n medical procedures.

Finally, we consider that the government sets that, on average, the maximum waiting time in the system for the i_{th} medical procedure can not exceed t_i, i.e., a maximum average waiting time guarantee is provided.[7] Moreover, these times differ across treatments (applying, for example, an urgency criterium) and we suppose, without loss of generality, that $t_1 \geq t_2 \geq \ldots \geq t_n$.

In the following subsections, we will study the costs required to fulfill the government's objective under the two scenarios mentioned above. In order to do it, we will assume that the costs are proportional to the amount of patients treated per unit of time. We interpret this as having costs that are linear in the capacity of the server. In our case, an increase in the capacity could be understood as having the operating theatre opened more hours, to be able to serve more patients.

[6] As in this section we are only studying the direct costs derived from the operating theatre, we exclude from our analysis the post-operative time spent at hospital.

[7] There exists evidence about the implementation of this kind of measures, by some Public Health Administrations in Europe. For instance, a maximum waiting time guarantee was introduced in Sweden in 1992 to shorten waiting times (see Hanning and Wimblad Spånberg (2000)). In Spain, the Ministry of Health and Consumption has recently design a program (Programa Avance INSALUD) that tries to ensure an average waiting time guarantee to those patients demanding elective surgery.

3.1 Different Operating Theatres

Since the costs of operating are proportional to the amount of patients treated, the overall costs arising from the n operating theatres are just the sum of the individual costs. The explicit shape of the individual costs is the following:

$$C_i = k\mu_i(t_i);$$

with $k \in R_{++}$.

We have to set μ_i (the potential amount of patients from medical procedure i that leave, on average, the theatre per unit of time) in order to guarantee the corresponding legal maximum waiting time (t_i). What indirectly we are setting is the number of hours that the theatre should stay opened per day. Formally:

$$W_i = \frac{1}{\mu_i - \lambda_i} = t_i \iff \mu_i(t_i) = \frac{1}{t_i} + \lambda_i. \tag{3}$$

Hence, the costs per medical procedure are:

$$C_i = k\left(\frac{1}{t_i} + \lambda_i\right); \; \forall i \in \{1; 2; ....; n\}. \tag{4}$$

As we can see these costs are increasing in the ratio at which the patients arrive (λ_i), and decreasing in the maximum average waiting time guarantee (t_i). The two features are reasonable, the more patients that arrive and the lower the average time we can keep them waiting, the higher will be the costs.

The overall costs from keeping n operating theatres opened are, therefore:

$$C^N = \sum_{i=1}^{n} C_i = k\left(\sum_{i=1}^{n} \frac{1}{t_i} + \sum_{i=1}^{n} \lambda_i\right). \tag{5}$$

3.2 A Single Operating Theatre

In this scenario, as there is only one operating theatre, the total costs will be given by:

$$C^1 = k\mu(T);$$

where $\mu(T)$ is the potential number of patients treated, on average, per unit of time from any medical procedure, and T is the lowest value that the maximum average time guarantee takes across treatments. Formally:

$$T = \min\{t_i / i \in N\}. \tag{6}$$

This means that if the system has enough capacity to guarantee the legal average time t_i for the i_{th} medical procedure, then it has to be able to serve also any medical procedure j with $j < i$ according to its legal maximum average time (recall that if $j < i$ then $t_j > t_i$).

Proceeding analogously as in the previous subsection, we compute $\mu^1(T)$:

$$W = \frac{1}{\mu - \sum_{i=1}^{n} \lambda_i} = T \iff \mu^1(T) = \frac{1}{T} + \sum_{i=1}^{n} \lambda_i. \tag{7}$$

Therefore, the overall costs are:

$$C^1 = k\left(\frac{1}{T} + \sum_{i=1}^{n} \lambda_i\right). \tag{8}$$

3.3 Comparing the Costs in both Scenarios

We proceed now to compare the costs obtained in the two analyzed situations. The aim is to check if there is any kind of saving, understood as lower aggregate costs, in the scenario in which the medical procedures share the use of the operating theatre.

Proposition 1 *Sharing the use of the operating theatre leads to a cost reduction.*

