BANKRUPTCY GAMES AND THE IBN EZRA'S PROPOSAL*

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ABSTRACT

This paper follows the interpretation of the bankruptcy problems in terms of TU games given in O'Neill (1982). In this context we propose the analysis of the *Transition Game* associated to each bankruptcy problem. We explore an old solution described by Ibn Ezra in the XII century. Firstly, we study the extension of the Ibn Ezra's proposal by O'Neill (1982), the *Minimal Overlap* solution. We provide a characterization of this value and show that it can be understood as the composition of the *Ibn Ezra* solution and the *Constrained Equal Loss* rule. Secondly, we introduce a new way of extending the Ibn Ezra's proposal, the *Generalized Ibn Ezra* solution, by imposing that the general distribution principle in which is inspired remains fixed. The characterization of our proposal clarifies the analogies and differences between the two ways of generalizing the Ibn Ezra's proposal.

Keywords: Bankruptcy Problems, Cooperative Games, Ibn Ezra's proposal, Minimal Overlap solution.

Journal of Economic Literature Classification Numbers: C71, D63, D71.

1. Introduction

The analysis of bankruptcy situations tries to prescribe how to ration an amount of a perfectly divisible resource among a group of agents according to a profile of demands which, in aggregate, exceed the quantity to be distributed.

The main illustrative example, but not the unique, to show how important it is to model this family of problems comes from the study of bankruptcy situations, which can be explained as follows. An individual, to be called the bankrupted, has not enough money to pay back all her creditors. The question to be answered is how the bankrupted's belongings should be shared among her creditors, according the credit conceded by each agent to their common debtor. A second family of problems which can be mathematically formalized in terms of a bankruptcy-like framework comes from the analysis of taxation systems (Young [22]). Just to introduce it, let us consider the following simplified situation. Usually, each government's expenditures have to be financed by taxes which are payed by the agents belonging to the government's jurisdiction. Let us imagine that the only way to reach this objective comes from the design of an income taxation system. It seems to be clear that, in a static framework, the government's expenditures should not exceed the agents' aggregate income. Therefore, the problem to be solved is how much each contributor should pay if the government is constrained to meet a budged equilibrium. The last example that we want to mention comes from the paper by O'Neill [16], which studies problems concerning how to share a deceased's inheritance among his heiresses, according to their incompatible rights on the deceased's belongings.

In the economic literature it can be found two main approaches to the study of bankruptcy situations. The first one, namely the axiomatic approach, was introduced by Young [21] and follows the next structure. Let us consider a set of properties that any bankruptcy rule must fulfill to be considered a fair proposal to solve bankruptcy situations. Then the problem to be studied is to describe the family of bankruptcy rules satisfying these properties. Usually one can find a trade-off between the set of properties to be employed and the size of the family of rules to be supported. In fact, the employ of much properties might lead to impossibility results, whereas the employ of a few of properties might characterize a huge family of solutions. The *most attractive* results are those employing the lower number of properties perfectly determining a unique solution. This approach was used, among others, by Chun [3], Herrero and Villar [10], Moulin [13], O'Neill [16] and Young [22] to characterize the *Proportional Solution*; Aumann and Maschler [1] developed an axiomatic study of the *Consistent Contested Garment Solution,* also analyzed in Herrero and Villar [10]; the *Constrained Equal Awards Solution* was studied by Dagan [6], Herrero [9], Herrero and Villar [10, 11] and Moulin [13]; and finally, different characterizations of the *Constrained Equal Loss Solution* appear in Herrero [9], Herrero and Villar [10, 11] and Moulin [13].[1]

[1] The papers by Thomson [20] and Moulin [14] provide two excellent surveys on this matter.

The second approach to the analysis of bankruptcy problems comes from an interpretation given in the O'Neill's [16] seminal paper. This author proposes a relationship between this family of problems and a particular class of Transferable Utility Cooperative Games, TU games from now on. To be more specific, the idea beyond O'Neill's [16] suggestion can be interpreted as follows. Let us imagine that any group of creditors could play the role of the bankrupted. Think of a bankrupted firm and that such a set of agents is buying this firm. In such a case, what a set of creditors (or coalition) can guarantee itself, is what left once they faced the debts that the firm have with the agents not in such a group. Just to introduce some group rationality in such an argument, we also assume that no coalition will like to share a negative amount, i.e. no group of agents is likely to buy the firm if the price that they collectively pay is higher than the credit the conceded. This idea was borrowed by some authors to justify some bankruptcy solutions because they coincide with some well-behaved values for cooperative games. For instance, Aumann and Maschler [1] propose the use of the *Consistent Contested Garment Solution*, also known as the *Talmudic Solution*, on the basis that it coincides with the Nucleolus of the related TU game. The *Random Arrival Solution*, proposed in O'Neill [16] coincides with the Shapley value of this game. This relationship between bankruptcy problems and TU games was also explored, among others, by Curiel et al. [5], Dutta and Ray [7] and Potters et al. [17].

The aim of this paper is the study of an old sharing method for bankruptcy problems attributed to Rabbi Abraham Ibn Ezra in the XII century. This author proposed a serial procedure to describe a method to ration agents' demands in a bankruptcy situation. In fact, the procedure described by Ibn Ezra is, in spirit, similar to a rule for cost sharing problems described by Moulin and Shenker [15], known as the serial cost sharing rule. This similarity becomes evident in environments were the good to be produced is a public good, being its consumption partially excludable. (See Moulin [12].)

The description provided by Ibn Ezra was formulated for the special case in which the agent whose demand is the highest exactly claims to be the owner of all the resources. This is, for instance, the case analyzed by Moulin [12]. Nevertheless, this particular case does not hold in a huge class of problems.

Our objective is to analyze how to extend the idea by Ibn Ezra to any bankruptcy problem. This question has received little attention in the economic literature: O'Neill [16] proposed the Minimal Overlap Solution as a way to generalize the Ibn Ezra's proposal. This rule was also studied by Chun and Thomson [4].

Throughout this paper we follow the (cooperative) game-theoretical approach of bankruptcy problems to characterize some solutions. The main concept that we introduce is *Transition Game*, which is the "difference" TU game arising when the estate increases.

With the help of the Transition Games we establish an appealing argument for the understanding of the Minimal Overlap rule. We show that this rule is the only anonymous value satisfying a property called *Core-Transition Responsiveness.* This property asks that the sharing of any "extra" estate could not be improved by any coalition related to what the transition game allows them.

Surprisingly enough, and in contrast with our characterization result, we also show that the Minimal Overlap solution has a very strong shift on its philosophy since it can be seen as a composition of the Ibn Ezra proposal and the *Constrained Equal Loss rule.*

This finding leads us to propose an alternative way of extending the Ibn Ezra's proposal, the *Generalized Ibn Ezra* rule, by imposing ourselves the underlying general distribution principle in which is inspired to remain fixed. Our characterization of this value is also based on the idea of transition game and it clarifies the analogies and differences between the two ways of generalizing the Ibn Ezra's proposal. Particularly, the axioms that we will employ in this result, together with Anonymity, are *Transitional Dummy* and *Worth-Generators Composition.* Transitional Dummy is a weak version of the usual Dummy axiom, but related to the case in which some coalition plays the role of clan in the sense of Potters et al. [17]. The axiom Worth-Generators Composition is a particular form of composition, which is very related to the non-emptiness of the transitional worth-generator coalition in a sense that we will made precis in this paper.

The organization of the rest of the paper is the following. First, Section 2 introduces some formalisms. In particular it describes bankruptcy problems, bankruptcy games and their relationship. Section 3 is devoted to present the proposal given by Ibn Ezra. Section 4 presents the Minimal Overlap rule. Section 5 introduces the concept of transition game and provides a characterization for the Minimal Overlap rule. Section 6 shows that the Minimal Overlap rule can be seen as the composition of the Ibn Ezra proposal and the Equal Loss Constrained rule. Section 7 is devoted to introduce the Generalized Ibn Ezra solution, our proposal of extending the illustrative situation given by this author. This section concludes by showing that the iterative procedure in which is based the Generalized Ibn Ezra solution is well defined. The introduction of the axioms and our main result concerning the Generalized Ibn Ezra solution is the aim of Section 8. Our main conclusions are summarized in Section 9. Finally, the technical proofs are relegated to the Appendixes.

2. Preliminaries

2.1. Bankruptcy problems

We can identify a bankrupted as an entity which cannot face all the debts it contracted. Therefore, and following this illustrative interpretation,

Definition 2.1. *A bankruptcy problem is characterized by a finite set of potential agents, $N = \{1, \dots, i, \dots, n\}$, or creditors; a positive real number E, representing the value of the bankrupted's estate; and the description of each creditor's claim,* *to be synthesized by $c \in \mathbb{R}^n_+$, whose aggregate worth is greater than the estate. Provided that we are not going to consider changes on the creditors' population, we can summarize a bankruptcy problem by the duplex (E, c) where*

$$E < \sum_{i=1}^{n} c_i.$$

Let $\mathcal{B}$ denote the set of all the bankruptcy problems,

$$\mathcal{B} = \left\{ (E, c) \in \mathbb{R}_{++} \times \mathbb{R}^n_+ : E < \sum_{i=1}^{n} c_i \right\}.$$

Given a bankruptcy problem, a recommendation for it consists in establishing how much of the credit each creditor will recover.

A solution for bankruptcy problems is a function which specifies a recommendation for each bankruptcy situation satisfying three conditions. The first one is that the estate is completely distributed among the creditors; the second condition introduces a limit on each creditor's loses, it being not higher than the credit conceded to the bankrupted; finally, the third one is that the amount that each creditor should recover must not be greater than the credit she conceded to the bankrupted. Formally,

Definition 2.2. *A solution for bankruptcy problems is a function φ which associates a recommendation to each bankruptcy problem*

$$\varphi : \mathcal{B} \to \mathbb{R}^n_+$$

such that for each $(E, c) \in \mathcal{B}$,

(*a*) $\sum_{i=1}^{n} \varphi_i(E, c) = E$, *and*

(*b*) $0 \leq \varphi_i(E, c) \leq c_i$ *for each creditor i.*

2.2. Bankruptcy problems and cooperative games

Given a set of agents N, a TU game involving N can be described as a function V associating a real number to each subset of agents, or *coalition*, S contained in N. Formally, a TU game is a pair (N, V), where

$$V : 2^N \to \mathbb{R}.$$

Given a coalition $S \subseteq N$, $V(S)$ is commonly called the worth of coalition S, and denotes the quantity that agents in S can guarantee to themselves if they cooperate.

We can describe a value for TU games as a function selecting, for each TU game a share of the worth among the agents in such a game. Formally, let $\mathcal{G}$ be a family of TU games referred to a fixed set of agents, say N.

$$\psi : \mathcal{G} \to \mathbb{R}^n$$

is a value on $\mathcal{G}$ if for each TU game (N, V) in $\mathcal{G}$,

$$\sum_{i \in N} \psi_i (N, V) = V (N) .$$

O'Neill [16] proposed a way for describing bankruptcy problems in terms of TU games. His proposal comes from the following interpretation of what agents can do when faced a bankruptcy problem. Just to introduce the rationale behind the proposal by this author, let us consider a given set of creditors, or coalition, S. If these agents pay the claims that the bankrupted owes to the others, they are free to share the rest of the estate among them as they want. Nevertheless, no coalition will wish to pay the others more than their own claims; i.e., no coalition will wish to share a negative amount of money.

