CHOICE FUNCTIONS: RATIONALITY RE-EXAMINED

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A B S T R A C T

On analyzing the problem that arises whenever the set of maximal elements is large, and a selection is then required (see Peris and Subiza, 1998), we realize that logical ways of selecting among maximals violate the classical notion and axioms of rationality. We arrive at the same conclusion if we analyze solutions to the problem of choosing from a tournament (where maximal elements do not necessarily exist). So, in our opinion the notion of rationality must be discussed, not only in the traditional sense of external conditions (Sen, 1993) but in terms of the internal information provided by the binary relation.

KEYWORDS: Rationality; Choice Functions; Maximal Elements.

1 Introduction.

One of the most common models employed in economic and social sciences is that of describing individual choices by means of a maximization problem: the individual makes choices by selecting, from each feasible set of alternatives, those which maximize his own preference relation. There are two different ways in which such a procedure may fail to be useful:

1. The set of maximal elements is too large, and a selection among this set is required.

2. The set of maximal elements is empty.

Consider the following examples that illustrate such cases.

Example 1 An individual chooses according to the binary relation R defined on $X = \{a, b, c, d, e\}$ by:

$$aPe, bPc, bPd, cPd$$

(the non-mentioned relationships being indifferences), but when more than one maximal exists he selects those which he prefers over the highest number of alternatives (see Peris & Subiza, 1998). Thus, for instance,

$$F(\{a, b, c, e\}) = \{a, b\};\ F(\{a, b, c\}) = \{b\}.$$

Example 2 An individual chooses according to the binary relation R defined on $X = \{a, b, c, d, e\}$ by:

$$aPb, aPd, aPe, bPc, cPa, cPd, cPe, dPe$$

(the non-mentioned relationships being indifferences), but when maximals do not exist, he chooses the Copeland winners. Thus, for instance,

$$F(\{a, b, c, d, e\}) = \{a, c\};\ F(\{a, c\}) = \{c\}.$$

If we observe the above choice functions, and ask ourselves if they are rational, in the sense that if a binary relation R^* exists such that

$$F(A) = Maximals \ of \ R^* \ in \ A \qquad for \ all \ A \subseteq X,$$

the answer is *no.*

The rationality of a choice function is a familiar theme in social choice theory and has been extensively studied. Basically, most of the results are devoted to finding conditions (which, in a sense, may be interpreted as individual coherence of choice or, in Sen's words, *internal consistency of choice* (Sen, 1993)) that ensure the rationality of the choice function. Such a notion, however, has been discussed from many different points of view (see, for instance, Sen (1993) where this author argues "against a priori imposition of requirements of internal consistency of choice"). As Sen points out, the reasons for violations of rationality are "easily understandable when the external context is spelled out". In other words, the choice function violates internal consistency due to some *external* conditions (good manners, additional information about the menu, freedom of choice, ...) which are independent from the real preferences.

In our context, however, those violations may be due to choosing by using additional information from the binary relation, which is not affected by external conditions. The previous examples show how rationality conditions are not fulfilled by *reasonable* choice functions which are not influenced by external contexts. So, as pointed out by Schwartz (1986), "how reasonable is rationality?". In fact, we can consider the individuals in our examples to be, in a certain sense, "more than rational", since they are able to distinguish, among maximals, and select the "better ones", or to find the "best elements"

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when the set of maximals is empty. It seems clear that rational individuals should choose maximal elements, but what rationality, in the classical sense, entails is that:

the individual chooses maximals, all maximals, and nothing but maximals.

In this paper, we are interested in defining weaker notions of rationality, so that reasonable choice functions (as the ones defined in the previous examples), which fail to be rational in the usual sense, may fulfill them.

2 Preliminaries.

Throughout the paper X denotes the finite set of all conceivable alternatives, whereas $\mathcal{P}(X)$ represents the family of all non-empty subsets of X; each $A \in \mathcal{P}(X)$ is called an issue (or agenda) and R denotes a complete and reflexive binary relation defined on X. From R the two following relations (the symmetric and asymmetric part, respectively) are defined as usual,

indifference: $xIy \Leftrightarrow xRy \text{ and } yRx$;

strict preference: $xPy \Leftrightarrow xRy \text{ and } not(yRx)$.