Proof. Using Equations (5) and (8) and, taking into account (6), it is straightforward to compute the sign of the difference between the costs keeping opened n operating theatres and the costs maintaining only one which serves all the medical procedures. We obtain:

$$C^N - C^1 = k\left(\sum_{i=1}^{n-1} \frac{1}{t_i}\right) > 0,$$

since both k and t_i $\forall i \in \{1, 2, ..., n\}$ are strictly positive. ■

Hence, we can see that it is possible to make savings if the different surgical procedures cooperate and share the use of the operating theatre. Let us explain the reason for this. When each medical procedure maintains its own server, it has to suffer not only a cost that is proportional to the average number of patients demanding surgical attention ($k\lambda_i$), but also a fixed cost depending on the maximum average waiting time guarantee for the medical procedure ($\frac{1}{t_i}$). This is due to the randomness of the process we are dealing with. Both the number of patients' arrivals and the number of patients departures, are measured in expected terms, since we are working with variables distributed according to random processes. Therefore, every operating theatre should maintain some additional capacity, to prevent a situation in which a number of patients higher than the expected one would arrive in a certain instant of time, or an operation gets complicated, requiring some extra time. If there is cooperation among the medical procedures, they maintain this necessary additional capacity just supporting together the fixed extra cost of the most priority one. The degree of priority is understood in our model as the average waiting time guarantee, and the lower the waiting time guarantee the higher the priority degree.

We can interpret the lower necessary capacity in terms of optimal risk-sharing among treatments. When a medical procedure is on its own, it has to cover all the risks, understanding them as an excessive arrival of patients or a hard intervention. This means that it has to fix supplementary capacity to be able of guaranteeing the legal average waiting time, even when the circumstances are worse than expected during a certain period of time.

However, when this medical procedure shares the use of the operating theatre with others, the "bad luck" in one treatment in a given day, may get compensated with the good one of another. It is not so important if one medical procedure performs badly one day because it can take some extra time from any other which has been more lucky in the realization of the uncertainty. This is a similar phenomenon to risk spreading. As we can compensate the results among medical procedures, we can serve the demand in the legal average time with less installed capacity (and therefore, at a lower cost).

Since the scenario in which cooperation appears is cheaper than the other one, what is interesting now is how to distribute among the medical procedures this benefit from working together or, equivalently, what is the optimal tariff that each treatment should pay for the use of the service. In the following section, we will model this problem as a cost-sharing cooperative game and we will compute an optimal fee.

4 OPTIMAL COST SHARING

Summarizing, the problem we are facing is the following: we have an operating theatre used by patients coming from different surgical procedures and we try to divide across them the costs of using the service. Then, we have to study how the operating theatre costs should be allocated to the medical procedures through an optimal tariff.

We shall construct now a cost-sharing game. Let us consider that the players are the different surgical procedures, $N = \{1, \ldots, n\}$. The cost-sharing game is defined as follows: $c : 2^N \to \mathbb{R}$, assigns to any non-empty coalition S of medical procedures the minimum cost $c(S)$ under which the time guarantee is fulfilled, for all the surgical procedures in S. For the empty coalition we have $c(\emptyset) = 0$. Since sharing the operation theatre always conveys to a cost reduction, this minimal cost will coincide with that necessary to keep functioning a single operation theatre, shared by the pathologies in set S. Namely,

$$c(S) = k\left(\frac{1}{T_S} + \sum_{i \in S} \lambda_i\right);$$

where $T_S = \min\{t_i : i \in S\}$ is the shortest average time guarantee within S; namely, the average time guarantee established for the most urgent procedure in set S. Note that previous cost function can be divided into two parts:

$$c(S) = \left(\frac{k}{T_S}\right) + \left(k \sum_{i \in S} \lambda_i\right);$$

there is a variable expense, which is proportional to the number of patients of each medical procedure that demand the service $c^v(S) = \left(k \sum_{i\in S} \lambda_i\right)$, and a fixed cost $c^f(S) = \left(\frac{K}{T_S}\right)$, which is independent of the medical procedure the agents belong to. So, our cost-sharing game is the sum of two other games, $c = c^v + c^f$.

We shall adopt the recommendation given by the Shapley value of the game as a way of distributing the costs among the surgical procedures. This solution has the following properties:

1. It is optimal, namely, it recommends to go for the largest possible cost reduction. In our case, for the use of a single operating theatre. It divides the total cost $c(N) = C^1(N)$ among the pathologies.

2. It is linear, namely, in order to solve a game which is the sum of two other games, we simply solve them separately, and add. In our case, $Sh(c) = Sh(c^v) + Sh(c^f)$.

3. It is symmetric, namely, if two procedures are indistinguishable in cost terms, they should contribute the same amount to the total cost.

4. It is fair, namely, we cannot manipulate the outcome by introducing artificial procedures with zero cost.

Our cost-sharing game is the sum of two other cost-sharing games: the variable cost-sharing game, c^v, and the fixed cost-sharing game, c^f. Because of property (2), $Sh(c) = Sh(c^v) + Sh(c^f)$.

It turns out that the variable cost-sharing game, c^v, is a linear game. In this game there are no cost reductions due to cooperation between different procedures. Consequently, $Sh_i(c^v) = k\lambda_i$, for all $i \in N$.