The above arguments allow us to associate to each bankruptcy problem $B = (E, c) \in \mathcal{B}$ a TU game (N, V_B), where the function V_B captures the idea above mentioned:

Definition 2.3. *Let $B = (E, c)$ be a bankruptcy problem. We define the cooperative game induced by B, called* bankruptcy game, *as the pair (N, V_B), where the function*

$$V_B : 2^N \to \mathbb{R},$$

associates, to each coalition $S \subseteq N$, the real value

$$V_B (S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} c_i \right\} . \tag{2.1}$$

2.3. Cooperative games and bankruptcy problems

The considerations established in Section 2.2 above allow us to translate most of the results relative to values for TU games into bankruptcy theory. The way to do this is very simple; let us consider a value for TU games, say ψ. Hence we can interpret ψ as a solution for bankruptcy problems just identifying $\psi (B)$ with $\psi (N, V_B)$.

The question that we consider in this section is just the opposite one. Let φ be a solution for bankruptcy problems, then, is there a value for TU games ψ such that for any bankruptcy problem $B = (E, c)$, $\varphi (E, c) = \psi (N, V_B)$? The answer to this question is clearly negative. To illustrate this fact, pointed out in Curiel et al. [5], let us consider the following example.

Example 2.4. *Let $N = \{1,2,3\}$, and consider the following two bankruptcy problems:*

$$B = (E,c) \quad = \quad (18;(6,12,18))$$

$$B' = (E,c') \quad = \quad (18;(6,12,36)).$$

Notice that $(N,V_B) = (N,V_{B'})$. Therefore, any bankruptcy solution φ consistent with a value for TU games ψ, must satisfy that $\psi(N,V_B) = \varphi(E,c) = \varphi(E,c') = \psi(N,V_{B'})$.

Note that this example is useful to show that some solutions for bankruptcy problems are not consistent with any value for TU games. For instance, the Proportional Solution, applied to the two problems above, gives the following results

$$\varphi^p(E,c) = (3,6,9) \neq (2,4,12) = \varphi^p\left(E,c'\right).$$

Definition 2.5. *Let φ be a solution for bankruptcy problems. We will say that φ is a value for bankruptcy problems, or a bankruptcy value, if there is a value for TU games ψ, such that for any bankruptcy problem $B = (E,c)$*

$$\varphi(E,c) = \psi(N,V_B).$$

The fact pointed out by Example 2.4 above allows us to state the following remark[2], whose straightforward proof is omitted.

Remark 1. *Let φ be a solution for bankruptcy problems. φ is a value for bankruptcy problems if and only if for any bankruptcy problem (E,c) we have that*

$$\varphi(E,c) = \varphi(E,\tilde{c})$$

where the claims vector $\tilde{c}$ is such that for each agent i, $\tilde{c}_i = \min\{c_i, E\}$.

Curiel et al. [5] showed that any bankruptcy value will recommend a distribution of the estate belonging to the Core of the corresponding TU game. In fact, from the positive point of view, the Core is one of the most important solution concepts for cooperative TU games whose formal definition is due to Gillies [8] and Shapley [19]. The intuitive idea of a Core imputation is that no set of agents could collectively improve their share by their own cooperation. Formally,

Definition 2.6. *Let (N,V) be a TU game. The Core of (N,V), denoted by $\mathbb{C}(N,V)$, is the set of imputations that cannot be objected by any coalition:*

$$\mathbb{C}(N,V) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = V(N), \sum_{i\in S} x_i \geq V(S) \;\; \forall S \subset N\}.$$

[2]The condition established in this remark is commonly known as *Invariance under Claims Truncation.*

3. The Ibn Ezra's proposal

Rabbi Abraham Ibn Ezra introduces an example with four agents. The problem he proposes can be described as a bankruptcy problem where E is 120 and agents' claims are 30, 40, 60 and 120. The agents are respectively Judah, Levi, Simeon and Reuben. Ibn Ezra's recommendation[3] is the following:

> In accordance with the view of the Jewish sages, the three older brothers say to Judah, "Your claim is only 30 $\left(\frac{1}{4}\right)$, but all of us have an equal claim on them. Therefore, take $7\frac{1}{2}$, which is one quarter and depart". Each one of the brothers takes a similar amount. Then Reuben says to Levi, "Your claim is only 40 $\left(\frac{1}{3}\right)$. You have already received your share of 30 which all four of us claimed; therefore take a $\frac{1}{3}$ of the (remaining) 10 and go". Thus Levi's is $10\frac{5}{6}$ (that is, $30 \times \frac{1}{4}$ plus $10 \times \frac{1}{3}$) ... Reuben also says to Simeon, "Your claim is for only half of the estate which is 60, while the remaining half is mine. Now you have already received your share of the 40, so that the amount at issue between us is 20 -take half of that and depart". Thus Simeon's share is $20\frac{5}{6}$ (i.e., $30 \times \frac{1}{4}$ plus $10 \times \frac{1}{3}$ plus $20 \times \frac{1}{2}$) and Reuben's share is $80\frac{5}{6}$ (i.e., $30 \times \frac{1}{4}$ plus $10 \times \frac{1}{3}$ plus $20 \times \frac{1}{2}$ plus 60×1).

The Ibn Ezra's recommendation can be understood as follows: Let us consider that from de total amount to share $[0, E]$, each agent i demands the specific part of the state $[0, c_i]$; once claims are arranged on specific units of the estate in this way, Ibn Ezra recommends for each unit equal division among all agents demanding it. Let us observe that the Ibn Ezra's recommendation can be easily extended to any bankruptcy problem (E, c) satisfying that the highest claim is the total amount to be shared. Let us denote by $\mathcal{B}_{IE}$ this family of bankruptcy problems,

$$\mathcal{B}_{IE} = \left\{ (E, c) \in \mathbb{R}_{++} \times \mathbb{R}_{+}^{n} : E = \max_{i \in N} c_i \right\}$$

For notational convenience, let us assume that agents' claims are increasingly ordered, i.e.

$$c_i \leq c_j \text{ whenever } i < j.$$

Under these considerations, we can define the Ibn Ezra's solution on $\mathcal{B}_{IE}$ as the function φ^{IE} that associates to each agent bankruptcy problem (E, c) on $\mathcal{B}_{IE}$ and agent i the amount

$$\varphi_i^{IE}(E, c) = \sum_{k=1}^{i} \frac{c_k - c_{k-1}}{n - k + 1}$$

[3] This quotation is borrowed from O'Neill [16] whom attributes it to Rabinovitch [18].

with $c_0 = 0$.[4]

A *natural* extension of Ibn Ezra's recommendation to the family of bankruptcy problems

$$\mathcal{B}_{EIE} = \left\{ (E, c) \in \mathcal{B} : E \leq \max_{i \in N} c_i \right\},$$

in order to get a value for bankruptcy problems, gives the expression

$$\varphi_i^{IE}(E, c) = \sum_{k=1}^{i} \frac{\min\{c_k, E\} - \min\{c_{k-1}, E\}}{n - k + 1} \tag{3.1}$$

as the share for agent i corresponding to the bankruptcy problem (E, c) in $\mathcal{B}_{EIE}$, when agents' claims are increasingly ordered.

4. The Minimal Overlap Value

This section introduces a formal definition of the *Minimal Overlap value.* This bankruptcy value chooses awards vectors that minimize "extent of conflict" over each unit available. The rationale used by O'Neill [16] to propose this procedure is the following: First, arrange claims on specific parts of the state in such a way that starting from the highest claim, and in decreasing order, there is minimal overlap between them; then, for each unit, apply equal division among all agents demanding it. This value was also analyzed by Chun and Thomson [4] who studied some of its characteristics and provided a precise formula to compute it.

These authors propose to compute the Minimal Overlap value by associating to each bankruptcy problem $(E, c) \in \mathcal{B}$ equal division among all creditors claiming a specific part of the estate, where the arrangement of claims is the following:

(a) If there is some creditor j such that $c_j \geq E$. Then, each creditor $i \in N$ such that $c_i \geq E$ claims $[0, E]$ and each other creditor h claims $[0, c_h]$.

(b) If $E > c_i$ for each creditor i, then, there is a unique $t \in [0, E]$ such that:

 (1) Each creditor $i \in N$ such that $c_i \geq t$ claims $[0, t]$ as well as a part of $[t, E]$ of size $c_i - t$, with no overlap between these claims; and
 (2) Each creditor h such that $t > c_h$ claims $[0, c_h]$.

Before stating a formal definition of the Minimal Overlap value, and just to simplify the exposition, from now on we will concentrate on the family of bankruptcy problems whose claims are increasingly ordered

$$\mathcal{B}_O = \{(E, c) \in \mathcal{B} : c_i \leq c_j \text{ whenever } i < j\}.$$

Note that there is no loss of generality in our analysis by assuming that bankruptcy problems belong to $\mathcal{B}_O$, rather than being in $\mathcal{B}$. (See Remark 2 below.)

[4] From now on, and for notational convenience, we will consider $c_0 = 0$.

Definition 4.1. *The Minimal Overlap value is the function* $\varphi^{mo} : \mathcal{B}_O \to \mathbb{R}^n_+$ *which associates, to any bankruptcy problem* (E, c) *and creditor* i*, the share of the estate*

$$\varphi_i^{mo}(E, c) = \sum_{j=1}^{i} \frac{\min\{c_j, t\} - \min\{c_{j-1}, t\}}{n - j + 1} + \max\{c_i - t, 0\},$$

where t *is such that*

(*a*) $t = E$ *if* $E < c_n$*, and*

(*b*) t *is the unique solution for the equation* $\sum_{k=1}^{n} \max\{c_k - t, 0\} = E - t$*, if* $E \geq c_n$.

Remark 2. *Note that, for any bankruptcy problem* (E, c) *in* $\mathcal{B} \setminus \mathcal{B}_O$ *there is a permutation*[5] π *such that* $(E, \pi(c))$ *is in* $\mathcal{B}_O$*. Hence we can compute*

$$\varphi^{mo}(E, c) = \pi^{-1}[\varphi^{mo}(E, \pi(c))]$$

The next example illustrates how to compute the *Minimal Overlap Value*.

