A TURNPIKE THEOREM FOR A FAMILY OF FUNCTIONS*

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ABSTRACT

We prove a unifom turnpike theorem for a family of optimal growth problems arising from a family functions. More precisely, we show that there exists a uniform $\pm^0$ such that, for any problem in the family, and for any $\pm$ ¸ $\pm^0$, the optimal path exhibits convergence to the steady state.

Key words: Optimal Growth; Turnpike; Complicated Dynamics.

1. Introduction

Many dynamic models use a structure that leads to a deterministic reduced form of optimal growth models with discounting. Sometimes the attempt is to show that cyclical behavior, or even chaotic behavior, is compatible with full rationality within a perfect foresight framework, but at other times for exactly the opposite purpose: to demonstrate that a well behaved economy is the most reasonable result in a perfect foresight framework with full rationality, when there are no "exogenous stochastic shocks" perturbing it. Not surprisingly, the discount factor turns out to be a crucial parameter in this discussion and this leads one to investigate the precise nature of the relation between erratic behavior of optimal paths and the discount factor.

In the literature on optimal growth theory we have results that shed some light on this question. In the first instance, we have the celebrated "Turnpike Theorems" (McKenzie [7], Scheinkman [16], and more recently, Montrucchio [8], for instance) where it is shown that, under standard assumptions, if the discount factor is large enough, there is convergence to the steady state (so complicated paths can be ruled out). Given this result, one can ask: Is it true that for all values of the discount factor we have global convergence to the steady state? The answer to this question is no: Boldrin and Montrucchio [1] prove an "Indeterminacy Theorem," where it is shown that any C^2-function can be a policy function of an optimal growth problem, provided the discount factor is small enough. They adopt a constructive approach that gives a systematic way to construct the desired model. Furthermore, they provide an example of a two sector model for which the policy function is the function $h : [0,1] \to [0,1]$ given by $h(x) = 4x(1 - x)$ (the logistic map) which is known to display complicated dynamics. But in this example the discount factor is $\delta \simeq 0.01$, which implies a stationary interest rate on capital of $r \simeq 99$. One could well ask whether such a value of the discount factor is due to Boldrin-Montrucchio's technique of proof, or it is due to an inherent property of the class of optimal growth problems considered; it turns out that the second possibility is the answer. Sorger [17] proves a "Minimum Impatience Theorem," where it is shown that, given any trajectory $\{k_t\}$, if it is the optimal solution for some optimal growth problem, then the discount factor should be lower than some value that depends on $\{k_t\}$, but nothing else. This property is used to get an upper bound of 0.475 for the logistic map, which implies that the logistic map cannot be the optimal policy function of a standard optimal growth problem with a discount factor greater than or equal to 0.475.

The Minimum Impatience Theorem indicates that there is a strong incompatibility between erratic behavior of paths and patience: if we fix the "type" of chaos, then we should expect that the agent cannot be as patient as we want (precise results of a similar flavor can be found in Montrucchio and Sorger [9], Nishimura and Yano [13], and Sorger [18]). Given that we know that in the case of a fixed technology and a fixed felicity function, there is indeed incompatibility between erratic behavior of paths and patience, we are led to the question of what can happen if we allow the model to change as we change the discount factor. Notice that to relax the apparent incompatibility between erratic behavior and patience, it would be sufficient to show the existence of a family of standard optimal growth problems displaying some relevant kind of erratic behavior for any $0 < \delta < 1$; for instance, ergodic chaos for any $0 < \delta < 1$. To our knowledge there are at least two studies in this line, the paper by Nishimura, Sorger and Yano [11], and the paper by Nishimura and Yano [12]. In the first one the authors construct a family of strictly concave optimal growth problems displaying ergodic chaos for any $0 < \delta < 1$, but they also show that the chaos "tends" to zero as the discount factor tends to one (this statement is made more precisely in that paper); in the second one, a similar result is proved, but the family is not strictly concave (the felicity functions are linear). In both papers we have that optimal paths are not interior, a feature that is very undesirable because it implies zero consumption during the periods in which the activities lie on the boundary.

One can ask under what conditions the behavior obtained in the examples given in [11] and [12] can be ruled out; after all, from both the Minimum Impatience Theorem and the Turnpike Theorem, we get the intuition that when the discount factor is large enough, we should only have "nice" behavior of paths. In fact, one might think that if we restrict attention to models with interior optimal paths, [1] plus some uniform degree of concavity of felicity functions, one might be

[1] Very recently in Nishimura, Shigoka and Yano [10], it has been proven that there is a family of optimal growth problems with interior solutions displaying topological chaos for any value of the discount factor. Clearly this example shows that interiority of optimal paths is not a sufficient assumption to rule out topological chaos for any value of the discount factor. However, it is well known that the probability of observing this type of chaos could be zero (see, for instance, Collet and Eckmann [3]). For this reason, we concentrate more on the case of ergodic chaos than on the case of topological chaos. As far as we know, the existence of a family of optimal growth problems with interior solutions displaying ergodic chaos for any value

able to rule out the kind of phenomena obtained in [11] and [12]. We give here an answer to this question, which is a turnpike theorem for a family of felicity functions. The Theorem can be stated in words, as follows: there exists a uniform value of the discount factor such that, for any problem in the family, and for any value of the discount factor greater than or equal to the uniform value, the optimal path exhibits convergence to the steady state (Theorem 3 below).

To prove our Uniform Turnpike Theorem we use a method of proof that is an extension of that in McKenzie [7], so that it is a very different method from that used by Montrucchio [8] and Scheinkman [16], which, on the other hand, under our framework, does not seem to be very suitable to prove a uniform turnpike theorem (see Remark 1).

The plan of the paper is as follows. In Section 2 we present the set-up, the assumptions and we define the family U with which we will work. In Section 3 we give the proof of the Turnpike Theorem for the family U. Section 4 presents some conclusions and open problems.

2. Preliminaries

As our work is heavily based on McKenzie [7], we present part of the set-up, equations, and results given in that paper.

Take $D \subset \Re^n_+ \times \Re^n_+$ where $n \geq 1$, $u : D \to \Re$; and $\delta \in (0;1]$: The set D is the technology, the function u is the felicity function and δ is the discount factor. We say that a sequence $\{h_t\} \subset \Re^n_+$ is a path if $(h_t; h_{t+1}) \in D$; for all $t \in \mathbb{N}$:[2] We define an optimal path from a capital stock $x \in \Re^n_+$, as a path $\{k_t\}$ such that: $k_0 = x$, and

$$\limsup_{T \to \infty} \sum_{t=0}^{T} \left[\delta^{t+1} u(h_t; h_{t+1}) - \delta^{t+1} u(k_t; k_{t+1}) \right] \leq 0$$

for all paths $\{h_t\}$, such that $h_0 = x$:

of the discount factor is still an open question.

[2] $\mathbb{N} = \{0; 1; 2; \ldots\}$

This dynamic optimization problem is called the reduced form in optimal growth theory. For a discussion of the connection between this definition and a primitive form of a standard optimal growth problem see, for instance, [7].

If $\delta < 1$ we will call the model quasi-stationary, while if $\delta = 1$ we will call it stationary.

A stationary optimal path $k_t = k$ for all $t \in \mathbb{N}$ is called an optimal steady state (OSS).

Note that if $\delta \in (0; 1)$ and u is, for instance, bounded, the definition of an optimal path is simply a path that maximize a discounted felicity sum. The more general definition allows us to deal with the stationary case, which will play a fundamental role in this paper.

All members of the family of optimal growth problems that we will define later are assumed to satisfy the following assumptions on the technology and felicity functions.

The technology D will be a set in $\Re^{2n}_{+}$ such that:

A1 D is closed and convex, and if $(x; y) \in D$; then for every $(z; w) \in \Re^{2n}_{+}$ such that $z \geq x$, $0 \leq w \leq y$ we have that $(z; w) \in D$:

A2 For every $\xi \in \Re_{++}$, there is $\zeta \in \Re_{+}$, such that: $(x; y) \in D$ and $\|x\| < \xi$, implies $\|y\| < \zeta$.

A3 There is $(\hat{x}; \hat{y}) \in D$ for which $\hat{y} > \hat{x}$ (existence of an expansible stock).

A4 There are $(M; \gamma) \in \Re^{2}_{++}$, $\gamma < 1$; such that: $(x; y) \in D$ and $\|x\| > M$; implies $\|y\| < \gamma \|x\|$; (bounded paths).

Throughout the paper the technology D will be fixed and assumed to satisfy A1-A4.

To avoid details related with derivatives we will suppose that all felicity functions considered are defined on $\Re^{2n}_+$, and to have our problem well defined we consider the restriction on D. For the restriction $u : D \to \Re$ we impose

A5 u is continuous, concave, and if $(x, y) \in D$, then for every $(z, w) \in \Re^{2n}_+$ such that $z \geq x$, $w \leq y$ we have that $u(z, w) \geq u(x, y)$ (free disposal).

It is important to notice that assumptions A1-A5 are standard in optimal growth theory. For a discussion of assumptions A1-A5 see [7].

Now for $\delta < 1$ and (u, D) satisfying A1-A5, we define the value function $V_t^{u,\delta}(x)$ which values a capital stock at time t by the felicity sums that can be obtained from it in the future:

$$V_t^{u,\delta}(x) := \sup\left(\lim_{T \to \infty} \sum_{\tau=t+1}^{T} \delta^{\tau} u(h_{\tau-1}, h_{\tau})\right)$$

over all paths $\{h_t\}$ with $h_t = x$. We say that $V_t^{u,\delta}(x)$ is well defined if the sup defined above is finite or $+\infty$. Now we define

$$L := \left\{ x \in \Re^n_+ \mid \text{there exists a path } \{h_t\} \text{ such that } h_0 = x \right\}.$$

Note that $V_t^{u,\delta}(x) \in \Re$ for all $x \in L$ and all $t \in \mathbb{N}$, and that if $\{k_t\}$ is an optimal path from x, then $V_t^{u,\delta}(x) = \delta^t \sum_{\tau=1} \delta^{\tau} u(k_{\tau-1}, k_{\tau})$. We will say that prices $\{p_t\}$ support an optimal path $\{k_t\}$ if the following conditions are satisfied:

$$\begin{aligned} &\delta^{t+1} u(k_t, k_{t+1}) + p_{t+1}k_{t+1} - p_t k_t \geq \\ &\delta^{t+1} u(x, y) + p_{t+1} y - p_t x, \text{ for all } (x, y) \in D \end{aligned} \tag{1}$$

and

$$V_t^{u,\delta}(k_t) - p_t k_t \geq V_t^{u,\delta}(y) - p_t y, \text{ for all } y \in L, \tag{2}$$

for all $t \in \mathbb{N}$.[3]

[3] We stress here that in general more subtle definitions are necessary. Recall that we concentrate on the quasi-stationary model for the definition of the value function and supporting prices, but it is possible, by means of an appropriate normalization of the felicity function, to define the value function and supporting prices for the stationary model as well. Even more general situations can be allowed for. See, for instance, [7].

Prices $\{p_t\}$ satisfying (1) and (2) for all $t \in \mathbb{N}$ are called full Weitzman Prices. By (1) the prices support the felicity function. By (2) they support the value function. We note that condition (2) implies that the prices $\{p_t\}$ are Malinvaud prices [6], that is, k_t has minimal value at p_t over the set of capital stocks $y \in L$ such that $V_t^{u,\delta}(k_t) = V_t^{u,\delta}(y)$.

