A PROCEDURE FOR SHARING RECYCLING COSTS

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Abstract

This paper examines a situation in which the production activities of different agents, in a common geographical location, create waste products that are either of a similar biological or chemical composition or offer commercially compatible combinations. What we propose here, therefore, is a cost-sharing model for the of recycling of their waste products. We concentrate, however, on the specific case in which the agents' activities are heterogeneous.

We first examine, from a normative point of view, the cost-sharing rule, which we shall call the multi-commodity serial (MCS) rule. We introduce a property, that we call Cost-Based Equal Treatment, and we demonstrate that the unique rule verifying the Serial Principle and this property is the MCS rule. We then deal with the analysis of the agents' strategic behavior when they are allowed to select their own production levels, in which case the total cost is then split, in accordance with the MCS rule. We show that there is only one Nash equilibrium, which is obtained from an interactive elimination of dominated strategies.

Keywords: Cost Sharing Rules, Serial Cost Sharing, Dominance Solvability.

Journal of Economic Literature Classification Numbers: C71, C72, D62, D63.

1. Introduction

Advanced societies are continually seeking economically feasible ways of reducing on the amount of pollution they create. One of the most practical solutions is the imposition of clean-up models (recycling) for both urban and industrial residues. In this paper, we concentrate on the specific case in which the polluters, whether consumers or firms, are located within a given geographical area and their residues are all sent to one common dump. In such a situation, there is generally the possibility of converting much of their waste into "residual products" that could be commercialized for other uses. In many industrial areas, several firms dump their waste matter into one common pit. In such a cases, the cost involved in the eventual cleaning up of such sludge can be quite high. It is already well known that several waste products from different industrial processes can be recycled into the production of new agricultural products, principally fertilizers, which can then be used within the same geographical area. Simply spreading such by-products on barren areas of land would help to improve their fertility. Both the US and the EEC impose restrictions on the dumping of industrial effluents that are generally based on limits to their heavy metal content. The reason given for the imposition of such restrictions is that whilst the cleaning up of organic matter is relative inexpensive, and in certain cases even costless, the cleaning up of chemical wastes is generally quite expensive, especially when such residues contain heavy metals. In other words, the clean-up cost is directly related to the sort of chemical elements they contain. Furthermore, the composition of a firm's inorganic residues correlates highly with its field of activity. For instance, Chromium is found in the residues of leather-producing factories. Textiles and toy factories produce Cadmium, Zinc, Nickel and Copper among other elements.

Whenever it is decided that the cost of cleaning up a common pit must be covered by the polluters themselves, it immediately becomes necessary to establish just how the total cost should be split among the different agents. One property that such a cost-sharing model should always satisfy is a sensitivity to the proportion of each individual agents' contribution in relation to the total cost. The cost distribution that is finally imposed should be directly related to the cost involved in reducing, converting or recycling the individual residues of each firm. This paper focuses on the problem of designing a cost-sharing model that reflects, as faithfully as possible, this desired sensitivity to each agent's individual contribution to the total cost.

There is a long history to the study of cost-sharing in joint-projects. The most common approach to the problem has been the normative study of cost-sharing (or surplus-sharing) procedures. Cost-sharing problems can be modeled as cooperative games with transferable utility. A viable solution for these problems, in other words, a cost-sharing rule, would be a value function for this transferable utility game.

The literature published so far on the matter provides us with an interesting study of different examples of economic situations in which the key question is how the cost of a joint-project is to be shared. Billera, Heath and Ranaan (1978) have studied the pricing of telephone systems. Cost-sharing solutions, inspired by price systems had be also studied for airport runways (Littlechild and Owen (1973)), irrigation networks (Aadland and Kolpin (1998)), or public facilities (Loehman and Whinston (1974)), among others. The reader is referred to the surveys by Tauman (1988) or Young (1994) for more examples on this matter.

The particular model that we are interested in is the classical Aumann-Shapley pricing model. Each individual i, in a set of n agents, demands q_i units of some (perfectly divisible) personalized good. Given their demands, $q = (q_1, \ldots, q_i, \ldots, q_n)$, the agents should split the cost of production, according to a function C, whose domain is $\mathbb{R}^n_+$.

Following this formulation of cost-sharing problems as atomless cooperative games, Moulin and Shenker (1992) introduced the serial cost-sharing rule. The main difference between Moulin and Shenker's model and that of Aumann-Shapley, is that the goods that the agents demand are homogeneous in the former model, whereas this is not necessarily the case in the latter.

In this paper, we re-formulate the serial cost-sharing rule and apply it to the original Aumann-Shapley model. This problem was recently analyzed by Friedman and Moulin (1999). These authors present a generalization for the serial cost-sharing rule to the case in which goods are not homogeneous. Their proposal reflects the original formulation given by Aumann and Shapley (1974), based on measuring the marginal cost along a path. The rule proposed by Friedman and Moulin also reflects the serial principle implicit in the Moulin-Shenker formula. In fact, when all of the agents' demands coincide, the Aumann-Shapley and the Friedman-Moulin rules propose the same proportional share of the total cost. Loosely speaking, we can say that both of the above-mentioned mechanisms propose sharing the total cost according to the measure of the marginal cost along a path which depends exclusively on the agents' demands. Figures 1.1 and 1.2 show the paths for the Aumann-Shapley and the Friedman-Moulin rules in the case of there being just two agents, and when the agents' demands are summarized in the vector $(3;1)$.

In this paper, we propose a cost-sharing procedure, which we shall call **multi-commodity serial** rule, and which is formulated in accordance with the original serial idea presented by Moulin and Shenker. This rule proposes a sharing of the total cost by measuring the marginal cost along a path. The main difference between the interpretation of our proposal and those of Aumann-Shapley and Friedman-Moulin, is in the way the path to be used is defined. Our path is cost-dependent (as we shall explain later on), whereas the paths presented by the above-mentioned authors are not.

The justification of a certain path, for the purpose of interpretation, is made on the basis of how heterogeneous goods should be compared. In our opinion, one aspect that must be taken into consideration is the fact that the problem is formulated in terms of a cost to be shared. We therefore believe that any way of comparing the goods must be formulated in terms of the particular cost function that characterizes the problem. Examples 2.2 and 2.3 should help us to clarify this aspect.

Our first aim, therefore, is to present a cost-sharing rule that reflects the comparison of heterogeneous goods according to the cost function that the agents face. The way in which we compare two different goods is based on the cost of producing each of these goods separately. As such, given two goods, i and j, we shall consider that q_i units of i is equivalent to q_j units of j whenever the cost of producing only q_i units of i coincides with that of producing just q_j units of j.

Let us now analyze of the properties that characterize our rule. We find that the multi-commodity serial rule is the only cost-sharing procedure that satisfies **Cost-Based Equal Treatment** and **the Serial Principle**. (See Theorem 3.3). The first property establishes that in the case of two goods i and j, and when the two agents produce q_i and q_j respectively, they must contribute equally to the total cost of production, assuming that q_i and q_j are equivalent. The second property was formally defined by Sprumont (1998) and, loosely speaking, establishes that an agent's contribution to the total cost does not depend on the production levels of other agents who might produce more (according to the above comparison).

