A discusión

A NEW APPROACH FOR BOUNDING AWARDS IN BANKRUPTCY PROBLEMS*

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ABSTRACT

The solution for the "Contested Garment Problem" proposed in the Babylonic Talmud, one of the most important sources of inspiration for solving situations where demand overcomes supply of some resources, suggests that each agent should receive at least some part of the available amount when facing these situations. This idea has been underlied the theoretical analysis of bankruptcy problems from its beginning (O'Neill, 1982) to present day (Dominguez and Thomsom, 2006). In this context, starting from the fact that a society establishes its own set of "Commonly Accepted Equity Principles", we propose a new award bound by providing each agent her minimum amount according to all the admissible bankruptcy rules for such a society. Moreover, we analyze the recursive application of this new bound, since it will not exhaust the resources, in general.

Keywords: Bankruptcy problems, bankruptcy rules, lower bound, recursive process.

1. Introduction

A bankruptcy problem reflects a situation where a group of agents claim more quantity of a good than available. According to that, a bankruptcy rule prescribes how to share out an amount of a perfectly divisible resource, called estate, among a group of agents, depending on a profile of demands whose aggregate overcomes its supply. In this context two natural questions arise: *How should the available resources be rationed among claimants? Should each agent have guaranteed a level of awards?*

The main goal of the two approaches to the study of bankruptcy problems: the axiomatic and the game theoretical methods, has been identifying bankruptcy rules by means of appealing properties. Following this line, many authors have found reasonable establishing some bound on awards. In fact, the formal definition of a solution for bankruptcy problems includes, by demanding that no agent gets more than her claim and less than zero, both an upper and a lower bounds on awards. In 1982, O'Neill [15] provides a new lower bound on awards called *Respect of Minimal Right,* which requires that each claimant receives at least the available amount of the estate after the other claimants have been fully compensated, or 0 if this amount is negative. Later, Herrero and Villar [10, 11] introduce two properties that bound awards, called *Sustainability* and *Exemption. Sustainability* says that, if we truncate all the claims by an agent i's claim and the bankruptcy problem becomes feasible, then agent i will receive all her claim. *Exemption* requires that when equal division provides agent i more than her claim, agent i should not be rationed. After that, Moulin [14] defines a new restriction on awards, called *Lower Bound,* which imposes that each agent has the amount corresponding to the egalitarian division guaranteed except those who demand less, in which case their demand is met in full. Afterwards, Moreno-Ternero and Villar [12] present a weaker notion of Moulin's *Lower Bound*, named *Securement*, which says that each agent should receive at least the nth part of her claim truncated at the amount to divide. Finally, Dominguez [7] proposes the *Min Lower Bound*, which modifies the previous one by substituting each agent's claim by the lowest one.

Apart from *Respect of Minimal Right*, property which is implied by the formal definition of a bankruptcy rule, the rest of the proposed limits on awards have been justified by their own reasonability or appeal. Our goal is to establish restrictions on awards taking as starting point a set, P, of 'basic' requirements, called *'Commonly Accepted Equity Principles',* on which a society could willingly agree. Then we consider the ordinary meaning of guarantee over all the bankruptcy rules satisfying properties in P as follows. By applying to a bankruptcy problem all *Socially Admissible Bankruptcy* rules we determine the agent's *Minimal Safety* as the lower amount she gets among those ones provided by such rules. Finally, we define the associated bound on awards, *Respect of Minimal Safety*, by demanding that each agent receives, at least her *Minimal Safety.*

Since, in general, the aggregate guaranteed amount by means of our *Minimal Safety* will not exhaust the available quantity of resources, we propose and analyze its recursive application, that is, the limit of the following procedure. At the initial step, each agent receives her *Minimal Safety* according to the original bankruptcy problem. At the second step, we redefine the residual bankruptcy problem, in which the estate is the leftover resources, and the claims are adjusted down by the amounts just given; then each agent receives her *Minimal Safety* of such a residual bankruptcy problem, and so on. We call the bankruptcy rule obtained in this way the *Recursive Minimal Safety* rule. This kind of process is not new, in fact it has already been used for introducing bankruptcy rules by Alcalde et al. [2], who generalize the Ibn Ezra's proposal, and by Dominguez and Thomson [8], who propose the *Recursive* rule by using the Moreno-Ternero and Villar's concept of boundedness, among other authors.

In this paper we apply the previous methodology to different sets of *'Commonly Accepted Equity Principles'*. First of all, we propose as basic properties the set P_1, composed by *Resource Monotonicity*, *Super-Modularity* and *Midpoint Property*. *Resource Monotonicity* demands that if the estate increases, then all individuals should receive at least as much as they did initially. *Super-Modularity* requires that if the estate increases, then individuals with higher claims should receive a greater part of the increment than those ones with lower claims. *Midpoint Property* says that if the estate is equal to the sum of the half-claims, then every individual should get her half-claim. In this case we find out that the *Minimal Safety* is the minimum of the two extreme bankruptcy solutions in this set, the *Piniles'* rule (Piniles' [17]) and its dual. Moreover, we prove that the *Recursive Minimal Safety* rule retrieves the *Dual Piniles'* rule.

Secondly, considering a new set of equity principles, P_2, we study the consequences of weakening the requirements of the previous set of properties by both eliminating *Midpoint Property*, and by substituting *Super-Modularity* by *Order Preservation*, which demands that a bankruptcy rule provides higher awards and greater losses to agents with higher claims. Then, we show that the associated *Minimal Safety* is the minimum of two different extreme bankruptcy solutions, the *Constrained Equal awards* rule (many authors, see Thomson [19]) and its dual, the *Constrained Equal Losses* rule (Maimonides 12th Century, among others). Besides this, we demonstrate that the *Recursive Minimal Safety* rule retrieves the *Constrained Equal Losses* rule.

The results previously mentioned could be written as follows: 'The recursive application of the *Minimal Safety* recovers, in the set of all admissible bankruptcy rules according to both P_1 and P_2, one of its extremes, that one providing more awards to the higher claimants'. Then, the analysis of the generalization of this statement arises as a natural question.

With this aim, we define the equity principles set P_3 consisting of *Resource Monotonicity*, *Order Preservation* and *Midpoint Property.* That is, a more permissive situation than the first one, since we require *Order Preservation* instead of *Super-Modularity,* but more restrictive than the second one, because we add *Midpoint Property* to it.

Surprisingly enough, since $P_2 \subset P_3 \subset P_1$, we show that not only this generalization is not possible, but also that the way of sharing awards by means of the recursive application of the *Minimal Safety* does not satisfy the equity principles which this process is based on.

The rest of the paper is organized as follows: Section 2 introduces the model used to analyze bankruptcy problems. Section 3 proposes the new approach for bounding awards based on the concept of *Minimal Safety* and defines its recursive application. Section 4 applies previous ideas to different sets of *'Commonly Accepted Equity Principles'* providing new basis to classical bankruptcy rules. Section 5 shows the incompatibility of the proposed process with some 'appealing' set of equity principles. Our main conclusions are summarized in Section 6. Finally, technical proofs are relegated to the Appendices.

2. Preliminaries

A bankruptcy problem is a situation where the agents' demand of a good exceeds its supply. Formally,

Definition 2.1. *A* ***bankruptcy problem*** *is a vector* $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ *such that*

$$E \leq \sum_{i \in N} c_i.$$

E is known as the estate, and represents the perfectly divisible good quantity that should be distributed among the agents in $N = \{1, ..., i, ..., n\}$. Each agent $i \in N$ has a claim c_i on the estate, which together could be greater than the available amount. Therefore, the resources to share should be rationed among agents.

Let B denote the set of all bankruptcy problems,

$$B = \left\{ (E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+ : E \leq \sum_{i \in N} c_i \right\}.$$

From now on, and for notational convenience, we will denote by C the sum of the agents' claims, $C = \sum_{i \in N} c_i$, by L the total amount of losses to distribute among the agents, $L = C - E$, and by B_0 the set of bankruptcy problems in which claims are increasingly ordered,

$$B_0 = \{(E, c) \in B : c_i \leq c_j \text{ for } i < j\}.$$

A bankruptcy rule associates for each bankruptcy problem a distribution of the available amount among the group of claimants. Next, we present this concept formally and define the bankruptcy rules that will be used in the following sections, emphasizing their dual relations.

Definition 2.2. *A* ***bankruptcy rule*** *is a function, $\varphi : B \to \mathbb{R}^n_+$, such that for each $(E,c) \in B$, (a) $\sum_{i \in N} \varphi_i(E,c) = E$ (efficiency) and*

(b) $0 \leq \varphi_i(E,c) \leq c_i$ for each $i \in N$ (non-negativity and c-boundedness).

The *Constrained Equal Awards* rule (Maimonides 12th Century, among others) recommends equal gains to all claimants subject to no one receiving more than her claim.

Constrained Equal Awards rule, φ^{CEA}: for each $(E,c) \in B$ and each $i \in N$, $\varphi_i^{CEA}(E,c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

Piniles' rule (Piniles' [17]) assigns the *Constrained Equal Awards* rule when the available amount is less than the half-sum of the claims. Otherwise, first each agent receives her half-claim, then the *Constrained Equal Award* rule is re-applied to divide the remainder but only taking into account the agents' half-claims.

Piniles' rule, φ^{Pin}: for each $(E,c) \in B$ and each $i \in N$,

$$\varphi_i^{Pin}(E,c) \equiv \begin{cases} \varphi_i^{CEA}(E, c/2) & \text{if } E \leq C/2 \\ c_i/2 + \varphi_i^{CEA}(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}.$$

The *Constrained Egalitarian* rule (Chun et al. [4]) is inspired by the *Uniform* rule, a solution to the problem of fair division when the preferences are single-peaked. It makes the minimal adjustment in the formula of the *Uniform* rule taking the half-claims as the peak and guaranteeing that awards are ordered as claims are.

Constrained Egalitarian rule, φ^{Ce}: for each $(E,c) \in B$ and each $i \in N$,

$$\varphi_i^{Ce}(E,c) \equiv \begin{cases} \varphi_i^{CEA}(E, c/2) & \text{if } E \leq C/2 \\ \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \geq C/2 \end{cases}$$

where δ is chosen so that $\sum_{i \in N} \varphi_i^{Ce}(E,c) = E$.

Given a bankruptcy rule φ, its dual shares losses in the same way as φ divides the available amount (Aumann and Maschler [1]).

The **dual bankruptcy rule** of φ, denoted by φ^d, assigns for each $(E,c) \in B$ and each $i \in N$, $\varphi_i^d(E,c) = c_i - \varphi_i(L,c)$.

Note that for all bankruptcy solution φ, its dual is well defined since given $(E,c) \in B$, we have that $L \in \mathbb{R}_+$ and $C > L$, therefore $(L,c) \in B$.

Moreover, $\sum\limits_{i\in N}\varphi_i(L,c)=L$ and $0\leq\varphi_i(L,c)\leq c_i$ for each $i\in N$ imply $\sum\limits_{i\in N}\varphi_i^d(E,c)=$ $=E$ and $0\leq\varphi_i^d(E,c)\leq c_i$ for each $i\in N$, that is, φ_i^d is a bankruptcy solution.

The *Constrained Equal Losses* rule, discussed by Maimonides (Aumann and Maschler [1]), is the dual of the *Constrained Equal Awards* rule (Herrero [9]). Specifically, it chooses the awards vector at which losses from the claims vector are the same for all the agents subject to no-one receiving a negative amount.

Constrained Equal Losses rule, φ^{CEL}: for each $(E,c)\in B$ and each $i\in N$, $\varphi_i^{CEL}(E,c)\equiv\max\{0,c_i-\mu\}$, where μ is chosen so that $\sum\limits_{i\in N}\max\{0,c_i-\mu\}=E$.

Dual Piniles' rule assigns the *Constrained Equal Losses* rule when the available amount is less than the half-sum of the claims. Otherwise, first each agent receives her half-claim, then the *Constrained Equal Losses* rule is re-applied to divide the remainder but only taking into account the agents' half-claims.

Dual Piniles' rule, φ^{DPin}: for each $(E,c)\in B$ and each $i\in N$,

$$\varphi_i^{DPin}(E,c)=\begin{cases} c_i/2-\min\{c_i/2,\lambda\} & \text{if } E\leq C/2 \\ c_i/2+(c_i/2-\min\{c_i/2,\lambda\}) & \text{if } E\geq C/2 \end{cases},$$

where λ is such that $\sum\limits_{i\in N}\varphi_i^{DPin}(E,c)=E$.

Dual Constrained Egalitarian rule gives the half-claims a central role and makes the minimal adjustment in the formula of the *Dual Uniform* rule which guarantees that losses are ordered as claims are.

Dual Constrained Egalitarian rule, φ^{DCe}: for each $(E,c)\in B$ and each $i\in N$,

$$\varphi_i^{DCe}(E,c)\equiv\begin{cases} c_i-\max\{c_i/2,\min\{c_i,\delta\}\} & \text{if } E\leq C/2 \\ c_i-\min\{c_i/2,\delta\} & \text{if } E\geq C/2 \end{cases}$$

where δ is chosen such that $\sum\limits_{i\in N}\varphi_i^{DCe}(E,c)=E$.

Next, we introduce some properties of bankruptcy rules which, subsequently, will be interpreted as *'Commonly Accepted Equity Principles'*, and we present the notion of *Self-Duality* between bankruptcy rules.

Resource Monotonicity, property considered by Curiel et al. [5] and Young [20], among several authors, demands that if the estate increases, then all individuals should receive at least as much as they did initially.

Resource Monotonicity: for each $(E,c)\in B$ and for each $E'\in\mathbb{R}_+$ such that $C\geq E'>E$, then $\varphi_i(E',c)\geq\varphi_i(E,c)$, for each $i\in N$.