Example 4.2. *Let consider the next three-agents bankruptcy problem where* $E = 40$*, and* $c = (18, 22, 24)$*. Notice that for this problem* $t = 12$*, since each unit of* $[t, E] = [12, 40]$ *will be claimed by only one creditor, that is*

$$(18 - 12) + (22 - 12) + (24 - 12) = 28 = 40 - 12.$$

Therefore, one of the arrangements of claims on specific part of the state according to the proposal by Chun and Thomson [4] is the following:

(1) *Creditor 1 claims* $[0, 12]$ *and* $[12, 18]$,

(2) *Creditor 2 claims*$[0, 12]$ *and* $[18, 28]$*, finally*

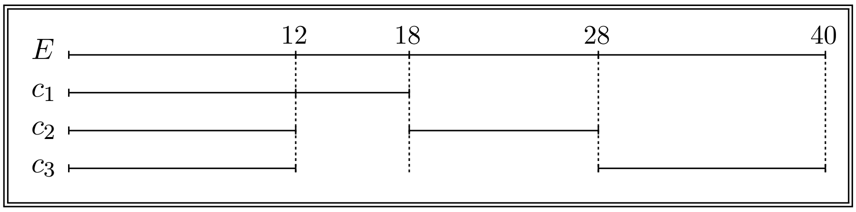
(3) *Creditor 3 claims* $[0, 12]$ *and* $[28, 40]$,

which, by applying equal division over each unit, yields the following recommendation:

$$\begin{aligned} \varphi_1^{mo}(E, c) = &\quad \tfrac{12}{3} + 6 = \quad 10 \\ \varphi_2^{mo}(E, c) = &\quad \tfrac{12}{3} + 10 = \quad 14 \\ \varphi_3^{mo}(E, c) = &\quad \tfrac{12}{3} + 12 = \quad 16 \end{aligned}$$

[5] Given a set of agents N, we say that $\pi : N \to N$ is a permutation on N if π is bijective. Throughout the rest of the paper, and abusing notation, $\pi(c)$ will denote the claims vector obtained by applying permutation π to its components, i.e. i-th component for $\pi(c)$ is c_j whenever $j = \pi(i)$.

Graphically, the arrangement of the claims for this bankruptcy problem is



5. Transition Games and the Minimal Overlap Value

As it is well known the Core is, in essence, a very solid solution concept, although in general neither its existence nor its uniqueness can be guaranteed. As we mentioned in Section 2.3, Curiel et al. [5] showed that the Core of bankruptcy games is non-empty but extremely large since any bankruptcy solution, according to Definition 2.2, belongs to the Core of the associated bankruptcy game.

Following the game-theoretical interpretation of bankruptcy problems given by O'Neill [16], this section introduces the concept of *Transition Game*, a new TU game associated to, ceteris paribus, increments of the estate in bankruptcy situations. In contrast with the previous assertion, we show that there is a unique anonymous bankruptcy value proposing estate distributions in the Core of the transition game: the Minimal Overlap value.

Just to explain the idea beyond the transition game, let us consider an increment of the estate and the associated bankruptcy games. Let $B = (E, c)$ and $B' = (E', c)$, whose only difference is the estate to be shared among the agents. Let us suppose that $E' > E$. Note that for any coalition S in N,

$$V_{B'}(S) \geq V_B(S),$$

and it must be the case that the above inequality becomes strict for some coalition S, in particular for the grand coalition N.

Therefore, when considering such a situation, another TU game arises, assigning to each coalition S in N the additional worth that agents in such coalition can guarantee to themselves when the estate increases, we call this game the transition game.

Definition 5.1. *Given two bankruptcy problems in $\mathcal{B}$, $B = (E, c)$ and $B' = (E', c)$ with $E' > E$, the transition game $\left(N, W_{(B',B)}\right)$ relative to this estate increment, $E' - E$, is the associated TU game where the function $W_{(B',B)}$ is defined by:*

$$W_{(B',B)}(S) = V_{B'}(S) - V_B(S) \quad \textit{For all } S \subseteq N.$$

To characterize the Minimal Overlap value we just need to employ two axioms, namely *Anonymity* and *Core-Transition Responsiveness.*

Anonymity imposes each agent's reward to depend on the entire structure of the problem rather than her identity.

Axiom 1. *Anonymity*

Let φ be a value for bankruptcy games. We say that φ satisfies Anonymity if for each bankruptcy problem $B = (E, c)$, and any permutation π,

$$\pi\left[\varphi\left(E,c\right)\right] = \varphi\left(E,\pi\left(c\right)\right).$$

Core-Transition Responsiveness is established in terms of increments of the estate. It demands to any bankruptcy value that the distribution of such an increment could not be improved by any coalition, related to what the transition game allows them.

Axiom 2. *Core-Transition Responsiveness*

We say that a bankruptcy value φ satisfies the Core-Transition Responsiveness *property if for any two bankruptcy problems, $B = (E, c)$ and $B' = (E', c)$ with $E' > E$,*

$$\left[\varphi\left(E',c\right) - \varphi\left(E,c\right)\right] \in \mathbb{C}\left(N, W_{(B',B)}\right) \tag{5.1}$$

whenever

$$\mathbb{C}(N, W_{(B',B)}) \neq \emptyset. \tag{5.2}$$

The following examples illustrate the restrictions that Axiom 2 imposes on bankruptcy values.

Example 5.2. *Let $N = \{1,2,3\}$, $c = (2,5,7)$, $E = 4$, and $E' = 6$. Let us consider the bankruptcy problems $B = (E,c)$, and $B' = (E',c)$. The TU games associated to each bankruptcy problem and the transition game are described by the functions V_B, $V_{B'}$ and $W_{(B',B)}$ respectively, where*

S	$V_B(S)$	$V_{B'}(S)$	$W_{(B',B)}(S)$
$\{i\}$	0	0	0
$\{1,2\}$	0	0	0
$\{1,3\}$	0	1	1
$\{2,3\}$	2	4	2
$\{1,2,3\}$	4	6	2

Therefore Axiom 2 establishes that

$$\begin{aligned}
\varphi_i\left(E',c\right) - \varphi_i\left(E,c\right) &\geq 0 \text{ for all } i \text{ in} N,\\
\sum_{i=1,3}\left[\varphi_i\left(E',c\right) - \varphi_i\left(E,c\right)\right] &\geq 1,\\
\sum_{i=2,3}\left[\varphi_i\left(E',c\right) - \varphi_i\left(E,c\right)\right] &\geq 2, \text{ and}\\
\sum_{i\in N}\left[\varphi_i\left(E',c\right) - \varphi_i\left(E,c\right)\right] &= 2.
\end{aligned}$$

The above inequalities imply

$$\begin{array}{rcl} \varphi_1\left(E',c\right) & = & \varphi_1\left(E,c\right), \\ \varphi_3\left(E',c\right) & \geq & 1+\varphi_3\left(E,c\right), \text{ and} \\ 0 & \leq & \varphi_2\left(E',c\right) = 2-\left[\varphi_3(E^{'},c)-\varphi_3\left(E,c\right)\right]+\varphi_2\left(E,c\right). \end{array}$$

Let us observe that agent 1 *will not participate in the share of the extra estate, and agent* 3 *will not receive less than one half of such an increment.*

The following example points out the need of imposing Condition (5.2) in Axiom 2.

Example 5.3. *Let* $N=\{1,2,3\}$, $c=(2,5,7)$, $E=6$, *and* $E'=9$. *Let us consider the bankruptcy problems* $B=(E,c)$, *and* $B'=(E',c)$. *The TU games associated to each bankruptcy problem and the transition game are described by the functions* V_B, $V_{B'}$ *and* $W_{(B',B)}$ *respectively, where*

S	$V_B(S)$	$V_{B'}(S)$	$W_{(B',B)}(S)$
$\{1\}$	0	0	0
$\{2\}$	0	0	0
$\{3\}$	0	2	2
$\{1,2\}$	0	2	2
$\{1,3\}$	1	4	3
$\{2,3\}$	4	7	3
$\{1,2,3\}$	6	9	3

Note that, in this case, the restrictions $x_3\geq 2$, $x_1+x_2\geq 2$ *and* $x_1+x_2+x_3=3$ *are incompatible. Therefore, the Core of the transition game associated to the increment of the estate* $E^{'}-E$ *is empty, and Axiom 2 does not imposes any restriction on how to share the* 3 *units of extra estate.*

We now present our main result. It establishes that the Minimal Overlap value is fully characterized by the two axioms above.

Theorem 5.4. *Let* φ *be a value for bankruptcy games.* φ *satisfies Anonymity and Core-Transition Responsiveness if, and only if,* $\varphi\equiv\varphi^{mo}$.

Proof. See Appendix 1. ∎

6. The Process Behind the Minimal Overlap Value

In this section we provide a negative result relative to the interpretation of what the minimal overlap solution proposes. In particular, we show that the Minimal Overlap value can be described as a mixture of two different principles of equity, which is, a priori, uncorfomable. Our next result explains how such a mixture is done.

Proposition 6.1. *Let $(E, c) \in \mathcal{B}$ be a bankruptcy problem. Then, the Minimal Overlap value gives to agent i the amount*

$$\varphi_i^{mo}(E, c) = \varphi_i^{IE}\left(E', c\right) + \varphi_i^{cel}\left(E - E', c - \varphi^{IE}\left(E', c\right)\right) \tag{6.1}$$

where $E' = \min\{E, c_n\}$, and φ^{cel} is the Constrained Equal Loss Rule, i.e., for any bankruptcy problem (E, c) and agent i

$$\varphi_i^{cel}(E, c) = \max\{0, c_i - \lambda\}$$

with λ being the solution to

$$\sum_{i=1}^{n} \max\{0, c_i - \lambda\} = E.$$

Proof. See Appendix 2. ∎

Therefore, as the previous result points out, the procedure described by the Minimal Overlap value has a very strong shift on its philosophy: Up to a certain amount of estate we should follow the recommendations by Ibn Ezra and, after it, we should divide the extra estate trying to equalize agents' loses. From our point of view this is a negative finding since the composition of two distinct well-defined rules can be understood as a broken in the underlying general distribution principle in which a rule is inspired. As we show in the next section, our proposal on generalizing the recommendation by Ibn Ezra does not consider any change on the interpretation of how the estate should be shared depending on the level of estate.