The *transitive closure* of the asymmetric part of a binary relation R is denoted by P^∞ and is defined by:

$xP^\infty y \Leftrightarrow \exists\, x_1, x_2, ..., x_{k-1}, x_k \in A$ such that $x = x_1 P x_2 P ... P x_{k-1} P x_k = y$.

The set of maximal elements of a complete binary relation R on a subset $A \in \mathcal{P}(X)$ will be denoted by

$$M(A, R) = \{x \in A \mid xRy \text{ , } for \text{ } all \text{ } y \in A\}.$$

We will use the following types of binary relations. Let R be a binary relation defined on X, it is said to be:

– A *preorder* if $xRyRz$ implies xRz, for all $x, y, z \in X$.

– An *interval-order* if $xPyRzPt$ implies xPt, for all $x, y, z, t \in X$.

– A *semiorder* if it is an interval-order such that whenever $xPyPz$, for any t then xPt or tPz, for all $x, y, z, t \in X$.

– *Quasi-transitive* if $xPyPz$ implies xPz, for all $x, y, z \in X$.

– *Acyclic* if $x_1Px_2P...Px_k$ implies $not(x_kPx_1)$, for all $x_1, x_2, ..., x_k \in X$.

A *choice function* is a functional relationship, $F : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ such that, for every $A \in \mathcal{P}(X)$, $F(A)$ is a non-empty subset of A, which represents those outcomes chosen by the individual or society. The choice function F is said to be *rational*, if there is a complete binary relation R such that for all $A \in \mathcal{P}(X)$, $F(A) = M(A, R)$. It must be noted that, in our framework, this condition implies that the binary relation R is acyclic. Moreover, if a choice function F is rational, the binary relation which rationalizes it coincides with the *base relation* R_F defined as follows:

$$aR_Fb \text{ if and only if } a \in F(\{a, b\}) \qquad for\ all\ a, b \in X,$$

so the notion of rationality can be rewritten as:

$$F(A) = M(A, R_F) \qquad \forall\ A \subseteq X.$$

(This condition is also known in the literature as *binariness* (Deb, 1983), or *binary choice property (BICH)* (Schwartz, 1986)).

Different axioms have been used in order to characterize rational choice functions (see Moulin, 1985). The following one will be useful in our discussion:

Sen: for all $B \in \mathcal{P}(X)$ and all $a \in B$:

$$a \in F(B) \Leftrightarrow \{a \in F(\{a, b\}), \text{ for all } b \in B\}.$$

A choice function is rational if and only if it satisfies *Sen*. The next axiom

is a necessary condition for a choice function to be rational,

$$Chernoff\colon B \subseteq B' \Rightarrow F(B') \cap B \subseteq F(B) \qquad for\ all\ B, B' \in \mathcal{P}(X).$$

The notion of rationality has been relaxed, in the literature, in different ways. For instance, Deb (1983) introduced the notion of *sub-rational* choice function F, if for some order R, $M(A, R) \subseteq F(A)$, $for\ all\ A \in \mathcal{P}(X)$ (see Moulin, 1985). In contrast to the classical notion of rationality, if a choice function is sub-rational, the binary relation which provides this property does not necessarily coincide with the base relation, although in this case the base relation is acyclic and $M(A, R) \subseteq M(A, R_F)$. A condition which characterizes sub-rationality is given by Deb (1983):

(Deb)   for all $B \in \mathcal{P}(X)$, there is $x^* \in F(B)$ such that

$$x^* \in A \subseteq B \Rightarrow x^* \in F(A)$$

3   Reconsidering rationality 1: choosing among maximals.

We begin this section with a survey of some specific ways of choosing, among the elements in $M(X, R)$, the "better ones" by using "internal" additional information obtained from the binary relation R.