Let $\hat{\delta} \in (0,1)$ be such that for any $\delta \in \left(\hat{\delta}, 1\right)$, we have $\delta\tilde{y} > \tilde{x}$. Then we have the following:

Lemma 1. For any $1 \geq \delta \geq \hat{\delta}$ and (u, D) satisfying A1-A5 there exists $(k^{u,\delta}, q^{u,\delta}) \in \Re^{2n}_{+}$, $q^{u,\delta} \neq 0$ and $(k^{u,\delta}, k^{u,\delta}) \in D$, which satisfy:

$$u(k^{u,\delta}, k^{u,\delta}) \geq u(x, y) \text{ for all } (x, y) \in D, \text{ such that } \delta y - x \geq (\delta - 1)k^{u,\delta} \tag{3}$$

and

$$\begin{aligned} u(k^{u,\delta}, k^{u,\delta}) + q^{u,\delta}(k^{u,\delta} - \delta^{-1}k^{u,\delta}) \geq \\ u(x, y) + q^{u,\delta}(y - \delta^{-1}x), \text{ for all } (x, y) \in D \end{aligned} \tag{4}$$

Proof: See [7], lemmata 6.1 and 7.2. ∎

The point $(k^u, k^u) \in D$ is called the turnpike in optimal growth theory.

Henceforth we will use the notation k^u or $k^{u,1}$ when we refer to the stationary model, and similarly for q^u, that is, when we "evaluate" the parameter δ at $\delta = 1$ we are referring to the stationary model, and whenever we have that δ is missing this will mean $\delta = 1$.

Note that (4) says that for any $1 \geq \delta \geq \hat{\delta}$, prices $p_t^{u,\delta} = \delta^t q^{u,\delta}$, $t \in \mathbb{N}$, support the felicity function in the sense of (1).

We consider two further assumptions:

A6 $u(x, y) + \frac{1}{2}B(x^2 + y^2)$ is concave at (k^u, k^u), for some $B \in \Re_{++}$.[4]

[4] For $x \in R^n_+$ we define $x^2 = x'x$.

A7 i) for all $\delta \in \left(\hat{\delta}, 1\right)$, $(k^{u,\delta}, k^{u,\delta}) \in \operatorname{int} D$;

ii) for all $k_0 \in S := \left\{ x \in \mathbb{R}^n_+ \mid (x, x) \in D \right\}$ and for all $\delta \in \left(\hat{\delta}, 1\right)$, any optimal path $\left\{ k_t^{u,\delta} \right\}$ from k_0 satisfies that: $(k_{t-1}^{u,\delta}, k_t^{u,\delta}) \in \operatorname{int} D$ for all $t \geq 1$.

Remark 1 Montrucchio [8] proves , as does Scheinkman [16] , a Local Turnpike Theorem (Lemma 16 in Scheinkman's proof), and they then obtain the Turnpike Theorem, using a Visit Lemma. The general idea is as follows. Firstly, given a felicity function u, it is proven that there is an $\varepsilon > 0$ such that if $k_0 \in \left\{ x \in \mathbb{R}^n_+ \mid \left\| x - k^{u,\delta} \right\| < \varepsilon \right\}$, then any optimal path from k_0 converges to the steady state (a Local Turnpike Theorem). Secondly, it is proven that given any $\varepsilon > 0$, there is $\delta(\varepsilon) \in (0, 1)$, such that if $1 > \delta > \delta(\varepsilon)$, any optimal path $\left\{ k_t^{u,\delta} \right\}$ from $x \in S$ has the property that there exists a t such that $k_t^{u,\delta} \in \left\{ x \in \mathbb{R}^n_+ \mid \left\| x - k^{u,\delta} \right\| < \varepsilon \right\}$ (a Visit Lemma). Hence the Turnpike Theorem follows at once. Now, in the general case of a family of felicity functions, ε may depend on the felicity function u, thus it is not possible to choose a uniform ε over the family. Indeed, for the case of a family of felicity functions, $\varepsilon(u)$ may be arbitrarily close to zero as we take different felicity functions u in the family under consideration, and hence the $\delta(\varepsilon(u))$ given in the Visit Lemma may be arbitrarily close to 1. So in general it is not clear to us how this method of proof can be used to prove a Turnpike Theorem for a Family of Felicity Functions. In the case of Montrucchio's proof, one possibility to overcome this obstacle would be to assume that Assumption A in [8] holds for any member of the family under consideration, which implies that there exists an $\varepsilon > 0$ such that $B((k^{u,\delta}, k^{u,\delta}); \varepsilon) = \left\{ x \in \mathbb{R}^{2n}_+ \mid \left\| x - (k^{u,\delta}, k^{u,\delta}) \right\| < \varepsilon \right\} \subset D$ for all $\delta \geq \hat{\delta}$ and for any felicity function u in the family under consideration. We stress here that Assumption A in Montrucchio's paper cannot be dispensed with. Nevertheless, on the other hand, to assume that there exists an $\varepsilon > 0$ such that $B((k^{u,\delta}, k^{u,\delta}); \varepsilon) = \left\{ x \in \mathbb{R}^{2n}_+ \mid \left\| x - (k^{u,\delta}, k^{u,\delta}) \right\| < \varepsilon \right\} \subset D$ for all $\delta \geq \hat{\delta}$ and for any felicity function u in the family under consideration would be a very strong assumption, which we do not require.

Assumption A6 implies that u is strictly concave at the turnpike (k^u, k^u). Further, it puts a lower bound on the degree of concavity. In fact it implies (under differentiability) that the absolute value of all the eigenvalues of the Hessian matrix $D^2u(k^u; k^u)$ cannot be lower than B (Lemma 8 below). Assumptions similar to A6 can be found in Cass and Shell [2], Montrucchio [8], Rockafellar [14], and Santos [15].

Note that strict concavity at $(k^u; k^u)$ is a standard assumption in turnpike theory as well as assumption A7, although it is implied by a weaker assumption ($(k^u; k^u) \in \operatorname{int} D$; see [7]) in a model with a fixed u.

It is possible to prove the following:

Theorem 1 For any $(u; D)$ satisfying A1-A7, the stock of capital and the price $(k^{u;\delta}; q^{u;\delta}) \in \Re^{2n}_{+}$ given in Lemma 1, satisfy the following:

i) for any $\delta \in \left[\hat{\delta}; 1\right)$

$$V_t^{u;\delta}(k^{u;\delta}) - \delta^t q^{u;\delta} k^{u;\delta} \geq V_t^{u;\delta}(x) - \delta^t q^{u;\delta} x; \text{ for all } x \in L \tag{5}$$

for all $t \in \mathbb{N}$:

ii) for any $\left[\hat{\delta}; 1\right)$ the path $k_t^{u;\delta} = k^{u;\delta}$ for all $t \in \mathbb{N}$; is an OSS.

iii) for any $\delta \in \left[\hat{\delta}; 1\right)$ and for any $k_0 \in L$ there exists an optimal path $\left\{k_t^{u;\delta}\right\}$ from k_0: If $k_0 \in \operatorname{int} L$; any optimal path from k_0 can be supported by full Weitzman prices $p_t^{u;\delta} := \delta^t q_t^{u;\delta}$ in the sense of (1) and (2).[5]

Proof: See [7], theorems 6.1 and 7.1, and a comment on page 1312. ¤

Note that Lemma 1 and Theorem 1 imply that for any $\hat{\delta} \leq \delta < 1$, prices $p_t^{u;\delta} = \delta^t q^{u;\delta}$; $t \in \mathbb{N}$; are full Weitzman prices that support the optimal stationary path $k_t^{u;\delta} = k^{u;\delta}$ for all $t \in \mathbb{N}$: Also note that part ii) in Theorem 1 says that the turnpike is in fact an OSS in the stationary model. There is a strong relation between the quasi-stationary model and the stationary model. For example, we

[5] We will use the notation $\left\{k_t^{u;\delta}\right\}$ for an optimal path that is not an OSS and $\left\{\delta^t q_t^{u;\delta}\right\}$ for the corresponding Weitzman supporting prices, whereas $\left\{k^{u;\delta}\right\}$ and $\left\{\delta^t q^{u;\delta}\right\}$, when we refer to the OSS.

will have as a particular case of our Lemma 5 that $\lim_{\delta \to 1} k^{u,\delta} = k^u$ (see [16] (the Visit Lemma, and its Corollary), and [7]; lemma 8.2, for different proofs of this fact), thus if $u \in C^2$, A6 implies that $D^2u(k^u, k^u)$ is negative definite and, therefore, $D^2u(k^{u,\delta}, k^{u,\delta})$ is negative definite for δ large enough as well. Hence u is strictly concave at $(k^{u,\delta}, k^{u,\delta})$ for δ large enough.

Take $\delta \in \left[\hat{\delta}, 1\right)$, (u, D) satisfying A1-A7, and $k_0 \in \operatorname{int} L$. Let $\alpha_t^{u,\delta}$ be defined as follows:

$$
\begin{aligned}
\alpha_t^{u,\delta} &:= \left[\delta^t u(k^{u,\delta}, k^{u,\delta}) + \delta^t q^{u,\delta}\left(k^{u,\delta} - \delta^{-1} k^{u,\delta}\right)\right] - \\
&\quad \left(\delta^t u(k_{t-1}^{u,\delta}, k_t^{u,\delta}) + \delta^t q^{u,\delta}\left(k_t^{u,\delta} - \delta^{-1} k_{t-1}^{u,\delta}\right)\right);
\end{aligned} \tag{6}
$$

that is, as the value loss in period t suffered by $\left\{k_t^{u,\delta}\right\}$ relative to the path $k_t = k^{u,\delta}$ for all $t \in \mathbb{N}$, at prices $p_t^{u,\delta} = \delta^t q^{u,\delta}$. Note that $\alpha_t^{u,\delta} \geq 0$ for all t by Lemma 1. Similarly, let $\beta_t^{u,\delta}$ be defined as follows:

$$
\begin{aligned}
\beta_t^{u,\delta} &:= \left(\delta^t u(k_{t-1}^{u,\delta}, k_t^{u,\delta}) + \delta^t q_t^{u,\delta} k_t^{u,\delta} - \delta^{t-1} q_{t-1}^{u,\delta} k_{t-1}^{u,\delta}\right) - \\
&\quad \left(\delta^t u(k^{u,\delta}, k^{u,\delta}) + \delta^t q_t^{u,\delta} k^{u,\delta} - \delta^{t-1} q_{t-1}^{u,\delta} k^{u,\delta}\right);
\end{aligned} \tag{7}
$$

that is, as the value loss in period t suffered by $k_t = k^{u,\delta}$ for all $t \in \mathbb{N}$ relative to the path $\left\{k_t^{u,\delta}\right\}$ at prices $p_t^{u,\delta} = \delta^t q_t^{u,\delta}$. Again, we have that $\beta_t^{u,\delta} \geq 0$ by (1) (part iii) in Theorem 1). Adding $\alpha_t^{u,\delta}$ and $\beta_t^{u,\delta}$ we have:

$$
\begin{aligned}
\alpha_t^{u,\delta} + \beta_t^{u,\delta} &= \delta^t \left(q^{u,\delta} - q_t^{u,\delta}\right)\left(k^{u,\delta} - k_t^{u,\delta}\right) - \\
&\quad \delta^{t-1}\left(q^{u,\delta} - q_{t-1}^{u,\delta}\right)\left(k^{u,\delta} - k_{t-1}^{u,\delta}\right)
\end{aligned} \tag{8}
$$

Now we have

$$
V_t^{u,\delta}(k^{u,\delta}) - \delta^t q^{u,\delta} k^{u,\delta} = V_t^{u,\delta}(k_t^{u,\delta}) - \delta^t q^{u,\delta} k_t^{u,\delta} + \lambda_t^{u,\delta} \tag{9}
$$

with $\lambda_t^{u,\delta} \geq 0$ by i) in Theorem 1, and

$$
V_t^{u,\delta}(k_t^{u,\delta}) - \delta^t q_t^{u,\delta} k_t^{u,\delta} = V_t^{u,\delta}(k^{u,\delta}) - \delta^t q_t^{u,\delta} k^{u,\delta} + \lambda_t^{1\,u,\delta} \tag{10}
$$

with $\lambda_t^{1,u,\delta} \geq 0$ by iii) in Theorem 1.