Order Preservation, property introduced by Aumann and Maschler [1], requires respecting the claims order, i.e., if agent i's claim is at least as large as agent j's claim, she should receive and lose at least as much as agent j does respectively.

Order Preservation: for each $(E, c) \in B$ and each $i, j \in N$ such that $c_i \geq c_j$, then $\varphi_i(E, c) \geq \varphi_j(E, c)$ and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$.

Super-Modularity, property introduced by Dagan et al. [6], demands that if the estate increases, then individuals with higher claims should receive a greater part of the increment than those with lower claims.

Super-Modularity: for each $(E, c) \in B$, all $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C \geq E' > E$ and $c_i \geq c_j$, then $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$.

Midpoint Property, introduced by Chun, Schummer and Thomson [4], says that if the estate is equal to the sum of the half-claims, then every individual should get her half-claim.

Midpoint Property: for each $(E, c) \in B$ such that $E = C/2$, $\varphi_i(E, c) = c_i/2$, for each $i \in N$.

Self-Duality, implies that a bankruptcy rule treats symmetrically the problem of dividing 'what is available' and the problem of sharing 'what is missing', so that the amount received by any agent will be independent of the interpretation of the problem when dividing gains or losses.

Self-Duality: for each $(E, c) \in B$ and each $i \in N$, $\varphi_i(E, c) = c_i - \varphi_i(L, c)$.

To conclude this Section, we present the idea of duality between properties, which has been analyzed by many authors (see, for instance, Herrero and Villar [10] and Moulin [13]).

Given two properties, $\mathcal{P}$ and $\mathcal{P}'$, we say that they are **dual** if whenever a bankruptcy rule, φ, satisfies $\mathcal{P}$, its dual bankruptcy rule, φ^d, satisfies $\mathcal{P}'$. A property, $\mathcal{P}$, is **Self-Dual** when it coincides with its dual.

It is straightforward to check that all the properties previously introduced, *Resource Monotonicity, Order Preservation, Super-Modularity,* and *Midpoint Property*, are *Self-Dual,* fact that will be used later on.

3. A new approach: Bounding awards from equity principles.

As we have mentioned in the Introduction, most of the lower bounds on awards that have been proposed in the Economic Literature have been justified by their own reasonability. A clear exception is *Respect of Minimal Right,* which requires that each claimant receives at least the available amount of the estate after the other claimants have been fully compensated, or 0 if this amount is negative. This property, as Thomson [19] pointed out, is a consequence of efficiency, non-negativity and claim boundedness altogether (See Definition 2.2).

In this Section we introduce a new method for bounding awards based on a set of *'Commonly Accepted Equity Principles'* by a society. With this aim and considering such a set of basic properties, next we propose the following extension of a bankruptcy problem.

Definition 3.1. *A* ***Bankruptcy Problem with Legitimate Principles*** *is a vector* (E, c, P_t) *where* $(E, c) \in B$ *and* P_t *denotes a set of principles on which a society has agreed.*

From now on, let P be the set of all subsets of properties on bankruptcy rules, and let B_P be the set of all *Bankruptcy Problems with Legitimate Principles*,

$$B_P = \left\{ (E, c, P_t) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times P : E \leq \sum_{i \in N} c_i \right\}.$$

In this context, a *Socially Admissible Bankruptcy* rule is a bankruptcy rule which satisfies all the properties imposed by the society.

Definition 3.2. *A* ***Socially Admissible Bankruptcy rule*** *is a function,* $\overline{\varphi} : B_P \to \mathbb{R}_+^n$*, such that for each* $(E, c, P_t) \in B_P$*,*

(a) $\sum_{i \in N} \overline{\varphi}_i(E, c, P_t) = E$*,*

(b) $0 \leq \overline{\varphi}_i(E, c, P_t) \leq c_i$ *for each* $i \in N$*, and*

(c) $\overline{\varphi}$ *satisfies all properties in* P_t*.*

Let Φ denote the set of all bankruptcy rules and let $\Phi(P_t)$ be the subset of bankruptcy rules satisfying P_t.

Taking this kind of extended bankruptcy problems as a starting point, we propose a new lower bound on awards based on the application of the ordinary meaning of guarantee. That is, each agent will receive, at least, her *Minimal Safety,* which is the lower amount among those ones provided by all the bankruptcy rules satisfying the selected properties. Formally,

Definition 3.3. *Given* (E, c, P_t) *in* B_P*, the* ***Minimal Safety,*** s*, is for each* $i \in N$*,*

$$s_i(E, c, P_t) = \min_{\varphi \in \Phi(P_t)} \{\varphi_i(E, c)\}.$$

At this point, using the idea of guarantee previously introduced, our new lower bound on awards, called *Respect of Minimal Safety,* demands that each claimant receives at least her *Minimal Safety.*

Definition 3.4. *Given* $P_t \in P$*, a bankruptcy rule* φ *satisfies* ***Respect of Minimal Safety*** *if for each* $(E, c) \in B$ *and each* $i \in N$*,* $\varphi_i(E, c) \geq s_i(E, c, P_t)$*.*

Since, in general, the aggregate amount of a lower bound on awards will not exhaust the available quantity of resources, properties requiring composition from such a lower bound arise in a natural way. These properties ask the awards vector to be equivalently obtainable (i) directly, or (ii) by first assigning to each agent her lower bound on awards, adjusting claims down by these amounts, and finally, applying the rule to divide the remainder. The following definition gathers this idea applied to our bound on awards.

Definition 3.5. *Given $P_t \in P$, a bankruptcy rule φ satisfies **Minimal Safety First** if for each $(E, c) \in B$ and each $i \in N$,*

$$\varphi_i(E, c) = s_i(E, c, P_t) + \varphi_i(E - \sum_{i \in N} s_i(E, c, P_t), c - s(E, c, P_t)).$$

Although many of the proposed lower bound on awards are respected by most of the bankruptcy rules, the composition from such lower bounds is quite demanding. For instance, *Respect of Minimal Right* is satisfied by any bankruptcy rule, however nor the *Proportional,* nor the *Constrained Equal Awards* neither the *Minimal Overlap* rules satisfy *Minimal Right First* (See Thomson [19]). In fact, imposing this kind of composition or equivalently applying a recursive method from a lower bound on awards, has been used to propose new bankruptcy rules. Domínguez and Thomson [8] introduce, in this way, a new bankruptcy rule, named the *Recursive* rule*,* by requiring the composition from *Securement*, the Moreno-Ternero and Villar's concept of boundedness.

Next, following the previous ideas, we define the recursive application of our *Minimal Safety*, which will be called the *Recursive Minimal Safety Process.*

Definition 3.6. *The **Recursive Minimal Safety Process,** RMS**,** associates for each $(E, c, P_t) \in B_P$ and each $i \in N$,*

$$[RMS(E, c, P_t)]_i = \sum_{k=1}^{\infty} s_i(E^k, c^k, P_t),$$

$$\text{where } (E^1, c^1) \equiv (E, c) \text{ and for } k \geq 2,$$

$$(E^k, c^k) \equiv (E^{k-1} - \sum_{i \in N} s_i(E^{k-1}, c^{k-1}, P_t), c^{k-1} - s(E^{k-1}, c^{k-1}, P_t)).$$

According to this process, an agent will receive the sum, whenever it is well defined, of the amounts she gets in each of the following stages. At the initial step, we compute the *Minimal Safety* of the original bankruptcy problem for each agent and we give it to each claimant. At the second step, we redefine the residual bankruptcy problem, in which the estate is the remaining resources and the claims are adjusted down by the amounts just given, then we again give each agent her *Minimal Safety* of such a residual bankruptcy problem, and so on.

Let us note that, in general, it can be ensured that the *Recursive Minimal Safety Process* neither converges, nor provides a *Socially Admissible Bankruptcy* rule, but when that happens, we will call it the *Recursive Minimal Safety* rule[1].

Definition 3.7. *The* ***Recursive Minimal Safety*** *rule,* $\bar{\varphi}^R$**,** *associates for each* $(E, c, P_t) \in B_P$ *and each* $i \in N$, $\bar{\varphi}_i^R(E, c, P_t) = [RMS(E, c, P_t)]_i$, *whenever*

$$\text{(i) } \sum_{i \in N} \left(\sum_{k=1}^{\infty} s_i(E^k, c^k, P_t) \right) = E, \text{ and (ii) } \bar{\varphi}^R \text{ satisfies all properties in } P_t.$$

4. Two equity principles sets

In this Section we consider two possible choices of '*Commonly Accepted Equity Principles'* by a society to apply the approach introduced previously for bounding awards.

Specifically,

$P_1 = \{$*Resource Monotonicity*, *Super-Modularity* and *Midpoint Property*$\}$ and
$P_2 = \{$*Resource Monotonicity* and *Order-Preservation*$\}$.

We think that all the equity principles compounding the set P_1 could be generally accepted. In fact, most of the proposed solutions satisfy *Resource Monotonicity* and *Super-Modularity* and, in words of Aumann and Maschler [1], 'it is socially unjust for different creditors to be on opposite sides of the halfway point, $C/2$'.

At this point it would be worthwhile to dwell on the meaning of the above mentioned properties. With respect to *Resource Monotonicity* simply let us comment that it has not been proposed any bankruptcy rule violating this property, otherwise we could have situations where more available amount would cause disadvantages to some agent.

A *Super-Modular* rule allocates each additional dollar in an 'order preserving' manner. Therefore, this property requires that if the estate increases, then individuals with higher claims should receive a greater part of the increment than those with lower claims. Apart from the *Constrained Egalitarian* rule and its dual, most of the proposed bankruptcy rules satisfy this principle.

Finally, *Midpoint Property* can be interpreted as the requirement of *Self-Duality* of a rule for the particular case in which the estate coincides with the sum of the half-claims, since in this situation both gains and losses amounts to share are identical. Above of being obviously satisfied by all the *Self-Dual* bankruptcy rules, this property is also fulfilled by some interesting non *Self-Dual* ones, such as the generalizations of the *Talmud* rules introduced by Hokari and Thomson [16].

[1] Let us note that non-negativity and claim boundedness are satisfied by construction. Moreover, it can be checked (by adapting the proof of the Remark 3) that whenever the *Minimal Safety* provides, in each step, a positive amount to some agent, efficiency is met.

The conditions constituting P_2, *Resource Monotonicity* and *Order-Preservation*, are weaker than the previous ones, since they are obtained from the P_1 set by both eliminating *Midpoint Property* and substituting *Super-Modularity* by *Order Preservation.* These principles not only have been understood by many authors as the minimal requirements of fairness (see for instance Young [20]), but also they are satisfied by all the bankruptcy rules proposed in the Economic Literature. In this sense, let us point out that *Order-Preservation*, which requires that higher claimants receive and support at least the awards and losses provided to the lower ones, respectively, is a weakened version of *Super-Modularity.*

Next results are used to find out the bankruptcy rules which mark out the region of admissible path of awards satisfying properties in P_1. Bosmans [3] introduced the first one, and starting from it, the second Theorem can be obtained straightforwardly taking into the fact that all the properties considered are *Self-Dual* and the concept of dual bankruptcy rule.

Theorem 4.1. *(Bosmans and Lauwers, 2007)*
Let φ be any bankruptcy rule satisfying properties in P_1, then for each $(E, c) \in B$ the Piniles' rule is the only rule such that the gap between the smallest and the largest amount any claimant receives is the smallest, and the variance of the amounts received by all the claimants is the smallest.

Theorem 4.2. *Let φ be any bankruptcy rule satisfying properties in P_1, then for each $(E, c) \in B$ the Dual Piniles' rule is the only rule such that the gap between the smallest and the largest loss any claimant supports is the smallest, and the variance of the losses supported by all the claimants is the smallest.*

Analogously, following Schummer and Thomson [18], we provide the results used to define the bankruptcy rules marking out the region of admissible path of awards satisfying properties in P_2.

Theorem 4.3. *(Schummer and Thomson, 1997)*
Let φ be any bankruptcy rule, then for each $(E, c) \in B$ the Constrained Equal Awards rule is the only rule such that the gap between the smallest and the largest amount any claimant receives is the smallest, and the variance of the amounts received by all the claimants is the smallest.

Theorem 4.4. *Let φ be any bankruptcy rule, then for each $(E, c) \in B$ the Constrained Equal Losses rule is the only rule such that the gap between the smallest and the largest loss any claimant supports is the smallest, and the variance of the losses supported by all the claimants is the smallest.*

At this point is worth noticing that, since the *Constrained Equal Losses* and the *Constrained Equal Awards* rules satisfy *Resource Monotonicity* and *Order Preservation*, these results will remain valid when restricting the bankruptcy rules to those belonging to P_2.

Figures 1 and 2 represent graphically the previous results for two-person bankruptcy problems. Figure 1 shows the bankruptcy rules marking out the area of all admissible paths satisfying properties in P_1. Analogously, in Figure 2 it can be observed the paths of those rules delimiting the area in P_2. Let us note that any bankruptcy rule which is out of the corresponding area will not satisfy any of the equity principles compounding the sets P_1 or P_2, respectively. For instance, in Figure 2, any solution providing a sharing in the area of the triangle defined by the vectors $(0,0)$, $(0,c_1)$ and (c_1,c_1) will recommend greater awards to the lowest claimant, in contradiction with *Order Preservation.* In the Figure 1, any solution satisfying *Midpoint Property* and providing agent one more than her half claim when the estate is lower than the sum of the half-claims, will not satisfy *Resource Monotonicity.*

Next two lemmas, whose proofs are omitted since they follow straightforwardly from the above theorems, determine the *Minimal Safety* for both P_1 and P_2. In this regard, let us note that given a bankruptcy problem, $(E^*,c) \in B$, the admissible sharings will be some point belonging to the segment defined by the vectors $\varphi^{Pin}(E^*,c)$ and $\varphi^{DPin}(E^*,c)$, when considering P_1. Analogously, the vectors $\varphi^{CEA}(E^*,c)$ and $\varphi^{CEL}(E^*,c)$ are the extremes of the segment of admissible sharings if society agrees on P_2.