If the preferences of an individual happens to be a preorder, then the maximal elements in a feasible set are *equivalent*, in the sense that if x and y are maximal elements, they are both simultaneously indifferent, or preferred, to any other element. In such a case, no additional information can be obtained from the binary relation in order to choose among the maximal elements. This is not the case of more general kinds of binary

relations, which, as pointed out by Luce (1956), represent the individual's behavior in a better way. In Luce (1956) one way of selecting among the maximal elements of a binary relation R is presented. In order to define this selection, a previous result would be required.

Definition 1 *(Luce, 1956). Given the binary relation R defined on A, the binary relation R^* is defined as follows: $xR^*y \Leftrightarrow not(yP^*x)$, where*

$$xP^*y \Leftrightarrow \begin{cases} xPy, \ or \\ \exists \ z \ such \ that \ xPz, zIy, \ or \\ \exists \ z \ such \ that \ zIx, zPy. \end{cases}$$

Theorem 1 *(Luce, 1956) The binary relation R defined on A is a semiorder if and only if R^* is a preorder. Moreover, $\emptyset \neq M(A, R^*) \subseteq M(A, R)$.*

Definition 2 *Let R be a complete and reflexive binary relation defined on A. Luce's maximals are the maximal elements of R^*, $LM(A, R) = M(A, R^*)$.*

The problem with Luce's selection, when R is a general preference relation (interval-orders, quasiorders or acyclic binary relations), is that such a selection may be empty-valued. In order to propose a nonempty selection of the set of maximal elements in those cases , in Peris and Subiza (1998) two other ways of choosing among maximals are presented (*undominated maximals* and *strong maximals*).

Definition 3 *Let R be a complete, reflexive and acyclic binary relation defined on A, and let $a, b \in A$. It is said that a **dominates** b **in** A (aD_Ab), if*

$$\text{for all } x \in A, \qquad bPx \Rightarrow aPx; \; bIx \Rightarrow aRx; \; and$$

$$\text{there is } z \in A, \text{ such that} \qquad \{aPz, zRb\} \text{ or } \{aIz, zPb\}.$$

*Then, the **set of undominated maximals in** A consists of*

$$UM(A, R) = \{x \in A \mid xRy \text{ for all } y \in A; \text{ and for no } z \in A : zD_Ax\}$$

It is obvious, from this definition, that undominated maximals are a selection of the set of maximal elements. Moreover, if the binary relation is an interval-order, the elements in $UM(A, R)$ are *equivalent,* that is, if $x, y \in UM(A, R)$ then xPz if and only if yPz, and xIz if and only if yIz, for all $z \in A$.

An alternative way of choosing among maximal elements, consists of selecting those maximals which are preferred (direct or indirectly) to the greatest number of alternatives.

Definition 4 *Let R a complete, reflexive and acyclic binary relation defined on A. To each alternative a in A we assign the number:*

$$u(a; A) = \#\{x \in A \mid aP^\infty x\}$$

*The set of **strong maximals of** R on A consists of*

$$SM(A, R) = \{x \in A \mid for \; no \; y \in A : u(y; A) > u(x; A)\}$$

Function $u(x; A)$ in the above definition is a specific weak-utility function representing the binary relation R, that is, satisfying that

$$xPy \Rightarrow u(x; A) > u(y; A),$$

and it is now obvious that the alternatives that maximize a weak-utility function are maximal elements; strong maximals are therefore a selection of the set of maximal elements.

Every strong maximal is undominated, so this is a more discriminating selection. The next result summarizes the relationship between maximals, Luce's maximals, undominated maximals and strong maximal elements, depending on the type of relation we consider.