Notice that the fixed cost-sharing game, c^f, is an analogous problem to the one appearing in the "airport game". In our case, instead of needing different landing track capacities for different types of planes, we deal with several operating theatre capacities, depending on the maximum average waiting time guarantee set by the government for the different surgical operations. This is a concave game. Consequently, the Core of the game is not empty, and the Shapley value allocation belongs to it. Taking into account (6) we know that:

$$\frac{k}{t_1} \leq \frac{k}{t_2} \leq \dots \leq \frac{k}{t_n} = \frac{k}{T}.$$

So, the fixed cost of that operating theatre with enough capacity to serve all the procedures depends essentially on the time guarantee of the procedure with the highest priority level (the n_{th} medical treatment in our case).

Baker (1965) and Thompson (1971) proposed a simple cost allocation rule to solve this type of cost-sharing problems. Littlechild and Owen (1973) showed that the above mentioned cost allocation coincides with the cost allocation based on the Shapley value. We can express this rule in the following way: each procedure contributes equally to the cost needed in order to maintain opened an operating theatre for the medical treatment with the least priority; then the contribution of the procedure with the least priority level is completely computed. Now, all remaining procedures also contribute equally to the additional cost needed to maintain opened a theatre for the treatment next in the finite order. This way, the second procedure contribution is completed, and so on.

Formally, the cost that should be charged to the j_{th} procedure is given by:

$$
\begin{aligned}
Sh_1(c^f) &= \frac{k}{nt_1} \\
Sh_2(c^f) &= k\left[\frac{1}{nt_1} + \frac{1}{(n-1)}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)\right] \\
Sh_3(c^f) &= k\left[\frac{1}{nt_1} + \frac{1}{(n-1)}\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + \frac{1}{(n-2)}\left(\frac{1}{t_3} - \frac{1}{t_2}\right)\right] \\
&\cdots \\
Sh_n(c^f) &= k\left[\frac{1}{nt_1} + \frac{1}{(n-1)}\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + \frac{1}{(n-2)}\left(\frac{1}{t_3} - \frac{1}{t_2}\right) + \cdots + \left(\frac{1}{t_n} - \frac{1}{t_{n-1}}\right)\right].
\end{aligned} \tag{9}
$$

Consequently, if in a certain period of time, we receive a set of patients M; where $M = M_1 \cup \cdots \cup M_n$ and M_i stands for the set of patients for procedure i, $m_i = \#M_i$; we have the following result:

Proposition 2 An optimal schedule of fees for any user $j \in M$ of the operating theatre $\left(\sigma_j^*(c)\right)$ is given by:

$$
\begin{array}{ll}
\sigma_j^*(c) = \frac{k}{m_1}\left[\lambda_1 + \frac{1}{nt_1}\right] & \text{if } j \in M_1 \\
\sigma_j^*(c) = \frac{k}{m_2}\left[\lambda_2 + \frac{1}{nt_1} + \frac{1}{(n-1)}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)\right] & \text{if } j \in M_2 \\
\cdots\cdots\cdots & \cdots\cdots \\
\sigma_j^*(c) = \frac{k}{m_n}\left[\lambda_n + \frac{1}{nt_1} + \frac{1}{(n-1)}\left(\frac{1}{t_2} - \frac{1}{t_1}\right) + \cdots + \left(\frac{1}{t_n} - \frac{1}{t_{n-1}}\right)\right] & \text{if } j \in M_n.
\end{array} \tag{10}
$$

Under our construction, M corresponds to the aggregate capacity set for the surgical theatre, hence $M = \mu(T)$. Analogously, the number of patients of each medical procedure will be computed as a fraction of this total capacity. Formally:

$$m_i = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} \mu(T).$$

5 EFFECTS ON THE POST-OPERATIVE COSTS

We have to take into account that a surgical intervention generates more costs than the direct ones derived from the operation. In almost every case, the patient has to spend some time in the hospital recovering, what is called "Post-Operative Time".[8] This time differs across

[8] There are exceptions, like the interventions for miopic reduction, in which the patient leaves the hospital just after being operated.

medical procedures, and has a random component, because not everybody reacts equally to an intervention.

The possibilities of fulfill a certain maximum average waiting time guarantee depend not only on the capacity of the operating theatre, but also by the availability of beds to treat the patients during their post-operative time. Actually, there is a second set of servers in the system: the beds to be used on post-operative patients. We may think of the operating theatre as the source of patients for this second set of servers. Once a patient exits the theatre, he should enter a bed. Thus, the system will work properly only if there are enough available beds for the patients leaving the theatre.