Lemma 4.5. *Given (E,c,P_1) in B_P, the* ***Minimal Safety,*** *s, is for each $i \in N$,*

$$s_i(E,c,P_1) = \min\left\{\varphi_i^{Pin}(E,c), \varphi_i^{DPin}(E,c)\right\}.$$

Lemma 4.6. *Given (E,c,P_2) in B_P, the* ***Minimal Safety,*** *s, is for each $i \in N$,*

$$s_i(E,c,P_2) = \min\left\{\varphi_i^{CEA}(E,c), \varphi_i^{CEL}(E,c)\right\}.$$

The following theorems prove that the recursive application of the *Minimal Safety* retrieves, for P_1 and P_2 the *Dual Piniles'* and the *Constrained Equal Losses* rules, respectively.

Theorem 4.7. *For each $(E,c,P_1) \in B_P$, $\overline{\varphi}^R(E,c,P_1) = \varphi^{DPin}(E,c)$.*

Proof. See Appendix 2. ∎

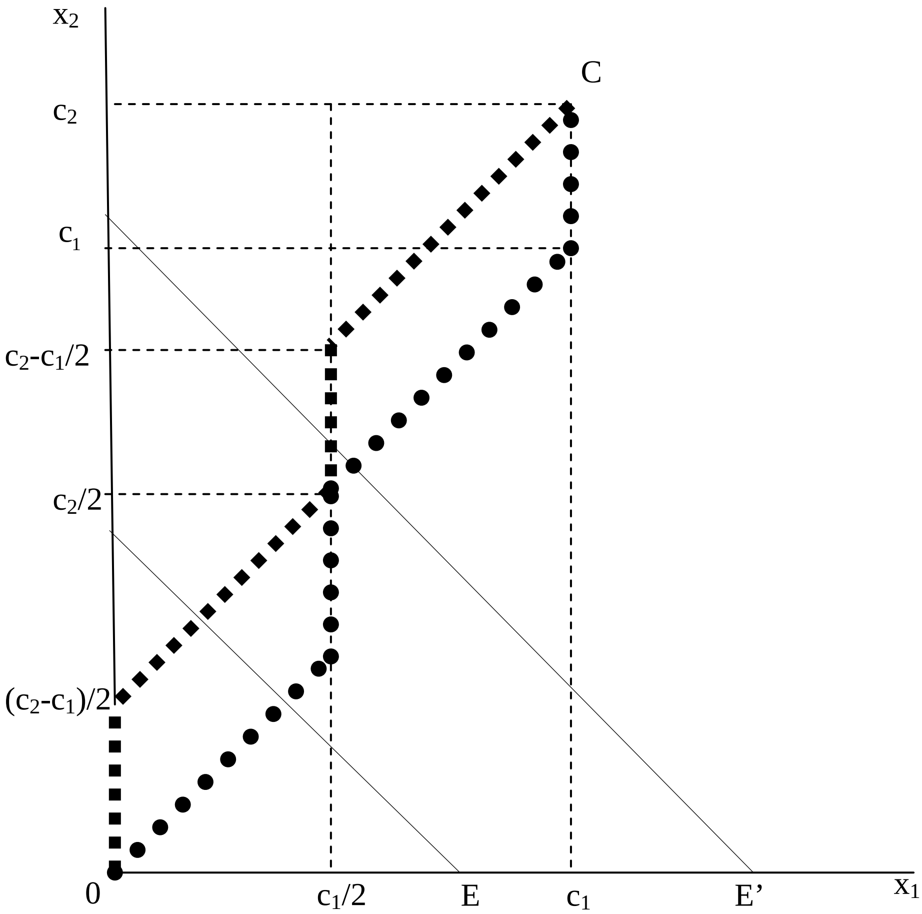

FIGURE 1: Socially Admissible Bankruptcy rules set in P1. The black lines represent all the possible sharing of two different levels of the estate (E and E'). The dots and the squares show φ^{Pin} and φ^{DPin}, which are the bankruptcy rules marking out the area of all the admissible path of awards satisfying the properties in P_1. The cross point between the estate lines and φ^{Pin} is the sharing of the resources between the two claimants provided by this rule (analogously for φ^{DPin}).

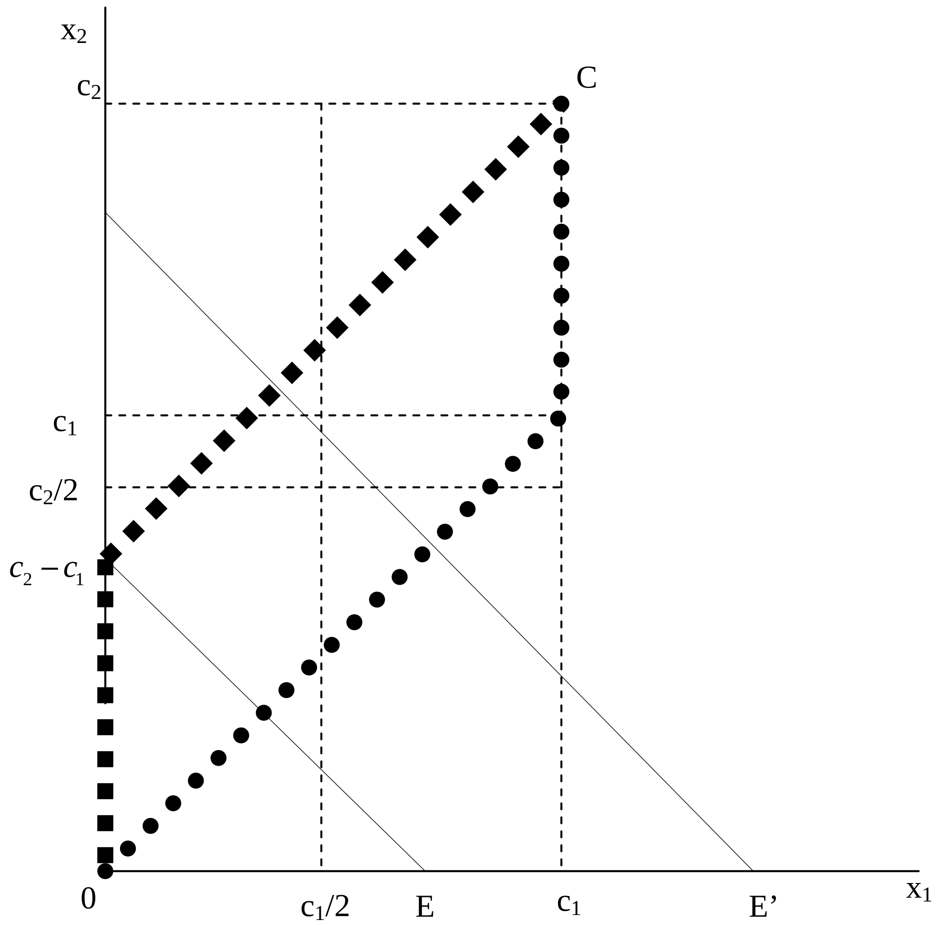

FIGURE 2: Socially Admissible Bankruptcy rules set in P2. The black lines represent all the possible sharing of two different levels of the estate (E and E’). The dots and the squares show φ^{CEA} and φ^{CEL}, which are the bankruptcy rules marking out the area of all the admissible path of awards satisfying the properties in P2. The cross point between the estate lines and the φ^{CEA} is the sharing of the resources between the two claimants provided by this rule (analogously for φ^{CEL}).

Theorem 4.8. *For each* $(E, c, P_2) \in B_P$, $\bar{\varphi}^R(E, c, P_2) = \varphi^{CEL}(E, c)$.

Proof. See Appendix 3. ∎

To conclude this Section, let us note that, in the two considered sets of properties the associated *Minimal Safety* is defined as the minimum of both a classical bankruptcy rule and its dual. These rules represent the extreme and opposite ways of sharing awards among conflicting claims in the set of *Socially Admissible Bankruptcy* rules according to the imposed requirements. Moreover we have proved, contrary to the first intuition which would be to get something in the middle of these extreme rules when applying the recursive procedure, that the corresponding *Recursive Minimal Safety* rule retrieves one of the such extremes, the one favoring the largest claims. In this sense, our results can be interpreted as new basis for old bankruptcy rules. So that, a natural question, analyzed in the next Section, would be:

'*For any appealing equity principles set,*
Would the recursive application of its Minimal Safety recover one of the extremes which define the area of all the Socially Admissible Bankruptcy rules?'

5. An incompatibility result

In this Section we show that, in general, the recursive application of our new lower bound on awards does not provide one of the extreme bankruptcy rules satisfying the considered equity principle set. Even more and surprisingly enough, we find out that the bankruptcy rule obtained by means of our procedure does not always satisfy the properties which it is based on, that is, it would not be a *Socially Admissible Bankruptcy* rule.

Let us consider the set of equity principles P_3 by adding to that ones defining P_2 the *Midpoint Property.* So that we are proposing an 'intermediate' situation more permissive than P_1, since we require *Order Preservation* instead of *Super-Modularity*, but more restrictive than P_2. That is,

P_3 ={*Resource Monotonicity*, *Order Preservation* and *Midpoint Property*}.

The following result, due to Chun, Schummer and Thomson [4], is used to find out the one of the extremes bankruptcy rule satisfying the properties in P_3.

Theorem 5.1. *(Chun, Schummer and Thomson, 2001)*

Let φ *be any bankruptcy rule satisfying Resource Monotonicity and Midpoint Property, then for each* $(E, c) \in B$ *the Constrained Egalitarian rule is the only rule such that the gap between the smallest and the largest amount any claimant receives is the smallest, and the variance of the amounts received by all the claimants is the smallest.*

Taking into account the previous result, the fact that all the considered properties are *Self-Dual* and the concept of dual bankruptcy rule, the next theorem, whose proof we omit since it is obvious, determines the another extreme bankruptcy rule satisfying properties in P_3.

Theorem 5.2. *Let φ be any bankruptcy rule satisfying Resource Monotonicity and Midpoint Property, then for each $(E, c) \in B$ the Dual Constrained Egalitarian rule is the only rule such that the gap between the smallest and the largest loss any claimant supports is the smallest, and the variance of the losses supported by all the claimants is the smallest.*

At this point is worth noticing that the *Dual Constrained Egalitarian* and the *Constrained Egalitarian* rules satisfy *Order Preservation.* Therefore, the previous results will remain valid when restricting the bankruptcy rules to those belonging to P_3.

The Figure 3 represents graphically the previous result for two-person bankruptcy problems.

Next lemma follows straightforwardly from the above theorems and determines the *Minimal Safety* for P_3.

Lemma 5.3. *Given (E, c, P_3) in B_p, the* **Minimal Safety,** *s, is for each $i \in N$,*

$$s_i(E, c, P_3) = \min \left\{ \varphi_i^{Ce}(E, c), \varphi_i^{DCe}(E, c) \right\}.$$

In this context, we show that, although for the two-person bankruptcy problems the recursive application of the *Minimal Safety* for P_3 retrieves the *Dual Constrained Egalitarian* rule, this fact can not be generalized.

Theorem 5.4. *For each two-person Bankruptcy Problem with Legitimate Principles in B_P with $P_t = P_3$ and each $i \in \{1, 2\}$, $\overline{\varphi}_i^R(E, c, P_3) = \varphi_i^{DCe}(E, c)$.*

Proof. See Appendix 4. ■

Proposition 5.5. *There is a bankruptcy problem, $(E, c) \in B$, such that $RMS(E, c, P_3) \neq \varphi^{DCe}(E, c)$.*

Proof. See Appendix 5. ■

Our next proposition points out that the composition of 'appealing' equity principles and 'natural' processes for finding solutions does not always guarantee desirable results. Particularly, it emphasizes both the need of being very careful when establishing the equity principles of the society if the procedure seems appropriated, and the need of searching processes which respect these principles, if they are considered irremovable.