7. The Generalized Ibn Ezra Value

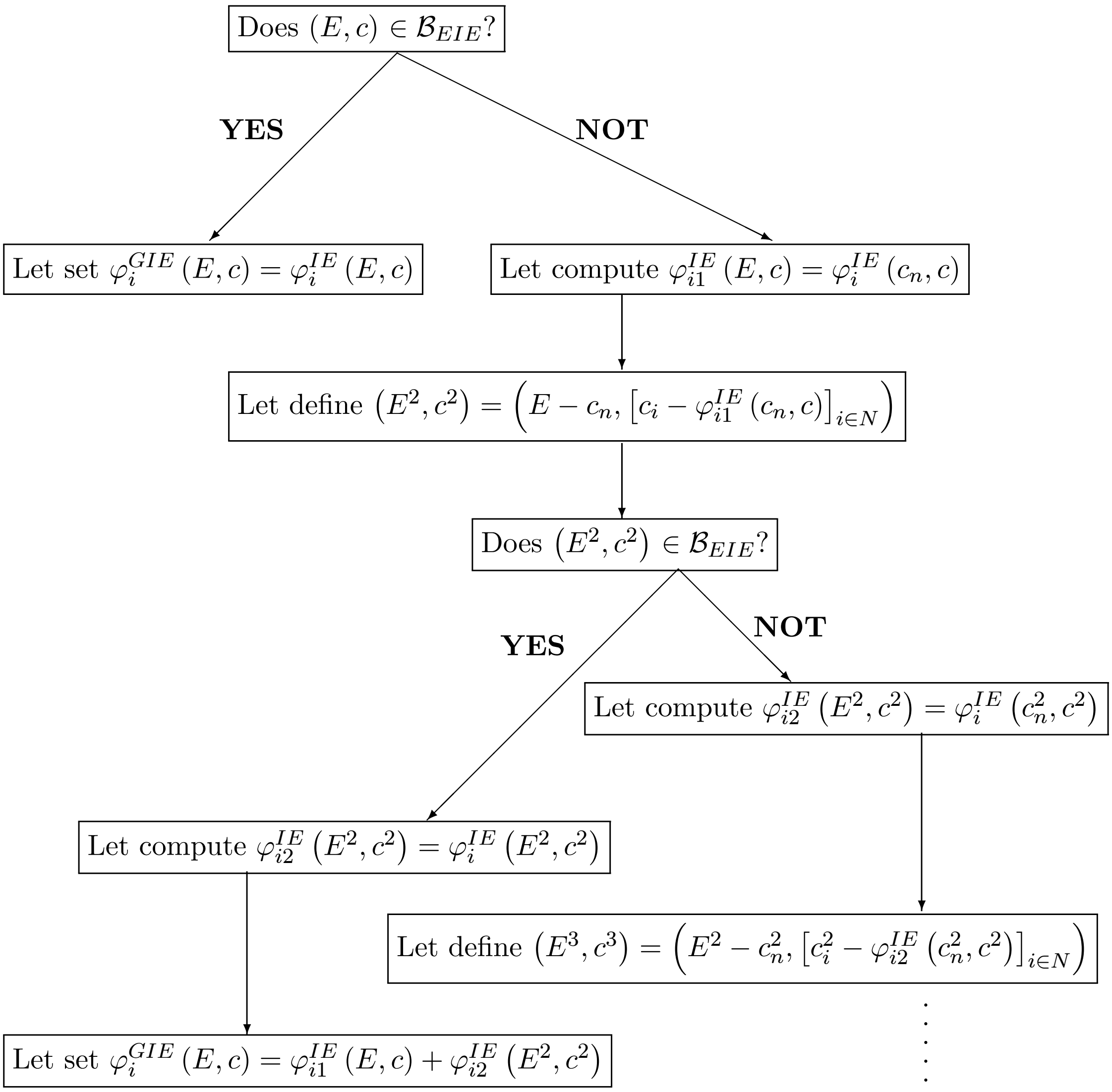
Throughout this section we made precise of our proposal to extend the arguments provided in the examples by Ibn Ezra. In particular, we impose that the general principle in which the recommendations by this author are inspired should remain fixed.

Just to explain our idea, let us consider a bankruptcy problem (E, c). For expositional convenience, let us assume that creditors' claims are increasingly ordered, and E is greater than the highest creditors' claim, say c_n. Then, our proposal is, in a first stage, to share c_n among all the creditors according to the proposal by Ibn Ezra. After this first stage, we can describe a second bankruptcy problem (E', c') with $E' = E - c_n$, and for each creditor i, $c_i' = c_i - \varphi_i^{IE}(c_n, c)$. When analyzing (E', c'), we have two possibilities. The first one is that $(E', c') \in \mathcal{B}_{EIE}$. In such a case, we apply the Formula (3.1) to (E', c') and add this value to the previous one:

$$\varphi_i^{GIE}(E, c) = \varphi_i^{IE}(c_n, c) + \varphi_i^{IE}\left(E', c'\right).$$

On the other hand, if $(E', c') \notin \mathcal{B}_{EIE}$, we compute the Ibn Ezra value of (c'_n, c'), and analyze the *residual* bankruptcy problem (E'', c''), where $E'' = E' - c'_n$, and $c'' = c' - \varphi_i^{IE}(c'_n, c')$. If we extend this procedure ad infinitum, we get the expression of what we call the Generalized Ibn Ezra Value.

The following diagram shows how the Generalized Ibn Ezra Value is computed.



The above diagram is synthesized by Definition 7.1. Before stating a formal definition of our extension of Ibn Ezra's value, and just to simplify the exposition, from now on we will concentrate on the family of bankruptcy problems $\mathcal{B}_O$ whose claims are increasingly ordered. Note that there is no loss of generality in our analysis by assuming that bankruptcy problems belong to $\mathcal{B}_O$, rather than being in $\mathcal{B}$. (See Remark 2 in Section 4.)

Definition 7.1. *The Generalized Ibn Ezra Value*

The Generalized Ibn Ezra Value is the function $\varphi^{GIE} : \mathcal{B}_0 \to \mathbb{R}^n_+$ *which associates to each bankruptcy problem* (E, c) *the vector* $\varphi^{GIE}(E, c)$ *whose* i*-th component is*

$$\varphi_i^{GIE}(E, c) = \sum_{t=1}^{\infty} \varphi_i^{PIE}\left(E^t, c^t\right) \tag{7.1}$$

where the above elements are described as follows:

$$\left(E^1, c^1\right) = (E, c),$$

for each $t > 1$

$$E^t = E^{t-1} - \min\left\{c_n^{t-1}, E^{t-1}\right\} = \max\left\{E^{t-1} - c_n^{t-1}, 0\right\}$$

and

$$c_i^t = c_i^{t-1} - \varphi_i^{PIE}\left(E^{t-1}, c^{t-1}\right),$$

and finally the function φ_i^{PIE} *is described by:*

$$\varphi_i^{PIE}\left(E^t, c^t\right) = \sum_{k=1}^{i} \frac{\min\left\{c_k^t, E^t\right\} - \min\left\{c_{k-1}^t, E^t\right\}}{n-k+1}, \text{ with } c_0^t = 0.$$

The next example shows how to compute the Generalized Ibn Ezra value.

Example 7.2. *Let consider the next three-agents bankruptcy problem where* $E = 41$*, and* $c = (18, 22, 24)$*; the computations yielding the Generalized Ibn Ezra Value for such a case are*

t	E^t	c_1^t	c_2^t	c_3^t	φ_1^{PIE}	φ_2^{PIE}	φ_3^{PIE}
$t=1$	41	18	22	24	$6 = \frac{18}{3}$	$8 = 6 + \frac{22-18}{2}$	$10 = 8 + \frac{24-22}{1}$
$t=2$	17	12	14	14	$4 = \frac{12}{3}$	$5 = 4 + \frac{14-12}{2}$	$5 = 5 + \frac{14-14}{1}$
$t=3$	3	8	9	9	1	1	1
$t=4$	0	7	8	8	0	0	0

which yield the following recommendation

$$\varphi_1^{GIE}(E, c) = 11 \qquad \varphi_2^{GIE}(E, c) = 14 \qquad \varphi_3^{GIE}(E, c) = 16.$$

Relative to the above computations, let us observe that

(a) *given that at $t = 3$, each creditor's claim exceeds the estate, it should be divided equally among the creditors, and*

(b) *since for $t \geq 4$ the estate is zero, the expression $\varphi_i^{PIE}\left(E^t, c^t\right)$ should be zero for each agent i and any $t \geq 4$. Therefore, in this example, the process described in expression (7.1) stops at $t = 3$.*

A first question that we should propose, relative to the expression (7.1), concerns the convergence of

$$\sum_{t=1}^{\infty} \varphi_i^{PIE}\left(E^t, c^t\right).$$

Our next result states that this sum is not only convergent, but it is reached for t finite.

Proposition 7.3. *Let (E, c) be a bankruptcy problem in $\mathcal{B}_O$. Then, there is a positive integer $\tilde{t} < \infty$ such that*

$$\sum_{t=1}^{\infty} \varphi_i^{PIE}\left(E^t, c^t\right) = \sum_{t=1}^{\tilde{t}} \varphi_i^{PIE}\left(E^t, c^t\right).$$

Proof. See Appendix 3. ∎

8. Characterizing the Generalized Ibn Ezra Value

In this section we characterize the Generalized Ibn Ezra value. To introduce the axioms used in our characterization, let us consider two bankruptcy problems $B = (E, c)$ and $B' = (E', c)$ with $E' > E$ and the associated transition game $\left(N, W_{(B',B)}\right)$. Observe that the worth of any coalition in the transition game is non-negative, but the relative position of some coalitions might present different configurations. We are going to consider the following three types of agents which arise from the analysis of the transition game $\left(N, W_{(B',B)}\right)$:

(i) *Transitional dummies*,

(ii) *Transitional worth-generators*, and

(iii) *Transitional pivotal agents.*

By transitional dummy we refer to any agent being dummy, in a weak sense, relative to the transitional game $\left(N, W_{(B',B)}\right)$. That is, an agent whose marginal contribution to the grand coalition is null. Formally,

Definition 8.1. *Given a transitional game* $\left(N, W_{(B',B)}\right)$*, we say that agent* i *is a transitional dummy, if*

$$W_{(B',B)}\left(N\backslash\{i\}\right) = W_{(B',B)}\left(N\right).$$

Let $D\left(N, W_{(B',B)}\right)$ *denote the set of transitional dummies for* $\left(N, W_{(B',B)}\right)$*.*

On the other hand, we say that agent i is transitional worth-generator relative to $\left(N, W_{(B',B)}\right)$, if he is indispensable for any coalition to generate a positive worth in such a transitional game. Formally,

Definition 8.2. *Given a transitional game* $\left(N, W_{(B',B)}\right)$*, we say that* $i \in N$ *is transitional worth-generator if for each coalition* $S \subseteq N$*,*

$$W_{(B',B)}\left(S\right) > 0 \Rightarrow i \in S.$$

Let $G\left(N, W_{(B',B)}\right)$ *denote the set of transitional worth-generators for* $\left(N, W_{(B',B)}\right)$*.*

We want to mention that the idea of transitional worth-generators is closely related to the notion of "clan", introduced by Potters et al. [17], but applied to changes on the amount to be shared rather than the static consideration made by these authors.

The third of the above mentioned categories includes all the agents being not transitional dummy neither transitional worth-generators. Let us observe that the transitional pivotal agents are not *indispensable* to reach a positive worth in the transitional game, but it is to reach the maximum worth that a coalition could generate. Formally,

Definition 8.3. *Given a transitional game* $\left(N, W_{(B',B)}\right)$*, we say that agent* i *is transitional pivot if*

$$W_{(B',B)}\left(N\backslash\{i\}\right) \neq W_{(B',B)}\left(N\right) \text{ and } \exists S \subseteq N, i \notin S, \text{ such that } W_{(B',B)}\left(S\right) > 0.$$

Let $P\left(N, W_{(B',B)}\right)$ *denote the set of transitional pivots for* $\left(N, W_{(B',B)}\right)$*.*

The next examples illustrate the partition of the set of agents above mentioned.

Example 8.4. *Let* $N = \{1,2,3\}$*,* $c = (20, 50, 70)$*,* $E = 40$*, and* $E' = 60$*. Let us consider the bankruptcy problems* $B = (E, c)$*, and* $B' = (E', c)$*. The TU games associated to each bankruptcy problem and the transition game are described by the functions* V_B*,* $V_{B'}$ *and* $W_{(B',B)}$ *respectively, where*

S	$V_B(S)$	$V_{B'}(S)$	$W_{(B',B)}(S)$
$\{i\}$	0	0	0
$\{1,2\}$	0	0	0
$\{1,3\}$	0	10	10
$\{2,3\}$	20	40	20
$\{1,2,3\}$	40	60	20

In this case $\{1,3\}$, $\{2,3\}$ *and* $\{1,2,3\}$ *are the coalitions that improve when the estate increases. Their unique common element is agent* 3*, therefore the transition worth-generator coalition will consist of agent* 3, $G\left(N, W_{(B',B)}\right) = \{3\}$. *Note that* $W_{(B',B)}\left(\{1,2,3\}\right) = W_{(B',B)}\left(\{2,3\}\right) \neq W_{(B',B)}\left(\{1,j\}\right)$ *for any* $j = 2,3$. *Hence* $D\left(N, W_{(B',B)}\right) = \{1\}$. *Finally,* $P\left(N, W_{(B',B)}\right) = \{2\}$.