Combining (9) and (10) we have:

$$\delta^t \left(q^{u,\delta} - q_t^{u,\delta}\right)\left(k^{u,\delta} - k_t^{u,\delta}\right) = -\left(\lambda_t^{1,u,\delta} + \lambda_t^{u,\delta}\right) \tag{11}$$

Setting $L^{u,\delta}(t) := \left(q^{u,\delta} - q_t^{u,\delta}\right)\left(k^{u,\delta} - k_t^{u,\delta}\right)$; we have

$$L^{u,\delta}(t) \geq 0 \tag{12}$$

for all $t \in \mathbb{N}$ by (9), (10) and (11). Also, by definition and (11), we have:

$$L^{u,\delta}(t) = -\delta^{-t}\left(\lambda_t^{1,u,\delta} + \lambda_t^{u,\delta}\right) \tag{13}$$

hence

$$L^{u,\delta}(t) \geq -\delta^{-t}\lambda_t^{u,\delta} \tag{14}$$

because $\lambda_t^{1,u,\delta} \geq 0$. On the other hand we have:

$$L^{u,\delta}(t) - \delta^{-1}L^{u,\delta}(t-1) = \delta^{-t}\left(\alpha_t^{u,\delta} + \alpha_t^{u,\delta}\right) \geq 0 \tag{15}$$

by (8), Lemma 1 and (1) applied to $\left\{k_t^{u,\delta}\right\}$.

Note that $\lambda_t^{1,u,\delta} + \lambda_t^{u,\delta}$ can be interpreted as the accumulated loss at time t, in terms of the felicity sums obtained, when $\left\{k_t^{u,\delta}\right\}$ stays away from the OSS $k^{u,\delta}$; therefore, $\delta^{-t}\left(\lambda_t^{1,u,\delta} + \lambda_t^{u,\delta}\right)$ is the present value of the loss. Condition (12) says that this present value is non-negative for all $t \in \mathbb{N}$, and we will show that, depending on the initial value $L^{u,\delta}(0)$, and with an appropriate value of δ, the difference equation (15) and the uniform concavity of the family of felicity functions that we will define (Lemma 6 and its Corollary), $-L^{u,\delta}(t)$ will be strictly decreasing period by period by a constant amount if the two paths stay away from each other by a constant amount. But $-L^{u,\delta}(t)$ is non-negative for all $t \in \mathbb{N}$, so this cannot occur for ever, implying that $\left\{k_t^{u,\delta}\right\}$ must "approach" $\left\{k^{u,\delta}\right\}$. This reasoning will apply to all members of the family, provided the discount factor is large enough. Roughly, this is the key reasoning leading to the main theorem of this paper, which is a uniform turnpike theorem for the family that we will define (Theorem 3 below).

Now define $X_1 = L \cap \left\{ x \in \Re^n_+ \mid |x| \le M \right\}$. Note that no path $\{h_t\}$ can be outside of X_1 for ever, that is, for any path $\{h_t\}$ there will exist $l_{\{h_t\}} \in \mathbb{N}$ such that $h_{l_{\{h_t\}}} \in X_1$; because if this is not the case, we will have $|h_t| < \gamma^t |h_0|$ for all $t \in \mathbb{N}$ by A4; which is clearly a contradiction. Hence, as our interest is in the long run behavior of optimal paths, there is no loss of generality if we restrict our analysis to paths $\{h_t\}$ such that $h_0 \in X_1$. On the other hand, we can show that there exists a compact, convex set, say X, such that, if $\{h_t\}$ is a path such that $h_0 \in X_1$, then $(h_t; h_{t+1}) \in X$ for all $t \in \mathbb{N}$. Indeed, take a path with h_0 in X_1; then $|h_1| \le M_1$ for some $M_1 > 0$ by A2; let $\bar{M} = \max\{M; M_1\}$ and make the inductive assumption: $|h_t| \le \max\{M; M_1\}$. Then, if $|h_t| \le M$, assumption A2 yields $|h_{t+1}| \le M_1$, and hence $|h_{t+1}| \le \bar{M}$; but, if $|h_t| > M$, we have $|h_{t+1}| < \gamma |h_t| < |h_t| \le \bar{M}$ (by A4); and thus $|h_{t+1}| \le \bar{M}$. Therefore $|h_t| \le \bar{M}$ for all $t \in \mathbb{N}$ by induction. Taking

$$X = \left\{ (x; y) \in D \mid |x|; |y| \le \bar{M} \right\}$$

we have the assertion proved. So, without loss of generality, we will restrict the analysis to the set X.

Let $N_1 \in \Re_{++}$ and define (for a fixed $B \in \Re_{++}$ in A6)

$$U = \left\{ \begin{array}{c} u : D \to \Re \mid (u; D) \text{ satisfying A1-A7 and } u \in C^4; \\ \text{with } |u_X|_4 \le N_1 \end{array} \right\}$$

where

$$|u_X|_4 := \sup_{(x;y) \in X} \left\{ |u(x;y)| + |Du(x;y)| + \left|D^2 u(x;y)\right| + \left|D^3 u(x;y)\right| + \left|D^4 u(x;y)\right| \right\};$$

(a uniform bound over functions and derivatives).

Remark 2 It is important to notice that in spite of the fact that the uniform bound over functions and derivatives seems to be a very strong condition, it is not at all. Indeed, if we take any $u' : D \to \Re$ such that $u' \in C^4$ (u' non-constant, so that $|u'_X|_4 \ne 0$); and then we define $u : D \to \Re$ given by $u(x;y) = \frac{u'(x;y)}{|u'_X|_4}$; therefore the optimal growth problems defined with u'

and u as the felicity functions have the same optimal paths as solutions, so for the sake of the study of the possible solutions of a given family of C^4-functions, there is no any loss of generality if we assume that $|u_X|_4 = 1$ for any member of the family. Nevertheless, in our framework we have chosen to make the assumption explicitly, because the assumption A6 is not necessarily invariant after this normalization, that is, if we have an arbitrary family U' as following

$$U' = \left\{ u' : D \to \mathbb{R} \mid (u, D) \text{ satisfying A1-A7 and } u' \in C^4 \right\};$$

for some $B' \in \mathbb{R}_{++}$ in A6, then the normalized family

$$U = \left\{ u : D \to \mathbb{R} \mid u = \frac{u'}{|u'_X|_4},\ u' \in U' \right\}$$

may not satisfy A6 for any $B \in \mathbb{R}_{++}$ (uniformly), although it will satisfy A1-A5 and A7.

Note that for any $u \in U$ and $\delta \in \left(\hat{\delta}, 1\right)$, we have that $\operatorname{int} L \neq \emptyset$, since $S = \left\{ x \in \mathbb{R}^n_+ \mid (x, x) \in D \right\} \subset L$, and $\operatorname{int} S \neq \emptyset$ by A3. Therefore, since the proof of the main theorem of this paper is based on the existence of optimal paths that can be supported by full Weitzman prices, it will suffice to take $k_0 \in \operatorname{int} S$ (note that $S \subset X_1$, by (A4)), and then, by iii) in Theorem 1, we will have that, for all $u \in U$ and $\delta \in \left(\hat{\delta}, 1\right)$, there exists an optimal path $\left\{k_t^{u,\delta}\right\}$ from k_0 that can be supported by full Weitzman prices. Furthermore, due to differentiability and concavity of u, and interiority of the path, we will have for all $u \in U$ and $\delta \in \left(\hat{\delta}, 1\right)$:

$$D_1u(k^{u,\delta}, k^{u,\delta}) = \delta^{-1}q^{u,\delta}, \quad D_2u(k^{u,\delta}, k^{u,\delta}) = -q^{u,\delta} \tag{16}$$

$$D_1u(k_{t-1}^{u,\delta}, k_t^{u,\delta}) = \delta^{-1}q_{t-1}^{u,\delta} \quad D_2u(k_{t-1}^{u,\delta}, k_t^{u,\delta}) = -q_t^{u,\delta}, \tag{17}$$

where

$$D_1u(k^{u,\delta}, k^{u,\delta}) = (\partial/\partial x)u(x, k^{u,\delta})|_{x=k^{u,\delta}}, \quad D_1u(k_{t-1}^{u,\delta}, k_t^{u,\delta}) = (\partial/\partial x)u(x, k_t)|_{x=k_{t-1}^{u,\delta}},$$

and similarly for $D_2u(k_{t-1}^{u,\delta}, k_t^{u,\delta})$ and $D_2u(k^{u,\delta}, k^{u,\delta})$. Thus, henceforth we will suppose that $k_0 \in \operatorname{int} S$.

With these preliminaries in place, we proceed to obtain a turnpike theorem for the family $\mathcal{U}$.

3. A turnpike theorem for the family $\mathcal{U}$

As we restrict the analysis to the set X, henceforth we will assume that all felicity functions are defined on X.

Lemma 2. The family $\mathcal{U}$ is (uniformly) equicontinuous, that is: for all $\varepsilon > 0$, there is $\delta > 0$ such that: $|x_1 - x_2| \leq \delta$ implies $|u(x_1) - u(x_2)| \leq \varepsilon$; for all $u \in \mathcal{U}$; and all $(x_1; x_2) \in X \times X$.

Proof: Routine and omitted. ¤

Remark 3 We note that a similar result can be proved for the family of the first derivatives, and for the family of the second derivatives, that is, for the following families:

$$\mathcal{U}_i = \{D_i u : X \to \Re^n \mid u \in \mathcal{U}\}$$
$$\mathcal{U}_{ij} = \left\{ D_{ij} u : X \to \Re^{n^2} \mid u \in \mathcal{U} \right\};$$ [6]

where $i; j \in \{1; 2\}$.

Henceforth whenever we have $\{u_n\} \subset \mathcal{U}$; and either $u_n \to u$ or $\lim_{n \to \infty} u_n = u$; the limit is taken in the norm of the sup; that is $|u| = \sup_{x \in X} |u(x)|$; and similarly when we deal with sequences in the families $\mathcal{U}_i$ $(i \in \{1; 2\})$ and $\mathcal{U}_{ij}$ $(i; j \in \{1; 2\})$.

[6] $D_{ij} u(x; y) := D_j (D_i u)(x; y)$

Lemma 3. i) Let $\{u_n\} \subset U$ be a sequence. Then there exists a concave function $u : X \to \mathbb{R}$ $(u \in \bar{U})$, [7] and a subsequence $\{u_{n_s}\}$ of $\{u_n\}$ such that $\lim_{s \to \infty} u_{n_s} = u$.
ii) Let $\{u_n\} \subset U$ be a sequence and let $u : X \to \mathbb{R}$ be the function such that $\lim_{n \to \infty} u_n = u$. Suppose $x_n \to x$ with $\{x_n\} \subset X$ and $x \in X$. Then $u_n(x_n) \to u(x)$.

Proof: i) Since U is bounded by definition and is equicontinuous by Lemma 2, the set U is relatively compact[8] by the Arzela-Ascoli theorem (see Dieudonné [4], for instance). Then the existence of a function $u \in \bar{U}$ and a subsequence $\{u_{n_s}\}$ such that $u = \lim_{s \to \infty} u_{n_s}$ follows at once. It is well known that the limit in the norm of the sup of concave functions is a concave function, so u is concave.

ii) Routine and omitted. ¤

Remark 4 In Lemma 3 we have shown that the family U is relatively compact. Similarly, because of the uniform bound over functions and derivatives and Remark 3, using repeatedly the Arzela-Ascoli Theorem, it is possible to show that the following families are relatively compact:

$$U_i = \{D_i u : X \to \mathbb{R}^n \mid u \in U\} \tag{18}$$

and

$$U_{ij} = \left\{ D_{ij} u : X \to \mathbb{R}^{n^2} \mid u \in U \right\} \tag{19}$$

for $i, j \in \{1, 2\}$. These facts will be used several times in the following proofs.