Theorem 2 *(Peris & Subiza, 1998). Let R be a complete and reflexive binary relation defined on the (finite) set A. Then, if R is*

a) acyclic, $M(A, R) \supseteq UM(A, R) \supseteq SM(A, R) \neq \emptyset$

b) an interval-order, $M(A, R) \supseteq UM(A, R) = SM(A, R) \neq \emptyset$

c) a semiorder, $M(A, R) \supseteq LM(A, R) = UM(A, R) = SM(A, R) \neq \emptyset$

d) a preorder, $M(A, R) = LM(A, R) = UM(A, R) = SM(A, R) \neq \emptyset$

Definition 5 *Let R be a complete and reflexive binary relation defined on the (finite) set X. Then, it is possible to define the following choice functions:*

*a) If R is acyclic, **the undominated choice function** assigns for all $A \in \mathcal{P}(X)$, the set $UM(A, R)$*

*b) If R is acyclic, **the strong maximal choice function** assigns for all $A \in \mathcal{P}(X)$, the set $SM(A, R)$*

*c) If R is a semiorder, **Luce's choice function** assigns for all $A \in \mathcal{P}(X)$, the set $LM(A, R)$*

As $LM(A, R) = M(X, R^*)$, it is obvious that Luce's choice function is rational. Nevertheless, a choice function coming from an individual who

is not only able to obtain the maximal elements, but also to differentiate among them by choosing either undominated or strong maximals, does not satisfy the usual axioms of rationality, so it would be criticized for being non rational, even non sub-rational. The following example shows this fact.

Example 3 Let $X = \{a, b, c, d\}$ and the binary relation on X defined by:

$$aPc, \ bPd,$$

being indifferences the remainder pairwise relations. If we define the choice function

$$F(A) = UM(A, R) = SM(A, R), \text{ for all } A \in \mathcal{P}(X),$$

we obtain:

$$F(\{a, b, c, d\}) = \{a, b\}; \ F(\{a, b, c\}) = \{a\}; \ F(\{a, b, d\}) = \{b\};$$

$$F(\{a, c, d\}) = \{a\}; \ F(\{b, c, d\}) = \{b\}; \ F(\{a, b\}) = \{a, b\};$$

$$F(\{a, c\}) = \{a\}; \ F(\{a, d\}) = \{a, d\}; \ F(\{b, c\}) = \{b, c\};$$

$$F(\{b, d\}) = \{b\}; \ F(\{c, d\}) = \{c, d\}; \ F(\{x\}) = \{x\} \text{ for all } x,$$

which does not satisfy either *Chernoff* or *Deb.*

Our objective is, therefore, to propose a weaker notion of rationality, that is adequate for considering individuals who can, *in a logical way*, select some of their maximal elements. In order to do so, it seems natural to relax the classical notion of rationality by asking for the existence of a complete binary relation R such that for all $A \in \mathcal{P}(X)$, $F(A) \subseteq M(A, R)$. It is clear that this binary relation has to be acyclic, since the existence of maximal elements in every feasible set is ensured by the non-emptiness of the choice function. A closer look at the above generalization, will convince us that it is void because every choice function fulfills it (consider the binary relation

that makes all the elements indifferent).

A possible approach, in order to generalize rationality in a non-trivial way, would require that the condition $F(A) \subseteq M(A, R)$, apply to some particular binary relation. In so doing, as we have already mentioned, if a choice function F is rational, then the binary relation which rationalizes it coincides with the base relation. We use this binary relation in the next definition.

Definition 6 *A choice function F is called* **basically-rational (b-rational** *in what follows) if for all $A \in \mathcal{P}(X)$,*

$$F(A) \subseteq M(A, R_F).$$

It is clear that rationality implies b-rationality. Moreover, as in the case of rational choice functions, b-rationality implies that the base relation R_F is acyclic. The converse however is not true, as we show in the following example.

Example 4 Let $X = \{a, b, c\}$, and the choice function defined by:

$$F(X) = \{a, b\}; F(\{a, b\}) = \{a\}; F(\{a, c\}) = \{a\};$$

$$F(\{b, c\}) = \{b\}; F(\{x\}) = \{x\} \qquad \text{for all } x \in X.$$

The base relation is acyclic (in fact, it is the order $aP_F bP_F c$) but $F(X)$ is not contained in $M(X, R_F)$, so F is not *b-rational*.

The above example also shows that b-rationality is not a trivial condition, in the sense that not all choice functions are b-rational. The idea of this definition is to require that the individual be completely rational in

pairwise comparisons; that is to say, when the feasible set has just two elements, the individual must choose both of them if and only if he considers these elements to be *"equally good"* for him.