The second question is the study of the agents' behavior when the cost is shared according to our rule. We propose a model in which the agents decide their own individual production levels and the total cost is shared according the multi-commodity serial rule. When the agents behave strategically, we show that there can only be an equilibrium outcome after an iterative elimination of the dominated strategies. This result is similar to the strategic conclusions for the serial cost-sharing in Moulin and Shenker (1992), when agents produce homogeneous goods.

The rest of our paper is structured as follows: Section 2 introduces the basic model and definitions. In section 3 we present a formal definition for the multi-commodity serial rule and characterize it as the only cost-sharing rule that satisfies Cost-Based Equal Treatment and the Serial Principle. A study of the agents' strategic behavior is done in Section 4, our conclusions are presented in Section 5, and finally, some technical proofs are given in Appendix 1.

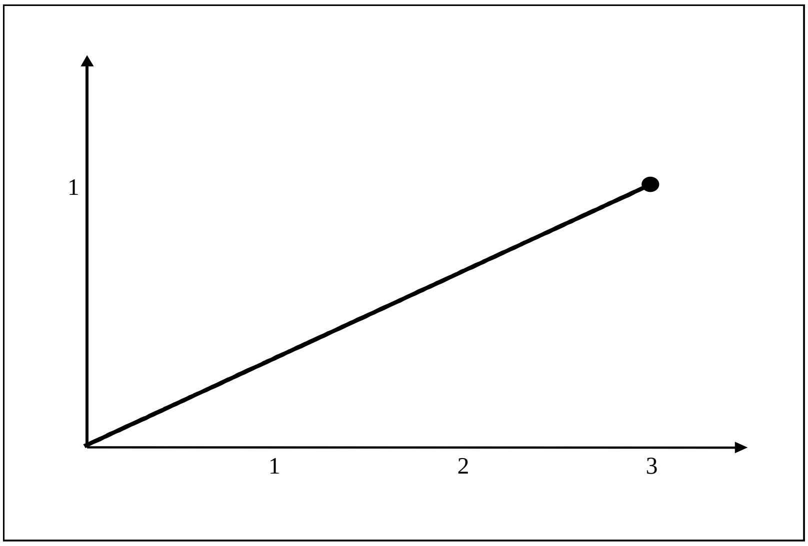

Figure 1.1: Aumann-Shapley Path

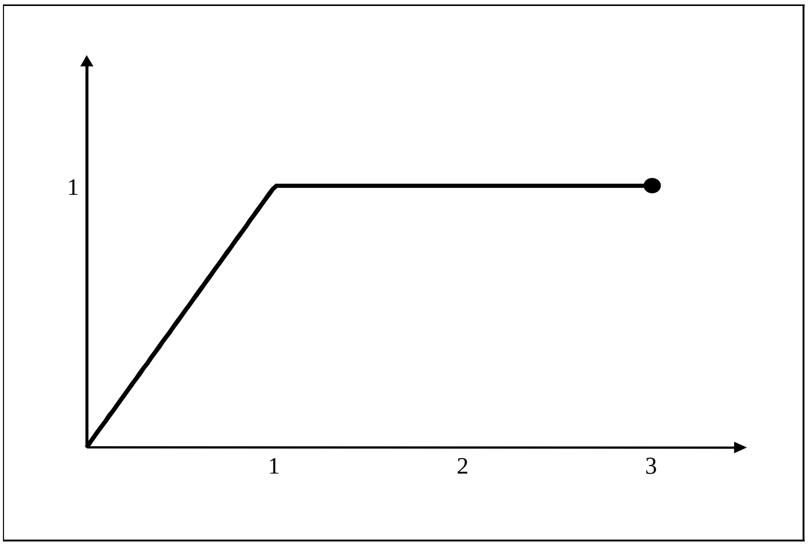

Figure 1.2: Friedman-Moulin Path

2. The framework

Let $N = \{1, \ldots, i, \ldots, n\}$ be a finite, non-empty set of agents, who we shall also call firms. Each agent i produces a certain good, such that, given a vector of agents' production $q \in \mathbb{R}^n_+$, we can identify its i-th component with agent i's production level. As a consequence of the agents' production activities, the environment is being polluted and a clean-up plan is imposed, the cost of which will be divided among the agents who generate the wastes. Let $C : \mathbb{R}^n_+ \to \mathbb{R}$ be a function which associates the cost of recycling the residuals with the production level of each firm. We assume that C is continuous, strictly increasing in each good i, $C(0) = 0$ and $\lim_{q_i \to \infty} C_i(q) = \lim_{q_j \to \infty} C_j(q)$ for each $i, j \in N$, where $C_h(q)$ is the evaluation of function C at the point whose h-th component is q_h, and all other components, except the h-th, is zero. Note that $C_h(q)$ can be interpreted as the cost of producing only agent h's demand, regardless of the other agents' demands, so that we refer to this expression as agent h's stand-alone cost at q. Let $\mathcal{C}$ denote the set of functions that satisfy the above properties.

Given a cost-sharing problem - i.e. a cost function C, and the firms' production levels q- we shall now propose a cost-sharing method for the recycling the agents' residuals based on their production levels. So, we shall describe a vector $x =$

$(x_1, \ldots, x_i, \ldots, x_n)$ such that $\sum_{i \in N} x_i = C(q)$. Our main interest is in defining a general procedure that provides a solution to any specific cost-sharing problem. These procedures are known as sharing rules.

Definition 2.1. A sharing rule is a function $X : \mathcal{C} \times \mathbb{R}^n_+ \to \mathbb{R}^n$ that associates a vector $X(C; q)$ with each cost function C in $\mathcal{C}$ and each production level of $q \in \mathbb{R}^n_+$ so that $\sum_{i \in N} X_i(C; q) = C(q)$.

The choice of a given cost-sharing procedure is generally based on the properties it satisfies. In this section, we shall present two properties, both of which can be defined as the result of a comparison of different schemes, by the agents, based on their production levels, to ensure its fair treatment. As we shall see (in Theorem 3.3), both properties, together, characterize the multi-commodity serial rule. This way of splitting the total cost can be considered an extension of Serial Cost Sharing (Moulin and Shenker (1992)) to the case of heterogeneous demands.

In the case of homogeneous goods, it is usually assumed that cost sharing mechanisms have to satisfy a symmetry property, namely: if two agents' demands coincide, then they should have the same share of the cost. This comparison can not be extended, in a trivial way, to the heterogeneous case, although we shall propose that the cost function does provide us with a way of making such comparisons. In general, when agents have to share the cost of joint-consumption, it is quite common for the agents to use the cost of their own consumption to argue how much each one should pay. Furthermore, no agent agrees to pay more share of the cost than any other whose consumption is more expensive that her own consumption. In accordance with our notation, we shall state the following axiom:

Axiom 1. : Let X be a cost-sharing rule. We shall say that it satisfies Cost-Based Equal Treatment (CBET) if, and only if, $X_i(C; q) = X_j(C; q)$ for all cost functions $C \in \mathcal{C}$ and production levels $q \in \mathbb{R}^n_+$ such that $C_i(q) = C_j(q)$.