Lemma 4. Let $\{u_n\} \subset U$ be an arbitrary sequence. Then there exists a concave function $u : X \to \mathbb{R}$, $u \in \bar{U}$, points k^u and q^u in $\mathbb{R}^n_+$, and a subsequence $\{u_{n_s}\}$ of $\{u_n\}$ such that:
i) $\lim_{s \to \infty} u_{n_s} = u$, $(k^u, k^u) \in X$ and $u(x, y) + \frac{1}{2}B(x^2 + y^2)$ is concave at (k^u, k^u); and,
ii) the following conditions are satisfied:

$$u(k^u, k^u) \geq u(x, y) \text{ for all } (x, y) \in D \text{ such that } y - x \geq 0 \tag{20}$$

[7] We denote by $\bar{U}$ the closure of U relative to the norm of the sup.

[8] A family of functions $F = \{u : X \to \mathbb{R}\}$ is said to be relatively compact if $\bar{F}$ is compact (relative to the norm of the sup).

and

$$\begin{gathered} u(k^u,k^u) \geq u(x,y) + q^u(y - x) \text{ for all } (x,y) \in X, \text{ and} \\ u(k^u,k^u) > u(x,y) + q^u(y - x) \text{ for all } (x,y) \in X, \\ \text{such that } (x,y) \neq (k^u,k^u). \end{gathered} \tag{21}$$

Proof: The existence of a subsequence $\{u_{n_s}\}$ of $\{u_n\}$ and a concave function $u \in \mathcal{U}^1$ such that $\lim_{s \to \infty} u_{n_s} = u$ is the content of i) in Lemma 3. Note that, without loss of generality, we can suppose $u_n \to u$.

Now, $\{k^{u_n}\} \subset X_1$ by A4, and $q^{u_n} = D_1 u_n(k^{u_n},k^{u_n})$ for any $n \geq 1$ by (16). Therefore, $\{k^{u_n}\}$ and $\{q^{u_n}\}$ are bounded because X_1 is compact and because of the uniform bound over functions and derivatives. Hence, there exists a point $(k^u,q^u) \in X_1 \times \Re^n_+$ such that, without loss of generality, $k^{u_n} \to k^u$ and $q^{u_n} \to q^u$ (taking subsequences, if necessary). Note that $(k^u,k^u) \in X$.

To prove that $u(x,y) + \frac{1}{2}B(x^2 + y^2)$ is concave at (k^u,k^u) we will proceed by contradiction. Hence, we suppose this is not the case, that is, there exists $(x,y) \in X$ with $(x,y) \neq (k^u,k^u)$ and $\lambda \in (0,1)$ such that if $v_\lambda = (x_\lambda,y_\lambda) = \lambda(x,y) + (1-\lambda)(k^u,k^u)$,

$$\begin{gathered} \left\{\lambda u(x,y) + \lambda B\tfrac{1}{2}(x^2 + y^2) + (1-\lambda)u(k^u,k^u) + B(1-\lambda)(k^u)^2\right\} > \\ u(v_\lambda) + \tfrac{1}{2}B(x_\lambda^2 + y_\lambda^2) \end{gathered} \tag{22}$$

Take $v^n_\lambda := \lambda(x,y) + (1-\lambda)(k^{u_n},k^{u_n})$. Then (22) implies that for all n large enough we have

$$\begin{gathered} \left\{\lambda u_n(x,y) + \lambda B\tfrac{1}{2}(x^2 + y^2) + (1-\lambda)u_n(k^{u_n},k^{u_n}) + B(1-\lambda)(k^{u_n})^2\right\} > \\ u_n(v^n_\lambda) + \tfrac{1}{2}B\left((x_\lambda^{(k^{u_n},k^{u_n})})^2 + (y_\lambda^{(k^{u_n},k^{u_n})})^2\right) \end{gathered} \tag{23}$$

where $(x_\lambda^{(k^{u_n},k^{u_n})}, y_\lambda^{(k^{u_n},k^{u_n})}) = v^n_\lambda$; because from Lemma 3 we have that $u_n(k^{u_n},k^{u_n}) \to u(k^u,k^u)$, $u_n(x,y) \to u(x,y)$ and $\lim_{n\to\infty} u_n(v^n_\lambda) = u(v_\lambda)$ as well. But (23) is in contradiction with assumption A6, then $u(x,y) + \frac{1}{2}B(x^2 + y^2)$ is concave at (k^u,k^u).

This proves the first part of the Lemma.

To show that $u(k^u,k^u) \geq u(x,y)$ for all $(x,y) \in D$ such that $y \geq x$, we will suppose this is not the case to get a contradiction; first note that u is indeed defined at $(x,y) \in D$ for $y \geq x$ because we have that $(x,y) \in X$ by A4; now suppose there is $(x, y) \in D$ such that $u(x, y) > u(k^u, k^u)$, hence $u_n(x, y) > u_n(k^{u_n}, k^{u_n})$ for n large enough by Lemma 3, which contradicts (3) in Lemma 1.

Last but not least, we will show that (k^u, q^u) satisfy the property (21). We will proceed by contradiction. For any $n \in \mathbb{N}$ we have that

$$u_n(k^{u_n}, k^{u_n}) \geq u_n(x, y) + q^{u_n}(y - x) \text{ for all } (x, y) \in D \tag{24}$$

by Lemma 1. Now suppose there is $(x, y) \in X$ such that $u(k^u, k^u) - u(x, y) - q^u(y - x) < 0$; but as $u_n \to u$ and $u_n(k^{u_n}, k^{u_n}) \to u(k^u, k^u)$ by Lemma 3, we have

$$\{u_n(k^{u_n}, k^{u_n}) - (u_n(x, y) + q^{u_n}(y - x))\} < 0 \tag{25}$$

for n large enough. But (24) and (25) are in contradiction for all n large enough. The strict inequality in (21) for $(x, y) \neq (k^u, k^u)$ follows at once from the weak inequality and the fact that the function $(x, y) \to u(x, y) + q^u(y - x)$ is strictly concave at (k^u, k^u) (because u is strictly concave at (k^u, k^u)).

This proves the second part of the lemma. $\square$

Lemma 5. Let $\{u_n\} \subset U$ be a sequence such that $u_n \to u$ for some $u \in \bar{U}$ and $\{\delta_n\}$ such that $\delta_n \to 1$, then $k^{u_n, \delta_n} \to k^u$.

Proof: The proof is by contradiction. Suppose the lemma does not hold, then there exists a subsequence $\left\{k^{u_{n_s}, \delta_{n_s}}\right\}$ of $\left\{k^{u_n, \delta_n}\right\}$ and a point $\bar{k} \in X_1$ such that $k^{u_{n_s}, \delta_{n_s}} \to \bar{k}$ with $\bar{k} \neq k^u$. Now, since $q^{u_n, \delta_n} = \delta_n D_1 u_n \left(k^{u_n, \delta_n}, k^{u_n, \delta_n}\right)$ for any $n \geq 1$ by (16), $\left\{q^{u_n, \delta_n}\right\}$ is bounded because of the uniform bound over functions and derivatives. Then, there is a subsequence $\left\{q^{u_{n_s}, \delta_{n_s}}\right\}$ and a point $\bar{q}$ such that $q^{u_{n_s}, \delta_{n_s}} \to \bar{q}$. Again, without loss of generality, we suppose $k^{u_n, \delta_n} \to \bar{k}$ and $q^{u_n, \delta_n} \to \bar{q}$. Now, for all $n \in \mathbb{N}$ we have that

$$u_n(k^{u_n, \delta_n}, k^{u_n, \delta_n}) \geq u_n(x, y) + q^{u_n, \delta_n}(y - x) \text{ for all } (x, y) \in D \tag{26}$$

by Lemma 1. Hence, taking limits

$$u(\bar{k}, \bar{k}) - u(x, y) - \bar{q}(y - x) \geq 0 \text{ for all } (x, y) \in X \tag{27}$$

But (27) implies $u(\bar{k}, \bar{k}) \geq u(k^u, k^u)$; now, by Lemma 4, u is strictly concave at (k^u, k^u) and condition (20) is satisfied, then $u(\bar{k}, \bar{k}) = u(k^u, k^u)$, and therefore

$\bar{k}^{1} = k^u$, contradiction. Then the lemma is proved. $\square$

Note that as a consequence of Lemma 5, we have that, with u fixed, $\lim_{\delta\to 1} k^{u,\delta} = k^u$, as was noted above.

The following lemma and its corollary will play a fundamental role in the proof of the Uniform Turnpike Theorem. The lemma will ensure a uniform positive value loss for any path that departs from the OSS; this will imply that this cannot occur for ever, a fact that, with the help of the corollary, will force optimal paths to remain close to the OSS. The lemma is due to the uniform strict concavity of the family around the turnpike for δ large enough, and the corollary is due to the uniform strict concavity of the family of the value functions around the turnpike for δ large enough (as occurs in models with a fixed u, the Corollary 1 is implied by Lemma 6).

Lemma 6. (A Generalized Atsumi-Radner Lemma) Let $q^{u,\delta}$ and $k^{u,\delta}$ be as in Lemma 1. Define $f^{u,\delta} : X \to \mathbb{R}$ by $f^{u,\delta}(x,y) = u(k^{u,\delta},k^{u,\delta}) - u(x,y) + q^{u,\delta}(k^{u,\delta} - y) - \delta^{-1}q^{u,\delta}(k^{u,\delta} - x)$. Then, for any $\varepsilon > 0$ there exists $\rho(\varepsilon) > 0$ and $\hat{\delta} \leq \delta^1(\varepsilon) < 1$, such that: $\|x - k^{u,\delta}\| \geq \varepsilon$ and $(x,y) \in X$, implies $f^{u,\delta}(x,y) \geq \rho(\varepsilon)$, for all $u \in U$ and $1 > \delta \geq \delta^1(\varepsilon)$.

Proof: Suppose the lemma is false; then, there exists $\varepsilon > 0$ such that for every n such that $1 - \frac{1}{n} \geq \hat{\delta}$ there exists δ_n such that $1 > \delta_n \geq 1 - \frac{1}{n}$; $u_n \in U$ and $(x_n, y_n) \in X$ such that $\|x_n - k^{u_n,\delta_n}\| = \varepsilon$, and $\frac{1}{n} \geq f^{u_n,\delta_n}(x_n,y_n) \geq 0$ (we have put $\|x_n - k^{u_n,\delta_n}\| = \varepsilon$, because f^{u_n,δ_n} is convex for all n). Now, by Lemma 4, we can suppose that there exists a function $u : X \to \mathbb{R}$, and points k^u and q^u in $\mathbb{R}^n$ that satisfy conditions (20) and (21), such that $u_n \to u$, $k^u = \lim_{n\to\infty} k^{u_n,1}$ and $q^u = \lim_{n\to\infty} D_1u_n(k^{u_n,1},k^{u_n,1})$. On the other hand, as U_1 is relatively compact (see Remark 4) and the sequence

$$\left\{x_n, y_n, k^{u_n,\delta_n}, q^{u_n,\delta_n}\right\}$$

is bounded since,

$$(x_n,y_n) \in X, (k^{u_n,\delta_n},k^{u_n,\delta_n}) \in X_1 \times X_1 \text{ and } q^{u_n,\delta_n} = \delta_n D_1u_n\left(k^{u_n,\delta_n},k^{u_n,\delta_n}\right)$$

for any $n \geq 1$, we have that there exists points $(\bar{k}, \bar{q}) \in \mathbb{R}^{2n}_{+}$, and $(\bar{x}, \bar{y}) \in X$ such that, without loss of generality (taking sub-subsequences, if necessary), we have

$$\left(x_n, y_n, k^{u_n,\delta_n}, q^{u_n,\delta_n}\right) \to \left(\bar{x}, \bar{y}, \bar{k}, \bar{q}\right)$$

By Lemma 5 $\left(\bar{k}, \bar{k}\right) = (k^u, k^u)$ and from Lemma 1 we have

$$\begin{aligned} u_n(k^{u_n,\delta_n}, k^{u_n,\delta_n}) + q^{u_n,\delta_n}(k^{u_n,\delta_n} - \delta_n^{-1}k^{u_n,\delta_n}) \geq \\ u_n(x, y) + q^{u_n,\delta_n}(y - \delta_n^{-1}x); \text{ for all } (x, y) \in D \end{aligned} \tag{28}$$

for all n large enough. Hence, taking limits

$$\begin{aligned} u(k^u, k^u) \geq u(x, y) + \bar{q}(y - x) \text{ for all } (x, y) \in X; \text{ and} \\ u(k^u, k^u) > u(x, y) + \bar{q}(y - x) \text{ for all } (x, y) \in X; \\ \text{such that } (x, y) \neq (k^u, k^u); \end{aligned} \tag{29}$$

once again, because u is strictly concave at (k^u, k^u).