To compute the number of servers (beds) we need for the adequate functioning of the hospital, we make again use of queueing theory. If patients exit the operation theatre at a rate λ; they have a length of stay in the server of $\frac{1}{\mu}$; and there are b servers (beds) in the system, the probability that a patient has to wait for a server to be free is given by the Erlang's C formula. Denoting by N the number of patients in the system, we have:

$$P(\text{queueing}) = P(N \geq b) = \frac{\left(\frac{(b\rho)^b}{b!}\right)\left(\frac{1}{1-\rho}\right)}{\left[\sum_{k=0}^{b} \frac{(b\rho)^k}{k!} + \left(\frac{(b\rho)^b}{b!}\right)\left(\frac{1}{1-\rho}\right)\right]}; \tag{11}$$

where $\rho = \frac{\lambda}{b\mu} < 1$ is the necessary and sufficient condition for ergodicity in the system.

We can set a maximum value to this probability, and then, by solving previous equation, we can estimate the number of beds needed, b:

Erlang's formula can be used to compute the required number of beds under different scenarios: (1) If the procedures share neither the theatre nor the beds; (2) If they share the theatre and they do not share the beds; (3) If they do not share the theatre but they do share the beds; and (4) If they share both theatre and beds.

Obviously, and since we cannot analytically solve by Erlang's formula previous values, it will be only used for computational purposes. In Section 6 we deal with a numerical example, and there we illustrate the method to compare all mentioned scenarios.

Nonetheless, what we can do is to introduce in our analysis the impact of sharing the use of an operating theatre, on the average costs derived from the post-operative period, assuming that the procedures do not share the use of the beds. That is, we can compute the average number of beds required in scenarios 1 and 2 of those previously described.

5.1 Average Post-Operative Costs

Considered n fields of treatment, ordered inversely to their urgency (defined by t_i), so that the one with the highest priority (the one with the lowest average waiting time guarantee) is the

n^{th} medical procedure.

Let d_i with $i = 1, 2, \ldots n$, denote the average number of units of time that a patient of type i spends recovering at the hospital after the intervention. Again, the service time is exponentially distributed.

Then, the average number of beds required for medical procedure i in the absence of cooperation $\left(\bar{b}_i\right)$ depends on the capacity fixed by its operating theatre in the previous stage and on its patients expected post-operative time. Formally:

$$\bar{b}_i = \mu_i(t_i) d_i = \left(\frac{1}{t_i} + \lambda_i\right) d_i, \quad i = 1, 2, \ldots, n. \tag{12}$$

Therefore, the expected number of beds needed in the system in the absence of cooperation in the use of the operating theatre is:

$$\sum_{i=1}^{n} \bar{b}_i = \sum_{i=1}^{n} \frac{d_i}{t_i} + \sum_{i=1}^{n} \lambda_i d_i.$$

When the medical procedures share the use of the operating theatre, they need an amount of beds according to:

$$\bar{b}^N = \mu(T)\left(\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} d_i\right) = \left(\frac{1}{T} + \sum_{i=1}^{n} \lambda_i\right)\left(\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j} d_i\right). \tag{13}$$

In this situation, as all the patients are treated in the same theatre, when we compute the expected number of beds, we need to take into account the proportion of individuals of each type $\left(\frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}\right)$ treated in the considered period of time, and their corresponding average hospitalization times.

Finally, we assume that the post-operative costs are linear in the number of beds. This reduces the analysis of the costs to the computation of the required amount of beds.

Let us call $\Lambda = \sum_{j=1}^{n} \lambda_j$. Comparing scenarios 1 and 2, therefore, we obtain the following result.

Proposition 3 *Sharing the operating theatre reduces the average post-operative costs if and only if:*

$$\sum_{i=1}^{n} \lambda_i d_i < \Lambda T \sum_{i=1}^{n} \frac{d_i}{t_i}$$

Proof. We have to compute the difference $\sum_{i=1}^{n} \bar{b}_i - \bar{b}^N$.

We can rewrite (13) as:

$$\bar{b}^N = \frac{1}{T} \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} d_i + \sum_{i=1}^{n} \lambda_i d_i.$$

Then, $\sum_{i=1}^{n}\hat{\mu}_i - \hat{\mu}^N = \sum_{i=1}^{n}\frac{d_i}{t_i} + \sum_{i=1}^{n}\lambda_i d_i - \left(\frac{1}{T}\sum_{i=1}^{n}\frac{\lambda_i}{\Lambda}d_i + \sum_{i=1}^{n}\lambda_i d_i\right).$

$$\hat{\mu}^N < \sum_{i=1}^{n}\hat{\mu}_i \iff \frac{1}{T}\sum_{i=1}^{n}\frac{\lambda_i}{\Lambda}d_i < \sum_{i=1}^{n}\frac{d_i}{t_i}.$$

Namely,

$$\hat{\mu}^N < \textstyle\sum_{i=1}^{n}\hat{\mu}_i \iff \sum_{i=1}^{n}\lambda_i d_i < \Lambda T\sum_{i=1}^{n}\frac{d_i}{t_i}.$$