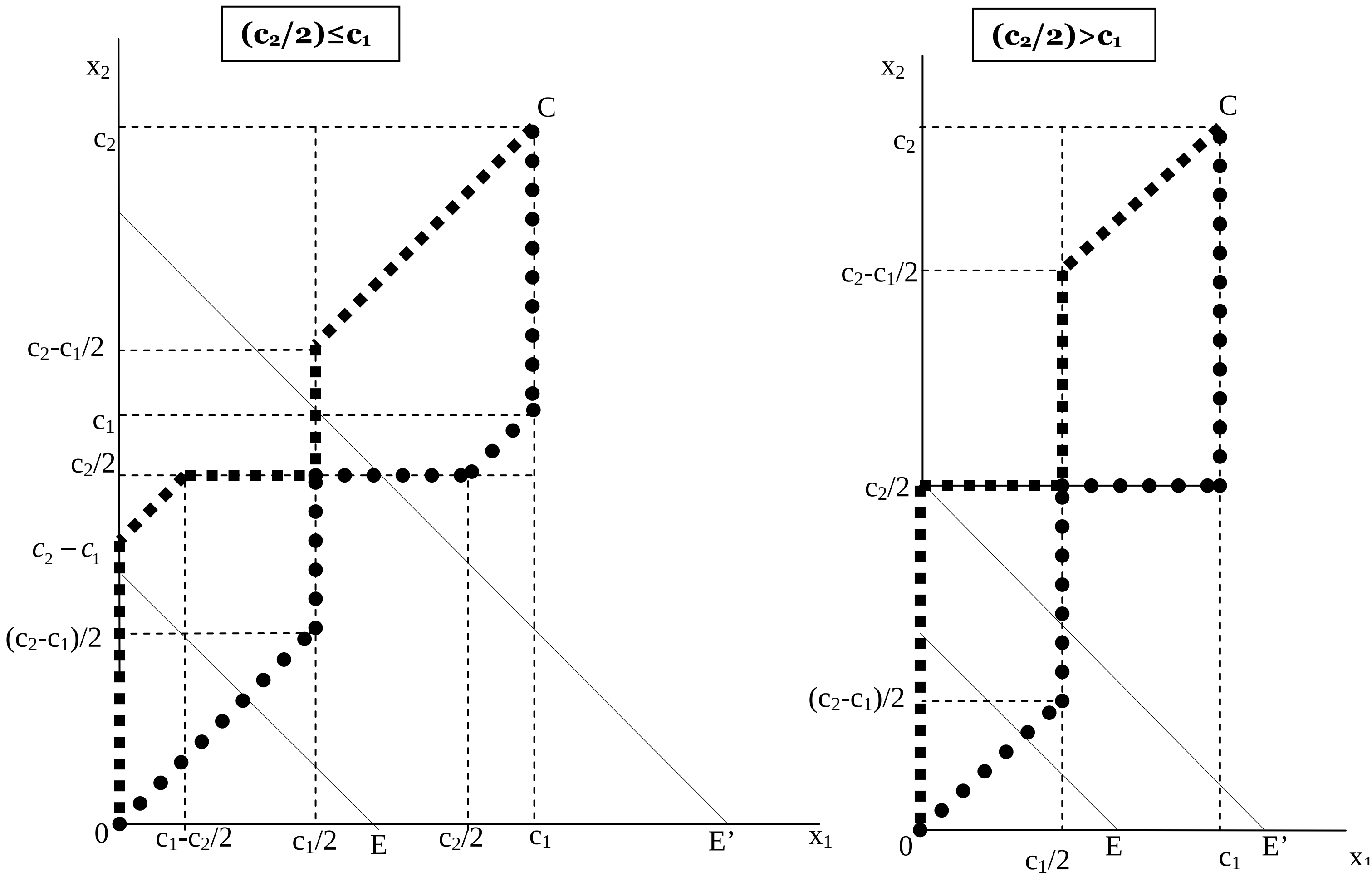

FIGURE 3: Socially Admissible Bankruptcy rules set in P3 The black lines represent all the possible sharing of two different levels of the estate (E and E'). The dots and squares show φ^{Ce} and φ^{DCe}, which are the bankruptcy rules marking out the area of all the admissible path of awards satisfying the properties in P3. The cross point between the estate lines and φ^{Ce} is the sharing of the resources between the two claimants provided by this rule (analogously for φ^{DCE}).

Proposition 5.6. *The Recursive Minimal Safety Process* for P_3 *does not satisfy Resource Monotonicity.*

Proof. See Appendix 5. ∎

Let us conclude this Section noting that, probably, it would not be difficult finding a society which accepts *Resource Monotonicity*, *Order Preservation* and *Midpoint Property,* willingly, and which considers fairly 'natural' our *Recursive Minimal Safety* process. However, we are sure that the result of this puzzle would not be accepted by any member of such a society, since it provides a bankruptcy rule which does not satisfy one of the equity principles upon which the society initially agreed to found its decisions; that is, *Resource Monotonicity*, one of the properties considered unquestionable in the Economic Literature.

6. Conclusions

In this paper we again take up a research line which has been underlaid the theoretical analysis of bankruptcy problems from its beginning: the search of a 'fair' minimum amount that each agent should receive when facing these situations.

In this context, our main contribution is a new method for bounding awards based on a set of *'Commonly Accepted Equity Principles'* by a society. Starting from this set, the award bound we propose, called *Minimal Safety*, is obtained by assigning each agent her minimum amount according to all admissible bankruptcy rules for such a society.

In general, once we allocate each agent her *Minimal Safety*, some part of the resources will still be available, fact that leads us to introduce the *Recursive Minimal Safety* rule, which lies in the recursive application of our new bound.

First, we apply the previous methodology to two different equity principles sets that, from our point of view, could be interpreted as social basic requirements. These are P_1={*Resource Monotonicity, Super-Modularity* and *Midpoint Property*} and P_2={*Resource Monotonicity* and *Order Preservation*}. In these cases we find out that the distribution recommended coincides with one of the extremes of the set of *Socially Admissible Bankruptcy* rules: the *Dual Piniles'* and the *Constrained Equal Losses* rules, both of those providing more awards to the higher claimants.

Next, we ascertain that not only the previous results can not be generalized, but also that the composition of both 'reasonable' equity principles and recursive process, a 'standard' way of exhausting the resources, does not always provide desirable results. To show this fact we do not need to define a complicated or artificial set of *'Commonly Accepted Equity Principles'.* Rather the contrary, by considering an 'intermediate' situation, P_3={*Resource Monotonicity, Order Preservation* and *Midpoint Property*}, we show that *Recursive Minimal Safety* rule does not satisfy *Resource Monotonicity*, one of the equity principles upon which the society initially agreed to found its decisions. Therefore, this result emphasizes the necessity of analyzing the consequences of the social agreements on both principles and procedure, since when put together could become meaningless.

Finally, let us note that the application of our analysis on losses is straightforward by using the idea of duality. In this context the starting point will be the same sets of '*Commonly Accepted Equity Principles*', P_1, P_2 and P_3, since all the considered properties are *Self-Dual.* Moreover, defining the *Minimal Safety* on losses, which could be called *Minimal Damage*, and applying it recursively the following results can be obtained. On the one hand, the *Piniles'* and the *Constrained Equal Awards* rules can be retrieved for P_1 and P_2 respectively. And on the other hand, for P_3, the *Constrained Egalitarian* rule arises for two-person bankruptcy problems by means of this process, but our approach does not guarantee a *Socially Admissible Solution* for the n-person case, since this procedure does not satisfy *Resource Monotonicity.*

To conclude this paper we would like to remark that our axiomatic analysis:

(i) Offers the understanding of old bankruptcy rules from a new angle.

(ii) Warns of the dangers that may involve the composition of 'a priori' appropriate pieces of a puzzle, and

(iii) Strengthens and complements the strategic study of the *Constrained Equal Losses* rule provided by Herrero [9], since totally different starting points, although with somehow similar mathematical modelization, retrieve the same bankruptcy solution; the axiomatic and the strategic approaches converge.

Therefore, the following questions remain opened: the analysis of the possible interpretation from the strategic point of view of the *Dual Piniles'* and the *Dual Constrained Egalitarian* rules; the search of new procedures which ensure the compatibility with socially accepted equity principles; and the analysis of conditions that should be imposed on the legitimate principle sets for guaranteeing their fulfillment when applying our recursive process.

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APPENDIX 1. *General Remarks*

Next we present three Remarks which will be used in the proofs provided in Appendices 2, 3 and 4. Moreover, from now on, $m \in \mathbb{N}$ will denote the m-*th* step of the *Recursive Minimal Safety Process*, that is,

$$(E^1, c^1) \equiv (E, c) \text{ and for } m \geq 2,$$

$$(E^m, c^m) \equiv (E^{m-1} - \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_t), c^{m-1} - s(E^{m-1}, c^{m-1}, P_t)).$$

The first Remark states that for any *Bankruptcy Problem with Legitimate Principles*, the amount of losses to distribute among agents is the same at every step of the *Recursive Minimal Safety Process.*

Remark 1. *For each $(E, c, P_t) \in B_P$ and each $m \in \mathbb{N}$, $L^m = L$.*

Proof. Let $(E, c, P_t) \in B_P$,
$L^m = C^m - E^m = \sum_{i \in N} \left(c_i - \sum_{k=1}^{m} s_i(E^k, c^k, P_t) \right) - \left(E - \sum_{i \in N} \sum_{k=1}^{m} s_i(E^k, c^k, P_t) \right) =$
$= C - E = L.$ ■

The second Remark establishes, for P_1, P_2 and P_3, that the order of the agents' claims is fixed along the different steps of the *Recursive Minimal Safety Process.*

Remark 2. *For each $(E, c, P_t) \in B_P$ with $t \in \{1, 2, 3\}$, if $c_i^m \leq c_j^m \Rightarrow c_i^{m+1} \leq c_j^{m+1}$.*

Proof. Let $(E, c, P_t) \in B_P$ with $t \in \{1, 2, 3\}$, and let denote $\left(\varphi^{Pin}, \varphi^{DPin}\right) =$
$= \left(\varphi^{L(P_1)}, \varphi^{U(P_1)}\right)$, $\left(\varphi^{CEA}, \varphi^{CEL}\right) = \left(\varphi^{L(P_2)}, \varphi^{U(P_2)}\right)$ and $\left(\varphi^{Ce}, \varphi^{DCe}\right) = \left(\varphi^{L(P_3)}, \varphi^{U(P_3)}\right)$.
Taking into account that all the above mentioned bankruptcy rules satisfy *Order Preservation*, we get for any $t \in \{1, 2, 3\}$:

(a) If $s_i^m(E, c, P_t) = \varphi_i^{L(P_t)}(E^m, c^m)$ and $s_j^m(E, c, P_t) = \varphi_j^{L(P_t)}(E^m, c^m)$, or alternatively, $s_i^m(E, c, P_t) = \varphi_i^{U(P_t)}(E^m, c^m)$ and $s_j^m(E, c, P_t) = \varphi_j^{U(P_t)}(E^m, c^m)$, by *Order Preservation*, $c_i^m - s_i^m(E^m, c^m, P_t) \leq c_j^m - s_j^m(E^m, c^m, P_t) \Rightarrow c_i^{m+1} \leq c_j^{m+1}$.

(b) If $s_i^m(E, c, P_t) = \varphi_i^{L(P_t)}(E^m, c^m)$ and $s_j^m(E, c, P_t) = \varphi_j^{U(P_t)}(E^m, c^m)$, by *Order Preservation*, $c_i^m - \varphi_i^{L(P_t)}(E^m, c^m) \leq c_j^m - \varphi_j^{L(P_t)}(E^m, c^m) \leq c_j^m - \varphi_j^{U(P_t)}(E^m, c^m)$ since, by definition of $s_j^m(E, c, P_t)$, $\varphi_j^{U(P_t)}(E^m, c^m) \leq \varphi_j^{L(P_t)}(E^m, c^m)$.

Therefore, $c_i^{m+1} \leq c_j^{m+1}$.

Let us note that the previous cases exhaust all the possibilities since if $s_i^m(E, c, P_t) = = \varphi_i^{U(P_t)}(E^m, c^m) \Rightarrow s_j^m(E, c, P_t) = \varphi_j^{U(P_t)}(E^m, c^m)$. ∎

The last Remark shows that, for any $t \in \{1, 2, 3\}$, the *Recursive Minimal Safety Process* converges.

Remark 3. *For each $(E, c, P_t) \in B_P$ with $t \in \{1, 2, 3\}$, $\sum_{i \in N} [RMS(E, c, P_t)]_i = E$.*

Proof. Let $(E, c, P_t) \in B_P$, with $t \in \{1, 2, 3\}$, and let denote by c^* and E^* the limits of the sequences of the claims and the estate arising in the *Recursive Minimal Safety Process,* respectively. Let us note that these limits exist since each agent's claim and the estate do not increase from a step to the next one, and both of them are lower bounded by zero. Next, we show by contradiction that $E^* = 0$. Let us suppose that $E^* > 0$, then by definition of limit and taking into account that for any $t \in \{1, 2, 3\}$, $s_j(E^*, c^*, P_t) > 0$, being j the highest claimant, we know that $\exists k \in \mathbb{N}$ such that $E^k - E^* \leq s_j(E^*, c^*, P_t)$. Since $\left(E^k, c^k, P_t\right)$ is a well-defined *Bankruptcy Problem with Legitimate Principles*, $s_j(E^k, c^k, P_t) > 0$, and given that all agents receive non-negative amounts, the estate to divide at the $(k+1)-$th step decreases by at least the j's *Minimal Safety.*

Therefore, $E^{k+1} \leq E^k - s_j(E^k, c^k, P_t) < E^k - s_j(E^*, c^*, P_t) \leq E^*$, which contradicts the definition of E^*. ∎

APPENDIX 2. ***Proof of Theorem 4.7***

This Appendix provides a formal proof of the following result:

Theorem 4.7. For each $(E, c, P_1) \in B_P$, $\bar{\varphi}^R(E, c, P_1) = \varphi^{DPin}(E, c)$.

Where, taking into account Lemma 4.5, Remark 3 and Definition 3.7, for each $(E, c) \in B$ and $i \in N$,

$$\bar{\varphi}_i^R(E, c, P_1) = \sum_{k=1}^{\infty} \min\{\varphi_i^{Pin}(E^k, c^k), \varphi_i^{DPin}(E^k, c^k)\}.$$

The proof of this result will be based on five Lemmas, but before presenting them, it is worth noticing the following two Facts. In all of them we will consider, without loss of generality, $(E, c) \in B_0$.

Fact 1. *Let us note that the definition of the Dual Piniles' rule can be written as follows, given $(E, c) \in B$, $i \in N$,*

$$\varphi_i^{DPin}(E, c) = \begin{cases} \frac{c_i}{2} - \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } E \leq C/2 \\ \frac{c_i}{2} + (\frac{c_i}{2} - \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } E \geq C/2 \end{cases},$$

where λ is such that $\sum\limits_{i \in N} \varphi_i^{DPin}(E, c) = E$.