In order to analyze how strong b-rationality is, some considerations can be made. Given a choice function we can consider the (non-empty) family of binary relations:

$$\mathcal{R}(F) = \{R \in R(X) \mid F(A) \subseteq M(A, R), \text{ for all } A \in \mathcal{P}(X)\}.$$

If there is a binary relation R^* such that its maximal elements coincide with $\bigcap\limits_{R \in \mathcal{R}(F)} M(A, R)$, such a relation will be the minimal one (with respect to the set-inclusion of its maximal elements) in $\mathcal{R}(F)$. There are some clear cases in which such a relation exists: the most obvious example is that of a rational choice function, where R^* coincides with the base relation. In the following Proposition we show that this relation always exists (and coincides with the revealed preference relation).

Proposition 1 *Let $F : \mathcal{P}(X) \to \mathcal{P}(X)$ be a choice function. Then, there exists a binary relation R^* such that:*

a) $F(A) \subseteq M(A, R^*)$ for all $A \in \mathcal{P}(X)$.

b) If for some binary relation R it is satisfied that

$$F(A) \subseteq M(A, R) \qquad \text{for all } A \in \mathcal{P}(X),$$

then $M(A, R^*) \subseteq M(A, R)$ for all $A \in \mathcal{P}(X)$.

Moreover, F is b-rational if and only if $R^* = R_F$.

Proof. Define relation R^* as follows (revealed preference relation):

$$x \, R^* \, y \Leftrightarrow \exists \, B \in \mathcal{P}(X) \text{ such that } x, y \in B \text{ and } x \in F(B).$$

From this definition, it is clear that if $a \in F(A)$, then $a \, R^* \, x$ for all $x \in A$,

so $a \in M(A, R^*)$ and $F(A) \subseteq M(A, R^*)$. If R is any binary relation such that $F(A) \subseteq M(A, R)$, then $a \in M(A, R^*)$ implies aR^*y for all $y \in A$. Thus, some $B_y \in \mathcal{P}(X)$ will exist such that, $a, y \in B_y$ and $a \in F(B_y) \subseteq M(B_y, R)$, which implies aRy for all $y \in A$, so $M(A, R^*) \subseteq M(A, R)$.

Finally, if $R^* = R_F$ it is obvious that the choice function is b-rational. Conversely, if F is b-rational, $M(\{x, y\}, R^*) \supseteq F(\{x, y\}) = M(\{x, y\}, R_F) \supseteq M(\{x, y\}, R^*)$, so $R^* = R_F$. ∎

From the above result, a question arises: can we presume that a choice function is b-rational or, if not R^* is the trivial relation (all alternatives are indifferent)?. Example 2 shows that this is not the case, since F is not b-rational and the binary relation R^* is:

$$aI^*b, aP^*c, bP^*c.$$

We know that rational choice functions can be characterized in terms of some coherence properties which involve the behavior of the choice function when the set presented for choice expands or contracts. The following axiom characterizes the b-rationality of a choice function:

Axiom 1 *(A1):*   *for all $B \in \mathcal{P}(X)$ and all $a \in B$:*
     $a \in F(B) \Rightarrow \{a \in F(\{a, b\}), \text{ for all } b \in B\}$.

One can readily see that condition *(A1)*, also called *Inverse Condorcet Property* in the literature, is part of Sen's property, so that it is a necessary condition (though not sufficient) for rational choice functions. This condition can be also interpreted as a weak Chernoff condition, applying only to binary subsets. As mentioned in Deb (1983), condition *(A1)* has a simple intuitive interpretation: "if x is picked in some set B it should never be rejected in

pairwise choice for all pairs which are subsets of B". The elemental proof of the characterization result is omitted.

Theorem 3 *A choice function is b-rational iff it satisfies (A1).*

Sub-rationality, as we have already mentioned, requires that a choice function contains a rational selection; the idea behind this notion (or a possible interpretation of it) is that the individual chooses some non-maximal alternatives due, for instance, to some lack of perception (as in the famous *coffee and sugar* example, Luce 1956). Our analysis has the converse intuition: it may be the case that the agent knows his maximal elements perfectly, and moreover, chooses among them in a specific way. The following example then, shows that sub-rationality and b-rationality are independent conditions.