Note that, in the case of homogenous goods, symmetry and CBET are equivalent terms. It seems natural, therefore, to take the cost function of each agent as a value for the comparison of their demands. In our novel way making these comparisons, we assume that two agents' demands are equivalent whenever the cost of producing each one's individual demands is the same, regardless of the cost of the other agents' productions.

The following examples demonstrate that the Aumann-Shapley Friedman-Moulin cost-sharing rules fail to satisfy CBET.

Example 2.2. Let $N = \{1; 2\}$, and let the cost function be $C(q_1; q_2) = (q_1 + 3q_2)^2$. It is clear that, by observing the symmetry notion in the case of homogeneous goods, the total cost should be split equally whenever $q_1 = 3q_2$. In fact, this is also the case with the Aumann-Shapley rule, although it is not so with the Friedman-Moulin rule. For instance, if we assume that $\bar{q} = (3; 1)$, then the Aumann-Shapley formula would be:

$$X_i^{AS}(C; \bar{q}) = \bar{q}_i \int_0^1 \frac{\partial C}{\partial q_i}(t\bar{q})\, dt$$

Which results in:

$$X_1^{AS}(C; \bar{q}) = X_2^{AS}(C; \bar{q}) = 18$$

The Friedman-Moulin rule, however, taking into account the fact that $\bar{q}_1 > \bar{q}_2$, we obtain the following expression:

$$\begin{aligned} X_1^{FM}(C; \bar{q}) &= \int_0^{\bar{q}_1} \frac{\partial C}{\partial q_1}(t; \min\{t; \bar{q}_2\})\, dt \\ X_2^{FM}(C; \bar{q}) &= \int_0^{\bar{q}_2} \frac{\partial C}{\partial q_2}(t; t)\, dt \end{aligned}$$

whose results are:

$$X_1^{FM}(C; \bar{q}) = 24, \text{ and } X_2^{FM}(C; \bar{q}) = 12$$

Note that the above example shows that the Friedman-Moulin rule does not satisfy re-scaling, i.e., the way in which the cost is shared depends on the type of units that are chosen to measure the levels of output. (For a formal definition of this property, the reader is referred to Axiom 3 in Tauman (1988) tauman.) It is well known that the Aumann-Shapley rule satisfies this property. This is no longer true, however, when the form in which two different measurements are related is not linear, as the next example demonstrates:

Example 2.3. Let $N = \{1; 2\}$, and let the cost function be $C(q_1; q_2) = (q_1 + q_2^2)^2$. We can assume that q_1 units of good 1 produce the same amount of pollution as q_2^2 units of good 2. The same argument that is used to justify the property of symmetry in the homogeneous case can be used to argue that the total cost should be split equally among the two agents whenever $q_1 = q_2^2$. This property, however, is not satisfied by the Aumann-Shapley rule. To demonstrate this, let us assume that $\bar{q} = (4; 2)$. The share proposed by the above rule is therefore:

$$X_1^{AS}(C; \bar{q}) = \frac{80}{3}, \text{ and } X_2^{AS}(C; \bar{q}) = \frac{112}{3}$$

As we shall see (Theorem 3.3) the multi-commodity serial rule satisfies Cost-Based Equal Treatment.

The second property that we deal with is the Serial Principle. This property was formally introduced by Sprumont (1998). The underlying idea of this axiom is the inter-personal comparisons made by the different agents among themselves, regarding the cost-sharing. We now present the intuition beyond this principle. Assuming that the agents use the cost function as a tool to made such comparisons; let us study the case of an agent who produces a certain quantity q_i, which, in turn, generates an external effect, in term of costs, to the others. The question to be settled is just how much each agent should accept to pay considering the externalities caused by that one agent. Notice that these externalities depend not only on this agent's demand, but also does on others' demands. This fact induces to consider reasonable that the share of the total cost corresponding to this agent would not be sensitive to increases on others' demands, when they are (comparatively) higher than the one made by such an agent.

Axiom 2. Let X be a cost-sharing rule. We say that it satisfies the Serial Principle (SP) if, and only if, for each agent $i \in N$, each cost function $C \in \mathcal{C}$ and any two production vectors $q, q' \in \mathbb{R}^n_+$,

$$\left\{ q'_j = q_j \text{ for } j = i \text{ and for all } j \text{ such that } X_j(C;q) < X_i(C;q) \right\}$$

and

$$\left\{ q'_j \geq q_j \text{ for all } j \text{ such that } X_j(C;q) \geq X_i(C;q) \right\}$$

imply that $X_i(C;q) = X_i(C;q')$.

3. The Multi-Commodity Serial Rule: Definition and Characterization

In this section we introduce the multi-commodity serial cost-sharing rule, and show that this is the only cost-sharing rule that satisfies CBET and SP, the two axioms introduced in the previous section. Before introducing our rule, however, we must present an additional notation. Given q and $q' \in \mathbb{R}^n$, let $q \wedge q'$ denote the minimum among these vectors, i.e., $q \wedge q' = q''$ such that, for each i, $q''_i = \min\{q_i; q'_i\}$. Given $q \in \mathbb{R}^n_+$, and agent i, let $L_i(q) = \{j \in N : C_j(q) < C_i(q)\}$.

Finally, for each q in $\mathbb{R}^n_+$, and agent i, let $\tilde{q}_i \in \mathbb{R}^n_+$ denote the vector that satisfies $C_j(\tilde{q}_i) = C_i(q)$ for all $j \in N$.[1]

Definition 3.1. The multi-commodity serial cost-sharing rule is a function $X^{mcs} : \mathcal{C} \times \mathbb{R}^n_+ \to \mathbb{R}^n$ which associates a vector $X^{mcs}(C, q)$ to each cost function C in $\mathcal{C}$, and to each production level $q \in \mathbb{R}^n_+$, such that for each $i \in N$,

$$X_i^{mcs}(C, q) = \sum_{h \in L_i(q) \cup \{i\}} \frac{1}{n - |L_h(q)|} \left[C(\tilde{q}_h \wedge q) - \max_{j \in L_h(q)} C\left(\tilde{q}_j \wedge q\right) \right] \quad (3.1)$$

where, for any set A, $|A|$ denotes its cardinality. By convention, we assume that $\max_{j \in L_i(q)} C\left(\tilde{q}_j \wedge q\right) = 0$ whenever $L_i(q) = \emptyset$.

The formula used to describe the multi-commodity serial rule in the above expression can be explained in a simple, intuitive way, with the help of an iterative argument. Let us assume that q is such that $C_1(q) \leq \cdots \leq C_i(q) \leq \cdots \leq C_n(q)$. In such a case, each agent has to pay $\frac{1}{n} C(\tilde{q}_1)$. Note, however, that this does not cover the total cost. The difference, $C(q) - C(\tilde{q}_1)$, is finally covered by all the agents except firm 1. To share this deficit among them, each agent from 2 to n, is charged an extra $\frac{1}{n-1}[C(\tilde{q}_2 \wedge q) - C(\tilde{q}_1)]$. It is clear that, in general, the total cost is not readily covered. The part that remains to be paid, $C(q) - C(\tilde{q}_2 \wedge q)$, must therefore be charged to agents 3 to n, complying with the above criterion. In other words, these agents must be charged an extra $\frac{1}{n-2}[C(\tilde{q}_3 \wedge q) - C(\tilde{q}_2 \wedge q)]$, which is the difference between the total cost paid by agents from 4 to n, and so forth. The next example will clarify this procedure.