Moreover, given $E^1 > C/2$ and $E^2 = E^1 - C/2$, the corresponding λ for (E^1, c) and (E^2, c), denoted by λ^1 and λ^2, respectively, will be the same, since

$$\lambda^1 : \sum_{i \in N} \min\left\{c_i/2, \lambda^1\right\} = C - E^1$$

and

$$\lambda^2 : \sum_{i \in N} \min\left\{c_i/2, \lambda^2\right\} = C/2 - E^2 = C - E^1.$$

It is straightforward to check that a way of computing this bankruptcy rule, that will be useful later on, is as follows.

Given $(E, c) \in B_0$, and for each $i \in N$, the amount of losses supported by agent i according to φ^{DPin} will be

$$\gamma_i = \begin{cases} \frac{c_i}{2} + \min\left\{c_i/2, \alpha_i\right\} & \text{if } E \leq C/2 \\ \min\left\{c_i/2, \alpha_i^*\right\} & \text{if } E \geq C/2 \end{cases},$$

where

$$\alpha_i = \left(C/2 - E - \sum_{j<i} \gamma_j \right) / (n - i + 1),$$

$$\alpha_i^* = \left(L - \sum_{j<i} \gamma_j \right) / (n - i + 1).$$

Therefore,

$$\varphi_i^{DPin}(E, c) = c_i - \gamma_i \ \forall i \in N.$$

Fact 2. *Taking into account the previous Fact and the Remark 1, we get:*

(a) Given $(E, c) \in B_0$ and $\forall i \in N$,

(i) when $E \leq C/2$, if $\gamma_i = c_i \Rightarrow \gamma_j = c_j \ \forall j < i$.

(ii) when $E \geq C/2$, if $\gamma_i = c_i/2 \Rightarrow \gamma_j = c_j/2 \ \forall j < i$.

(b) Given $(E, c) \in B_0$ and $\forall \ i \in N$,

(i) when $E \leq C/2$, if $\gamma_i = c_i/2 + \alpha_i \Rightarrow \alpha_i = \lambda$ and $\forall j > i, \alpha_j = \alpha_i$.

Therefore, $\gamma_i = c_i/2 + \lambda$.

(ii) when $E \geq C/2$, if $\gamma_i = \alpha_i \Rightarrow \alpha_i = \lambda$ and $\forall j > i, \alpha_j = \alpha_i$.

Therefore, $\gamma_i = \lambda$.

(c) At every step $m \in \mathbb{N}$, α_i^m will only depend on the initial bankruptcy problem, (E, c), and on the claims, at step m, corresponding to the agents $j \in N$ such that $j < i$.

Next, we provide the five Lemmas on which Theorem 4.7 is based. The first Lemma shows that the relationship among claims and λ is fixed.

Lemma 6.1. *For each $(E, c) \in B_0$ and each $m \in \mathbb{N}$,*

$$(a) \text{ if } c_i^m/2 < \lambda^m, \text{ then } c_i^{m+1}/2 < \lambda^m$$

and

$$(b) \text{ if } c_i^m/2 > \lambda^m, \text{ then } c_i^{m+1}/2 \geq \lambda^m.$$

Proof. The part (a) is obvious by considering Fact 1.

In order to prove (b) let us consider the two following cases.

(Case b.1) If $E \leq C/2$, let agent i be the first agent who receives a positive amount of awards at step $m \in \mathbb{N}$, i.e., (i) $s_i(E^m, c^m, P_1) > 0$ and (ii) $s_j(E^m, c^m, P_1) = = 0, \forall j < i$. By (i) and Fact 2, $c_i^m/2 > \lambda^m = \alpha_i^m$. Given (ii) and Definition 3.7, of $\bar{\varphi}^{RP_t}$, for any $t \in \{1, 2, 3\}$, $c_j^{m+1} = c_j^m$.

Taking into account Fact 2-(c), $\alpha_i^{m+1} = \alpha_i^m = \lambda^m < c_i^m/2$.

Moreover,

$$\begin{aligned} c_i^{m+1}/2 &= c_i^m/2 - \min\left\{\varphi_i^{DPin}(E^m, c^m), \varphi_i^{Pin}(E^m, c^m)\right\} \geq \\ &\geq c^m/2 - \varphi_i^{DPin}(E^m, c^m) = c_i^m/2 - (c_i^m/2 - \min\{c_i^m/2, \lambda^m\}) = \\ &= \min\{c_i^m/2, \lambda^m\} = \lambda^m. \end{aligned}$$

Therefore, $c_i^{m+1}/2 \geq \lambda^m = \alpha_i^m$.

(Case b.2) If $E \geq C/2$, taking into account that agents receive their half-claim and the remained estate is distributed as previously, see Fact 1, the conclusion follows straightforwardly. ■

The second Lemma says that λ is always the same.

Lemma 6.2. *For each $(E, c) \in B_0$ and each $m \in \mathbb{N}$, $\lambda^m = \lambda^{m+1}$, which solved $\sum_{i \in N} \varphi_i^{DPin}(E^m, c^m) = E^m$ and $\sum_{i \in N} \varphi_i^{DPin}(E^{m+1}, c^{m+1}) = E^{m+1}$, respectively.*

Proof. Let us note that given Fact 1, γ_i represents the amount of losses supported by each agent, and this will not change whenever α_i does not change. By construction, α_i will be the same whenever the relationship among claims and λ is fixed. By Remark 2, Fact 2-(b) and Lemma 6.1, we know that $\gamma_i^{m+1} = \alpha_i^{m+1} = \alpha_i^m = \gamma_i^m = \lambda^m$. ■

From now on, λ will denote λ^m, $\forall m \in \mathbb{N}$.

The third Lemma that we present says that if at some step $m \in \mathbb{N}$ the agent's i *Minimal Safety for* P_1 is $\varphi_i^{DPin}(E^m, c^m)$, then in the following steps her *Minimal Safety* for P_1 will be zero.

Lemma 6.3. *For each $(E, c) \in B_0$, if $\exists m \in \mathbb{N}$ such that $s_i(E^m, c^m, P_1) = \varphi_i^{DPin}(E^m, c^m)$ then,*

$$s_i(E^{m+h}, c^{m+h}, P_1) = 0, \text{ for each } h \in \mathbb{N}.$$

Proof. We are going to prove that if $s_i(E^m, c^m, P_1) = \varphi_i^{DPin}(E^m, c^m)$, then $s_i(E^{m+1}, c^{m+1}, P_1) = \varphi_i^{DPin}(E^{m+1}, c^{m+1}) = 0$.

Let $(E, c) \in B_0$ and $m \in \mathbb{N}$ such that

$$s_i(E^m, c^m, P_1) = \varphi_i^{DPin}(E^m, c^m).$$

(Case a) $E^m \leq C^m/2$ and $\lambda \geq c_i^m/2$, then

$$\varphi_i^{DPin}(E^m, c^m) = 0 \text{ and } c_i^{m+1}/2 = c_i^m/2 - \left(\varphi_i^{DPin}(E^m, c^m)\right) = c_i^m/2.$$

Therefore,

$$\varphi_i^{DPin}(E^{m+1}, c^{m+1}) = c_i^{m+1}/2 - \min\left\{c_i^{m+1}/2, \lambda\right\} = 0.$$

(Case b) $E^m \leq C^m/2$ and $\lambda < c_i^m/2$, then

$$\varphi_i^{DPin}(E^m, c^m) = c_i^m/2 - \min\{c_i^m/2, \lambda\} = c_i^m/2 - \lambda.$$

Thus,

$$\begin{aligned} c_i^{m+1}/2 &= c_i^m/2 - \left(\varphi_i^{DPin}(E^m, c^m)\right) = \\ &= c_i^m/2 - (c_i^m/2 - \lambda) = \lambda. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi_i^{DPin}(E^{m+1}, c^{m+1}) &= c_i^{m+1}/2 - \min\left\{c_i^{m+1}/2, \lambda\right\} = \\ &= \lambda - \min\{\lambda, \lambda\} = 0. \end{aligned}$$

(Case c) $E^m \geq C^m/2$, taking into account that agents receive their half-claim and the remained estate is distributed as previously, see Fact 1, the conclusion follows straightforwardly.

Let us note that considering Fact 1 again, we have that once $\varphi_i^{DPin}(E^m, c^m) = 0$, then $\varphi_i^{DPin}(E^{m+h}, c^{m+h}) = 0, \forall h \in \mathbb{N}$, so the agent's i *Minimal Safety* for P_1 will be from this step on always zero. ■

Next Lemma establishes that, if agent's i *Minimal Safety* for P_1 is, at every step, the amount provided by the *Piniles'* rule, then the total amount received by this agent will be at most her distribution corresponding to the *Dual Piniles'* rule applied to the initial problem.

Lemma 6.4. *For each $(E, c) \in B_0$, given $i \in N$, if $s_i(E^m, c^m, P_1) = \varphi_i^{Pin}(E^m, c^m)$ for each $m \in \mathbb{N}$, then*

$$\bar{\varphi}_i^R(E, c, P_1) = \sum_{k=1}^{\infty} s_i(E^k, c^k, P_1) \leq \varphi_i^{DPin}(E, c).$$

Proof. Let $(E, c) \in B_0$ and $i \in N$,

(Case a) First let us consider $E^m \leq C^m/2 \ \forall m \in \mathbb{N}$. If $s_i(E^m, c^m, P_1) =$ $= \varphi_i^{Pin}(E^m, c^m) \ \forall m \in \mathbb{N}$, then by Lemma 4.5,

$s_i(E^m, c^m, P_1) \leq \varphi_i^{DPin}(E^m, c^m) = c_i^m/2 - \lambda = c_i/2 - \sum_{k=1}^{m-1} s_i(E^k, c^k, P_1) - \lambda$, so that

$$\begin{aligned} s_i(E^m, c^m, P_1) + \sum_{k=1}^{m-1} s_i(E^k, c^k, P_1) &\leq c_i/2 - \lambda, \\ \sum_{k=1}^{m} s_i(E^k, c^k, P_1) &\leq \varphi_i^{DPin}(E, c). \end{aligned}$$

(Case b) If $E^m \geq C^m/2$, then $m = 1$. Now, taking into account that agents receive their half-claim and the remained estate is distributed as in case 1 (See Fact 1) the proof follows straightforwardly. ■

The last Lemma that we present states that once the agent's i *Minimal Safety* for P_1 is the amount provided by the *Dual Piniles'* rule, then the total amount received by this agent at that step will correspond with that one given by this bankruptcy rule applied to the initial problem.

Lemma 6.5. *For each $(E, c) \in B_0$ and each $i \in N$, if $\exists m^* > 1 \in \mathbb{N}$ such that $s_i(E^{m^*}, c^{m^*}, P_1) = \varphi_i^{DPin}(E^{m^*}, c^{m^*})$ and*

$$s_i(E^{m^*-1}, c^{m^*-1}, P_1) = \varphi_i^{Pin}(E^{m^*-1}, c^{m^*-1}),$$

then

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = \varphi_i^{DPin}(E, c).$$

Proof. Let $(E, c) \in B_0$.

(Case a) $E \leq C/2$.

$$\begin{aligned} s_i(E^{m^*}, c^{m^*}, P_1) &= \varphi_i^{DPin}(E^{m^*}, c^{m^*}) \text{ and} \\ s_i(E^{m^*-1}, c^{m^*-1}, P_1) &= \varphi_i^{Pin}(E^{m^*-1}, c^{m^*-1}), \end{aligned}$$

by construction, $\varphi_i^{Pin}(E^{m^*-1}, c^{m^*-1}) > 0$ and by Lemma 4.5, $\varphi_i^{DPin}(E^{m^*-1}, c^{m^*-1}) >$ > 0, given Fact 1, $c_i^{m^*-1}/2 > \lambda$ and by Lemma 6.2, $c_i^{m^*}/2 \geq \lambda$.

Then, at step m^*, $E^{m^*} \leq C^{m^*}/2$ and agent i has received

$$\begin{aligned}
\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) &= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + \varphi_i^{DPin}(E^{m^*}, c^{m^*}) = \\
&= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + \left[c_i^{m^*}/2 - \min\left\{c_i^{m^*}/2, \lambda\right\}\right] = \\
&= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1) + \left[\left(c_i/2 - \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_1)\right) - \lambda\right] = \\
&= c_i/2 - \lambda = \varphi_i^{DPin}(E, c).
\end{aligned}$$

(Case b) $E \geq C/2$. By Fact 1, we know that each agent receives at least half of her claim, so $\sum\limits_{k=1}^{m^*} s_i(E^k, c^k, P_1) = c_i/2 + \sum\limits_{k=1}^{m^*} s_i(\tilde{E}^k, c^k, P_1)$, where $\tilde{E} = E - C/2$.

Then if,

$$\begin{aligned}
s_i(\tilde{E}^{m^*}, c^{m^*}, P_1) &= \varphi_i^{DPin}(\tilde{E}^{m^*}, c^{m^*}) \text{ and} \\
s_i(\tilde{E}^{m^*-1}, c^{m^*-1}, P_1) &= \varphi_i^{Pin}(\tilde{E}^{m^*-1}, c^{m^*-1}),
\end{aligned}$$

by construction, $\varphi_i^{Pin}(\tilde{E}^{m^*-1}, c^{m^*-1}) > 0$, and by Lemma 4.5, $\varphi_i^{DPin}(\tilde{E}^{m^*-1}, c^{m^*-1}) >$ > 0. Therefore, by Fact 1 and Lemma 6.2, $c_i^{m^*-1}/2 > \lambda$, $\Rightarrow c_i^{m^*}/2 \geq \lambda$. Moreover, let us note that $\tilde{E}^{m^*} \leq C^{m^*}/2$.