On the other hand, since $\left\|x_n - k^{u_n,\delta_n}\right\| = \varepsilon$ for all $n \in \mathbb{N}$, taking limits, we have

$$(\bar{x}, \bar{y}) \neq (k^u, k^u) \text{ and } f^u(\bar{x}, \bar{y}) := u(k^u, k^u) - u(\bar{x}, \bar{y}) - \bar{q}(\bar{y} - \bar{x}) = 0; \tag{30}$$

but (29) is not consistent with (30). This completes the proof of the Lemma. $\square$

Let $X_2 = B(0; \bar{M}) \cap L$ and take $V_0^{u,\delta}|_{X_2}$. Note that $V_t^{u,\delta}(x) = \delta^t V_0^{u,\delta}(x)$ for all $t \in \mathbb{N}$ and that, by iii) in Theorem 1, for all $x \in X_2$ there exists an optimal path from x, say $\left\{k_t^{u,\delta}(x)\right\}$, and $V_0^{u,\delta}(x) = \sum_{l=0} \delta^{l+1} u(k_l^{u,\delta}(x), k_{l+1}^{u,\delta}(x))$.

Corollary 1. For any $\varepsilon > 0$ there exists $\hat{\delta} \equiv \delta^2(\varepsilon) < 1$ and $\sigma(\varepsilon) > 0$; such that: $\left\|x - k^{u,\delta}\right\| \geq \varepsilon$ and $x \in X_2$; implies $V_0^{u,\delta}(k^{u,\delta}) - V_0^{u,\delta}(x) - q^{u,\delta}(k^{u,\delta} - x) \geq \sigma(\varepsilon)$; for any $\delta^2(\varepsilon) \leq \delta < 1$ and for any $u \in U$.

Proof: The proof is by contradiction. Suppose there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ large enough, there exists δ_n; $x_n \in X_2$ and $u_n \in U$ such that $\left\|x_n - k^{u_n,\delta_n}\right\| \geq \varepsilon$, $\delta_n \to 1$ and

$$\frac{1}{n} > V_0^{u_n,\delta_n}(k^{u_n,\delta_n}) - V_0^{u_n,\delta_n}(x_n) - q^{u_n,\delta_n}(k^{u_n,\delta_n} - x_n) \geq 0 \tag{31}$$

On the other hand, as noted above, for all $x_n \in X_2$, there exists an optimal path from x_n, say $\left\{ k_t^{u_n,\delta_n}(x_n) \right\}$ and

$$V_0^{u_n,\delta_n}(x_n) = \sum_{l=0}^{\infty} \delta_n^{l+1} u_n(k_t^{u_n,\delta_n}(x_n), k_{t+1}^{u_n,\delta_n}(x_n))$$

and $V_0^{u_n,\delta_n}(k^{u_n,\delta_n}) = \sum_{l=0}^{\infty} \delta_n^{l+1} u(k^{u_n,\delta_n}, k^{u_n,\delta_n})$. Now, by Lemma 6, there exists $\rho(\varepsilon) > 0$ such that

$$\begin{aligned} & u(k^{u_n,\delta_n}, k^{u_n,\delta_n}) - u(k_t^{u_n,\delta_n}(x_n), k_{t+1}^{u_n,\delta_n}(x_n)) + \\ & q^{u_n,\delta_n}(k^{u_n,\delta_n} - k_{t+1}^{u_n,\delta_n}(x_n)) - \delta_n^{-1} q^{u_n,\delta_n}(k^{u_n,\delta_n} - k_t^{u_n,\delta_n}(x_n)) \geq \rho(\varepsilon) > 0 \end{aligned}$$

for n large enough and for all $t \in \mathbb{N}$. Hence, fixing n, multiplying by δ_n^{t+1} and summing over t:

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta_n^{t+1} \left[u(k^{u_n,\delta_n}, k^{u_n,\delta_n}) - u(k_t^{u_n,\delta_n}(x_n), k_{t+1}^{u_n,\delta_n}(x_n)) \right] + \\ & \sum_{t=0}^{\infty} \delta_n^{t+1} \left[q^{u_n,\delta_n}(k^{u_n,\delta_n} - k_{t+1}^{u_n,\delta_n}(x_n)) - \delta_n^{-1} q^{u_n,\delta_n}(k^{u_n,\delta_n} - k_t^{u_n,\delta_n}(x_n)) \right] \geq \\ & \rho(\varepsilon) \sum_{t=0}^{\infty} \delta_n^{t+1}. \end{aligned}$$

But

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta_n^{t+1} \left[q^{u_n,\delta_n}(k^{u_n,\delta_n} - k_{t+1}^{u_n,\delta_n}(x_n)) - \delta_n^{-1} q^{u_n,\delta_n}(k^{u_n,\delta_n} - k_t^{u_n,\delta_n}(x_n)) \right] = \\ & -q^{u_n,\delta_n}(k^{u_n,\delta_n} - x_n) \end{aligned}$$

and

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta_n^{t+1} \left[u(k^{u_n,\delta_n}, k^{u_n,\delta_n}) - u(k_t^{u_n,\delta_n}(x_n), k_{t+1}^{u_n,\delta_n}(x_n)) \right] = \\ & V_0^{u_n,\delta_n}(k^{u_n,\delta_n}) - V_0^{u_n,\delta_n}(x_n) \end{aligned}$$

hence,

$$V_0^{u_n,\delta_n}(k^{u_n,\delta_n}) - V_0^{u_n,\delta_n}(x_n) - q^{u_n,\delta_n}(k^{u_n,\delta_n} - x_n) \geq \rho(\varepsilon)\delta_n \frac{1}{1-\delta_n} \tag{32}$$

for n large enough. Then, combining (31) and (32) we get, for n large enough:

$$\frac{1}{n} > \rho(\varepsilon)\delta_n \frac{1}{1-\delta_n}$$

which is a contradiction, proving the corollary. □

We recall here that we have supposed that $k_0 \in \operatorname{int} S$.

Lemma 7. Take $L^{u,\delta}(t) = \left(q^{u,\delta} - q_t^{u,\delta}\right)\left(k^{u,\delta} - k_t^{u,\delta}\right)$ defined earlier. Then for any $\varepsilon > 0$ and $1 > \delta^3 > 0$ there exists $L_{\varepsilon,\delta^3} < 0$, $L_{\varepsilon,\delta^3} \in \mathbb{R}$, such that $\left\|k_t^{u,\delta} - k^{u,\delta}\right\| \geq \varepsilon$ implies $L^{u,\delta}(t) \geq \delta L_{\varepsilon,\delta^3}$, for all $1 > \delta \geq \delta^3$ and $u \in U$. Furthermore, $\lim_{\varepsilon \to 0} L_{\varepsilon,\delta^3} = 0$.

Proof: Simply take $L_{\varepsilon,\delta^3} = \min_{(x,k,q,q') \in \mathbb{R}^{4n}_+;\, |x-k| \geq \varepsilon;\, |x|,|k| \leq M;\, |q|,|q'| \leq N_1;\, \delta^3 \leq \delta \leq 1} (q - q')(x - k)\delta^{-1}$. □

Theorem 2 (A Generalized U-neighborhood Turnpike Theorem) For any $\varepsilon > 0$ there exists $N(\varepsilon)$ and $0 < \delta^4(\varepsilon) < 1$ such that: for all $\delta^4(\varepsilon) \leq \delta < 1$ and $u \in U$, we have $\left\|k_t^{u,\delta} - k^{u,\delta}\right\| \leq \varepsilon$ for all $t > N(\varepsilon)$.

Proof: First we prove that for any $\varepsilon > 0$ there exists some $N(\varepsilon)$ and $\delta^1(\varepsilon) \in (0,1)$ such that for any $\delta^1(\varepsilon) \leq \delta < 1$ and all $u \in U$ we have: $\left\|k_{t^{u,\delta}}^{u,\delta} - k^{u,\delta}\right\| \leq \varepsilon$ for some $t^{u,\delta} < N(\varepsilon)$, that is: a generalized Visit Lemma. From (15) we had $L^{u,\delta}(t) - \delta^{-1}L^{u,\delta}(t-1) = \delta^{-t}\left(\alpha_t^{u,\delta} + \hat{\alpha}_t^{u,\delta}\right)$. Now, by the definition of $\alpha_t^{u,\delta}$ (see (6)) and $f^{u,\delta}$ in Lemma 6, we have $\delta^{-t}\alpha_t^{u,\delta} = f^{u,\delta}(k_{t-1}^{u,\delta}, k_t^{u,\delta})$, and since $\delta^{-t}\hat{\alpha}_t^{u,\delta} \geq 0$ (see (7) and (1)), we have

$$L^{u,\delta}(t) - \delta^{-1}L^{u,\delta}(t-1) \geq f^{u,\delta}(k_{t-1}^{u,\delta}, k_t^{u,\delta}). \tag{33}$$

Take $\varepsilon > 0$ arbitrary, then, by Lemma 6, there exists $\rho(\varepsilon) > 0$, and $\delta^1(\varepsilon) \in (0,1)$ such that, for all $u \in U$ and $1 > \delta \geq \delta^1(\varepsilon)$, whenever we have $\left\|k_{t-1}^{u,\delta} - k^{u,\delta}\right\| > \varepsilon$, then $f^{u,\delta}(k_{t-1}^{u,\delta}, k_t^{u,\delta}) \geq \rho(\varepsilon)$. Hence, by (33), we have that for all $u \in U$ and $1 > \delta \geq \delta^1(\varepsilon)$

$$\left\|k_{t-1}^{u,\delta} - k^{u,\delta}\right\| > \varepsilon; \text{ implies } L^{u,\delta}(t) - \delta^{-1}L^{u,\delta}(t-1) \geq \rho(\varepsilon) \tag{34}$$

Now, let $N(\varepsilon) = 2 + \sup_{1 \geq \delta \geq \delta^1(\varepsilon);\, u \in U} \left(-2L^{u,\delta}(0)/\rho(\varepsilon)\right)$ and $\underline{L} = \inf_{1 \geq \delta \geq \delta^1(\varepsilon);\, u \in U} L^{u,\delta}(0)$. Note that since $L^{u,\delta}(0) = \left(q^{u,\delta} - q_0^{u,\delta}\right)\left(k^{u,\delta} - k_0\right)$, $N(\varepsilon)$ and $\underline{L}$ are well defined because of the uniform bound over functions and derivatives, and $k_0, k^{u,\delta} \in X_2$.