Example 3.2. Let $N = \{1, 2, 3\}$, $C(q) = \sqrt{q_1} + \sqrt{q_2} + q_3^2 + q_1 q_2 + q_1 q_3 + q_2 q_3$, and $q = (9, 16, 1)$. In such a case, $C_3(q) < C_1(q) < C_2(q)$. $\tilde{q}_3 = (1, 1, 1)$; $\tilde{q}_1 = \left(9, 9, \sqrt{3}\right)$, $\tilde{q}_2 = (16, 16, 2)$. Then, each agent is charged $\frac{1}{3} C(\tilde{q}_3) = 2$. But agents 1 and 2 have also got to pay $\frac{1}{2}[C(\tilde{q}_1 \wedge q) - C(\tilde{q}_3)] = \frac{1}{2}[C(9, 9, 1) - C(1, 1, 1)] = 50$, and finally, agent 2 is also charged the remainder of the outstanding cost, which is, $C(\tilde{q}_2 \wedge q) - C(\tilde{q}_1 \wedge q) = C(9, 16, 1) - C(9, 9, 1) = 71$. Then,

$$X^{mcs}(C, q) = (2 + 50, 2 + 50 + 71, 2) = (52, 123, 2)$$

The next theorem characterizes the multi-commodity serial rule by axioms 1 and 2.

[1] Under the assumptions made on $\mathcal{C}$, for any cost function $C \in \mathcal{C}$, each agent $i \in N$, and any production level for this agent q_i, $\tilde{q}_i$ is always unique.

Theorem 3.3. *The multi-commodity serial rule is the only cost-sharing rule that satisfies CBET and SP in $\mathcal{C}$.*

Proof. It is clear that the multi-commodity serial rule verifies the two axioms.

On the other hand, let $X : \mathcal{C} \times \mathbb{R}^n_+ \to \mathbb{R}^n$ be a cost sharing rule satisfying axioms 1 and 2. To demonstrate that there is no other rule but the multi-commodity serial rule that satisfies these two axioms, we shall proceed by induction. Let $q \in \mathbb{R}^n_+$ be a vector of agents' demands and $C \in \mathcal{C}$ a cost function. Without loss of generality, and for notational convenience, let us first assume that

$$C_1(q) \leq \cdots \leq C_i(q) \leq \cdots \leq C_n(q). \tag{3.2}$$

This assumption implies that $q^e_1 \leqq q$,[2] hence SP means that $X_1(C;q) = X_1(C;q^e_1)$. Moreover, CBET implies that $X_1(C;q^e_1) = \frac{1}{n}C(q^e_1)$. Note that, by SP, this means that

$$X_1(C;q') = \frac{1}{n}C(q^e_1) \text{ for any } q' = q^e_1 \text{ such that } q'_1 = q_1$$

Now, consider the vector $q^e_2 \wedge q$. In this vector, the first component coincides with q_1 whereas the other components coincide with those of q^e_2. By SP, we know that $X_2(C;q) = X_2(C;q^e_2 \wedge q)$. Furthermore, by CBET, $X_i(C;q^e_2 \wedge q) = X_2(C;q^e_2 \wedge q)$ for all $i \geq 2$. Since the cost has to be fully covered, it should be satisfied that $C(q^e_2 \wedge q) = \sum_{i \in N} X_i(C;q^e_2 \wedge q) = X_1(C;q^e_2 \wedge q) + \sum_{i>1} X_i(C;q^e_2 \wedge q) = X_1(C;q) + (n-1)X_2(C;q^e_2 \wedge q)$, so that by considering Agent 1's contribution, Agent 2's share would have to be

$$X_2(C;q) = \frac{1}{n-1}\left(C(q^e_2 \wedge q) - \frac{1}{n}C(q^e_1)\right)$$

therefore,

$$X_2(C;q) = \frac{1}{n-1}\left[C(q^e_2 \wedge q) - C(q^e_1)\right] + \frac{1}{n}C(q^e_1)$$

and, following the convention that $q^e_0 = 0$, agent 2's share can therefore be expressed as

$$X_2(C;q) = \sum_{k=1}^{2} \frac{1}{n-k+1}\left[C(q^e_k \wedge q) - C\left(q^e_{k-1} \wedge q\right)\right]$$

[2] We use the following notation for vectorial comparissons. $q \leqq q'$ means that $q_i \leq q'_i$ for all i; $q \leq q'$ means that $q_i \leq q'_i$ for all i, and $q_j < q'_j$ for some j and, finaly $q < q'$ means that $q_i < q'_i$ for all i.

We now assume the following induction hypothesis: For all agent $i = 1, \dots, h-1$,

$$x_i(C;q) = \sum_{k=1}^{i} \frac{1}{n-k+1}\left[C(q_k^e \wedge q) - C\left(q_{k-1}^e \wedge q\right)\right]$$

Let us argue what the share of agent h should be. Note that, by Axiom 2, $x_h(C;q) = x_h(C;q_h^e \wedge q)$. Hence, by Axiom 1, $x_j(C;q_h^e \wedge q) = x_h(C;q_h^e \wedge q)$ for all $j \geq h$. Since $C(q_h^e \wedge q) = (n-h)\, x_h(C;q_h^e \wedge q) + \sum_{i<h} x_i(C;q_h^e \wedge q)$. So, $x_h(C;q_h^e \wedge q) = \frac{1}{n-h}\left[C(q_h^e \wedge q) - \sum_{i<h} x_i(C;q_h^e \wedge q)\right]$. Hence, by the induction hypothesis, agent h's share has to satisfy

$$x_h(C;q_h^e \wedge q) = \frac{1}{n-h}\left[C(q_h^e \wedge q) - \sum_{i<h}\sum_{k=1}^{i} \frac{1}{n-k+1}\left[C(q_k^e \wedge q) - C\left(q_{k-1}^e \wedge q\right)\right]\right].$$

Henceforth,

$$x_h(C;q) = \sum_{k=1}^{h} \frac{1}{n-k+1}\left[C(q_k^e \wedge q) - C\left(q_{k-1}^e \wedge q\right)\right]$$

which is the expression of the multi-commodity serial cost sharing rule given in Definition 3.1 ■

4. Strategic Behavior and the Multi-Commodity Serial Rule

The previous section has shown that, based on the agents' production levels, the share of their cost, in accordance with the multi-commodity serial rule, satisfies some nice properties, with regard to the equity aspect. It tells us nothing, however, about how the agents select their own production levels. In this section, we shall do a game-theoretical study of the agents' production decisions. Let us imagine that each agent decides its own output level, and let us assume that the total cost is shared in accompliance with the multi-commodity serial rule, and this is unanimously accepted by all the agents. The question that arises is: Is it possible to know the production level that each will have? Or, more precisely, is there a theoretical game analysis of agents' behavior that could provide us with an accurate prediction of the agents' individual choices? This section will provide positive answers to both of these questions.