This completes the proof. ■

Previous result can be read in the following way: It is not always true that sharing the use of the operating theatre can lead to smaller average costs in the second stage of the treatment process (post-operative time). Formula $\sum_{i=1}^{n}\lambda_i d_i < \Lambda T\sum_{i=1}^{n}\frac{d_i}{t_i}$ can be read as follows: On the left hand side we have the aggregate expected time of hospitalization in a certain period of time of all individuals entering the queue (if they were going to be served). On the right hand side we can think of a similar hospitalization time, in which the average stay is given by $d = \sum_{i=1}^{n}\frac{Td_i}{t_i}.$ There are savings in the costs of the post-operative period if and only if $\sum_{i=1}^{n}\lambda_i d_i < \Lambda d.$

The explanation for this result has to do with the fact that the number of beds is proportional to the capacity installed in the previous stage. When the medical procedures cooperate they agree to set a common capacity, which is composed by a variable factor depending on each and every medical procedure (λ_i) and a constant additional term determined by the most demanding one $\left(\frac{1}{T}\right)$. We already proved in Section 3 that this cooperation ensures savings in the direct costs of the interventions. However, the less urgent field has the potential capacity to perform more interventions per unit of time, than if it were working on its own (since it is guaranteeing a lower average waiting time to its patients). Depending on its rate of patients' arrivals, it will make use of all this additional capacity or not, and this will determine the resulting amount of expected beds needed. Following this reasoning, we can see how the condition found in Proposition 3 to guarantee savings, is more difficult to be fulfilled the smaller the ratio $\frac{T}{t_i}$ for each and every medical treatment. This ratio provides us with a measure of how much the capacity of the pathologies, which are not first in the priority order, increases with respect to the reference non-cooperative situation.

We will now provide two sufficient conditions that ensure savings in the average post-operative costs, from sharing the operating theatre, and which have a clearer intuition. The next one is an immediate consequence of Proposition 3:

Corollary 4 If $\Lambda T > \lambda_i t_i, \forall i = 1, 2, \ldots, n,$ sharing the operating theatre reduces the average post-operative costs.

This condition is enough for making savings in the required number of beds, since it ensures that all fields of specialization (for the one with the highest priority it is trivial), use in expected terms a fraction of the total capacity that is smaller than the one it would set when being alone. Whereas the capacity of medical procedure i when being alone was $\mu_i(t_i) = \frac{1}{t_i} + \lambda_i$; when cooperating its share of the global one is given by $\frac{\lambda_i}{\Lambda}\frac{1}{T} + \lambda_i$. Therefore, the smaller expected capacity required implies a reduction in the average amount of beds for post-operative treatment.

When some of the conditions of Corollary 1 do not hold, it means that some medical procedures would demand more beds. If this is the case, the effect over the total average costs is ambiguous, since the extra demand of some medical procedures could be compensated with the smaller requirement of others. This possibility is analyzed in the following Corollary.

Corollary 5 If $d_n > \frac{\sum_{i=1}^{n-1}\lambda_i d_i}{\Lambda - \lambda_n}$, sharing the operating theatre reduces the average post-operative costs.

Proof. If:

$$d_n > \frac{\sum_{i=1}^{n-1}\lambda_i d_i}{\Lambda - \lambda_n}, \text{ then } d_n > \frac{\sum_{i=1}^{n-1}\lambda_i}{\Lambda - \lambda_n} d_i \left(1 - \frac{t_n \Lambda}{t_i \lambda \beta}\right); \text{ and thus,}$$
$$d_n > \sum_{i=1}^{n-1} d_i \left(\frac{\lambda_i}{\sum_{i=1}^{n-1}\lambda_i} - \frac{t_n}{t_i}\frac{\Lambda}{\sum_{i=1}^{n-1}\lambda_i}\right) \iff d_n \frac{\sum_{i=1}^{n-1}\lambda_i}{\Lambda} >$$
$$> \sum_{i=1}^{n-1} d_i \left(\frac{\lambda_i}{\Lambda} - \frac{t_n}{t_i}\right) \iff \sum_{i=1}^{n-1} d_i \left(\frac{\lambda_i}{t_n\Lambda} - \frac{1}{t_i}\right) < \frac{d_n}{t_n}\left(1 - \frac{\lambda_n}{\Lambda}\right) \iff$$
$$\iff \frac{1}{t_n}\frac{\lambda_n}{\Lambda} d_n + \frac{1}{t_n}\sum_{i=1}^{n-1}\frac{\lambda_i}{\Lambda} d_i < \frac{d_n}{t_n} + \sum_{i=1}^{n-1}\frac{d_i}{t_i} \iff$$
$$\iff \frac{1}{T}\sum_{i=1}^{n}\lambda_i d_i < \Lambda \sum_{i=1}^{n}\frac{d_i}{t_i}$$

that is, Proposition 3 holds. ■

Corollary 2 states that, if the medical procedure with the highest priority has a longer expected hospitalization time than the average hospitalization time of the rest, then by cooperating in the use of the operating theatre we reduce the expected amount of required beds.