Now we will show that the reasoning given by McKenzie ([7], page. 1313) can be adapted to our problem: Let $\bar{\delta}^2(\varepsilon)$ be such that

$$(\delta^{-1} - 1)\bar{L} > -\rho(\varepsilon)/2 \text{ for all } 1 > \delta \geq \bar{\delta}^2(\varepsilon); \tag{35}$$

then for all $u \in U$ and $1 > \delta \geq \bar{\delta}^2(\varepsilon)$, we have

$$(\delta^{-1} - 1)L^{u,\delta}(0) > -\rho(\varepsilon)/2 \tag{36}$$

Suppose $\left\|k_0 - k^{u,\delta}\right\| > \varepsilon$, then, summing (34) for $t = 1$ and (36), for all $1 > \delta \geq \max\left\{\delta^1(\varepsilon), \bar{\delta}^2(\varepsilon)\right\}$ and $u \in U$ we have

$$L^{u,\delta}(1) - L^{u,\delta}(0) > \rho(\varepsilon)/2 \tag{37}$$

Then

$$(\delta^{-1} - 1)L^{u,\delta}(1) > -\rho(\varepsilon)/2 \tag{38}$$

also holds, because if $(\delta^{-1} - 1)L^{u,\delta}(1) \leq -\rho(\varepsilon)/2$, using (36), we would obtain $\left[L^{u,\delta}(1) - L^{u,\delta}(0)\right]\left[\delta^{-1} - 1\right] < 0$, so $L^{u,\delta}(1) - L^{u,\delta}(0) < 0$ since $\left[1 - \delta^{-1}\right] < 0$, contradicting (37). Then we have proved the following statement:

$$\begin{gathered}\left\|k_0 - k^{u,\delta}\right\| > \varepsilon \text{ and } (\delta^{-1} - 1)L^{u,\delta}(0) > -\rho(\varepsilon)/2; \text{ implies} \\ L^{u,\delta}(1) - L^{u,\delta}(0) > \rho(\varepsilon)/2 \text{ and } (\delta^{-1} - 1)L^{u,\delta}(1) > -\rho(\varepsilon)/2.\end{gathered} \tag{39}$$

Thus we may apply induction to obtain $L^{u,\delta}(t) - L^{u,\delta}(t-1) > \rho(\varepsilon)/2$, if $\left\|k_l - k^{u,\delta}\right\| > \varepsilon$ for all $0 \leq l < t$. Hence we have proved the following statement: for all $u \in U$ and $1 > \delta \geq \max\left\{\delta^1(\varepsilon), \bar{\delta}^2(\varepsilon)\right\}$, we have

$$\begin{gathered}\left\|k_l - k^{u,\delta}\right\| > \varepsilon \text{ for all } 0 \leq l < t; \text{ implies } L^{u,\delta}(l) - L^{u,\delta}(l-1) > \rho(\varepsilon)/2 \\ \text{for all } 1 \leq l \leq t.\end{gathered} \tag{40}$$

From (40), and since $L^{u,\delta}(t) \leq 0$ for all $t \in \mathbb{N}$; if we fix (u, δ);

$$\left\|k_t - k^{u,\delta}\right\| > \varepsilon \tag{41}$$

cannot occur for ever, because $L^{u,\delta}(t)$ would become positive. Indeed, if we denote by $N(u, \delta, \varepsilon)$ the maximal number of consecutive periods that (41) is satisfied, then

$$L^{u,\delta}(N(u, \delta, \varepsilon)) - L^{u,\delta}(0) > N(u, \delta, \varepsilon)\rho(\varepsilon)/2$$

hence, $0 \ge N(u,\delta,\varepsilon)\rho(\varepsilon)/2 + L^{u,\delta}(0)$, and then $N(u,\delta,\varepsilon) \le -2L^{u,\delta}(0)/\rho(\varepsilon) < N(\varepsilon)$. Then, taking $\bar\delta^1(\varepsilon) = \max\left\{\delta^1(\varepsilon), \bar\delta^2(\varepsilon)\right\}$ we have proved the following statement:

$$\begin{gathered}\text{for all } u \in U \text{ and } 1 > \delta \ge \bar\delta^1(\varepsilon), \\ \text{there exists } t^{u,\delta} < N(\varepsilon), \text{ such that } \left\|k^{u,\delta}_{t^{u,\delta}} - k^{u,\delta}\right\| \le \varepsilon,\end{gathered} \tag{42}$$

as required.

Set $\delta^3 = \hat\delta$ (recall that $\hat\delta \in (0,1)$ is such that, $\delta\bar{y} > \bar{x}$ for any $\delta \ge \hat\delta$). Then $\left\|k^{u,\delta}_t - k^{u,\delta}\right\| \le \varepsilon$ implies $L^{u,\delta}(t) \ge \delta L_{\varepsilon,\delta^3}$, by Lemma 7, and $L^{u,\delta}(t+1) \ge L_{\varepsilon,\delta^3}$, by (15). Hence we have:

$$\left\|k^{u,\delta}_t - k^{u,\delta}\right\| \le \varepsilon, \text{ implies } L^{u,\delta}(t+1) \ge L_{\varepsilon,\delta^3} \text{ for all } 1 > \delta \ge \delta^3 \text{ and } u \in U. \tag{43}$$

On the other hand, by Corollary 1, for any ε' we know that there exists $\sigma(\varepsilon') > 0$ and $\hat\delta \le \delta^2(\varepsilon') < 1$ such that for all $u \in U$ and $1 > \delta \ge \delta^2(\varepsilon')$, we have

$$V_0^{u,\delta}(k^{u,\delta}) - V_0^{u,\delta}(k^{u,\delta}_{t+1}) - q^{u,\delta}(k^{u,\delta} - k^{u,\delta}_{t+1}) < \sigma(\varepsilon'), \text{ implies } \left\|k^{u,\delta}_{t+1} - k^{u,\delta}\right\| < \varepsilon'. \tag{44}$$

Also we note that

$$L^{u,\delta}(t+1) \ge L_{\varepsilon,\delta^3}, \text{ implies } \left(V_0^{u,\delta}(k^{u,\delta}) - V_0^{u,\delta}(k^{u,\delta}_{t+1}) - q^{u,\delta}(k^{u,\delta} - k^{u,\delta}_{t+1})\right) \le -L_{\varepsilon,\delta^3} \tag{45}$$

since $\delta^{t+1}L^{u,\delta}(t+1) \le -\lambda_{t+1}$ (recall (14)) and by (9)

$$\lambda_{t+1} = \delta^{t+1}\left(V_0^{u,\delta}(k^{u,\delta}) - V_0^{u,\delta}(k^{u,\delta}_{t+1}) - q^{u,\delta}(k^{u,\delta} - k^{u,\delta}_{t+1})\right);$$

where we use the fact that $V_t^{u,\delta}(x) = \delta^t V_0^{u,\delta}(x)$ for all $x \in L$. Hence

$$L^{u,\delta}(t+1) \le -\left(V_0^{u,\delta}(k^{u,\delta}) - V_0^{u,\delta}(k^{u,\delta}_{t+1}) - q^{u,\delta}(k^{u,\delta} - k^{u,\delta}_{t+1})\right)$$

for all $t \in \mathbb{N}$, and thus (45) follows directly from this inequality.

Take ε' arbitrary, and ε so that $-L_{\varepsilon,\delta^3} < \sigma(\varepsilon')$. Letting

$$\bar\delta^2(\varepsilon') = \max\left\{\delta^3, \bar\delta^1(\varepsilon), \delta^2(\varepsilon')\right\};$$

we have proved that, given ε' arbitrary, there exists $\bar\varepsilon$ and $N(\varepsilon)$, such that for all $u \in U$ and $1 > \delta \geq \bar\delta^2(\varepsilon')$, there is $t^{u,\delta} < N(\varepsilon)$ so that $\left\| k^{u,\delta}_{t^{u,\delta}} - k^{u,\delta}\right\| \leq \bar\varepsilon$ and $\left\| k^{u,\delta}_{t^{u,\delta}+1} - k^{u,\delta}\right\| < \varepsilon'$; since, by (42) there is $t^{u,\delta} < N(\varepsilon)$ so that $\left\| k^{u,\delta}_{t^{u,\delta}} - k^{u,\delta}\right\| \leq \varepsilon$, (43) implies that $L^{u,\delta}(t^{u,\delta}+1) \geq L_{\varepsilon,\bar\delta^3}$, (45) implies that
$V_0^{u,\delta}(k^{u,\delta}) - V_0^{u,\delta}(k^{u,\delta}_{t+1}) - q^{u,\delta}(k^{u,\delta} - k^{u,\delta}_{t+1}) \leq -L_{\varepsilon,\bar\delta^3} < \sigma(\varepsilon')$; and finally (44) implies that $\left\| k^{u,\delta}_{t^{u,\delta}+1} - k^{u,\delta}\right\| < \varepsilon'$. We note that if it were the case that $\varepsilon' \leq \bar\varepsilon$, then we are done, because necessarily we have $\left\| k^{u,\delta}_{t^{u,\delta}+l} - k^{u,\delta}\right\| \leq \varepsilon'$ for all $l \geq 1$. If $\varepsilon' > \bar\varepsilon$, we have that, for all $\bar\delta^2(\varepsilon') \leq \delta < 1$ and $u \in U$,

$$\left\| k^{u,\delta}_{t^{u,\delta}+l} - k^{u,\delta}\right\| > \varepsilon, \text{ implies } L^{u,\delta}(t^{u,\delta}+l+1) - \delta^{-1}L^{u,\delta}(t^{u,\delta}+l) \geq \rho(\varepsilon) \quad \text{for all } l \geq 1, \tag{46}$$

by (34). We stress now that (46) holds for $l = 1$.

Let $\bar\delta^3(\varepsilon)$ be such that $(\delta^{-1} - 1)L_{\varepsilon,\bar\delta^3} > -\rho(\varepsilon)/2$ for all $1 > \delta \geq \bar\delta^3(\varepsilon)$. Then, since $L^{u,\delta}(t^{u,\delta}+1) \geq L_{\varepsilon,\bar\delta^3}$ for all $1 > \delta \geq \bar\delta^2(\varepsilon')$ and $u \in U$ by (43), we have $(\delta^{-1} - 1)L^{u,\delta}(t^{u,\delta}+1) > -\rho(\varepsilon)/2$; and using (46) we get

$$L^{u,\delta}(t^{u,\delta}+2) - L^{u,\delta}(t^{u,\delta}+1) > \rho(\varepsilon)/2 \text{ for all } 1 > \delta \geq \max\left\{\bar\delta^2(\varepsilon'), \bar\delta^3(\varepsilon)\right\} \text{ and } u \in U;$$

hence $L^{u,\delta}(t^{u,\delta}+2) > L_{\varepsilon,\bar\delta^3}$ and $\left\| k^{u,\delta}_{t^{u,\delta}+2} - k^{u,\delta}\right\| \leq \varepsilon'$ by (44) and (45), since $-L_{\varepsilon,\bar\delta^3} < \sigma(\varepsilon')$. Now we can use the same argument to obtain $\left\| k^{u,\delta}_{t^{u,\delta}+\tau} - k^{u,\delta}\right\| \leq \varepsilon'$ for $\tau \geq 2$ so long as $\left\| k^{u,\delta}_{t^{u,\delta}+\tau-1} - k^{u,\delta}\right\| > \bar\varepsilon$. But $k^{u,\delta}_{t^{u,\delta}+l}$ must eventually satisfy $\left\| k^{u,\delta}_{t^{u,\delta}+l} - k^{u,\delta}\right\| \leq \bar\varepsilon$ for some $1 \leq l < \infty$, or (46) would force $L^{u,\delta}(t^{u,\delta}+l)$ to be positive, which is a contradiction with (12). Once we have again $\left\| k^{u,\delta}_{t^{u,\delta}+\tau} - k^{u,\delta}\right\| \leq \bar\varepsilon$ for some $\tau < \infty$, a repetition of the argument shows that, $\left\| k^{u,\delta}_{t^{u,\delta}+l} - k^{u,\delta}\right\| \leq \varepsilon'$ for all $l \geq 1$; taking $\delta^4(\varepsilon') = \max\left\{\bar\delta^2(\varepsilon'), \bar\delta^3(\varepsilon)\right\}$, we have the result. ∎

Lemma 8. Let λ^u denote any characteristic root of $D^2u(k^u, k^u)$. Then for any $u \in U$, we have $-\lambda^u \geq B$.