Example 5 Let $X = \{a, b, c\}$ and the choice function defined by:

$F(\{x\}) = \{x\}$, $x \in X$; $F(\{a, b\}) = a$; $F(\{a, c\}) = a$;

$F(\{b, c\}) = b$; $F(\{a, b, c\}) = \{a, b, c\}$.

It is sub-rational, but it does not satisfy *(A1)*.

Conversely, consider the choice function F defined on $X = \{a, b, c, d\}$ by:

$F(\{a, b, c, d\}) = \{b\}$; $F(\{a, b, c\}) = \{a\}$; $F(\{a, b, d\}) = \{a\}$;

$F(\{a, c, d\}) = \{a\}$; $F(\{b, c, d\}) = \{b\}$; $F(\{a, b\}) = \{a, b\}$;

$F(\{a, c\}) = \{a\}$; $F(\{a, d\}) = \{a\}$; $F(\{b, c\}) = \{b\}$;

$F(\{b, d\}) = \{b\}$; $F(\{c, d\}) = \{c\}$; $F(\{x\}) = \{x\}$, *for all x*.

This choice function is not sub-rational, yet it is easy to observe that *(A1)* holds.

When analyzing rationality, several additional conditions can be found in the literature that provide more information about the binary relation that rationalizes the choice function. For instance, the *Arrow axiom* characterizes rationality by means of a preorder, whereas *Aizerman, Chernoff and Expansion axioms* characterize rationality by means of a quasitransitive relation (see, for instance, Moulin (1985)). It is possible to analyze conditions in order to ensure such properties of the base relation in the case of b-rationality. The Aizerman axiom, and a weak modification of the Arrow axiom, are sufficient to imply, respectively, the quasitransitivity and transitivity of the base relation.

Aizerman: for all $A, B \in \mathcal{P}(X)$, $F(B) \subseteq A \subseteq B \Rightarrow F(A) \subseteq F(B)$.

Axiom 2 *(A2):* for all $a, b, c \in X$, and for all $\{x, y\} \subset \{a, b, c\}$,

$$F(\{a, b, c\}) \cap \{x, y\} \neq \emptyset \Rightarrow F(\{x, y\}) = F(\{a, b, c\}) \cap \{x, y\}.$$

Theorem 4 *Let $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ be a choice function satisfying (A1).*
a) If F satisfies Aizerman, then F is quasitransitive b-rational (R_F is quasitransitive).
b) If F satisfies (A2), then F is transitive b-rational (R_F is transitive).

Proof. We know that (A1) implies b-rationality. In order to prove the quasitransitivity in a), consider $x, y, z \in X$ such that $xP_F y, yP_F z$, that is,

$$F(\{x, y\}) = \{x\}, F(\{y, z\}) = \{y\}.$$

(A1) implies that the only possibility for $F(\{x, y, z\})$ is,

$$F(\{x, y, z\}) = \{x\},$$

and then *Aizerman* implies $F(\{x, z\}) = \{x\}$, that is $xP_F z$.

The proof of b) runs parallel to the one in Moulin (1985), Theorem 3, and it is omitted. ■

The following example shows a choice function satisfying *(A1)*, *(A2)* and *Aizerman*, which is not rational in the classical sense.

Example 6 Consider the choice function F defined on $X = \{a, b, c, d\}$ as follows:

$$F(X) = \{a, b, c\}; \; F(A) = A \text{ for all } A \in \mathcal{P}(X), \; A \neq X.$$

It is clear that F satisfies the axioms in Theorem 4, but it is not rational, since it does not satisfy Sen's axiom.

In the following result, we prove that the *undominated* and the *strong maximal* choice functions satisfy b-rationality.

Proposition 2 *1) The undominated choice function is b-rational.*

2) The strong maximal choice function is b-rational.