Then, at step m^* agent i has received

$$\begin{aligned}
\sum_{k=1}^{m^*} s_i(\tilde{E}^k, c^k, P_1) &= \sum_{k=1}^{m^*-1} s_i(\tilde{E}^k, c^k, P_1) + \varphi_i^{DPin}(\tilde{E}^{m^*}, c^{m^*}) = \\
&= \sum_{k=1}^{m^*-1} s_i(\tilde{E}^k, c^k, P_1) + \left[c_i^{m^*}/2 - \min\left\{c^{m^*}/2, \lambda\right\}\right] = \\
&= \sum_{k=1}^{m^*-1} s_i(\tilde{E}^k, c^k, P_1) + \left[\left(c_i/2 - \sum_{k=1}^{m^*-1} s_i(\tilde{E}^{m^*}, c^{m^*}, P_1)\right) - \lambda\right] = \\
&= c_i/2 - \lambda = \varphi_i^{DPin}(E, c).
\end{aligned}$$

Therefore, $\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = c_i/2 + \sum_{k=1}^{m^*} s_i(\tilde{E}^k, c^k, P_1) = c_i - \lambda = \varphi_i^{DPin}(E, c)$. ■

Theorem 4.7. For each $(E, c, P_1) \in B_P$, $\bar{\varphi}^R(E, c, P_1) = \varphi^{DPin}(E, c)$.

Proof. Let $(E, c) \in B$. Let's consider all of the possible cases.

(Case a) All of the agents claim the same amount. Then, by the Definition of the *Minimal Safety* for P_1, each agent receives the same amount and the total estate is completely distributed at the first step. This amount corresponds with that provided by $\varphi^{DPin}(E, c)$.

(Case b) There are at least two agents with different claims. By construction, given Fact 1 the agent with the smallest claim, say i, has as *Minimal Safety* for P_1, $s_i(E, c, P_1) = \varphi_i^{DPin}(E, c)$. Moreover, the agent with the highest claim, say j, has as *Minimal Safety* for P_1, $s_j(E^m, c^m, P_1) = \varphi_j^{Pin}(E^m, c^m)$ for all $m \in \mathbb{N}$. Now, by Lemmas 6.3 and 6.5 we know that for each agent $r \in N$ who at some step $m \in \mathbb{N}$ receive $\varphi_r^{DPin}(E^m, c^m)$ as *Minimal Safety* for P_1, we have that $\bar{\varphi}_r^R(E, c, P_1) = \varphi_r^{DPin}(E, c)$. For the rest of the agents, say $l \neq r$, by Lemma 6.4 $\bar{\varphi}_l^R(E, c, P_1) \leq \varphi_l^{DPin}(E, c)$. Then, since $\bar{\varphi}^R(E, c, P_1)$ exhausts the estate, by Remark 3, $\bar{\varphi}^R(E, c, P_1) = \varphi^{DPin}(E, c)$. ■

APPENDIX 3. *Proof of Theorem 4.8*

This Appendix provides a formal proof of the following result:

Theorem 4.8. For each $(E, c, P_2) \in B_P$, $\bar{\varphi}^R(E, c, P_2) = \varphi^{CEL}(E, c)$.

Where, taking into account Lemma 4.6, Remark 3 and Definition 3.7, for each $(E, c) \in B$ and each $i \in N$,

$$\bar{\varphi}_i^R(E, c, P_2) = \sum_{k=1}^{\infty} \min\{\varphi_i^{CEA}(E^k, c^k), \varphi_i^{CEL}(E^k, c^k)\}.$$

The proof of this result will be based on four Lemmas, but before presenting them, it is worth noticing the following two Facts. In all of them we will consider, without loss of generality, $(E, c) \in B_0$.

Fact 3. *Given $(E, c) \in B_0$, $i \in N$, the definition of the Constrained Equal Losses rule is,*

$$\varphi_i^{CEL}(E, c) = \max\{0, c_i - \mu\},$$

where

$$\mu \text{ is such that } \sum_{i \in N} \max\{0, c_i - \mu\} = E.$$

Therefore, μ can be understood as the losses supported by the agents who receive a positive amount of awards applying the Constrained Equal Losses rule. It is straightforward to check that a way of computing this bankruptcy rule, that will be useful later on, is as follows.

Given $(E, c) \in B_0$, and for each $i \in N$, the amount of losses supported by agent i according to φ^{CEL} will be

$$\gamma_i = \min\{c_i, \alpha_i\},$$

where

$$\alpha_i = \left(L - \sum_{j<i} \gamma_j\right) / (n - i + 1).$$

Therefore,

$$\varphi_i^{CEL}(E, c) = c_i - \gamma_i, \forall i \in N.$$

Fact 4. *Taking into account the previous Fact and the Remark 1 we get:*

(a) *Given $(E, c) \in B_0$, $i \in N$, if $\gamma_i = c_i \Rightarrow \gamma_j = c_j \ \forall j < i$.*

(b) *Given $i \in N$, if $\gamma_i = \alpha_i \Rightarrow \alpha_i = \mu$ and $\forall j > i, \alpha_j = \alpha_i$. Therefore $\gamma_i = \mu$.*

(c) *At every step $m \in \mathbb{N}$, α_i^m will only depend on the initial bankruptcy problem, (E,c), and on the claims, at step m, corresponding to the agents $j \in N$ such that $j < i$.*

Next, we provide the four Lemmas on which Theorem 4.8 is based.

Lemma 6.6. *For each $(E,c) \in B_0$ and each $m \in \mathbb{N}$, $\mu^{m+1} = \mu^m$.*

Proof. Let agent i be the first agent who receives a positive amount of awards at step $m \in \mathbb{N}$, i.e., (i) $s_i(E^m, c^m, P_2) > 0$ and (ii) $s_j(E^m, c^m, P_2) = 0, \forall j < i$. By (i) and Fact 4, $c_i^m > \mu^m = \alpha_i^m$. Given (ii) and Definition 3.7, of $\bar{\varphi}^{RP_t}$, for any $t \in \{1,2,3\}$, $c_j^{m+1} = c_j^m$. Taking into account Fact 4-(c), $\alpha_i^{m+1} = \alpha_i^m = \mu^m < c_i^m$. Moreover,

$$\begin{aligned} c_i^{m+1} &= c_i^m - \min\left\{\varphi_i^{CEL}(E^m, c^m), \varphi_i^{CEA}(E^m, c^m)\right\} \geq \\ &\geq c_i^m - \varphi_i^{CEL}(E^m, c^m) = c_i^m - (c_i^m - \mu^m) = \\ &= \mu^m = \alpha_i^{m+1}. \end{aligned}$$

Therefore, by Remark 2 and Fact 4-(b), $\gamma_i^{m+1} = \alpha_i^{m+1} = \mu^{m+1}$. ■

From now on, μ will denote μ^m, $\forall m \in \mathbb{N}$.

The second Lemma that we present says that if at some step $m \in \mathbb{N}$ the agent's i *Minimal Safety* for P_2 is $\varphi_i^{CEL}(E^m, c^m)$, then in the following steps her *Minimal Safety* for P_2 will be zero.

Lemma 6.7. *For each $(E,c) \in B_0$, if $\exists m \in \mathbb{N}$ such that $s_i(E^m, c^m, P_2) = \varphi_i^{CEL}(E^m, c^m)$ then,*

$$s_i(E^{m+h}, c^{m+h}, P_2) = 0, \text{ for each } h \in \mathbb{N}.$$

Proof. We are going to prove that if $s_i(E^m, c^m, P_2) = \varphi_i^{CEL}(E^m, c^m)$ then $s_i(E^{m+1}, c^{m+1}, P_2) = \varphi_i^{CEL}(E^{m+1}, c^{m+1}) = 0$.

Let $(E,c) \in B_0$ and $m \in \mathbb{N}$ such that

$$s_i(E^m, c^m, P_2) = \varphi_i^{CEL}(E^m, c^m) = c_i^m - \min\{c_i^m, \mu\}.$$

Then,

$$c_i^{m+1} = c_i^m - \varphi_i^{CEL}(E^m, c^m) = c_i^m - (c_i^m - \min\{c_i^m, \mu\}) = \min\{c_i^m, \mu\}.$$

Therefore,

$$\begin{array}{rcl}\varphi_i^{CEL}(E^{m+1}, c^{m+1}) & = & c_i^{m+1} - \min\left\{c_i^{m+1}, \mu\right\} = \\ & = & \min\left\{c_i^m, \mu\right\} - \min\left\{\min\left\{c_i^m, \mu\right\}, \mu\right\} = \\ & = & \min\left\{c_i^m, \mu\right\} - \min\left\{c_i^m, \mu\right\} = 0.\end{array}$$

Let's note that given Fact 3, once $\varphi_i^{CEL}(E^m, c^m) = 0$, then $\varphi_i^{CEL}(E^{m+h}, c^{m+h}) = = 0, \forall h \in \mathbb{N}$, so the agent's i *Minimal Safety* for P_2 will be from this step on always zero. ■

Next Lemma establishes that, if agent's i *Minimal Safety* for P_2 is, at every step, the amount provided by the *Constrained Equal Awards* rule, then the total amount received by this agent will be at most her distribution corresponding to the *Constrained Equal Losses* rule applied to the initial problem.

Lemma 6.8. *For each* $(E, c) \in B_0$*, given* $i \in N$*, if* $s_i(E^m, c^m, P_2) = \varphi_i^{CEA}(E^m, c^m)$ *for each* $m \in \mathbb{N}$*, then*

$$\overline{\varphi}_i^R(E, c, P_2) = \sum_{k=1}^{\infty} s_i(E^k, c^k, P_2) \leq \varphi_i^{CEL}(E, c).$$

Proof. Let $(E, c) \in B_0$ and $i \in N$,

If $\forall m \in \mathbb{N}$, $s_i(E^m, c^m, P_2) = \varphi_i^{CEA}(E^m, c^m)$, then by Lemma 4.6, $s_i(E^m, c^m, P_1) \leq \leq \varphi_i^{CEL}(E^m, c^m) = c_i^m - \mu = c_i - \sum_{k=1}^{m-1} s_i(E^k, c^k, P_2) - \mu$, so that

$$\begin{array}{rcl}s_i(E^m, c^m, P_2) + \displaystyle\sum_{k=1}^{m-1} s_i(E^k, c^k, P_2) & \leq & c_i - \mu, \\ \displaystyle\sum_{k=1}^{m} s_i(E^k, c^k, P_2) & \leq & \varphi_i^{CEL}(E, c).\end{array}$$

■

The last Lemma that we present says that once the agent's i *Minimal Safety* for P_2 is the amount provided by the *Constrained Equal Losses* rule, then the total amount received by this agent at that step will correspond with that one given by the *Constrained Equal Losses* applied to the initial problem.

Lemma 6.9. *For each* $(E, c) \in B_0$ *and each* $i \in N$, *if* $\exists m^* > 1 \in \mathbb{N}$ *such that* $s_i(E^{m^*}, c^{m^*}, P_2) = \varphi_i^{CEL}(E^{m^*}, c^{m^*})$ *and* $s_i(E^{m^*-1}, c^{m^*-1}, P_2) = \varphi_i^{CEA}(E^{m^*-1}, c^{m^*-1})$, *then* $\sum\limits_{k=1}^{m^*} s_i(E^k, c^k, P_2) = \varphi_i^{CEL}(E, c)$.

Proof. Let $(E, c) \in B_0$.

$$\begin{aligned} s_i(E^{m^*}, c^{m^*}, P_2) &= \varphi_i^{CEL}(E^{m^*}, c^{m^*}) \text{ and} \\ s_i(E^{m^*-1}, c^{m^*-1}, P_2) &= \varphi_i^{CEA}(E^{m^*-1}, c^{m^*-1}), \end{aligned}$$

since $\varphi_i^{CEA}(E^{m^*-1}, c^{m^*-1}) > 0$, $\varphi_i^{CEL}(E^{m^*-1}, c^{m^*-1}) > 0$. Therefore $c_i^{m^*-1} > \mu$ and by Lemma 6.6, $c_i^{m^*} \geq \mu$.

Then, at m^* agent i has received

$$\begin{aligned} \sum_{k=1}^{m^*} s_i(E^k, c^k, P_2) &= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_2) + \varphi_i^{CEL}(E^{m^*}, c^{m^*}) = \\ &= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_2) + \left[c_i^{m^*} - \min\left\{c_i^{m^*}, \mu\right\}\right] = \\ &= \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_2) + \left[\left(c_i - \sum_{k=1}^{m^*-1} s_i(E^k, c^k, P_2)\right) - \min\left\{c_i^{m^*}, \mu\right\}\right] = \\ &= c_i - \min\left\{c_i^{m^*}, \mu\right\} = c_i - \mu. \end{aligned}$$

Therefore

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_2) = \varphi_i^{CEL}(E, c).$$

∎

Theorem 4.8. For each $(E, c, P_2) \in B_P$, $\bar{\varphi}^R(E, c, P_2) = \varphi^{CEL}(E, c)$.

Proof. Let $(E, c) \in B$. Let's consider all of the possible cases.

(Case a) All of the agents claim the same amount. Then, by the Definition of *Minimal Safety* for P_2 , each agent receives the same amount and the total estate is completely distributed at the first step. This amount corresponds with that provided by $\varphi^{CEL}(E, c)$.