We shall now describe the mechanism that the agents face. As assumed in Section 2, there is a set N of firms which produce n different goods and face a cost function C in $\mathcal{C}$. Each agent has preferences defined on $\mathbb{R}^2_+$. A bundle for agent i, (q_i, x_i) is interpreted as the situation in which firm i produces q_i and has to assume a cost of x_i units. Agent i's preferences are assumed to be non-decreasing in q_i, non-increasing in x_i, nowhere locally satiated, continuous, convex and representable by utility functions. We shall denote the set of preference orderings we have just defined by $\mathcal{U}$. We shall use, throughout, a utility representation $U_i(q_i, x_i)$ instead of the cumbersome binary relation notation. Finally, we shall introduce some additional assumptions relating to the cost function and to the agents' preferences:

Assumption 1. C is a smooth and strictly convex function.

Assumption 2. C satisfies cross-monotonicity, i.e., for each $\tilde{q} \in \mathbb{R}^l_{++}$, and any two agents i and j in N,

$$\frac{\partial C}{\partial q_i}(\tilde{q}) \text{ is increasing in } q_j \tag{4.1}$$

Assumption 3. The utility function is bounded: For each agent i, there exists a production level, say $\hat{q}_i$ such that

$$U_i(\hat{q}_i, C_i(\hat{q}_i^e)) < U_i(0, 0)$$

We must comment, here, on our assumptions. Convexity is also assumed in Moulin and Shenker (1992) serial, and it guarantees the existence of a best-reply correspondence for each agent. Smoothness is a technical assumption, which is adopted to simplify the proofs presented in this section. Remark 1 in the Appendix 1 provides arguments on how to proceed with these proofs in non-smooth environments. Assumption 2 only imposes that each agent's marginal cost must increase when the another's demand increases. Note that Assumption 2 is satisfied by polynomial cost functions satisfying monotonicity and convexity. Finally, with regard to Assumption 3, we must note that it is not a strong assumption. To be more specific, it is only satisfied when the cost function is not upper-bounded, (i.e., for each agent i, $\lim_{q_i \to \infty} C_i(q) = \infty$).

We shall now study the agents' behavior when they are faced with the multi-commodity serial cost-sharing mechanism, and individual strategies are based on their own demands. In other words, given N, the set of agents with utility functions U_i in $\mathcal{U}$, and a fixed cost function C in $\mathcal{C}$ we define the game $\Gamma(C, U_1, \ldots, U_i, \ldots, U_n)$, in which each firm i selects an output level q_i. Each agent's strategy space is therefore $\mathbb{R}_+$. Finally, given a vector of agents' strategies, q, each agent i pays $X_i^{mcs}(C; q)$ and obtains an utility level of $U_i(q_i; X_i^{mcs}(C; q))$.

Moulin and Shenker (1992), analyze the case of homogeneous goods and show the existence of cost sharing mechanisms, inducing dominance solvable game forms. They define a cost-sharing rule, which they call a serial rule, and which coincides with the multi-commodity serial rule when goods are homogeneous. Our next result extends this same idea to the case of heterogeneous goods.

Theorem 4.1. *Let C be a cost function in $\mathcal{C}$, and let us assume that each agent's utility function U_i is in $\mathcal{U}$. Under assumptions 1-3, the game $\Gamma(C; U_1; \ldots; U_i; \ldots; U_n)$ is dominance solvable.*[3]

Before providing a formal proof of Theorem 4.1, we shall provide two lemmas that are essential to our argument for the construction of a sophisticated equilibrium of $\Gamma(C; U_1; \ldots; U_i; \ldots; U_n)$.

Lemma 4.2. (Moulin and Shenker (1992))

Let $h_1(\lambda)$ and $h_2(\lambda)$ be two increasing and strictly convex functions, form $\mathbb{R}_+$ into itself, and which coincide up to λ_0,

$$h_1(\lambda) = h_2(\lambda) \text{ for all } \lambda, 0 \leq \lambda \leq \lambda_0.$$

For every utility function U_i in $\mathcal{U}$, the (unique) maximizers of $U_i(\lambda; h_j(\lambda))$ on $\mathbb{R}_+$, denoted by λ_j, $j = 1; 2$; are on the same side of λ_0,

$$\lambda_1 \geq \lambda_0 \iff \lambda_2 \geq \lambda_0, \text{ and } \lambda_1 = \lambda_0 \iff \lambda_2 = \lambda_0$$

Before enunciating the next lemma, let us introduce another notation. Let i be an agent, S a set of agents, $\emptyset \subseteq S \subseteq N \setminus \{i\}$, and let $\bar{q}$ be a production level. We shall now construct the function $G : \mathbb{R}_+ \to \mathbb{R}_+^n$ which selects, for any production level of Agent i, say q_i, a vector whose j-th component is q_{ij}^e, the j-th component of q_i^e, if j does not belong to S, and the minimum between $\bar{q}_j$ and q_{ij}^e if $j \in S$.

[3] Dominance Solvability was first introduced by Moulin (1979). This equilibrium concept is a special case of sophisticated equilibria, introduced by Farquharson (1969). The idea of dominance solvability for a game is that the Nash equilibrium outcome is unique under iterated elimination of dominated strategies. The reader is referred to Moulin (1979) for a formal definition.

Lemma 4.3. Let C be a cost function in $\mathcal{C}$ that satisfies Assumption 1. Then, for each set of agents S, $\emptyset \subseteq S \subseteq N \setminus \{i\}$, and production level $\hat{q}$, the function f defined by $f(q_i) = X_i^{mcs}(C; G(q_i))$ is a continuous and strictly convex function on q_i.

Note that continuity of the multi-commodity serial rule comes from the continuity of the cost function and the expression 3.1. Strict convexity of f comes from the strict convexity of C. A formal proof of this strict convexity is provided in Appendix 1.

We can now present a formal proof of Theorem 4.1. But let us first introduce the following notation. Let $q_i \in \mathbb{R}$. We shall denote by q_i^{∞} a vector whose i-th component is q_i and is infinity for all the others.

Proof of Theorem 4.1.

Throughout the proof, C will be a strictly convex function in $\mathcal{C}$, and utility functions $(U_1; \dots; U_i; \dots; U_n)$ in $\mathcal{U}^n$. We first define, inductively, an outcome $(\hat{q}_i; \hat{x}_i)$, $i \in N$, and we show that $\hat{q} = (\hat{q}_1; \dots; \hat{q}_n)$ is a Nash Equilibrium for the game $\Gamma(C; U_1; \dots; U_i; \dots; U_n)$. We then show that there is only one outcome left after the successive elimination of strictly dominated strategies in that game.