The longer is the recovering time at hospital, the higher is the impact of an increase in the capacity of a medical procedure on the amount of beds. This makes that when d_n is sufficiently high, the decrease in the expected number of beds needed for medical procedure n surely exceeds the possible increase in the requirements of the others.

6 A NUMERICAL EXAMPLE

In this Section, we provide a numerical example to illustrate, for a particular case, the main features of our analysis. The basis of our example is real data obtained from a small hospital, concerning the average number of patients' arrivals and their average post-operative time.

In order to set the maximum average time guarantee for the different procedures, we have taken into account the actual time spent by the patients in the waiting lists of these pathologies.

We will consider that the priority of a procedure corresponds to the time guarantee, in such a way that given two treatments, the one with a shorter waiting time has priority over the other.

We analyze 6 medical procedures ($n = 6$), all of them of elective nature and with a very short hospitalization time (even null for some cases). The procedures employed are: cataract surgery, inguinal hernia operations, varicose veins, arthroscopies, hysterectomies and knee replacements. In the following table we provide the information required for the construction of the example; all the variables are measured in monthly terms:

Medical Procedures	λ_i	t_i	d_i
Knee Replacements (r)	12	4	0.266
Cataract Surgery (c)	129	2	0.043
Hysterectomies (hy)	19	1	0.243
Arthroscopies (a)	39	$\frac{1}{2}$	0.083
Inguinal Hernias (h)	33	$\frac{1}{3}$	0.074
Varicose Veins (v)	15	$\frac{1}{3}$	0.083

6.1 Operative costs

We first compute, using Equation (3), the optimal capacity that each surgical procedure should install on its own:

μ_r	μ_c	μ_{hy}	μ_a	μ_h	μ_v
$\frac{49}{4}$	$\frac{259}{2}$	20	41	36	18

Hence the overall capacity set, in the absence of cooperation is:

$$\sum_{i \in \{r,c,hy,a,h,v\}} \mu_i = 256.75.$$

When all the procedures share the use of the operating theatre, the capacity ($\mu(T)$) is set according to Equation (7), where T is given by $t_v = t_h = \frac{1}{3}$:

$$\mu(T) = 247.33.$$

Hence, if the procedures cooperate, we can reduce the installed capacity of the operating theatre in an amount of time per month similar to that necessary to perform around 10 interventions. As we have assumed the costs to be linear in the capacity installed, this will generate a proportional cost reduction.

The next step is to assign to the procedures and the patients their corresponding costs. Each surgical procedure will be charged with its variable cost (determined by its rate of patients' arrivals λ_i), plus a fraction of the fixed cost. In our case this fixed cost is $\frac{k}{t_v} = 3k$. The sharing is done following Equations (9) and (10), and it is given by:

Medical Procedures	$Sh_i(c^f)$	$Sh_i(c)$	$\sigma^*_j(c)$
Knee Replacements (r)	$\frac{1}{24}k$	12:041k	1:0020k
Cataract Surgery (c)	$\frac{11}{120}k$	129:091k	0:9993k
Hysterectomies (hy)	$\frac{13}{60}k$	19:216k	1:0103k
Arthroscopies (a)	$\frac{11}{20}k$	39:55k	1:0128k
Inguinal Hernias (h)	$\frac{21}{20}k$	34:05k	1:0305k
Varicose Veins (v)	$\frac{21}{20}k$	16:05k	1:0685k

The table shows how the part of the fixed costs assigned to each procedure $Sh_i(c^f)$ is increasing in the level of priority of the pathology. However, the sharing of the total costs $Sh_i(c)$ does not respect this ranking since it is affected also by the variable costs, i.e., the different rates of patients arrivals. In fact, we can see how the fixed costs are only a small fraction of the total costs of the service. Finally, in the last column we show which would be the part of the total costs corresponding to each patient, depending on the pathology he belongs to.

6.2 Post-operative Costs

We move now to the analysis of the postoperative period. As we mentioned in Section 5, we consider the beds as the servers of the system in this part of the process, and we can compute the required number of beds under different scenarios: (S_1) if the procedures share neither the theatre nor the beds; (S_2) if they share the theatre and they do not share the beds; (S_3) if they do not share the theatre but they do share the beds; and (S_4) if they share both theatre and beds.

First of all, it is immediate to check that condition in Proposition 3 is fulfilled, even though neither conditions of Corollary 1 nor Corollary 2 are.[9] Consequently, there are savings in the average number of beds when moving from scenario (S_1) to scenario (S_2).