Taking into account the previous definitions it is clear that the transition worth-generator coalition, if any, occupies a predominant position. However, this fact does not imply, in general, that such a coalition has a dictatorial power for sharing the increment of the estate. In the above example, the amount available from B to B' is 20 units and $G\left(N, W_{(B',B)}\right) = \{3\}$. The positions of agents 1 and 2 relative to the transitional worth-generator coalition are different. If agents 2 and 3 cooperate, they can get the total extra estate by themselves. This is because agent 1 belongs to $D\left(N, W_{(B',B)}\right)$. Whereas there is no coalition not including agent 2 that could get such worth. This implies that agent 2 must be taken into consideration to share the 20 extra units.

What this analysis suggests is that, if the set of transitional worth-generator agents is non-empty, these agents will exclude the transitional dummies from the share of the extra estate. The aim of the next axiom is the formalization of this idea.

Axiom 3. ***Transitional Dummy***

We say that a bankruptcy value φ *satisfies Transitional Dummy if for any two bankruptcy problems,* $B = (E, c)$ *and* $B' = (E', c)$ *with* $E' > E$ *such that* $G\left(N, W_{(B',B)}\right) \neq \emptyset$,

$$\varphi_i\left(E', c\right) - \varphi_i\left(E, c\right) = 0.$$

for each $i \in D\left(N, W_{(B',B)}\right)$.

The next example points out the need of having transitional worth-generator agents to apply our Transitional Dummy axiom.

Example 8.5. *Let* $N = \{1,2,3\}$, $c = (20, 50, 70)$, $E = 60$, *and* $E' = 90$. *Let us consider the bankruptcy problems* $B = (E, c)$, *and* $B' = (E', c)$. *The TU games associated to both bankruptcy problems, and the transition game, are described by the functions* V_B, $V_{B'}$ *and* $W_{(B',B)}$ *respectively, where*

S	$V_B(S)$	$V_{B'}(S)$	$W_{(B',B)}(S)$
$\{1\}$	0	0	0
$\{2\}$	0	0	0
$\{3\}$	0	20	20
$\{1,2\}$	0	20	20
$\{1,3\}$	10	40	30
$\{2,3\}$	40	70	30
$\{1,2,3\}$	60	90	30

In this case $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{1,2,3\}$ are the coalitions that improve when the estate increases, but they have no common element; therefore the transitional worth-generator coalition will be the empty set, $G\left(N, W_{(B',B)}\right) = \emptyset$. Let us observe that $D\left(N, W_{(B',B)}\right) = \{1,2\}$. Transitional Dummy would imply that agents 1 and 2 would not receive any from the extra estate. However, and given that agent 3 is not a transitional worth generator, it is the case that agents 1 and 2 generate worth by themselves. Our next axiom, called *Worth-Generators Composition,* takes into account this fact. In particular, this axiom concerns variations in the estate, from $B = (E, c)$ to $B' = (E'c)$ with $E < E'$, when the transitional worth-generator coalition is empty. Let us suppose that, in such a case, there is a unique intermediate value E^*, $E < E^* < E'$ such that , for any $E^1 < E^*$,

$$G\left(N, W_{(B',B)}\right) \neq \emptyset,$$

with

$$W_{(B^*,B^1)}(S) = \max\left\{0, E^* - \sum_{i\in N\setminus S} c_i\right\} - \max\left\{0, E^1 - \sum_{i\in N\setminus S} c_i\right\},$$

and for any $E^2 > E^*$,

$$G\left(N, W_{(B^2,B^*)}\right) = \emptyset,$$

with

$$W_{(B^2,B^*)}(S) = \max\left\{0, E^2 - \sum_{i\in N\setminus S} c_i\right\} - \max\left\{0, E^* - \sum_{i\in N\setminus S} c_i\right\},$$

being $B^* = (E^*, c)$, $B^1 = \left(E^1, c\right)$ and $B^2 = \left(E^2, c\right)$.

We propose a particular form of composition, relative to the estate E^*. Since E^* is the maximum value of the estate such that, for levels E^1 smaller than it, the transitional worth-generator coalition associated to the transition from E^1 to E^* is non-empty, we refer this property as Worth-Generators Composition.

Just to avoid ambiguities on the level of estate E^* previously described, let us state the next result, whose straightforward proof is omitted.

Proposition 8.6. *Let $B = (E, c)$ and $B' = \left(E', c\right)$ two bankruptcy problems in $\mathcal{B}$ with $E < E'$ such that $G\left(N, W_{(B',B)}\right) = \emptyset$. If there exists some intermediate value of the estate, E^*, $E < E^* < E'$, such that for any $E^1 < E^*$, $G\left(N, W_{(B^*,B^1)}\right) \neq \emptyset$, and for any $E^2 > E^*$, $G\left(N, W_{(B^2,B^*)}\right) = \emptyset$. Then, E^* is unique.*

Axiom 4. ***Worth-Generators Composition***

We say that a bankruptcy value φ satisfies Worth-Generators Composition if for any two bankruptcy problems in $\mathcal{B}$, $B = (E, c)$ and $B' = \left(E', c\right)$ with $E < E'$ such that $G\left(N, W_{(B',B)}\right) = \emptyset$ and, for any E^*, $E < E^* < E'$, with $G\left(N, W_{(B^*,B^1)}\right) \neq \emptyset$, for any $E^1 < E^*$ and $G\left(N, W_{(B^2,B^*)}\right) = \emptyset$ for any $E^2 > E^*$, then

$$\varphi\left(E', c\right) = \varphi\left(E^*, c\right) + \varphi\left(E' - E^*, c - \varphi\left(E^*, c\right)\right).$$

We now present our main result relative to the Generalized Ibn Ezra solution. It establishes that this value is fully characterized by anonymity and the two axioms above.

Theorem 8.7. *The Generalized Ibn Ezra value is the unique bankruptcy value satisfying Anonymity, Transitional Dummy and Worth-Generators Composition.*

Proof. See Appendix 4. ∎

Remark 3. *It is easy to see that the axioms used in our characterization are independent. Just to check it, let us note that*

(*i*) The Minimal Overlap value satisfies Anonymity and Transitional Dummy but does not fullfil Worth-Generators Composition.

(*ii*) Anonymity and Worth-Generators Composition are fulfilled by the Constrained Equal Awards value. Nevertheless, this value does not satisfy Transitional Dummy.

(*iii*) The next bankruptcy value $\tilde{\varphi}$ satisfies Transitional Dummy and Worth-Generators Composition, but does not Anonymity: Given a bankruptcy problem $(E, c) \in \mathcal{B}$, let define the permutation π, described in an inductive way by

$$\pi(n) = \max\{i : \tilde{c}_i \geq \tilde{c}_j \text{ for all } j \in N\},$$

where for each agent i, $\tilde{c}_i = \min\{E, c_i\}$. And, for $1 \leq k \leq n-1$

$$\pi(n-k) = \max\left\{i : \tilde{c}_i \geq \tilde{c}_j \text{ for all } j \in N \setminus \cup_{t=0}^{k-1}\{\pi(n-t)\}\right\}$$

Let us consider the solution $\tilde{\varphi} : \mathcal{B} \to \mathbb{R}^n_+$, defined as follows. For a bankruptcy problem $(E, c) \in \mathcal{B}$, let

$$\tilde{\varphi}_{\pi(n)}(E, c) = \min\left\{E, c_{\pi(n)}\right\},$$

and for each $1 \leq k \leq n-1$

$$\tilde{\varphi}_{\pi(n-k)}(E, c) = \min\left\{E - \sum_{j=n-k+1}^{n} \tilde{\varphi}_{\pi(j)}(E, c), c_{\pi(n-k)}\right\}.$$

9. Conclusions

This paper explored an old recommendation for bankruptcy problems proposed by Ibn Ezra. The instances used by this author satisfy that the creditor whose claim is the highest asks for the total estate of the bankrupted. The question that this paper analyzes is how to extend the arguments provided by Ibn Ezra to the general case in which creditors' claims are not restricted to be not lower than the estate.

As far as we know, the proposal by Ibn Ezra has been partially taken into account by Bergantiños and Méndez-Naya [2]. These authors proposed a characterization of the Ibn Ezra value on the basis of an Additivity Axiom, on the full problem (E, c), but restricted to the domain $\mathcal{B}_{IE}$, in which the estate coincides with the highest claim. On the other hand, the Minimal Overlap value was proposed by O'Neill [16] on the basis of minimizing the "extent of conflict" over each available unit. It generalizes the Ibn Ezra's proposal but there is no exhaustive analysis of it, except a precise formula to its computation, introduced recently by Chun and Thomson [4].

Following the (cooperative) game-theoretical interpretation of bankruptcy problems, this paper introduces the concept of *Transition Game*, a TU game associated to increments of the estate, and provides a new and appealing rationale on which is based the Minimal Overlap value: It is the unique anonymous bankruptcy value proposing estate distributions in the Core of the Transition Game.

In spite of the previous arguments on the Minimal Overlap value, and surprisingly enough, this paper describes the procedure behind the Minimal Overlap value as a mixture of two different principles of equity: Up to a certain estate we should follow the recommendations by Ibn Ezra and, after it, we should divide the extra estate trying to equalize agents' loses.

Given these antecedents, this paper also explores the possibility of designing a bankruptcy value to extend the example by Ibn Ezra and avoiding changes on the interpretation of how the estate should be shared depending on its magnitude. Our proposal, called the Generalized Ibn Ezra value, can be seen as an iterative procedure in which, at each stage, a part of the estate is shared among the creditors. This part is taken in such a way that the Ibn Ezra example is always replicated: we should divide among the creditors the highest claim whenever it is possible.

Moreover the paper proposes a characterization of this extension of the Ibn Ezra example, also based on the concept of transition games. This characterization clarifies the analogies and differences among the two ways of generalizing the Ibn Ezra's proposal.

Just to conclude, let us stress such analogies and differences between the Minimal Overlap and the Generalized Ibn Ezra values.

1. For any two-creditors bankruptcy problem, both values coincide with that

of the *Contested Garment* value (Dagan [6]). Therefore, both of them can be presented as extensions of the Contested Garment principle.

None of the two rules is population consistent. This is because the unique consistent generalization of the Contested Garment principle is the Talmudic value, explored by Aumann and Maschler [1].

2. Both values satisfy a "minimal requirement" relative to transition games: agents belonging to the set of transitional dummy players are excluded from the sharing of the "extra" estate.

3. The main distinction between both extensions of the Ibn Ezra solution is the following. On one hand, the Minimal Overlap value always provides recommendations belonging to the Core of the Transition Game. And, on the other hand, the recommendation of the Generalized Ibn Ezra value respect the distributions of any part of the estate according to the restrictions imposed by the above mentioned transitional game "minimal requirement".