Let q_{i1} be agent i's agreement demand, namely the unique solution to

$$\max_{q_i} U_i\left(q_i;\ X_i^{mcs}(C; q_i^e)\right) \tag{4.2}$$

Let us choose the agent i with the lowest $C_i\left(q_{i1}^e\right)$, whom we shall denote by π_1. We then set $\hat{q}_{\pi_1} = q_{\pi_1 1}$ and $\hat{x}_{\pi_1} = X_{\pi_1}^{mcs}\left(C; \hat{q}_{\pi_1}^e\right)$, and solve (for all agents except π_1), the program

$$\max_{q_i} U_i\left(q_i;\ X_i^{mcs}\left(C; \hat{q}_{\pi_1}^{\infty} \wedge q_i^e\right)\right) \tag{4.3}$$

Since the two functions $X_i^{mcs}(C; q_i^e)$ and $X_i^{mcs}\left(C; \hat{q}_{\pi_1}^{\infty} \wedge q_i^e\right)$ coincide up to $\hat{q}_{\pi_1}^e$, and are strictly convex with respect to q_i, we can apply Lemma 4.2. We therefore observe that the unique solution, namely q_{i2}, cannot be smaller than q_{i1}.

We now choose an agent i in $N \setminus \pi_1$ whose solution to Program (4.3) has the lowest $C_i\left(q_{i2}^e\right)$ and we denote this agent by π_2. We set $\hat{q}_{\pi_2}$ as the corresponding solution to (4.3) and $\hat{x}_{\pi_2} = X_{\pi_2}^{mcs}\left(C; \hat{q}_{\pi_1}^{\infty} \wedge \hat{q}_{\pi_2}^e\right)$.

To complete the inductive argument, let us assume that we have constructed the sequence $\left(\hat{q}_{\pi_i}; \hat{x}_{\pi_i}\right)$ up to $i = k$. We now compute, for all i in $N \setminus \bigcup_{j=1} \pi_j$, the solution $q_{i(k+1)}$ to the program

$$\max_{q_i} U_i \left(q_i; X_i^{mcs} \left(C; \left(\bigwedge_{j=1}^{k} q^1_{\pi_j} \right) \wedge q_i^e \right) \right) \tag{4.4}$$

Let i be an agent whose solution to Program (4.4), $q_{i(k+1)}$, has the lowest $C_i \left(q^e_{i(k+1)} \right)$, and let us denote this agent by π_{k+1}. Observe that $C_{\pi_k} \left(\hat{q}^e_{\pi_k} \right) \le C_{\pi_{k+1}} \left(q^e_{\pi_{k+1}} \right)$. This follows from Lemma 4.2. We take $\hat{q}_{\pi_{k+1}}$ as the solution to (4.4) for agent π_{k+1} and $\hat{x}_{\pi_{k+1}} = X^{mcs}_{\pi_{k+1}} \left(C; \left(\bigwedge_{j=1}^{k} \hat{q}^1_{\pi_j} \right) \wedge \hat{q}^e_{\pi_{k+1}} \right)$.

We now verify that $\hat{q}$ is a Nash Equilibrium of $\Gamma(C; U_1; \ldots; U_i; \ldots; U_n)$. Let us now select a player i and consider the two functions

$$h_1(\lambda_i) = X_i^{mcs} \left(C; \left(\bigwedge_{j \in S} \hat{q}^1_j \right) \wedge \lambda^e_i \right) \quad \text{where } S = \left\{ j \mid C_j \left(\hat{q}^e_j \right) \le C_i \left(\hat{q}^e_i \right) \right\}$$

$$h_2(\lambda_i) = X_i^{mcs} \left(C; \left(\bigwedge_{j \neq i} \hat{q}^1_j \right) \wedge \lambda^e_i \right)$$

Note that these two functions coincide at $[0; \hat{q}_i]$. Furthermore $\hat{q}_i$ maximizes $U_i(\lambda_i; h_1(\lambda_i))$ on R_+. Thus Lemma 4.2 implies that $\hat{q}_i$ also maximizes $U_i(\lambda_i; h_2(\lambda_i))$ on R_+, which is the Nash equilibrium property we require.

We shall now show that $\hat{q}$ is the only outcome that is left after the successive elimination of strictly dominated strategies. It is now sufficient to construct, for each agent i, a decreasing sequence of closed intervals $I_{i\nu}$ of vanishing length:

$$I_{i\nu} = [a_{i\nu}; b_{i\nu}] \supseteq \left[a_{i(\nu+1)}; b_{i(\nu+1)} \right] = I_{i(\nu+1)}$$

such that every strategy in $R_+ \setminus I_{i0}$ is strictly dominated in the initial game $\Gamma(C; U_1; \ldots; U_i; \ldots; U_n)$ and where a strategy in $I_{i\nu} \setminus I_{i(\nu+1)}$ is strictly dominated when each agent's strategy in that game is restricted to $I_{i\nu}$.

Since the agents' preferences are convex, and C is strictly convex, it holds that for each agent there is only one production level, $\bar{q}_{i0} > 0$, such that $U_i(\bar{q}_{i0}; C_i(\bar{q}^e_{i0})) = U_i(0; 0)$. Let us define $I_{i0} = [a_{i0}; b_{i0}] = [0; \bar{q}_{i0}]$.

Clearly, any strategy q_i for agent i that is greater than $\bar{q}_{i0}$ is dominated by her strategy $q^0_i = 0$. This is because $U_i(q_i; C_i(q^e_i))$ must be decreasing in $\bar{q}_{i0}$, and, since C satisfies cross-monotonicity, $C_i(q^e_i) \le X_i^{mcs}(C; \tilde{q})$ for any $\tilde{q}$ such that $\tilde{q}_i = q_i$.

The sequence $I_{i\nu}$ is now defined inductively as follows:

$a_{i(\nu+1)}$ and $b_{i(\nu+1)}$ are the lowest and the greatest solutions, respectively, to the equation

$$\begin{aligned} & U_i\left(\bar q_i; X_i^{mcs}\left(C; \left(\bigvee_{j\neq i} a_{j\nu}^1\right) \wedge \bar q_i^e\right)\right) = \\ & = \max_{q_i} U_i\left(q_i; X_i^{mcs}\left(C; \left(\bigvee_{j\neq i} b_{j\nu}^1\right) \wedge q_i^e\right)\right) \end{aligned} \qquad (4.5)$$

Observe that if every other agent j is using a strategy in the interval $[a_{j\nu}; b_{j\nu}]$, then agent i's share of the total cost is between $X_i^{mcs}\left(C; \left(\bigvee_{j\neq i} a_{j\nu}^1\right) \wedge q_i^e\right)$ and $X_i^{mcs}\left(C; \left(\bigvee_{j\neq i} b_{j\nu}^1\right) \wedge q_i^e\right)$. Therefore her strategy $q_{i\nu}$ which solves the right-hand side of (4.5), strictly dominates any strategy below $a_{i(\nu+1)}$ or above $b_{i(\nu+1)}$. One can easily observe, by induction, that $a_{i\nu}$ is non-decreasing in ν, that $b_{i\nu}$ is non-increasing in ν, and that $a_{i\nu} \leq q_{i\nu} \leq b_{i\nu}$.