Proof: By A6 we have that $x'D^2u(k^u;k^u)x + Bx^2 \leq 0$ for all $x \in \Re^{2n}$ and all $u \in U$. Take any $u \in U$, and let $\lambda_1, ..., \lambda_{2n}$ denote the characteristic roots of $D^2u(k^u;k^u)$. Now, as $D^2u(k^u;k^u)$ is symmetric, there exists an orthogonal matrix $P \in \Re^{4n^2}$ (that is, P satisfies that $P'P = PP' = I$, where I is the identity matrix of order $2n$) such that

$$P'D^2u(k^u;k^u)P = \Lambda$$

where the $2n \times 2n$ matrix $\Lambda = \left\{\tilde{\lambda}_{i,j}\right\}$, is such that $\tilde{\lambda}_{i,j} = 0$ if $i \neq j$, and $\tilde{\lambda}_{ii} = \lambda_i$ where λ_i is the ith eigenvalue of $D^2u(k^u;k^u)$. Let $z = P'x$ for $x \in \Re^{2n}$. Therefore, $z'z = x'PP'x = x'x$, and also $x'D^2u(k^u;k^u)x = z'[P'D^2u(k^u;k^u)P]z = z'\Lambda z = \sum_{i=1}^{2n} z_i^2\lambda_i$. Summing up, we have that for any $x \in \Re^{2n}$, if $z = P'x$, the following equalities are satisfied

$$\sum_{i=1}^{2n} z_i^2 = z'z = x'x, \text{ and}$$
$$x'D^2u(k^u;k^u)x = \sum_{i=1}^{2n} z_i^2\lambda_i$$

Recalling that $x'D^2u(k^u;k^u)x + Bx^2 \leq 0$ for all $x \in \Re^{2n}$, we have that

$$\sum_{i=1}^{2n} z_i^2\lambda_i + B\sum_{i=1}^{2n} z_i^2 \leq 0$$

for all $z \in \Re^{2n}$ (because P is orthogonal), and therefore $\lambda_i + B \leq 0$ for all $i = 1, 2, ..., 2n$, which yields the statement of the Lemma. ¤

Now define $h^{u,\delta} : X \to \Re$ given by

$$\begin{aligned} h^{u,\delta}(x,y) = & -(D_2u(x,y) - D_2u(k^{u,\delta};k^{u,\delta}))(y - k^{u,\delta}) - \\ & (\delta D_1u(x,y) - \delta D_1u(k^{u,\delta};k^{u,\delta}))(x - k^{u,\delta}). \end{aligned} \quad (47)$$

Note that

$$h^{u,\delta}(k_{t-1}^{u,\delta};k_t^{u,\delta}) = L^{u,\delta}(t) - L^{u,\delta}(t-1). \quad (48)$$

As we have already seen in Theorem 2, we had $h^{u,\delta}(k_{t-1}^{u,\delta};k_t^{u,\delta})$ positive whenever $\left|k_{t-1}^{u,\delta} - k^{u,\delta}\right|$ was strictly positive, as a consequence of Lemma 6, that is, as a consequence of the uniform strict concavity of the family U, and then we could force optimal paths to be close to the OSS, although this is not enough to have convergence, because we had $\delta^4 = \delta^4(\varepsilon)$, which means, the discount factor is not independent of ε. But under our assumptions $h^{u,\delta}(k_{t-1}^{u,\delta}, k_t^{u,\delta})$ will be uniformly strictly positive whenever the optimal path departs from the OSS by an arbitrarily small but strictly positive quantity, but this cannot occur for ever, because $L^{u,\delta}(t)$ would become positive, and hence we will have convergence to the OSS, and this reasoning will apply to all members of the family U, provided the discount factor is large enough, which will be shown in detail in Theorem 3. We still need the following results:

Lemma 9. For any $h^{u,\delta} : X \to \mathbb{R}$ defined as in (47), we have:

$$Dh^{u,\delta}(k^{u,\delta}, k^{u,\delta}) = 0$$

and

$$D^2h^{u,\delta}(k^{u,\delta}, k^{u,\delta}) = \begin{pmatrix} -2\delta D_{11}u(k^{u,\delta}, k^{u,\delta}) & -D_{12}u(k^{u,\delta}, k^{u,\delta})(1+\delta) \\ -D_{21}u(k^{u,\delta}, k^{u,\delta})(1+\delta) & -2D_{22}u(k^{u,\delta}, k^{u,\delta}) \end{pmatrix}$$

Proof: Routine and omitted. ■

Corollary 2. Let ϵ be such that $0 < B - \epsilon$, and let $\lambda_h^{u,\delta}$ the characteristic root of $-D^2h^{u,\delta}(k^{u,\delta}, k^{u,\delta})$ of minimal absolute value. Then there exists $0 < \delta^5 < 1$, such that: $-\lambda_h^{u,\delta} > 2(B - \epsilon)$, for all $u \in U$ and every $1 > \delta \geq \delta^5$.

Proof: The proof is by contradiction. Suppose there exists δ_n and $u_n \in U$ such that, $\delta_n \to 1$ and $-\lambda_{h_n}^{u_n,\delta_n} \leq 2(B - \epsilon)$ for all n large enough. Now, since U and U_{ij} for $i, j \in \{1, 2\}$ are relatively compact (see Lemma 3 and Remark 4), once again, without loss of generality, we can suppose that there exists functions $u \in U$ and $D_{ij}u \in U_{ij}$ for $i, j \in \{1, 2\}$, such that $u_n \to u$, and $D_{ij}u_n \to D_{ij}u$ for $i, j \in \{1, 2\}$.[9] Hence

$$D^2h^{u_n,\delta_n}(k^{u_n,\delta_n}, k^{u_n,\delta_n}) \to$$
$$D^2h^u(k^u, k^u) = -2\begin{pmatrix} D_{11}u(k^u, k^u) & D_{12}u(k^u, k^u) \\ D_{21}u(k^u, k^u) & D_{22}u(k^u, k^u) \end{pmatrix},$$

[9] Under our assumptions, if a sequence $\{u_n\}$ is such that $u_n \to u$, we have that the limit function u is in fact C^3, $D_1u_n \to D_1u$, and $D_{ij}u_n \to D_{ij}u$ for $i, j \in \{1, 2\}$.

where $D_{ij}u(k^u;k^u) = \lim_{l\to\infty} D_{ij}u_n(k^{u_n,\delta_n};k^{u_n,\delta_n})$ for $i,j \in \{1,2\}$, by Lemmata 9, 5 and 3. Let λ^u denote the characteristic root of $D^2u(k^u;k^u)$ of minimal absolute value, thus $2\lambda^u$ is the characteristic root of $-D^2h^u(k^u;k^u)$ of minimal absolute value. Then $\lambda_{h_n}^{u_n,\delta_n} \to 2\lambda^u$ and $-2\lambda^u \le 2(B-\epsilon)$. Now, define

$$D^2\bar{h}^n(\bar{k}^n;\bar{k}^n) := -2D^2u_n(k^{u_n};k^{u_n})$$

then, since $D_{ij}u_n \to D_{ij}u$ for $i,j \in \{1,2\}$, $D^2\bar{h}^n(\bar{k}^n;\bar{k}^n) \to D^2h^u(k^u;k^u)$ as well, by Lemma 5 and Lemma 3. But this implies that, if λ^{u_n} denotes the characteristic root of $D^2u_n(k^{u_n};k^{u_n})$ of minimal absolute value (hence $2\lambda^{u_n}$ is the characteristic root of $-D^2\bar{h}^n(\bar{k}^n;\bar{k}^n)$ of minimal absolute value), then $2\lambda^{u_n} \to 2\lambda^u$ as well. Therefore $-2\lambda^{u_n} \le 2(B-\epsilon)+\epsilon = 2B-\epsilon$ for all n large enough. Hence $-\lambda^{u_n} \le B-\epsilon/2 < B$ for all n large enough, which contradicts Lemma 8. $\square$

Now we will prove the main theorem of the paper. We show that under the assumption that $k_0 \in \operatorname{int} S$; we have a uniform $\delta' \in (0;1)$ such that for any $u \in U$ and for any $\delta \ge \delta'$, the optimal path from k_0 exhibits convergence to the steady state. Furthermore, the convergence is uniform over the family in the sense that, given any $\varepsilon > 0$ we get a uniform $N(\varepsilon)$ in the definition of the limit, for any optimal growth problem in the family U; that is: given any $\varepsilon > 0$ we have that, if $1 > \delta \ge \delta'$; for any $u \in U$, any optimal path $\left\{k_t^{u,\delta}\right\}$ from k_0 will enter the ball $B(k^{u,\delta};\varepsilon)$ at or before $N(\varepsilon)$. Formally:

Theorem 3 (A Turnpike Theorem for the family U). *There exists $0 < \delta' < 1$ such that, for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that $\left\|k_t^{u,\delta} - k^{u,\delta}\right\| \le \varepsilon$ for all $t > N(\varepsilon)$; for all $1 > \delta \ge \delta'$ and $u \in U$:*

Proof: Take ϵ such that $0 < B - \epsilon$ and let δ^5 denote the constant given in Corollary 2. Write $h^{u,\delta}(x;y) = \frac{1}{2}\bar{x}'D^2h^{u,\delta}\bar{x} + O^{u,\delta}(\bar{x})$ with $\bar{x} = (x;y) - (k^{u,\delta};k^{u,\delta})$ (recall that $Dh^{u,\delta}(k^{u,\delta};k^{u,\delta}) = 0$; by Lemma 9), hence for $\delta \ge \delta^5$ and all $u \in U$ we have

$$h^{u,\delta}(x;y) \ge (B-\epsilon)|\bar{x}|^2 + O^{u,\delta}(\bar{x})$$

because $\bar{x}'D^2h^{u,\delta}\bar{x} \ge -\lambda_h^{u,\delta}|\bar{x}|^2$, where $-\lambda_h^{u,\delta}$ is the characteristic root of $D^2h^{u,\delta}$ of minimal value. But $\left|O^{u,\delta}(\bar{x})\right| \le N_1|\bar{x}|^3$ for all $u \in U$ and $0 < \delta < 1$ (because of the bound over functions and derivatives). Then we have $h^{u,\delta}(x;y) \ge$

$(B-\bar{\varepsilon})|\Delta|^2 - N_1|\Delta|^3$. Let $\tilde{\varepsilon}$ be such that $0<\tilde{\varepsilon}<\frac{2}{3}\frac{(B-\bar{\varepsilon})}{N_1}$. Now consider the function $\varepsilon \to f(\varepsilon) = (B-\bar{\varepsilon})\varepsilon^2 - N_1\varepsilon^3$ defined on $[0,\tilde{\varepsilon}]$ and note that:
i) $f(\varepsilon) \geq 0$ for all $\varepsilon \in [0,\tilde{\varepsilon}]$. Also, $f(\varepsilon)=0$ implies $\varepsilon = 0$; and
ii) $f'(\varepsilon) > 0$ for all $\varepsilon \in (0,\tilde{\varepsilon}]$.