Proof. 1) Let R be a binary relation, and consider the choice function defined by:

$$F(A) = UM(A, R) \subseteq M(A, R) \text{ for all } A \in \mathcal{P}(X).$$

If we prove that R coincides with R_F, then the choice function is b-rational. Let $a, b \in X$ such that $aP_F b$; this implies $F(\{a, b\}) = \{a\}$, and therefore $UM(\{a, b\}, R) = \{a\}$, that is aPb. The converse is also true, so that we have $aP_F b$ if and only if aPb.

2) Analogous to part 1). ■

The following example shows that there are b-rational choice functions that cannot be expressed either as the undominated maximals, or as the strong maximals of the base relation.

Example 7 Consider the choice function F defined in $X = \{a, b, c, d\}$ as:

$F(\{x\}) = \{x\}$, for all $x \in X$; $F(X) = \{c\}$; $F(\{a, b, c\}) = \{a\}$;

$F(\{a, b, d\}) = \{a\}$; $F(\{a, c, d\}) = \{a\}$; $F(\{b, c, d\}) = \{c\}$;

$F(\{a, b\}) = \{a\}$; $F(\{a, c\}) = \{a, c\}$; $F(\{a, d\}) = \{a\}$;

$F(\{b, c\} = \{c\}$; $F(\{b, d\}) = \{b\}$; $F(\{c, d\}) = \{c\}$

Is is obvious that $F(A) \subseteq M(A, R_F)$, so the choice function is b-rational. Nevertheless, $F(X) \neq UM(X, R_F) = SM(X, R_F) = \{a, c\}$.

4 Reconsidering rationality 2: there are not maximal elements.

So far, we have tried to extend the notion of rationality by looking for a binary relation R in such a way that the choice function is a subset of the set of maximal elements of R. This fact implies that such a binary relation is acyclic, and there are well-known choice functions defined for more general binary relations This is the case, for instance, that of the solutions to the problem of choosing from *tournaments*: complete and assymetric binary relations P, where aPb is interpreted as *"alternative a beats alternative b"*. This kind of binary relation arises in many different models: sports competitions, biometric and psychometric models, collective choice,... (see Moulin, 1986).

An important approach to discussing the notion of rationality may be found in Schwartz (1986). He argues that if the choice function comes from the aggregation of several individual preferences (social choice functions), "the impossibility theorems show [rationality] to be unreasonable as a *gen-*

eral assumption". This fact gives sense to the analysis of the problem of choosing from binary relations which may have no maximal elements and, for instance, with this aim Schwartz (1986) introduces two choice functions, namely *GETCHA* and *GOCHA*. We now introduce these solutions.

Definition 7 *Given a binary relation R defined on X, and given $A \in \mathcal{P}(X)$, a subset B of A is said to be **dominant** in A, if B is nonempty and xPy for every $x \in B$ and every $y \in A - B$. Moreover, B is a **minimum dominant** subset of A if B is dominant and no proper subset of B is dominant.*

Schwartz (1986) proves that if P is an asymmetric binary relation then every set A has a unique minimum dominant subset, so we have the following definition.

Definition 8 *Given a binary relation R the **GETCHA** choice function is given by: for all $A \in \mathcal{P}(X)$,*

$\qquad G_e(A) = $ minimum dominant subset of A.

Definition 9 *Given a binary relation R defined on X, and given $A \in \mathcal{P}(X)$, a subset B of A is said to be **undominated** in A, if B is nonempty and $not(yPx)$ for every $x \in B$ and every $y \in A - B$. Moreover, B is a **minimum undominated** subset of A if B is undominated and no proper subset of B is undominated.*

Unlike the dominant sets, there is not single minimum undominated subset, so the following choice function is defined in terms of the union of minimum undominated subsets.

Definition 10 *Given a binary relation R the **GOCHA** choice function is defined by: for all $A \in \mathcal{P}(X)$,*

$$G_o(A) = \text{ the union of minimum undominated subsets of } A.$$

Another important solution function for choosing the "best" elements whenever maximals do not exist, is given by the notion of *uncovered set* (introduced by Miller (1977) and Fishburn (1977) for asymmetric binary relations, and extended for general binary relations in Peris and Subiza (1999)). Formally,

Definition 11 *Let R be a complete and reflexive binary relation defined on A, and let $a, b \in A$. It is said that a **covers** b **in** A (aC_Ab), if*

aPb*; and*

for all $x \in A$, $bPx \Rightarrow aPx$; $bIx \Rightarrow aRx$.