(Case b) There are at least two agents with different claims. By construction, in this case the agent with the smallest claim, say i, receives as *Minimal Safety* for P_2, $s_i(E, c, P_2) = \varphi_i^{CEL}(E, c)$. Moreover, the agent with the highest claim, say j, receives as *Minimal Safety* for P_2, $s_j(E, c, P_2) = \varphi_j^{CEA}(E^m, c^m)$ for all $m \in \mathbb{N}$. Now, by Lemmas 6.7 and 6.9 we know that for each agent $r \in N$ who at some step $m \in \mathbb{N}$ receives $\varphi_r^{CEL}(E^m, c^m)$ as *Minimal Safety* for P_2, we have that $\bar{\varphi}_r^R(E, c, P_2) = \varphi_r^{CEL}(E, c)$. For the rest of the agents, say $l \neq r$, by Lemma 6.8, $\bar{\varphi}_l^R(E, c, P_2) \leq \varphi_l^{CEL}(E, c)$. Then, since $\bar{\varphi}^R(E, c, P_2)$ exhausts the estate, by Remark 3, $\bar{\varphi}^R(E, c, P_2) = \varphi^{CEL}(E, c)$. ■

APPENDIX 4. *Proof of Theorem 5.4*

This Appendix provides a formal proof of the following result:

Theorem 5.4. For each two-person *Bankruptcy Problem with Legitimate Principles* in B_P with $P_t = P_3$ and each $i \in \{1,2\}$, $\bar{\varphi}_i^R(E,c,P_3) = \varphi_i^{DCe}(E,c)$.

Where, taking into account Lemma 5.3, Remark 3 and Definition 3.7, for each $(E,c) \in B$ and each $i \in N$,

$$\bar{\varphi}_i^R(E,c,P_3) = \sum_{k=1}^{\infty} \min\{\varphi_i^{Ce}(E^k,c^k), \varphi_i^{DCe}(E^k,c^k)\}.$$

At this point it is worth noticing the following two Facts, in which we will consider, without loss of generality, $(E,c) \in B_0$.

Fact 5. *Let us note that the definition of the Dual Constrained Egalitarian rule can be written as follows, given $(E,c) \in B_0$, $i \in N$,*

$$\varphi_i^{DCe}(E,c) \equiv \begin{cases} c_i - \max\{c_i/2, \min\{c_i,\delta\}\} & \text{if } E \leq C/2 \\ c_i - \min\{c_i/2, \delta\} & \text{if } E \geq C/2 \end{cases},$$

where δ is chosen such that $\sum_{i \in N} \varphi_i^{DCe}(E,c) = E$.

From the previous expression, it is obvious that agent one will receive nothing when $\min\{c_2 - c_1, c_2/2\} \geq E$.

The following Fact gives us the two conditions used in the proof of Theorem 5.4**.**

Fact 6. *Let $(E,c) \in B_0$ a two-person bankruptcy problem. Then given Fact 5 and the definition of the Dual Constrained Egalitarian rule, at any step $m \in \mathbb{N}$, $s_1(E^m,c^m,P_3) = = \varphi_1^{DCe}(E^m,c^m)$. Therefore:*

(i) Next inequality characterizes the fact that agent one has guaranteed nothing at any step $m \in \mathbb{N}$

$$s_1(E^m,c^m,P_3) = 0 \Leftrightarrow \min\{c_2^m - c_1^m, c_2^m/2\} \geq E^m. \tag{6.1}$$

(ii) The previous characterization for step $m \in \mathbb{N}$ implies the following conditions in terms of the bankruptcy problem at the step $m-1$,

$$E^m \leq c_2^m - c_1^m \Leftrightarrow E^{m-1} - s_1(E^{m-1},c^{m-1},P_3) - s_2(E^{m-1},c^{m-1},P_3) \leq$$
$$\leq c_2^{m-1} - s_2(E^{m-1},c^{m-1},P_3) - (c_1^{m-1} - s_1(E^{m-1},c^{m-1},P_3))$$

$$\text{so that, for } m=2,\ E^2 \leq c_2^2 - c_1^2 \Leftrightarrow E \leq c_2 - c_1 + 2s_1(E,c,P_3). \tag{6.2}$$

Moreover, $E^2 \leq c_2^2/2 \Leftrightarrow E - s_1(E,c,P_3) - s_2(E,c,P_3) \leq c_2/2 - s_2(E,c,P_3)/2$

$$\textit{so that, } E^2 \leq c_2^2/2 \Leftrightarrow E \leq c_2/2 + s_2(E,c,P_3)/2 + s_1(E,c,P_3). \tag{6.3}$$

.

Proof of Theorem 5.4

For each two-person bankruptcy problem, $(E,c) \in B_0$, we have by Fact5 that at any step $m \in \mathbb{N}$, $s_1(E^m,c^m,P_3) = \varphi_1^{DCe}(E^m,c^m)$, and $s_2(E^m,c^m,P_3) = \varphi_2^{Ce}(E^m,c^m)$. Given this, we show that agent one's *Minimal Safety* for P_3 at any step $m \geq 2$, will be zero, so the agent one's *Recursive Minimal Safety* rule for P_3 will correspond with the *Dual Constrained Egalitarian* rule. Then, since $\bar{\varphi}^R(E,c,P_3)$ exhausts the estate, given Remark 3, $\bar{\varphi}_2^R(E,c,P_3) = \varphi_2^{DCe}(E,c)$.

If $c_1 = c_2$, by the Definition of the *Recursive Minimal Safety* rule for P_3, each agent i receives the same amount at the initial step, and if $c_1 \neq c_2$, with $E = (c_1 + c_2)/2$ by *Midpoint Property* each agent i receives her half-claim, $c_i/2$. Therefore in both cases at the initial step the estate is completely distributed, and this amount correspond with that one given by the *Dual Constrained Egalitarian* rule.

Next we consider three possible cases when $c_1 \neq c_2$.

Case 1) Let $s_1(E,c,P_3) = 0$[2].

Then, by Condition 6.1 we have that $E \leq \min\{c_2 - c_1, c_2/2\}$. Now, in the following step $E^2 = E - s_2(E,c,P_3)$, $c_1^2 = c_1$ and $c_2^2 = c_2 - s_2(E,c,P_3)$. Therefore, again Condition 6.1 states that $s_1(E^2,c^2,P_3) = 0$ if and only if $E - s_2(E,c,P_3) \leq c_2 - c_1 - s_2(E,c,P_3)$,which follows from $E \leq c_2 - c_1$, and $E - s_2(E,c,P_3) \leq (c_2/2) - (s_2(E,c,P_3)/2)$,which follows from $E \leq c_2/2$.

Applying the previous reasoning from step 2 to the next one, and so on, we get $\bar{\varphi}_1^R(E,c,P_3) = 0$. Therefore, by Remark 3, $\bar{\varphi}^R(E,c,P_3) = (0,E) = \varphi^{DCe}(E,c)$.

In Cases 2 and 3, we will show that at $m = 2$ the agent's one *Minimal Safety* for P_3 will be zero. Then, Case 1 can be applied for the remainder *Bankruptcy Problem with Legitimate Principles*, so from $m = 2$ on, $s_1(E^{m+h},c^{m+h},P_3) = 0$, $\forall h \in \mathbb{N}$. Therefore, $\bar{\varphi}_1^R(E,c,P_3) = s_1(E,c,P_3)$.

Case 2) Let $s_1(E,c,P_3) > 0$, and $c_2/2 \geq c_2 - c_1$.

In this case the agents' *Minimal Safety* for P_3 can be placed in the following regions.

[2] This is only possible in Figure 4, region I.a, and Figure 5, regions I.b and II.b.

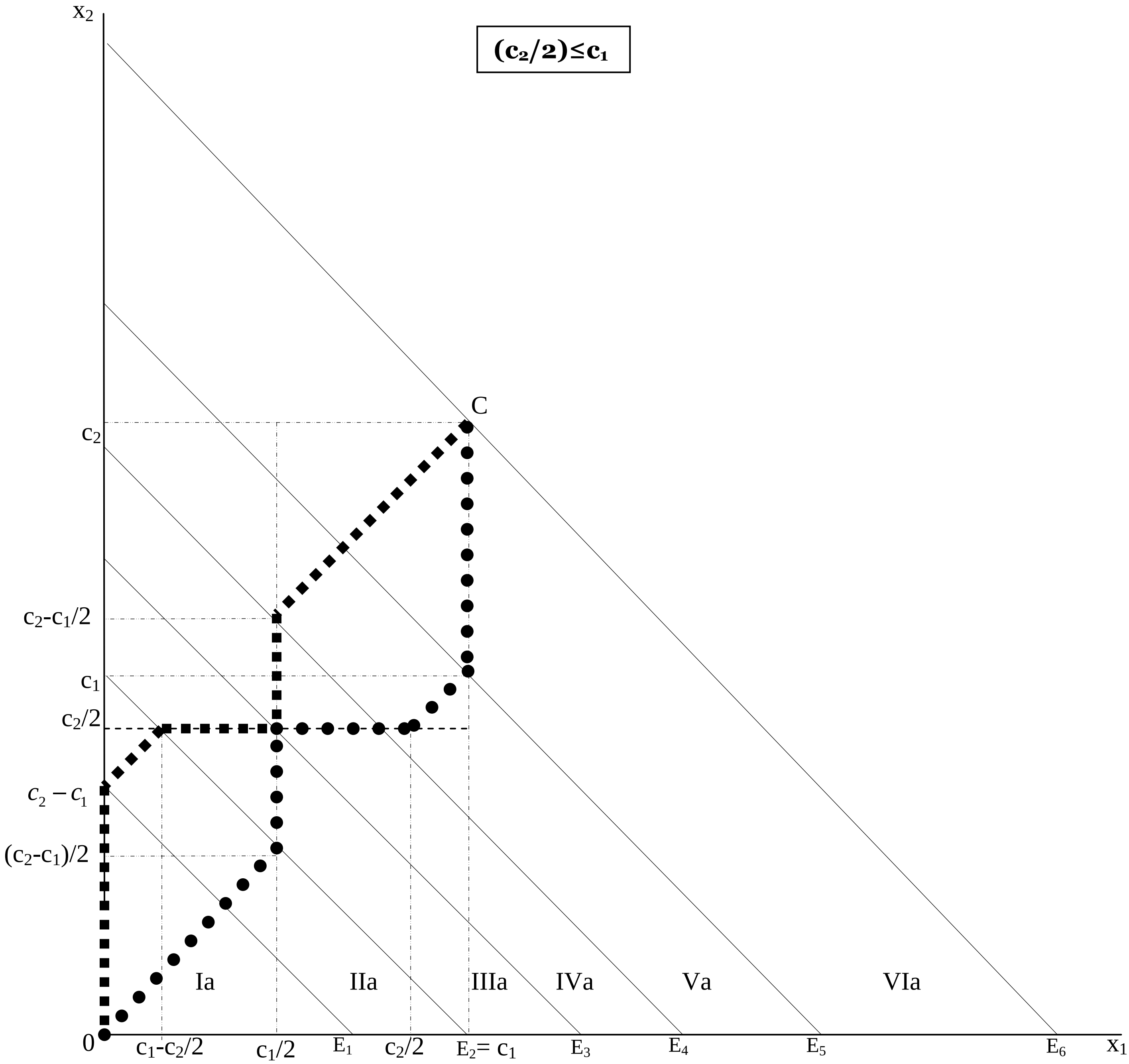

FIGURE 4: **Regions of the *Constrained Egalitarian* rule and its dual. Case a.** The black lines represent all the possible sharing of six different levels of the estate (E_1, E_2, E_3, E_4, E_5 and E_6). The dots and squares show φ^{Ce} and φ^{DCe}, which are the bankruptcy rules marking out the area of all the admissible path of awards satisfying the properties in P3. The regions are, starting from the estate zero, those areas bounded by an estate and by the next one.

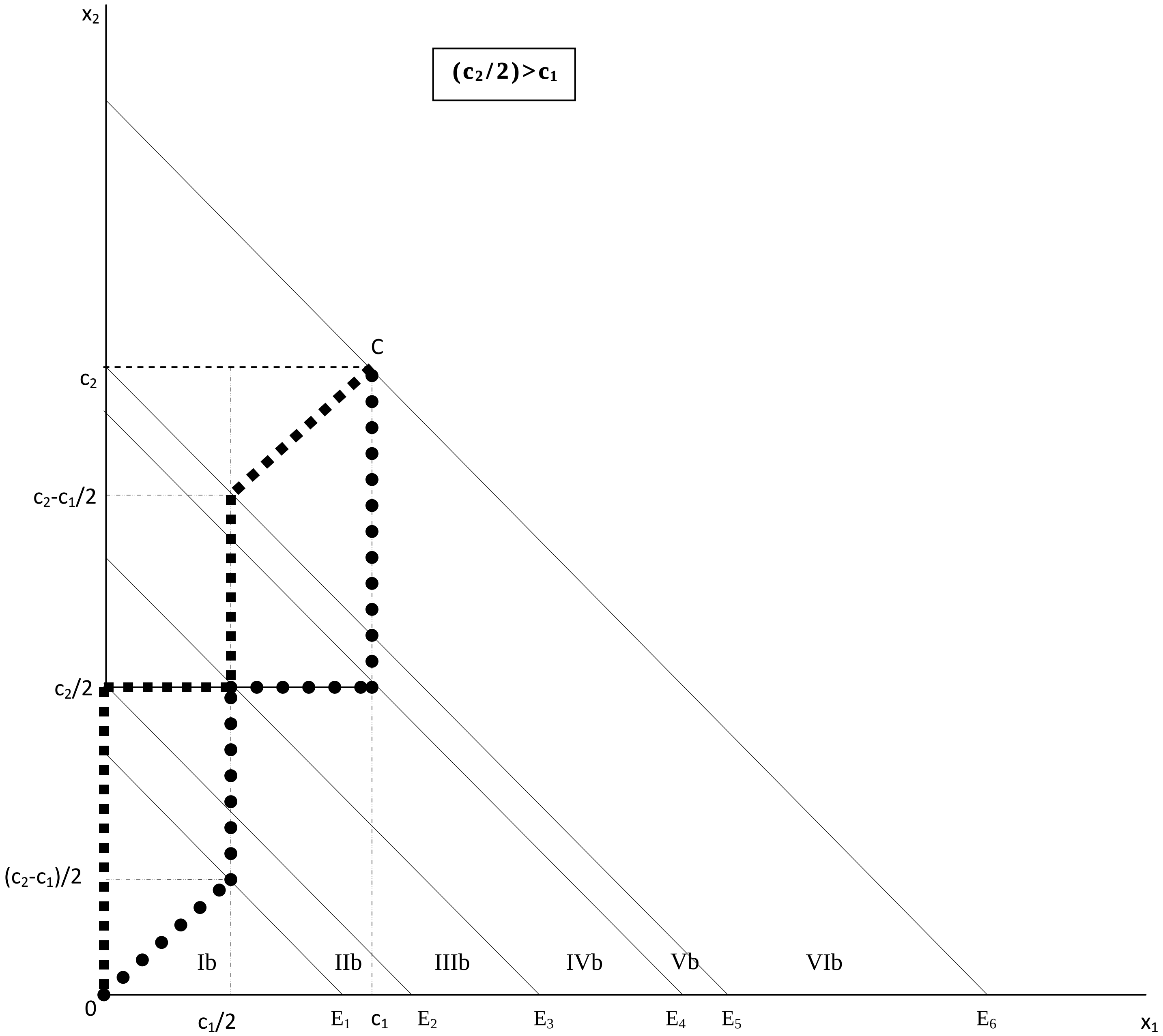

FIGURE 5: Regions of the *Constrained Egalitarian* rule and its dual. Case b. The black lines represent all the possible sharing of six different levels of the estate (E_1, E_2, E_3, E_4, E_5 and E_6). The dots and squares show φ^{Ce} and φ^{DCe}, which are the bankruptcy rules marking out the area of all the admissible path of awards satisfying the properties in P3. The regions are, starting from the estate zero, those areas bounded by an estate and by the next one.