Then, for all $0 \leq \varepsilon < \tilde{\varepsilon}$, we have that :

$$\tilde{\varepsilon} \geq |\Delta| \geq \varepsilon; \text{ implies } (B-\bar{\varepsilon})|\Delta|^2 - N_1|\Delta|^3 \geq (B-\bar{\varepsilon})\varepsilon^2 - N_1\varepsilon^3 \geq 0;$$

and, given any $0<\varepsilon<\tilde{\varepsilon}$, for all $1>\delta\geq\delta^5$ and $u\in U$, we have:

$$\tilde{\varepsilon} \geq |\Delta| \geq \varepsilon; \text{ implies } h^{u,\delta}(x,y) \geq (B-\bar{\varepsilon})\varepsilon^2 - N_1\varepsilon^3 > 0.$$

Now take $0<\varepsilon<\tilde{\varepsilon}$; hence, using (48), for all $1>\delta\geq\delta^5$ and $u\in U$, we have:

$$\begin{aligned} &\tilde{\varepsilon} \geq \left\|(k_{t-1}^{u,\delta},k_t^{u,\delta}) - (k^{u,\delta},k^{u,\delta})\right\| \geq \varepsilon; \\ &\text{implies } L^{u,\delta}(t) - L^{u,\delta}(t-1) \geq (B-\bar{\varepsilon})\varepsilon^2 - N_1\varepsilon^3 > 0; \end{aligned} \tag{49}$$

and also, using (48) again, for all $1>\delta\geq\delta^5$ and $u\in U$, we have:

$$\tilde{\varepsilon} \geq |\Delta_t|; \text{ implies } L^{u,\delta}(t) - L^{u,\delta}(t-1) \geq (B-\bar{\varepsilon})|\Delta_t|^2 - N_1|\Delta_t|^3 \geq 0 \tag{50}$$

where $\Delta_t = (k_{t-1}^{u,\delta},k_t^{u,\delta}) - (k^{u,\delta},k^{u,\delta})$.

Now take $N(\tilde{\varepsilon}/2)$ and $\delta^4(\tilde{\varepsilon}/2)$ as in Theorem 2, then for all $1>\delta\geq\delta' = \max\left\{\delta^5,\delta^4(\tilde{\varepsilon}/2)\right\}$ and all $u\in U$, we have that $\left\|k_t^{u,\delta}-k^{u,\delta}\right\| \leq \tilde{\varepsilon}/2$; for all $t \geq N(\tilde{\varepsilon}/2)$. Take $0<\varepsilon<\tilde{\varepsilon}$ arbitrary, then for all $1>\delta\geq\delta' = \max\left\{\delta^5,\delta^4(\tilde{\varepsilon}/2)\right\}$ and all $u\in U$, there exists $N(\varepsilon,u,\delta)$ such that $t>N(\varepsilon,u,\delta)$ implies $\left\|k_t^{u,\delta}-k^{u,\delta}\right\| \leq \varepsilon$, because if not, (49) and (50) would force $L^{u,\delta}(t)$ to be positive, which contradicts (12). Furthermore, we will show that, given any $0<\varepsilon<\tilde{\varepsilon}$ there is $N(\varepsilon) > N(\tilde{\varepsilon}/2)$ such that, $t>N(\varepsilon)$ implies $\left\|k_t^{u,\delta}-k^{u,\delta}\right\| \leq \varepsilon$, for all $1>\delta\geq\delta'$ and all $u\in U$. This will prove our theorem. Indeed, set

$$\tilde{N}(\varepsilon,u,\delta) = \left\{ t\in\mathbb{N};\ \left\|k_t^{u,\delta}-k^{u,\delta}\right\| > \varepsilon;\ t > N(\tilde{\varepsilon}/2) \right\}$$

and let $N(\varepsilon,u,\delta)$ denote the cardinality of $\tilde{N}(\varepsilon,u,\delta)$. Now let m_1 denote the cardinality of the set

$$M_1 = \left\{ t\in\mathbb{N};\ \left\|k_t^{u,\delta}-k^{u,\delta}\right\| \leq \varepsilon;\ t > N(\tilde{\varepsilon}/2) \text{ and } t \leq t_1 \right\};$$

where t_1 is the first time that $\left\|k_t^{u,\delta} - k^{u,\delta}\right\| > \varepsilon$ occurs for $t > N(\varepsilon/2)$. Similarly, let m_2 denote the cardinality of the set

$$M_2 = \left\{ t \in \mathbb{N};\ \left\|k_t^{u,\delta} - k^{u,\delta}\right\| \leq \varepsilon;\ t > N(\varepsilon/2) \text{ and } t_1 \leq t \leq t_2 \right\};$$

where t_2 is the second time that $\left\|k_t^{u,\delta} - k^{u,\delta}\right\| > \varepsilon$ occurs for $t > N(\varepsilon/2)$. Proceeding inductively we define $m_3, m_4, \ldots, m_{N(\varepsilon;u,\delta)}$. Now notice that (50) implies that whenever $t \notin \bar{N}(\varepsilon;u,\delta)$ and $t > N(\varepsilon/2)$, then $L^{u,\delta}(t) - L^{u,\delta}(t-1) \geq 0$; also, (49) implies that if $t \in \bar{N}(\varepsilon;u,\delta)$, then $L^{u,\delta}(t) - L^{u,\delta}(t-1) \geq (B-\eta)\varepsilon^2 - N_1\varepsilon^3 > 0$. Then we have

$$L^{u,\delta}\Big(N(\varepsilon;u,\delta) + \sum_{l=1}^{N(\varepsilon;u,\delta)} m_l + N(\varepsilon/2) + 1\Big) - L^{u,\delta}(N(\varepsilon/2)) \geq N(\varepsilon;u,\delta)\rho(\varepsilon)$$

where $\rho(\varepsilon) = 2(B-\eta)\varepsilon^2 - \bar{M}\varepsilon^3$. Hence $N(\varepsilon;u,\delta) \leq -L^{u,\delta}(N(\varepsilon/2))/\rho(\varepsilon)$; because of inequality (12). Setting $N(\varepsilon) = 2 + \sup_{1 \geq \delta \geq \delta', u \in U} -L^{u,\delta}(N(\varepsilon))/\rho(\varepsilon)$ ($N(\varepsilon)$ is well defined because of the uniform bound over functions and derivatives). This completes the proof of the Theorem. $\square$

Remark 5 Note that Theorems 2 and 3 hold true almost without changes if we assume the existence of full Weitzman prices for the optimal path considered. We can rewrite the results considering all those optimal paths for which there exist supporting prices (so it is not necessary to assume $k_0 \in \operatorname{int} S$; the condition that ensures the existence of supporting prices for all optimal paths, for all $u \in U$ and $1 > \delta > \hat{\delta}$). Of course, in this case we have weaker results. In fact, for a given $\varepsilon > 0$, we cannot ensure the uniform $N(\varepsilon)$ independent of k_0; neither for Theorem 2, nor for Theorem 3. More precisely, if $\left\{k_t^{u,\delta}\right\}$ is any optimal path from k_0 for which there exists supporting prices with $u \in U$ and $1 > \delta > \hat{\delta}$; then, for any $\varepsilon > 0$ there exists $\delta^*(\varepsilon)$ and $N(\varepsilon;k_0)$ for which Theorem 2 holds, and similarly, there exists δ' (still uniform) and $N(\varepsilon;k_0)$ for which Theorem 3 holds. In particular, if we add the assumption that $(0,0) \in D$, we have that $L = \mathbb{R}^n_+$, and hence both theorems hold true for any $k_0 \in \mathbb{R}^n_{++}$ in the sense described in this remark.

Remark 6 Theorem 2 admits weaker assumptions. In the first instance, observe that it can be proven assuming only C^1-functions instead of C^4-functions and assuming that the family is uniformly bounded in the sense that there exists some N_1 such that for any $u' \in U'$ we have $|u'_X|_1 \leq N_1$.[10] That is, take a family as the following

$$U' = \left\{ \begin{array}{c} u' : D \to \mathbb{R} \mid (u'; D) \text{ satisfies A1-A7, } u' \in C^1 \\ \text{and } |u'_X|_1 \leq N_1 \end{array} \right\}$$

then Theorem 2 holds for this class of families.

In the second instance, consider the following framework: replace A6 by the following assumption:

$$\text{A'6: } u(x; y) \text{ is strictly concave at } (k^u; k^u).$$

Now take following family:

$$U' = \left\{ \begin{array}{c} u' : D \to \mathbb{R} \mid (u'; D) \text{ satisfies A1-A5, A'6, A7} \\ \text{and } u' \in C^1 \end{array} \right\}$$

and then define the normalized family:

$$U = \left\{ u : D \to \mathbb{R} \mid u = \frac{u'}{|u'_X|_1};\ u' \in U' \right\}$$

and suppose that for any $u \in \overset{1}{U}$; u is strictly concave at $(k^u; k^u)$. Then Theorem 2 holds for this class of families.

Remark 7 Note the difference between these last two generalizations given in Remark 6. In the first, the set-up is similar to our Theorem 2, where we require A6 and the uniform bound over functions and derivatives explicitly. In the second, we normalize and we then ask that for any $u \in \overset{1}{U}$; u is strictly concave at $(k^u; k^u)$; once the family is normalized. This last condition is weaker than the former. Furthermore, any family as the following

$$U' = \left\{ \begin{array}{c} u' : D \to \mathbb{R} \mid (u'; D) \text{ satisfies A1-A7, } u' \in C^1 \\ \text{and } |u'_X|_1 \leq N_1 \end{array} \right\};$$

[10] We define $|u_X|_1 = \sup_{(x;y) \in X} \{|u(x; y)| + |Du(x; y)|\}$

satisfies that for any $u \in \bar{U}$; u is strictly concave at $(k^u; k^u)$; where $\bar{U}$ is the closure of the normalized family

$$U = \left\{ u : D \to \mathbb{R} \mid u = \frac{u'}{|u'_X|_1};\ u' \in U' \right\}:$$

4. Conclusions

The papers by Nishimura, Sorger and Yano [11] and Nishimura and Yano [12], make the point that the incompatibility between ergodic chaos and patience is not as strong as would appear from the standard Turnpike Theorem. Our result clarifies the extent to which an incompatibility exists. In fact, roughly speaking, interiority, smoothness ($C^4 -$ felicity functions), and, fundamentally, the uniform strict concavity of the family is enough to have uniform convergence to the steady state for any member of the family, hence the phenomena shown in [11] and [12] is ruled out. An interesting question here is to see if it is possible to improve this Uniform Turnpike Theorem assuming only $C^2 -$ felicity functions and/or relaxing the uniform strict concavity of the family (assumption A6). We think that this is not the case, but this can only be proved by means of an example, which we leave for further research.

Another question to investigate is to determine if the restriction to models with interior solutions is a sufficient condition to rule out the existence of a family of optimal growth problems displaying ergodic chaos for any value of the discount factor. We have already pointed out in footnote 1 that, to our knowledge, the existence of a family of optimal growth problems with interior solutions displaying ergodic chaos for any value of the discount factor is still an open question.

A more precise knowledge of the relation between the discount factor and the extent of chaos that optimal paths can display is a problem that we are currently investigating. We recall that in [11] the authors show in precise formulation that, in spite of having ergodic chaos for any $\delta \in (0; 1)$; the chaos disappears as the discount factor tends to one, and that a general result of a similar flavor can be found in [9], where it is shown that topological entropy tends to zero as the discount factor tends to one, in a family of strictly concave felicity functions. Also, very recently in [5], as a consequence of our Theorem 2, it is proven in precise formulation that, for a given $C^1 -$ family of felicity functions as in Remark 6, if it displays some kind of chaos for any value of the discount factor, the chaos has to be less and less important as the discount factor tends to one. It remains then to explore whether there is a more general result encompassing the example in [11], that is, if it is true that if we have a family of strictly concave optimal growth problems displaying ergodic chaos for any value of the discount factor, without assuming any other additional assumption over the family, then the chaos should be less and less relevant as the discount factor tends to one, what is also left for further research. This would give us a much clearer picture about the role played by discounting in permitting erratic trajectories.

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