Definition 12 *Given a binary relation R the **Uncovered** choice function is defined by:*

$$UC(A, R) = \{x \in A \mid \text{ for no } y \in A : yC_Ax\}.$$

Apart of the above mentioned choice functions, other ways of choosing in non acyclic binary relations have been introduced in the literature. We must mention the important notion of *minimal covering* (Dutta 1988; also extended for general binary relations in Peris and Subiza (1999)), as well as the *bipartisan set* (Laffond, Laslier and Le Breton, 1995), the *essential set* (Dutta and Laslier (1999)), and the *Copeland set* (Copeland, 1951, Henriet, 1985), among others.

None of these choice functions satisfy the classical notion of rationality. In order to extend such a notion to the context we are now analyzing, where maximal elements may or may not exist, it must be mentioned that it seems natural to assume that maximals, provided they exist, must be selected (*Condorcet consistency*). All the above mentioned solution functions satisfy Condorcet consistency and we can therefore define an extension of the notion of rationality in an opposite direction as in the previous section: by looking for a complete binary relation R such that, for all $A \in \mathcal{P}(X)$,

$$F(A) \supseteq M(A, R).$$

The idea is to ask for the same condition as in Deb (1983), without imposing acyclicity on the binary relation. It must be noted, however, that every choice function satisfies this condition: it is sufficient to define the binary relation P as follows:

$$xPy \qquad for\ all\ x, y \in X, x \neq y.$$

Since the set of maximal elements in every subset with at least two elements is empty, the condition is obviously fulfilled.

By following a parallel analysis as in the previous section, we ask that the base relation be the one which fulfills this condition.

Definition 13 *A choice function F is called **basically-sub-rational (bs-rational** in what follows) if for all $A \subseteq X$,*

$$F(A) \supseteq M(A, R_F).$$

The following axiom characterizes bs-rationality (the elemental proof of the result is omitted).

Axiom 3 *(A3):   for all $B \subseteq X$ and all $a \in B$:*

$$\{a \in F(\{a,b\}), \text{ for all } b \in B\} \Rightarrow a \in F(B).$$

Theorem 5 *A choice function is bs-rational iff it satisfies (A3).*

Note that this axiom, known in the literature as *Direct Condorcet Property*, is the *other* part of *Sen*'s axiom (the first half is axiom *(A1)*). This is not surprising, since *b-rationality* plus *bs-rationality* coincide with the classical notion of rational choice functions. The next result, taken from Deb (1983), shows how asking for any additional property on the base relation, gives rise to the notion of *sub-rationality*.

Theorem 6 *(Deb, 1983). A choice function $F : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is bs-rational with R_F being acyclic iff F satisfies Deb axiom.*

In the following result, we prove that the usual solution concepts we have introduced in this section define choice functions which are *bs-rational* (we only prove it for the *GETCHA, GOCHA* and *Uncovered* choice functions, although the result is also true for the *minimal covering* and the *essential set*).

Proposition 3 *1) The GETCHA choice function is bs-rational.*
2) The GOCHA choice function is bs-rational.
3) The Uncovered choice function is bs-rational.

Proof. It is easy to observe that, in all three cases, the base relation R_F coincides with the binary relation R which defines the choice functions.

1) It is clear that $M(A, R_F)$ is contained in every dominant subset of A, since there is not $x \in A$ such that $x P_F a$ for $a \in M(A, R_F)$. So $M(A, R_F)$ is contained in the minimum dominant subset.

2) If $a \in M(A, R_F)$, then it is obvious that $\{a\}$ is a minimum undominated subset, and $M(A, R_F)$ is contained in the union of the minimum undominated subsets.

3) Since a maximal element is not covered by any other, $M(A, R_F)$ is contained in the uncovered subset. ∎

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