Let a_i (b_i) denote the limit of $a_{i\nu}$ ($b_{i\nu}$ respect.), as ν tends to infinity. By continuity of the utilities and of X^{mcs}, Property (4.5) is maintained at the limit. Hence, for all i, the following holds for $\bar q_i = a$ or b.

$$\begin{aligned} & U_i\left(\bar q_i; X_i^{mcs}\left(C; \left(\bigvee_{j\neq i} a_j^1\right) \wedge \bar q_i^e\right)\right) = \\ & = \max_{q_i} U_i\left(q_i; X_i^{mcs}\left(C; \left(\bigvee_{j\neq i} b_j^1\right) \wedge q_i^e\right)\right) \end{aligned} \qquad (4.6)$$

We shall now verify that $a = b$. Let us first choose an agent with (one of) the lowest value $C_i(a_i^e)$, whom we shall denote by π_1. Since $C_{\pi_1}\left(a_{\pi_1}^e\right) \leq C_i(a_i^e) \leq C_i(b_i^e)$ for all i, the two functions

$$\begin{aligned} h_1(\lambda_{\pi_1}) &= X_{\pi_1}^{mcs}\left(C; \left(\bigvee_{j\neq \pi_1} a_j^1\right) \wedge \lambda_{\pi_1}^e\right) \\ h_2(\lambda_{\pi_1}) &= X_{\pi_1}^{mcs}\left(C; \left(\bigvee_{j\neq \pi_1} b_j^1\right) \wedge \lambda_{\pi_1}^e\right) \end{aligned}$$

coincide on $[0; a_{\pi_1}]$. Specifically:

$$X_{\pi_1}^{mcs}(C; a) = X_{\pi_1}^{mcs}\left(C; \left(\left(\bigwedge_{j\neq \pi_1} a_j^1\right) \wedge a_{\pi_1}^e\right)\right) = X_{\pi_1}^{mcs}\left(C; \left(\left(\bigwedge_{j\neq \pi_1} b_j^1\right) \wedge a_{\pi_1}^e\right)\right):$$

Hence, from (4.6), a_{π_1} maximizes $U_{\pi_1}(\lambda_{\pi_1}; h_2(\lambda_{\pi_1}))$ on R_+. By Lemma 4.2, a_{π_1} is also the only maximizer of $U_{\pi_1}(\lambda_{\pi_1}; h_1(\lambda_{\pi_1}))$ as well. From (4.6) again $U_{\pi_1}\left(a_{\pi_1}; h_1\left(a_{\pi_1}\right)\right) = U_{\pi_1}\left(b_{\pi_1}; h_2(b_{\pi_1})\right)$, hence $a_{\pi_1} = b_{\pi_1}$, as desired.

The proof of $a_i = b_i$ follows from an induction argument. Let there be an i such that $a_j = b_j$, for all j such that $C_j\left(a_j^e\right) < C_j\left(a_i^e\right)$. Then $X_i^{mcs}\left(C; \left(\bigvee_{j \neq i} a_j^1\right) \wedge \lambda_i^e\right) = X_i^{mcs}\left(C; \left(\bigvee_{j \neq i} b_j^1\right) \wedge \lambda_i^e\right)$ for all λ_i in $[0; a_i]$. The above argument can be replicated to show that $a_i = b_i$. ■

5. Conclusions

In this paper we have studied the problem of how the cost of recycling the residuals generated by several firms should be assigned. We have examined the problem from both, an axiomatic approach and a game theoretical approach.

The first question that we dealt with is how heterogeneous goods should be compared. Our proposal is the adoption of the cost function as the key to the comparison of different production levels. The main justification for this is the very nature of the problem: Since we have to share production costs, the particular functional properties of the cost function cannot be forgotten when comparing production levels. In keeping with this concept, we have proposed a cost-sharing rule that reflects the general idea of considering equivalent production levels that are associated to the same production cost.

We adapted the underlying idea of two classical properties, relative to equity in sharing costs, to the case in which comparisons are cost-dependent (in the sense that we explained above), and formulated the notions of Cost-Based Equal Treatment and the Serial Principle. Our study of such properties has shown that there is only one accurate method of sharing the cost of recycling residuals. We call this model the multi-commodity serial rule.

Finally, we examined the agents' behavior when each firm selects its own production level and knows that the total cost will be shared in accordance with the multi-commodity serial rule. We find that there is only one production level supported by an equilibrium.

6. Appendix 1

Axiom 2 introduced the notation of q_i^e as the production vector in which firm i considers that all the other firms produce at the same level as it does. We now show that, under Assumption 1, the function that selects, for each q_i, the production level for firm i, the vector q_i^e is differentiable.

Lemma 6.1. Let C be a smooth function in $\mathcal{C}$, and let $E : \mathbb{R}_+ \to \mathbb{R}^n$ be such that for any $\alpha \geq 0$, $E_i(\alpha) = \alpha$, and $C_i(E(\alpha)) = C_j(E(\alpha))$, for each j. Then, each E_j is differentiable in $\mathbb{R}_{++}$.

Proof. Let us now construct the auxiliary function $F : \mathbb{R}^n_+ \to \mathbb{R}$, defined by $F(q) = C_i(q) - C_j(q)$. Note that $F(q) = 0$ if, and only if, $q_j = E_j(q_i)$. Furthermore, since C is strictly increasing and differentiable, we have that, for any production level q, and any firm h,

$$\frac{\partial C(q)}{\partial q_h} > 0$$

The implicit function $F(q) = 0$ is therefore differentiable and so we get

$$\frac{\partial F(q)}{\partial q_h} \neq 0$$

for any q and h. We can therefore apply the Theorem of the implicit function at the surface $F(q) = 0$. By observing that $F(q) = 0$ if, and only if, $q_j = E_j(q_i)$, the result follows. ■

Lemma 6.2. Let $\tilde{q} \in \mathbb{R}^n_+$, and let S be a set of firms, $\emptyset \subseteq S \subset N$, such that $i \notin S$. Consider now, the function $\tilde{E} : \mathbb{R}_+ \to \mathbb{R}^n$ be such that for any $\alpha \geq 0$, $\tilde{E}_j(\alpha) = E_j(\alpha)$, for each j in $N \setminus S$, and $\tilde{E}_h(\alpha) = \tilde{q}_h$ for each $h \in S$. Then, the function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that $\phi(\alpha) = C\left(\tilde{E}(\alpha)\right)$ is convex for any smooth and strictly convex function C in $\mathcal{C}$.

Proof. We shall show that ϕ is always locally convex. This fact guarantees the convexity of such a function.