We first compute, by means of Equation (12) the average number of beds required in scenario (S_1). Also, we compute using (13) the average aggregate number of beds required in scenario (S_2), and with the help of the relative frequencies of arrivals, we assign the corresponding fraction of the total beds to each pathology.

[9] Conditions of Corollary 1 do not hold, since cataract surgery does not fulfill the requirement there. Conditions of Corollary 2 are fulfilled only for one of the two procedures with higher priority: the average hospitalization time of the inguinal hernias is smaller than the average post-operative period of the other pathologies.

However, as we stated in Section 5 this does not guarantee that every patient has a bed when he departs from the operating theatre. Using the Erlang's C formula, we can …x an upper bound for the probability of not having a bed ready when it is needed. Once this probability is …xed, we can solve for the minimum number of beds required to ful…ll it. We will denote this value by b_i.

To avoid an excessive expenditure of resources, and have a situation in which a large number of beds are "almost always" unused, we set the probability that a patient waits to :1: Notice that by …xing this upper bound we are in fact setting the expected waiting time of a patient to a very low level.

The following table shows the average number of beds ($\bar{b}_i$) and the number of beds required to guarantee that with probability .9 no patient has to wait (b_i), in scenarios (S_1) and (S_2).

Medical Procedures	$\bar{b}_i^{(S\_1)}$	$\bar{b}_i^{(S\_2)}$	$b_i^{(S\_1)}$	$b_i^{(S\_2)}$
Knee Replacements (r)	3:26	3:19	6:35	6:26
Cataract Surgery (c)	5:57	5:55	9:46	9:43
Hysterectomies (hy)	4:86	4:62	8:53	8:21
Arthroscopies (a)	3:42	3:25	6:57	6:40
Inguinal Hernias (h)	2:67	2:45	5:52	5:20
Varicose Veins (v)	1:5	1:25	3:75	3:35

Hence, when the procedures share neither the theatre nor the beds (S_1), the total average number of beds is:

$$\sum_{i \in \{r,c,hy,a,h,v\}} \bar{b}_i^{(S\_1)} = 21:28;$$

and the total maximum number of beds is:

$$\sum_{i \in \{r,c,hy,a,h,v\}} b_i^{(S\_1)} = 40:18:$$

One can see how the presence of randomness in the post-operative treatment makes that in order to ensure a negligible probability of waiting, the number of beds has to be almost doubled from the reference level (computed in expected terms).

We now see how the results di¤er when the pathologies share the use of the operating theatre (S_2). The average number of beds is:

$$\sum_{i \in \{r,c,hy,a,h,v\}} \bar{b}_i^{(S\_2)} = 20:31;$$

and the maximum number of beds is:

$$\sum_{i \in \{r,c,hy,a,h,v\}} b_i^{(S\_2)} = 38{:}85{:}$$

In this framework we face the same problem as in scenario 1. When we want to ensure a low probability of waiting, the required capacity almost doubles, and this means having a lot spare beds which are needed to cover the risks.

Moreover, we see how by sharing the operating theatre we can decrease both the average and the total post-operative costs, since it allows to save one bed per-month. Notice that, even if this seems to be a small improvement, we are dealing with pathologies that have a very short hospitalization time, and that therefore require few beds. Hence, the decrease in the number of beds is around 5%.

The next step is to repeat the analysis for the third and fourth scenarios, that is, when the medical procedures share the use of the beds, distinguishing among a situation where each one has its own operating theatre (S_3), and another in which there is full cooperation, both in the operation, and in the hospitalization period (S_4).

In these scenarios, since beds are shared, we consider the rate of patient's arrivals as the sum of the average of arrivals of patients coming from the di¤erent procedures, and the hospitalization time the average of the length of stay at hospital of the pathologies, weighted by the proportion of patients demanding a bed in each procedure. The results are summarized in the following table:

$\sum_{i \in \{r,c,hy,a,h,v\}} \bar{b}_i^{(S\_3)}$	$\sum_{i \in \{r,c,hy,a,h,v\}} b_i^{(S\_3)}$	$\sum_{i \in \{r,c,hy,a,h,v\}} \bar{b}_i^{(S\_4)}$	$\sum_{i \in \{r,c,hy,a,h,v\}} b_i^{(S\_4)}$
21:33	28:32	20:30	27:25

The results are really illustrative, and several insights can be highlighted. First, again, sharing the use of the operating theatre is always pro…table in terms of the average post-operative costs. When moving from scenario (S_3) to (S_4), we also save one bed per month.