4. Our characterization results do not require the employ of much properties used in the literature for Bankruptcy Theory. Nevertheless, it can be straightforwardly seen that both the Generalized Ibn Ezra and the Minimal Overlap values satisfy some appealing properties. Among others we would like to mention:

   (i) Order Preservation. For any $(E, c) \in \mathcal{B}$, and each i, j in N,

   $$c_i \leq c_j \text{ implies } \varphi_i(E, c) \leq \varphi_j(E, c)$$

   (ii) Continuity, both in the estate and in the claims.

   (iii) Claims Monotonicity and Estate Monotonicity.

   (iv) Supermodularity.

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APPENDIX 1.

This appendix will provide a formal proof for Theorem 5.4. Our objective will be reach by combining the results exhibited in Lemmata 1 and 2.

For notational convenience, we will consider a fix claims vector, $c \in \mathbb{R}^n_+$. Since we are interested on rules satisfying anonymity, we do no loss generality in considering that the components of c are increasingly ordered; i.e. for any i, j

$$c_i \geq c_j \text{ whenever } i \geq j.$$

Given c, let $\mathcal{E}$ denote the set of real numbers E such that (E, c) is a bankruptcy problem:

$$\mathcal{E} = \{E \in \mathbb{R}_+ : (E, c) \in \mathcal{B}\}$$

Given $E \in \mathcal{E}$, let t^E be the unique solution to

$$\sum_{i \in N} \max\left\{c_i - t^E, 0\right\} = E - t^E \tag{9.1}$$

if $E > c_n$, and $t^E = E$ otherwise.

The following three claims, whose straightforward proof is omitted, are useful to characterize the core of a transition game.

Claim 1. *Let $E' \in \mathcal{E}$ such that $t^{E'} = E'$, then $\mathbb{C}(N, W_{(B',B)}) \neq \emptyset$ for any two bankruptcy problems $B = (E, c)$ and $B' = (E', c)$ with $0 \leq E < E'$.*

Claim 2. *Given E, $E' \in \mathcal{E}$, such that $\max\left\{E, t^{E'}\right\} < E'$, let $V_B = (N, B)$ and $V_{B'} = (N, B')$ be the cooperative games associated to (E, c) and (E', c) respectively. Then $\mathbb{C}(N, W_{(B',B)}) \neq \emptyset$ if, and only if, $t^{E'} \geq E$.*

Claim 3. *Let $B = (E, c)$ and $B' = (E', c)$, with $E < E'$, be two bankruptcy problems such that $\mathbb{C}(N, W_{(B',B)}) \neq \emptyset$. Then $x \in \mathbb{C}(N, W_{(B',B)})$ if, and only if*

(a) *$x_i = 0$, for all i such that $c_i \leq E$,*

(b) *$0 \leq x_i \leq c_i - E$, for all i such that $E < c_i \leq E'$,*

(c) *$0 \leq x_i \leq E' - E$, for all i such that $c_i \geq E'$, and*

(d) *$\sum_{i \in N} x_i = E' - E$.*

We are now ready to proof that the Minimal Overlap Value satisfies Core-Transition Responsiveness. This is the aim of the following lemma.

Lemma 1. *Let $B = (E, c)$ and $B' = (E', c)$, with $E < E'$, two bankruptcy problems such that $\mathbb{C}(N, W_{(B',B)}) \neq \emptyset$. Then*

$$\left[\varphi^{mo}\left(E', c\right) - \varphi^{mo}\left(E, c\right)\right] \in \mathbb{C}(N, W_{(B',B)}).$$

Proof.
Let $B = (E, c)$ and $B' = (E', c)$, with $E < E'$, two bankruptcy problems such that $\mathbb{C}(N, W_{(B',B)}) \neq \emptyset$. Without loss of generality, let us assume that c's components are increasingly ordered.

Since $\mathbb{C}(N, W_{(B',B)}) \neq \emptyset$, Claims 1 and 2 inform us that

$$t^E = E \leq t^{E'} \leq E', \text{ and } E \leq c_n. \tag{9.2}$$

Hence, by Definition 4.1, we have that, for each creditor i

$$\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}\left(E, c\right) =$$

$$= \sum_{j=1}^{i} \frac{\min\left\{c_j, t^{E'}\right\} - \min\left\{c_j, E\right\} - \left[\min\left\{c_{j-1}, t^{E'}\right\} - \min\left\{c_{j-1}, E\right\}\right]}{n-k+1} +$$

$$+ \left[\max\left\{c_i - t^{E'}, 0\right\} - \max\left\{c_i - E, 0\right\}\right]$$

Let us select k such that, $c_{k-1} < E \leq c_k$. Note that, by the above expression we get that, for each i such that $i < k$,

$$\varphi_i^{mo}\left(E', c\right) = \varphi_i^{mo}\left(E, c\right),$$

and, for each agent $i \geq k$,

$$\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}\left(E, c\right) =$$

$$= \frac{\min\left\{c_k, t^{E'}\right\} - E}{n-k+1} + \sum_{j=k+1}^{i} \frac{\min\left\{c_j, t^{E'}\right\} - \min\left\{c_{j-1}, t^{E'}\right\}}{n-j+1} + \max\left\{c_i - t^{E'}, 0\right\}$$

Therefore, taking into account that the minimal overlap value is (weakly) increasing on the estate, for each i such that $i \geq k$,

$$0 \leq \varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}\left(E, c\right) \leq$$

$$\leq \min\left\{c_k, t^{E'}\right\} - E + \sum_{j=k+1}^{i} \left[\min\left\{c_j, t^{E'}\right\} - \min\left\{c_{j-1}, t^{E'}\right\}\right] + \max\left\{c_i - t^{E'}, 0\right\}$$

And, thus

$$0 \leq \varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}(E, c) \leq \min(c_i, t^{E'}) - E + \max\left\{c_i - t^{E'}, 0\right\} \tag{9.3}$$

Let consider the cases in Claim 3.

(a) $c_i \leq E$ then it is immediate that, by Equation (9.2),

$$\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}(E, c) = 0;$$

(b) $E < c_i \leq E'$. Let us consider the following two cases:

$(b.1)$ $c_i \leq t^{E'}$. By (9.3) we have that

$$\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}(E, c) \leq c_i - E.$$

$(b.2)$ $t^{E'} < c_i$. By (9.3) we get that

$$\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}(E, c) \leq t^{E'} - E + c_i - t^{E'} = c_i - E$$

(c) $E' < c_i$ in this case $t^{E'} = E'$ and using equation (9.3), we obtain that

$$\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}(E, c) \leq t^{E'} - E = E' - E$$

(d) Since the minimal overlap value provides recommendation for bankruptcy problems, it is immediate that

$$\sum_{i=1}^{n}\left[\varphi_i^{mo}\left(E', c\right) - \varphi_i^{mo}(E, c)\right] = E' - E \qquad \blacksquare$$

Lemma 2. *Let φ be a bankruptcy value satisfying Axioms 1 and 2. Then*

$$\varphi \equiv \varphi^{mo}$$

Proof.
Let φ be a bankruptcy vale, and let $B = (E, c)$ be a bankruptcy problem. Without loss of generality, we can assume that c is such that for any two creditors i, j, $c_i \leq c_j$ whenever $i \leq j$.

Let us assume that φ satisfies Axioms 1 and 2. We are going to consider the following cases, which exhausts all the possibilities:

[1] $E \leq c_1$.

Since φ is a value for bankruptcy games, we find that

$$\varphi(E, c) = \varphi(E, (E, \ldots, E, \ldots, E))$$

By Anonymity (ANON), we have that, for each creditor i

$$\varphi_i(E, c) = \frac{E}{n} = \varphi_i^{mo}(E, c)$$

[2] $c_1 < E \leq c_2$.

Since φ is a value for bankruptcy games, it should be the case that

$$\varphi(E, c) = \varphi(E, (c_1, E, \ldots, E, \ldots, E))$$

By Core-Transition Responsiveness (CTR), we find that

$$\varphi_1(E, (c_1, E, \ldots, E, \ldots, E)) = \varphi_1(c_1, (c_1, E, \ldots, E, \ldots, E))$$

Applying again that φ is a value for bankruptcy games, we have that

$$\begin{aligned} \varphi_1(c_1, (c_1, E, \ldots, E, \ldots, E)) &= \varphi_1(c_1, (c_1, c_1, \ldots, c_1)) = \\ &= \frac{c_1}{n} = \varphi_1^{mo}(E, c) \end{aligned}$$

By ANON it holds, for each i, $j \in N \setminus \{1\}$

$$\varphi_i(E, c) = \varphi_j(E, c)$$

Since

$$\sum_{i \in N} \varphi_i(E, c) = E$$

we find that, for each $i \neq 1$,

$$\varphi_i(E, c) = \frac{1}{n-1}\left(E - \frac{c_1}{n}\right) = \frac{E - c_1}{n-1} + \frac{c_1}{n} = \varphi_i^{mo}(E, c)$$

Note that, if $c_1 < E \leq c_2$, it holds that, for each $i \geq 2$

$$\varphi_i(E, c) - \varphi_i(c_1, c) = \varphi_2(E, c) - \varphi_2(c_1, c) = \frac{E - c_1}{n-1}$$

To provide an inductive argument, let us assume that for each $1 \leq i < j \leq n$

$$\varphi_i(E, c) - \varphi_i(c_{i-1}, c) = \frac{E - c_{i-1}}{n - i + 1} \tag{9.4}$$

whenever $c_{i-1} < E \leq c_i$.

[j] Let us consider that, for some $2 < j \leq n$, $c_{j-1} < E \leq c_j$.

Provided that φ is a value for bankruptcy games, it should be the case that

$$\varphi(E, c) = \varphi(E, (c_1, \ldots, c_{j-1}, E, \ldots, E))$$

Hence, by CTR, for each $i < j$

$$\varphi_i(E, (c_1, \ldots, c_{j-1}, E, \ldots, E)) = \varphi_i(c_i, (c_1, \ldots, c_{j-1}, E, \ldots, E))$$

Since φ is a value for bankruptcy games, we find that from (9.4), for each $i < j$

$$\varphi_i(E, c) = \varphi_i(c_i, (c_1, \ldots, c_{i-1}, E, \ldots, E)) = \sum_{k=1}^{i} \frac{c_k - c_{k-1}}{n-k+1}, \text{ with } c_0 = 0.$$

On the other hand, we find that since φ is a bankruptcy value,

$$\sum_{h \in N} \varphi_h(E, c) = E \tag{9.5}$$

and

$$\sum_{h \in N} \varphi_h(c_{j-1}, c) = c_{j-1} \tag{9.6}$$

Because φ is an anonymous value for bankruptcy games, we find that, for each $h > j$

$$\varphi_h(E, c) = \varphi_j(E, c), \text{ and } \varphi_h(c_{j-1}, c) = \varphi_j(c_{j-1}, c)$$

and by CTR we find that for each $k \leq j-1$

$$\varphi_k(E, c) = \varphi_k(c_{j-1}, c)$$

Therefore, from (9.5) and (9.6) we get

$$\begin{aligned} E - c_{j-1} &= \sum_{h \in N} [\varphi_h(E, c) - \varphi_h(c_{j-1}, c)] = \\ &= (n-j+1)\left[\varphi_j(E, c) - \varphi_j(c_{j-1}, c)\right] \end{aligned}$$

So

$$\begin{aligned} \varphi_j(E, c) &= \frac{E - c_{j-1}}{n-j+1} + \varphi_j(c_{j-1}, c) = \frac{E - c_{j-1}}{n-j+1} + \varphi_{j-1}(c_{j-1}, c) = \\ &= \frac{E - c_{j-1}}{n-j+1} + \sum_{k=1}^{j-i} \frac{c_k - c_{k-1}}{n-k+1}, \text{ with } c_0 = 0, \end{aligned}$$

i.e.

$$\varphi_j(E, c) = \varphi_j^{mo}(E, c)$$

for each $j \in N$ whenever $E \leq c_n$.