Region II.a) $c_2 - c_1 \leq E \leq c_1$. Then, $s_1(E,c,P_3) = (E+c_1-c_2)/2$ and $s_2(E,c,P_3) = E/2$. Conditions 6.2 and 6.3 impose $E \leq 2c_1$, which is true since in this region $E \leq c_1$. Therefore, $\bar{\varphi}^R(E,c,P_3) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = \varphi^{DCe}(E,c)$.

Region III.a) $c_1 \leq E \leq (c_1+c_2)/2$. Then, $s_1(E,c,P_3) = E - c_2/2$ and $s_2(E,c,P_3) = E - c_1/2$. Now, Conditions 6.2 and 6.3 impose $E \geq c_1$, which is obviously fulfilled in this region. Therefore, $\bar{\varphi}^R(E,c,P_3) = (E-(c_2/2), c_2/2) = \varphi^{DCe}(E,c)$.

Region IV.a) $(c_1+c_2)/2 \leq E \leq [(c_1+c_2)/2] + [(c_2-c_1)/2] = c_2$. Then, $s_1(E,c,P_3) = c_1/2$ and $s_2(E,c,P_3) = c_2/2$. Again, from Conditions 6.2 and 6.3 we need to show that $E \leq c_2$, which is the Estate-upper bound of this region, and that $E \leq (3c_2/4) + (c_1/2)$,which, taking into account the Estate-upper bound of this region, is true since $c_2/2 \geq c_2 - c_1$ in Case 2, which implies $c_1/2 \geq c_2/4$. Therefore, $\bar{\varphi}^R(E,c,P_3) = (c_1/2, E - c_1/2) = \varphi^{DCe}(E,c)$.

Region V.a) $c_2 \leq E \leq 2c_1$. Then, $s_1(E,c,P_3) = (E+c_1-c_2)/2$ and $s_2(E,c,P_3) = E/2$. Now, Conditions 6.2 and 6.3 impose $E \leq 2c_1$, which is obviously fulfilled in this region. Therefore, $\bar{\varphi}^R(E,c,P_3) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = \varphi^{DCe}(E,c)$.

Region VI.a) $2c_1 \leq E$. Then, $s_1(E,c,P_3) = (E+c_1-c_2)/2$ and $s_2(E,c,P_3) = E - c_1$. Here, Conditions 6.2 and 6.3 do not impose any restriction, so that, $\bar{\varphi}^R(E,c,P_3) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = \varphi^{DCe}(E,c)$.

Case 3) Let $s_1(E,c,P_3) > 0$, and $c_2/2 \leq c_2 - c_1$. In this case the agents' *Minimal Safety* for P_3 can be placed in the following regions.

Region III.b) $c_2/2 \leq E \leq (c_1+c_2)/2$. Then, $s_1(E,c,P_3) = E - c_2/2$ and $s_2(E,c,P_3) = E - c_1/2$. Conditions 6.2 and 6.3 impose $E \geq c_1$, inequality fulfilled since in this region $c_2/2 \leq c_2 - c_1$, implying $c_1 \leq c_2/2$. Therefore, $\bar{\varphi}^R(E,c,P_3) = (E - c_2/2, c_2/2) = \varphi^{DCe}(E,c)$.

Region IV.b) $(c_1+c_2)/2 \leq E \leq c_1 + c_2/2$. Then $s_1(E,c,P_3) = c_1/2$ and $s_2(E,c,P_3) = c_2/2$. Now, Conditions 6.2 and 6.3 impose $E \leq c_2$ and $E \leq 3c_2/4 + c_1/2$, and both inequalities are satisfied since in this region $c_2/2 \leq c_2 - c_1$ which implies $c_1 \leq c_2/2$. Therefore, $\bar{\varphi}^R(E,c,P_3) = (c_1/2, E - c_1/2) = \varphi^{DCe}(E,c)$.

Region V.b) $c_1 + c_2/2 \leq E \leq c_2$. $s_1(E,c,P_3) = c_1/2$ and $s_2(E,c,P_3) = E - c_1$. Conditions 6.2 and 6.3 impose $E \leq c_2$, which is the Estate-upper bound of this region. Therefore, $\bar{\varphi}^R(E,c,P_3) = (c_1/2, E - c_1/2) = \varphi^{DCe}(E,c)$.

Region VI.b) $c_2 \leq E$. Then, $s_1(E,c,P_3) = (E+c_1-c_2)/2$ and $s_2(E,c,P_3) = E - c_1$. Here, Conditions 6.2 and 6.3 do not impose any restriction, so that, $\bar{\varphi}^R(E,c,P_3) = ((E+c_1-c_2)/2, (E-c_1+c_2)/2) = \varphi^{DCe}(E,c)$.

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APPENDIX 5. ***Proof of Propositions 5.5 and 5.6.***

This Appendix provides a formal proof of the following results:

Proposition 5.5: There is a bankruptcy problem, $(E, c) \in B$, such that $RMS(E, c, P_3) \neq \varphi^{DCe}(E, c)$.

Proposition 5.6: The *Recursive Minimal Safety Process* for P_3 does not satisfy *Resource Monotonicity.*

Where $[RMS(E, c, P_3)]_i = \sum_{k=1}^{\infty} s_i(E^k, c^k, P_3)$.

At this point it is worth noticing the following Fact, in which we will consider, without loss of generality, $(E, c) \in B_0$.

Fact 7. *Let us note that the Dual Constrained Egalitarian rule (Fact 5), can be written as follows, given $(E, c) \in B_0$, $i \in N$,*

$$\varphi_i^{DCe}(E, c) \equiv \begin{cases} c_i - \gamma_i & \text{if } E \leq C/2 \\ c_i - \gamma_i & \text{if } E \geq C/2 \end{cases},$$

where γ_i is chosen such that $\sum_{i \in N} \varphi_i^{DCe}(E, c) = E$.

Therefore,

(Case a) If $E \leq C/2$, we can compute γ_i as:

$$\gamma_i = \begin{cases} c_i & \forall\ i < l \\ \max\{c_i/2, \alpha_i\} & \forall\ i \geq l \end{cases},$$

where agent l is that one such that $\sum_{j>l} \min\{c_j - c_l; c_j/2\} < E$, and either $\sum_{j>l-1} \min\{c_j - c_{l-1}; c_j/2\} \geq E$, either $l = 1$. Otherwise, $l = n$. Then, $\forall i \geq l$,

$$\alpha_i = \frac{L - \sum_{j=1}^{k-1} c_j - \sum_{j>i} \gamma_j}{i - l + 1}.$$

Note also that we should compute α from the highest claimant to the smallest one.

Fact 8. *(Case b) If $E \geq C/2$, γ_i will denote the losses supported by agent i when the losses from the claim vector are equal to all the agents subject to no-one obtaining less than her half-claim.*

Proposition 5.5: There is a bankruptcy problem, $(E, c) \in B$, such that $RMS(E, c, P_3) \neq \varphi^{DCe}(E, c)$.

Proof. Let us consider the following bankruptcy problem $(E, c) \in B = (21, (5; 19.5; 20))$.

Thus, given the definitions of the *Constrained Egalitarian* rule and its dual and Fact 7, we get,

At step $m = 1$:

$(E^1, c^1) = (21, (5; 19.5; 20))$.

$\varphi^{Ce}(E^1, c^1) = (2.5; 9.25; 9.25)$.

$\varphi^{DCe}(E^1, c^1) = (5 - \gamma_1; 19.5 - \gamma_2; 20 - \gamma_3)$.

$l = 1 : \min\{20 - 5; 10\} + \min\{19.5 - 5; 9.75\} = 19.75 < E = 21$.

$\alpha_3 = 23.5/3 = 7.83 \Rightarrow \gamma_3 = \max\{10; 7.83\} = 10$.

$\alpha_2 = (23.5 - 10)/2 = 6.75 \Rightarrow \gamma_2 = \max\{9.75; 6.75\} = 9.75$.

$\alpha_1 = 23.5 - 10 - 9.75 = 3.75 \Rightarrow \gamma_1 = \max\{2.5; 3.75\} = 3.75$.

So, $\varphi^{DCe}(E^1, c^1) = (1.25; 9.75; 10)$.

Therefore, $s(E^1, c^1, P_3) = (1.25; 9.25; 9.25)$.

At step $m = 2$:

$\left(E^2, c^2\right) = (1.25, (3.75; 10.25; 10.75))$.

$\varphi^{Ce}(E^2, c^2) = (0.416; 0.416; 0.416)$.

$\varphi^{DCe}(E^2, c^2) = (3.75 - \gamma_1^2; 10.25 - \gamma_2^2; 10.75 - \gamma_3^2)$.

$l = 1$:

$\min\{10.75 - 3.75; 5.375\} + \min\{10.25 - 3.75; 5.125\} = 10.5 > E^2 = 1.25$.

$l = 2 : \min\{10.75 - 10.25; 5.375\} = 0.5 < E^2 = 1.25$.

$\alpha_3^2 = (23.5 - 3.75)/2 = 9.875 \Rightarrow \gamma_3^2 = \max\{5.375; 9.875\} = 9.875$.

$\alpha_2^2 = 23.5 - 3.75 - 9.875 = 9.875 \Rightarrow \gamma_2^2 = \max\{5.125; 9.875\} = 9.875$.

$\gamma_1^2 = 3.75$.

So, $\varphi^{DCe}(E^2, c^2) = (0; 0.375; 0.875)$.

Therefore,

$s(E^2, c^2, P_3) = (0; 0.375; 0.416)$, and

$\sum_{k=1}^{2} s(E^k, c^k, P_3) = (1.25; 9.625; 9.666)$.

At step $m = 3$:

$\left(E^3, c^3\right) = (0.459, (3.75; 9.875; 10.334))$.

$\varphi^{Ce}(E^3, c^3) = (0.153; 0.153; 0.153)$.

$\varphi^{DCe}(E^3, c^3) = (3.75 - \gamma_1^3; 9.875 - \gamma_2^3; 10.334 - \gamma_3^3)$.

Let us note that $l = 2 : \min\{10.334 - 9.875; 5.167\} = 0.459 = E^3$, thus

$\alpha_3^3 = 23.5 - 3.75 - 9.875 = 9.875 \Rightarrow \gamma_3^3 = \max\{5.167; 9.875\} = 9.875$.

$\gamma_2^3 = 9.875$.

$\gamma_1^3 = 3.75$.

So, $\varphi^{DCe}(E^3, c^3) = (0; 0; 0.153)$.

Therefore,

$s(E^3, c^3, P_3) = (0; 0.375; 0.416)$, and

$\sum_{k=1}^{3} s(E^k, c^k, P_3) = (1.25; 9.625; 9.819)$.

And for every step $m \geq 3$, $\varphi^{DCe}(E^m, c^m) = (0; 0; E^m)$, since $E^m \leq c_3^m - c_2^m$.

Therefore,
$RMS(E, c, P_3) = (1.25; 9.625; 10.125) \neq \varphi^{DCe}(E, c) = (1.25; 9.75; 10)$. ■

Proposition 5.6: The *Recursive Minimal Safety Process* for P_3 does not satisfy *Resource Monotonicity.*
Proof. Let us consider the two following *Bankruptcy Problem with Legitimate Principles* $(E, c, P_3) = (22.25; (5; 19.5; 20), P_3)$, and $(E', c, P_3) = (21, (5; 19.5; 20), P_3)$.

In this case, given the definitions of the *Constrained Egalitarian* rule and its dual we get

$$RMS(E, c, P_3) = (2.5; 9.75; 10),$$

and in the previous example we have seen that

$$RMS(E, c, P_3) = (1.25; 9.625; 10.125).$$

Obviously, these two distributions contradict *Resource Monotonicity* since highest claimant receives less amount when the estate increases. ■