First, note that by Lemma 6.1, $\tilde{E}_j$ is differentiable for each j in S. This implies that, for any α,

$$\tilde{E}_j(\alpha + t) - \tilde{E}_j(\alpha) \approx \frac{d\tilde{E}_j(\alpha)}{d\alpha} t$$

for every θ sufficiently small, and $t \in]\alpha - \theta; \alpha + \theta[$. Furthermore, since C is strict convex, we have that, for each t and α,

$$\phi(\alpha + t) - \phi(\alpha) = C\left(\tilde{E}(\alpha + t)\right) - C\left(\tilde{E}(\alpha)\right) >$$

$$
\begin{aligned}
&> \sum_{j \in N \setminus S} \frac{\partial C\left(\bar{E}(\alpha)\right)}{\partial q_j} \left(\bar{E}_j(\alpha + t) - \bar{E}_j(\alpha)\right) \approx \\
&\approx \sum_{j \in N \setminus S} \frac{\partial C\left(\bar{E}(\alpha)\right)}{\partial q_j} \frac{d\bar{E}_j(\alpha)}{d\alpha} t = \frac{d\phi(\alpha)}{d\alpha} t
\end{aligned}
$$

Function ϕ is, therefore, a local convex function for all α, which means that it is convex. ■

Lemma 6.3. Let f_1 and f_2 be two strictly increasing, differentiable and strictly convex functions applying $\mathbb{R}_+$ in $\mathbb{R}$. Let a be such that $f_1(a) = f_2(a)$ and $f_2(x) < f_1(x)$ for each $x < a$. Then, the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$
f(x) = \begin{cases} f_1(x) & \text{if } x \leq a \\ f_2(x) & \text{if } x \geq a \end{cases}
$$

is strictly convex.

Proof. We should first note that, since $f_2(x) < f_1(x)$ for each $x < a$, the function $f_1 - f_2$ must be non-increasing in α, hence:

$$
\frac{df_1(a)}{dx} \leq \frac{df_2(a)}{dx}. \tag{6.1}
$$

Let us now construct the function

$$
g_1(x) = \begin{cases} f_1(x) & \text{if } x \leq a \\ f_1(a) + \frac{df_1(a)}{dx}(x - a) & \text{if } x \geq a \end{cases}
$$

Note that such a function is convex, and strictly convex for $x < a$. Moreover, since f_2 is strictly convex, we have that

$$
f_2(x) \geq f_2(a) + \frac{df_2(a)}{dx}(x - a)
$$

and, for $x > a$, and by considering Expression (6.1), we have that

$$
f_2(x) \geq f_2(a) + \frac{df_1(a)}{dx}(x - a) = g_1(x).
$$

Function f can therefore be expressed by

$$f(x) = \max\{g_1(x); f_2(x)\}$$

which is convex by the convexity of g_1 and f_2. Strict convexity comes from the fact that this property is satisfied by g_1 for $x < a$ and f_2 for values of x that are greater than a. ■

We can now provide a formal proof of Lemma 4.3.

Proof of Lemma 4.3.

Let $\hat{q} \in \mathbb{R}^n_+$ be a production vector, and let S be a set of firms, $\emptyset \subseteq S \subset N$, such that $i \notin S$. Consider now, the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(q_i) = X_i^{mcs}\left(C; \left(\left(\bigwedge_{j \in S} \hat{q}_j^1\right) \wedge q_i^e\right)\right). \tag{6.2}$$

We shall show that f is strictly convex for any set S. We shall proceed by induction on the cardinality of S.

Let us first assume that S is the empty set. Then, Expression (6.2) can be rewritten as

$$f(q_i) = \frac{1}{n} C(q_i^e).$$

Now, by applying Lemma 6.2, we have that f is strictly convex in q_i.

Let us now suppose that S contains only one agent. Without loss of generality, we shall assume that $S = \{1\}$. In such a case, f follows the expression

$$f(q_i) = \begin{cases} \frac{1}{n} C(q_i^e) & \text{if } \hat{q}_1^e \geq q_i^e \\ \frac{1}{n-1}[C(\hat{q}_1^1 \wedge q_i^e) - C(\hat{q}_1^e)] + \frac{1}{n} C(\hat{q}_1^e) & \text{if } \hat{q}_1^e < q_i^e \end{cases}$$

We must note, here, that f is continuous.

Let $f_1(q_i) = \frac{1}{n} C(q_i^e)$, and $f_2(q_i) = \frac{1}{n-1}[C(\hat{q}_1^1 \wedge q_i^e) - C(\hat{q}_1^e)] - \frac{1}{n} C(\hat{q}_1^e)$. Since C is strictly increasing, we have $f_2(q_i) < f_1(q_i)$ for each q_i such that $\hat{q}_1^e < q_i^e$; Moreover, $f_2(q_i) = f_1(q_i)$ when $q_i^e = \hat{q}_1^e$. Since C is strictly convex, the two functions f_1 and f_2 are also strictly convex. Then, Lemma 6.3 confirms that f is strictly convex.

To conclude our proof, we shall assume that f is strictly convex for any set S with a cardinal $k < n - 1$. We will show that it is also strictly convex for S having a cardinal k. Without loss of generality, let us now assume that $S = \{1; \ldots; k\}$, and $\hat{q}_1^e \geq \hat{q}_h^e$ for any h in $S \setminus \{1\}$. By the induction hypothesis, the function

$f^k(q_i) = X_i^{mcs}\left(C;\left(\bigvee_{h=2}^{k}\bar{q}_j^{-1}\right)\wedge q_i^e\right)$ is strictly convex in q_i. We shall define the function f_2 by

$$f_2(q_i) = \frac{1}{n-k}\left[C\left(\left(\bigwedge_{j\in S}\bar{q}_j^{-1}\right)\wedge q_i^e\right) - C\left(\left(\bigwedge_{j=2}^{k}\bar{q}_j^{-1}\right)\wedge \bar{q}_1^e\right)\right] + f^k(E_i(\bar{q}_1)),$$

where $E_i(\bar{q}_1)$ is the i-th component of $\bar{q}_1^e$, i.e., the value $\bar{q}_i$ for which $C_1(\bar{q}_1^e) = C_i(\bar{q}_i^e)$ holds. Combining the strict convexity of cost function C and the results established in Lemma 6.2, we confirm the strict convexity of function f_2.

Given the demands of the agents in $S\setminus\{1\}$, the function f can be expressed by

$$f(q_i) = \begin{cases} f^k(q_i) & \text{if } \bar{q}_1^e \geq q_i^e \\ f_2(q_i) & \text{if } \bar{q}_1^e < q_i^e \end{cases},$$

Note that f is continuous, $f_2(q_i) < f^k(q_i)$ for each q_i such that $\bar{q}_1^e < q_i^e$; moreover, $f_2(q_i) = f_1(q_i)$ when $q_i^e = \bar{q}_1^e$. By Lemma 6.3 we can therefore conclude that f is strictly convex. ■

Remark 1. Lemma 4.3 assumes the smoothness of the cost functions. We should point out, however, that such an assumption is only introduced to simplify the proofs that we have presented here, but is not indispensable for the formal proofs. In fact, Lebesgue's Theorem guarantees that the functions we have analyzed throughout this appendix are differentiable almost everywhere. Henceforth, given a point in which some function is not differentiable, we can analyze the local behavior of such a function at the referred point, relative to the subgradients of the function at that point. The question is: Which subgradient should be chosen? If we simply consider the limit of the gradients for any succession of points, in the function domain, as being increasing and converging to the point in question. The reader can easily verify that this argument yields the desired results. Obviously, if we do not assume differentiability, we lose the simplicity of the proofs. We can only hope that the reader will share our opinion that what is gained in simplicity out-weighs any possible loss from the introduction of this additional assumption.

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