[$n+1$] Finally, let us consider that $E > c_n$. In this case we know that there is a unique t, $0 \leq t < c_n$ such that $\sum_{j=1}^{n} \max\{0, c_j - t\} = E - t$.

By applying previous cases we have that

$$\varphi_i(t, c) = \sum_{k=1}^{i} \frac{\min\{c_k, t\} - \min\{c_{k-1}, t\}}{n - k + 1}.$$

Now, we consider the increment of the estate from t to E. Then, by CTR we get

$$\begin{aligned} \varphi_i(E, c) - \varphi_i(t, c) &= 0 \text{ for all } i \text{ such that } c_i \leq t \text{ and} \\ \varphi_i(E, c) - \varphi_i(t, c) &= c_i - t \text{ for all } i \text{ such that } c_i > t, \end{aligned}$$

which is the desired result

$$\varphi_i(E, c) = \left[\sum_{k=1}^{i} \frac{\min\{c_k, t\} - \min\{c_{k-1}, t\}}{n - k + 1}\right] + \max\{0, c_i - t\},$$

where $t = E$ if $E < c_n$ and otherwise t is such that

$$\sum_{j=1}^{n} \max\{0, c_j - t\} = E - t. \qquad \blacksquare$$

APPENDIX 2.

This appendix is devoted to providing a formal proof for Proposition 6.1.

First of all, and for the sake of concretion, let us remember the formula proposed by Chun and Thomson [4] for the Minimal Overlap Solution. Given a bankruptcy problem (E, c) in $\mathcal{B}_O$ the Minimal Overlap Solution assigns to agent i the amount

$$\varphi_i^{mo}(E, c) = \left[\sum_{k=1}^{i} \frac{\min\{c_k, t\} - \min\{c_{k-1}, t\}}{n-k+1}\right] + \max\{0, c_i - t\}, \tag{9.7}$$

where t is such that, if $E > c_n$,

$$\sum_{j=1}^{n} \max\{0, c_j - t\} = E - t,$$

and $t = E$, otherwise.

Proof of Proposition 6.1.

First, note that for any bankruptcy problem (E, c) in $\mathcal{B}_{EIE}$, our result follows from the description of the Minimal Overlap Solution given by Chun and Thomson [4].

Now, let us consider a bankruptcy problem $(E, c) \in \mathcal{B}_O \backslash \mathcal{B}_{EIE}$. Note that there is a unique agent k such that

$$c_{k-1} \leq t < c_k.$$

Then for $i < k$ we have that

$$\varphi_i^{mo}(E, c) = \sum_{j=1}^{i} \frac{c_j - c_{j-1}}{n-j+1} = \varphi_i^{IE}(c_n, c),$$

and for any agent i, $i \geq k$,

$$\begin{aligned} \varphi_i^{mo}(E, c) &= \sum_{j=1}^{k} \frac{c_j - c_{j-1}}{n-j+1} + \frac{t - c_{k-1}}{n-k+1} + c_i - t = \\ &= \sum_{j=1}^{k} \frac{c_j - c_{j-1}}{n-j+1} + \varphi_i^{IE}(c_n, c) + \frac{t - c_{k-1}}{n-k+1} + c_i - \varphi_i^{IE}(c_n, c) - t. \end{aligned}$$

Therefore,

$$\begin{aligned} &\varphi_i^{mo}(E, c) - \varphi_i^{IE}(c_n, c) = \\ &= c_i - \varphi_i^{IE}(c_n, c) - \left(t - \sum_{j=1}^{k} \frac{c_j - c_{j-1}}{n-j+1} - \frac{t - c_{k-1}}{n-k+1}\right) \geq 0. \end{aligned} \tag{9.8}$$

Let us denote by λ the expression

$$\lambda = t - \sum_{j=1}^{k} \frac{c_j - c_{j-1}}{n-j+1} - \frac{t - c_{k-1}}{n-k+1} = t - \varphi_k^{IE}\left(c_n, \left(c_1, ..., c_{k-1}, t, c_{k+1}, ..., c_n\right)\right) \geq 0.$$

Then, from (9.8) we have that

$$\sum_{i=k}^{n} \varphi_i^{mo}\left(E, c\right) - \sum_{i=k}^{n} \varphi_i^{IE}\left(c_n, c\right) = \sum_{i=k}^{n} \left(c_i - \varphi_i^{IE}\left(c_n, c\right) - \lambda\right).$$

Since

$$\sum_{i=k}^{n} \varphi_i^{mo}\left(E, c\right) = E - \sum_{i=1}^{k-1} \varphi_i^{mo}\left(c_n, c\right) = E - \sum_{i=1}^{k-1} \varphi_i^{IE}\left(c_n, c\right),$$

we obtain that

$$\begin{aligned} E - c_n &= E - \sum_{i=1}^{n} \varphi_i^{IE}\left(c_n, c\right) = \\ &= E - \sum_{i=1}^{k-1} \varphi_i^{IE}\left(c_n, c\right) - \sum_{i=k}^{n} \varphi_i^{IE}\left(c_n, c\right) = \\ &= \sum_{i=k}^{n} \left(c_i - \varphi_i^{IE}\left(c_n, c\right) - \lambda\right). \end{aligned}$$

Moreover, we have that, for any agent j, $j < k$,

$$c_j - \varphi_j^{IE}\left(c_n, c\right) = c_j - \varphi_j^{IE}\left(c_n, c^t\right) \leq t - \varphi_k^{IE}\left(c_n, c^t\right),$$

where

$$c^t = \left(c_1, ..., c_{k-1}, t, c_{k+1}, ..., c_n\right).$$

Then

$$\max\left\{c_i - \varphi_i^{IE}\left(c_n, c^t\right) - \lambda, 0\right\} = 0.$$

Similarly, we have that, for any agent i, $i \geq k$, given that

$$\varphi_i^{IE}\left(c_n, c\right) \leq \varphi_i^{IE}\left(c_n, c^t\right),$$

then

$$t - \varphi_k^{IE}\left(c_n, c^t\right) < c_i - \varphi_i^{IE}\left(c_n, c^t\right) \leq c_i - \varphi_i^{IE}\left(c_n, c\right).$$

Therefore

$$\max\left\{c_i - \varphi_i^{IE}(c_n, c) - \lambda, 0\right\} = c_i - \varphi_i^{IE}(c_n, c) - \lambda > 0.$$

And, hence

$$E - c_n = \sum_{i=1}^{n} \max\left\{c_i - \varphi_i^{IE}(c_n, c) - \lambda, 0\right\}.$$

Note that our result follows from the expression above. ∎

APPENDIX 3.

This appendix presents a formal proof for Proposition 7.3. This result establishes that the formula (7.1), introduced to define the Generalized Ibn Ezra Value, can be computed in finite iterations.

Before proving our result, we want to point out some facts which will help with the arguments employed throughout the proof.

Fact 1. *Let $(E,c) \in \mathcal{B}_O$ a bankruptcy problem. Let $\left(E^t, c^t\right)$ be the estate and claims vector at t-th stage in the description of the Generalized Ibn Ezra Value. Then $\left(E^t, c^t\right) \in \mathcal{B}_O$.*

The above fact can be straightforwardly shown with the help of Definition 7.1. Note that at any $t > 1$,

$$\begin{aligned} E^t &= E^{t-1} - \min\left\{c_n^{t-1}, E^{t-1}\right\} = E^{t-1} - \sum_{i=1}^{n} \varphi_i^{PIE}\left(E^{t-1}, c^{t-1}\right) < \\ &< \sum_{i=1}^{n} c_i^{t-1} - \sum_{i=1}^{n} \varphi_i^{PIE}\left(E^{t-1}, c^{t-1}\right) = \sum_{i=1}^{n} c_i^t. \end{aligned}$$

Hence if $\left(E^{t-1}, c^{t-1}\right) \in \mathcal{B}$, then $\left(E^t, c^t\right) \in \mathcal{B}$. Since $(E,c) \in \mathcal{B}_O \subset \mathcal{B}$, we find that $\left(E^t, c^t\right) \in \mathcal{B}$ for each t. Moreover, we have that, for any creditor i, $i \neq 1$, and each t,

$$\begin{aligned} c_i^{t+1} - c_{i-1}^{t+1} &= \left[c_i^t - \varphi_i^{PIE}\left(E^t, c^t\right)\right] - \left[c_{i-1}^t - \varphi_{i-1}^{PIE}\left(E^t, c^t\right)\right] = \\ &= c_i^t - c_{i-1}^t - \frac{c_i^t - c_{i-1}^t}{n-i+1} = \frac{n-i}{n-i+1}\left(c_i^t - c_{i-1}^t\right) \geq 0. \end{aligned}$$

Therefore, provided that $\left(E^t, c^t\right) \in \mathcal{B}$ for any t, we find that $\left(E^{t+1}, c^{t+1}\right) \in \mathcal{B}_O$ whenever $\left(E^t, c^t\right) \in \mathcal{B}_O$. Since $(E,c) \in \mathcal{B}_O$, the statement of Fact 1 follows.

Fact 2. *Let $(E,c) \in \mathcal{B}_O$ be a bankruptcy problem. For each creditor i such that $c_i > 0$, and any $t \geq 1$*

$$0 < c_i^{t+1} \leq c_i^t.$$

Note that the above fact is a direct consequence of the description of c^t given in Definition 7.1, provided that for each creditor i, $0 \leq \varphi_i^{PIE}\left(E^t, c^t\right) < c_i^t$ for any t.

Fact 3. *Let $(E,c) \in \mathcal{B}_O$ a bankruptcy problem. For each creditor i and any $t \geq 1$*

$$\varphi_i^{PIE}\left(E^t, c^t\right) = 0 \textit{ if, and only if, } c_i^t = 0 \textit{ or } E^t = 0.$$

Notice that this fact comes directly from Fact 2. We now deal with the proof of Proposition 7.3.