However the most interesting comparison is between scenarios (S_2) and (S_4) (or analogously between (S_1) and (S_3)). By doing so, we see how crucial it is to share the beds. First of all, it allows to decrease the extra capacity required to ensure a low probability of waiting by a 50%. Using as reference for instance scenarios (S_2) and (S_4), we see that the same probability of waiting can be ensured by setting only 7 extra beds in scenario (S_4), instead of 18 in scenario (S_2). The same occurs confronting situations (S_1) and (S_3).

Moreover, it yields a very important saving in the number of beds that have to be installed in order to ensure the given probability of waiting. Taking as reference the scenarios in which the pathologies share the use of the operating theatre, we see how if the medical procedures cooperate in the management of the post-operative period, the need for beds is reduced by nearly a 30% (approximately 11 beds).

The reason for this reduction can be explained by the same argument we used in Subsection 3.3 for the operating theatre costs. We are treating the beds as servers, and by allowing the different medical procedures share the use of the beds, we optimally spread risks among them. Therefore, we set a unique extra capacity of beds to account for the potential bad realizations of the random variables, instead of making each pathology have its own extra capacity. And this is shown to generate savings.

However, if we proceed to distribute the costs resulting of the cooperation (S_4), among the different medical procedures, the cost sharing game we would face is not an "airport game". Although we can identify which pathologies require a bigger capacity than others, since we are guaranteeing an almost zero probability of waiting for all the patients, the number of beds fixed by the most demanding procedure is not enough to ensure that nobody has to wait.

But we can compute the Shapley value of this cost-sharing game, just charging each procedure by averaging its marginal contributions to all coalitions containing it. The cost share of procedure i is computed as an average of the marginal cost (marginal number of beds) inflicted by procedure i to each and every coalition $(T\setminus\{i\})$ of other pathologies, and it is given by:

$$Sh_i(b) = \sum_{T\subseteq N; i\in T} \left(\frac{(N - \#T)!(\#T - 1)!}{N!}\right)(C(T) - C(T\setminus\{i\})).$$

We next present the results:

$Sh_r(b)$	$Sh_c(b)$	$Sh_{hy}(b)$	$Sh_a(b)$	$Sh_h(b)$	$Sh_v(b)$
3.8041	7.3502	6.1996	4.488	3.4834	1.9247

As we can see, the cost share assigned to each procedure is increasing in the number of beds that they required when they do not share the servers. Moreover, if we compare these costs shares with the ones is Scenario S_2, we see how the savings range between the 22% reduction for cataract surgery, and the 43% savings that varicose veins attain. In this example we observe that the most demanding pathology in terms of required beds (c) is the one that benefits less from cooperation, and conversely varicose veins (which necessity of beds is the minimum) enjoys the greatest fraction of the savings from cooperation.

7 CONCLUSIONS

Surgical waiting lists are a persistent and unsatisfactory phenomenon in the Public Health Services worldwide, since their inception. They have been the subject of a great deal of research.

In this paper we model the problem of the waiting lists to get surgical treatment making use of queueing theory. We considered that both, the arrival of patients to the waiting list and the process of treatment have random components. Therefore, the application of queueing theory results arises naturally.

The simplifying assumptions of considering an exponential distribution of the time between two subsequent arrivals and exponential service time distributions, were made for analytical convenience. The other extreme would be to assume arrivals and service times that are always constant. The realistic distribution is often somewhere in between (Worthington (1987)).

We concentrate ourselves on the costs that interventions generate, taking into account that the higher the resources spent by the hospital the shorter the resulting waiting lists. Our aim in this work has been two-fold.

On the one hand, we study the effects on the direct costs of an intervention that the use of a common operating theatre by the different medical procedures has. We show that sharing the use of the operating theatre leads to a cost reduction.

Afterwards, we study how these savings should be allocated to the medical procedures through a optimal tariff. Clearly, we are dealing with a cost allocation problem. Since the Shapley value is a well-known solution concept with good theoretical and computational properties, we propose it as the basis for the computation of the optimal fee per medical procedure.

On the other hand, we extend our analysis to the post-operative time. In order to fulfill the maximum average waiting time guarantee set by the government, it is necessary not only that the operating theatre works properly, but also that there is an enough supply of beds for the recovering of patients at hospital. Therefore, it is relevant to analyze the impact of cooperation among medical procedures on the post-operative costs.

We obtain that the sign of the effect that sharing the use of the operating theatre has on the average post-operative costs, depends on the characteristics of the treatments and can not be stated in general. We compute two sufficient conditions for making savings also in this second stage of the process.

Finally, we provide a numerical example to illustrate the main features of our model, on the basis of real data obtained from a small hospital concerning the average number of patients' arrivals and the average length of their post-operative time. We apply our theoretical analysis to this particular case and interpret the results that arise. In particular, we show that when procedures also cooperate in the managing of beds major savings are obtained.

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