Proof of Proposition 7.3.
Let $(E,c) \in \mathcal{B}_O$ be a bankruptcy problem. We shall show that there exists a positive integer $\tilde{t}$ such that

$$\varphi_i^{PIE}\left(E^{\tilde{t}}, c^{\tilde{t}}\right) = 0 \text{ for each agent } i.$$

Note that, since (E,c) is a bankruptcy problem, the above condition will hold in some $\tilde{t}$ such that $E^{\tilde{t}-1} \leq c_i^{\tilde{t}-1}$ for some agent i. In order to prove the statement of Proposition 7.3, let us assume that it is not true. Then, it should be the case that, for each creditor i and stage $t \geq 1$

$$E^t > c_i^t \tag{9.9}$$

This implies that, for creditor 1, at each stage $t \geq 1$,

$$\varphi_1^{PIE}\left(E^t, c^t\right) = \frac{c_1^t}{n}.$$

Moreover, we find that

$$c_1^{t+1} = \frac{n-1}{n} c_1^t \text{ for each } t \geq 1,$$

and therefore

$$c_1^t = \left(\frac{n-1}{n}\right)^{t-1} c_1.$$

Then, we have that

$$\varphi_1^{PIE}\left(E^t, c^t\right) = \frac{c_1^t}{n} = \left(\frac{n-1}{n}\right)^{t-1} \frac{c_1}{n}$$

Hence,

$$\varphi_1^{GIE}(E,c) = \sum_{t=1}^{\infty} \varphi_1^{PIE}\left(E^t, c^t\right) = \sum_{t=1}^{\infty} \left(\frac{n-1}{n}\right)^{t-1} \frac{c_1}{n} = c_1.$$

Note that, for creditor i, other than 1, we find that

$$\begin{aligned}
&\varphi_i^{PIE}\left(E^t, c^t\right) - \varphi_{i-1}^{PIE}\left(E^t, c^t\right) = \frac{c_i^t - c_{i-1}^t}{n-i+1} = \\
&= \frac{1}{n-i+1}\left[\left(c_i^{t-1} - \varphi_i^{PIE}\left(E^{t-1}, c^{t-1}\right)\right) - \left(c_{i-1}^{t-1} - \varphi_{i-1}^{PIE}\left(E^{t-1}, c^{t-1}\right)\right)\right] = \\
&= \frac{n-i}{(n-i+1)^2}\left(c_i^{t-1} - c_{i-1}^{t-1}\right) = \\
&= \frac{n-i}{(n-i+1)^2}\left[\left(c_i^{t-2} - \varphi_i^{PIE}\left(E^{t-2}, c^{t-2}\right)\right) - \left(c_{i-1}^{t-2} - \varphi_{i-1}^{PIE}\left(E^{t-2}, c^{t-2}\right)\right)\right] = \\
&= \frac{(n-i)^2}{(n-i+1)^3}\left(c_i^{t-2} - c_{i-1}^{t-2}\right) = \ldots = \frac{(n-i)^{t-1}}{(n-i+1)^t}\left(c_i - c_{i-1}\right).
\end{aligned}$$

Therefore, for $i > 1$,

$$\varphi_i^{GIE}(E,c) - \varphi_{i-1}^{GIE}(E,c) = \sum_{t=1}^{\infty} \frac{(n-i)^{t-1}}{(n-i+1)^t}(c_i - c_{i-1}) = c_i - c_{i-1}.$$

Since $\varphi_1^{GIE}(E,c) = c_1$, we find that, for each creditor i

$$\varphi_i^{GIE}(E,c) = c_i.$$

Now, let consider the sequence

$$\left\{\left(E^t, c^t\right)\right\}_{t=1}^{\infty}.$$

Note that by Definition 7.1, at any $t > 1$

$$c_i^t = c_i^{t-1} - \varphi_i^{PIE}\left(E^{t-1}, c^{t-1}\right) = c_i - \sum_{k=1}^{t-1} \varphi_i^{PIE}\left(E^k, c^k\right), \tag{9.10}$$

for each creditor i, and

$$E^t = E^{t-1} - c_n^{t-1} = E - \sum_{k=1}^{t-1} c_n^k = E - \sum_{k=1}^{t-1}\sum_{i=1}^{n} \varphi_i^{PIE}\left(E^k, c^k\right). \tag{9.11}$$

Taking limits in expressions (9.10) and (9.11), as t goes to infinity, we find that

$$\lim_{t\to\infty} c_i^t = 0,$$

and

$$\lim_{t\to\infty} E^t = E - \sum_{i=1}^{n} c_i < 0. \tag{9.12}$$

Since $E > 0$, the above fact implies that there is some positive integer $\tilde{t}$, such that for each $t \geq \tilde{t} + 1$, $E^t < 0$, and hence

$$E^{\tilde{t}} < c_n^{\tilde{t}}.$$

This contradicts our hypothesis in Equation (9.9). ∎

APPENDIX 4.

The aim of this appendix is to present a formal proof for Theorem 8.7. Note that, under Axiom 1, we can assume without loss of generality, that the bankruptcy problems to be analyzed belong to $\mathcal{B}_O$. Once our analysis is restricted in such a way, we can replace Axiom 1 by Equal Treatment of Equals.[6] Therefore, we will assume throughout this Appendix that the values to be considered satisfy such a property.

Proof of Theorem 8.7.
First, it can be straightforwardly seen that the *Generalized Ibn Ezra* value satisfies the three axioms: *Anonymity* (ANON), *Transitional Dummy* (TD) and *Worth-Generators Composition* (WGC).

On the other hand, let φ be a bankruptcy value satisfying Axioms 1, 3 and 4, and let $(E,c) \in \mathcal{B}_O$ a bankruptcy problem. We will show that

$$\varphi(E,c) = \varphi^{GIE}(E,c).$$

Let us consider the following cases, which exhausts all the possibilities:

Case 1.– $E \le c_1$.

Since φ is a bankruptcy value, we find that

$$\varphi(E,c) = \varphi(E,(E,\dots,E,\dots,E)).$$

By Equal Treatment of Equals (ETE), we have that, for each creditor i

$$\varphi_i(E,c) = \frac{E}{n} = \varphi_i^{IE}(E,c) = \varphi_i^{GIE}(E,c).$$

Case 2.– $c_1 < E \le c_2$.

Since φ is a bankruptcy value, it should be the case that

$$\varphi(E,c) = \varphi(E,(c_1,E,\dots,E,\dots,E)).$$

By TD, we find that

$$\varphi_1(E,(c_1,E,\dots,E,\dots,E)) = \varphi_1(c_1,(c_1,E,\dots,E,\dots,E)).$$

[6] We can define Equal Treatment of Equals as follows:

Let φ be a value for bankruptcy problems. We say that φ satisfies *Equal Treatment of Equals* if for each bankruptcy game $B=(E,c)$, any two agents, i, j; and any coalition $S \subseteq N\setminus\{i,j\}$

$$\text{if } V_B(S\cup\{i\}) = V_B(S\cup\{j\}) \text{ then } \varphi_i(E,c) = \varphi_j(E,c)$$

Applying again that φ is a bankruptcy value, we have that

$$\begin{aligned}\varphi_1(c_1,(c_1,E,\ldots,E,\ldots,E)) &= \varphi_1(c_1,(c_1,c_1,\ldots,c_1)) = \\ &= \frac{c_1}{n} = \varphi_1^{IE}(E,c) = \varphi_1^{GIE}(E,c).\end{aligned}$$

By ETE it holds, for each $i,\, j \in N\setminus\{1\}$

$$\varphi_i(E,c) = \varphi_j(E,c).$$

Since

$$\sum_{i\in N}\varphi_i(E,c) = E,$$

we find that, for each $i \neq 1$,

$$\varphi_i(E,c) = \frac{1}{n-1}\left(E - \frac{c_1}{n}\right) = \frac{E-c_1}{n-1} + \frac{c_1}{n} = \varphi_i^{IE}(E,c) = \varphi_i^{GIE}(E,c).$$

Note that, if $c_1 < E \leq c_2$, it holds that, for each $i \geq 2$

$$\varphi_i(E,c) - \varphi_i(c_1,c) = \varphi_2(E,c) - \varphi_2(c_1,c) = \frac{E-c_1}{n-1}.$$

To provide an inductive argument, let us assume that for each $1 \leq i < j \leq n$

$$\varphi_i(E,c) - \varphi_i(c_{i-1},c) = \frac{E-c_{i-1}}{n-i+1}, \tag{9.13}$$

whenever $c_{i-1} < E \leq c_i$.

Case j.– Let us consider that, for some $2 < j \leq n$, $c_{j-1} < E \leq c_j$. Provided that φ is a bankruptcy value, it should be the case that

$$\varphi(E,c) = \varphi(E,(c_1,\ldots,c_{j-1},E,\ldots,E)).$$

Hence, by TD, for each $i < j$

$$\varphi_i(E,(c_1,\ldots,c_{j-1},E,\ldots,E)) = \varphi_i(c_i,(c_1,\ldots,c_{j-1},E,\ldots,E)).$$

Since φ is a bankruptcy value, we find that from (9.13), for each $i < j$

$$\varphi_i(E,c) = \varphi_i(c_i,(c_1,\ldots,c_{i-1},E,\ldots,E)) = \sum_{k=1}^{i}\frac{c_k-c_{k-1}}{n-k+1}, \text{ with } c_0 = 0.$$

On the other hand, we find that since φ is a bankruptcy value,

$$\sum_{h\in N} \varphi_h (E, c) = E \tag{9.14}$$

and

$$\sum_{h\in N} \varphi_h (c_{j-1}, c) = c_{j-1}. \tag{9.15}$$

Because φ is an anonymous bankruptcy value, we find that, for each $h > j$

$$\varphi_h (E, c) = \varphi_j (E, c), \text{ and } \varphi_h (c_{j-1}, c) = \varphi_j (c_{j-1}, c),$$

and by TD we find that for each $k \leq j - 1$

$$\varphi_k (E, c) = \varphi_k (c_{j-1}, c).$$

Therefore, from (9.14) and (9.15) we get

$$\begin{aligned} E - c_{j-1} &= \sum_{h\in N} [\varphi_h (E, c) - \varphi_h (c_{j-1}, c)] = \\ &= (n - j + 1) \left[\varphi_j (E, c) - \varphi_j (c_{j-1}, c)\right]. \end{aligned}$$

So

$$\begin{aligned} \varphi_j (E, c) &= \frac{E - c_{j-1}}{n - j + 1} + \varphi_j (c_{j-1}, c) = \frac{E - c_{j-1}}{n - j + 1} + \varphi_{j-1} (c_{j-1}, c) = \\ &= \frac{E - c_{j-1}}{n - j + 1} + \sum_{k=1}^{j-i} \frac{c_k - c_{k-1}}{n - k + 1}, \text{ with } c_0 = 0, \end{aligned}$$

i.e.

$$\varphi_j (E, c) = \varphi_j^{IE} (E, c) = \varphi_j^{GIE} (E, c).$$

For each $j \in N$ whenever $E \leq c_n$.

Case $n + 1$.– Finally, let us consider that $E > c_n$.

Let us construct the sequence $\left\{c^t\right\}_{t=0}^{\infty}$ such that $c^0 = c$, and for $t \geq 1$ $c_i^t = \max\left\{0, c_i^{t-1} - \varphi_i \left(c_n^{t-1}, c^t\right)\right\}$. By WGC we find that

$$\varphi (E, c) = \varphi (c_n, c) + \varphi (E - c_n, c - \varphi (c_n, c)) = \sum_{t=0}^{\infty} \varphi \left(c_n^t, c^t\right).$$

By Cases 1 to n, we find that, for each integer t

$$\varphi \left(c_n^t, c^t\right) = \varphi^{IE} \left(c_n^t, c^t\right).$$

Therefore applying Proposition 7.3, we get the desired result

$$\varphi (E, c) = \sum_{t=0}^{\infty} \varphi \left(c_n^t, c^t\right) = \sum_{t=0}^{\infty} \varphi^{IE} \left(c_n^t, c^t\right) = \varphi^{GIE} (E, c),$$

which completes our characterization for the Generalized Ibn Ezra Value. ∎