

REPEATED GAMES WITH PROBABILISTIC HORIZON*

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ABSTRACT

Repeated games with probabilistic horizon are defined as those games where players have a common probability structure over the length of the game's repetition, T . In particular, for each t , they assign a probability p_t to the event that "the game ends in period t ".

In this framework we analyze Generalized Prisoners' Dilemma games in both finite stage and differentiable stage games. Our construction shows that it is possible to reach cooperative equilibria under some conditions on the distribution of the discrete random variable T even if the expected length of the game is finite. More precisely, we completely characterize the existence of sub-game perfect cooperative equilibria in finite stage games by the (first order) convergence speed: the behavior in the limit of the ratio between the ending probabilities of two consecutive periods. Cooperation in differentiable stage games is determined by the second order convergence speed, which gives a finer analysis of the probability convergence process when the first convergence speed is zero.

Leptokurtic distributions are defined as those distributions for which the (first order) convergence speed is zero and they preclude cooperation in finite stage games with probabilistic horizon. However, this negative result is obtained in differential stage games only for a subset of these distributions.

Keywords: Repeated Games, probabilistic horizon, cooperation.

JEL Classification System: C72

1 INTRODUCTION

Assumptions about the length of players' horizons often have profound implications for their behavior. As is well known, equilibria in infinitely repeated games are characterized by folk theorems. However, in instances where the stage game gives rise to a unique equilibrium, a large but finite horizon does not allow players to sustain anything other than the repetition of the stage game equilibrium.

Games with infinite horizon and constant discounting imply that the end of the game never gets any closer (in a probability sense). Yet, the expected horizons of agents do not remain constant, in general, over time. In this context the analysis of games with unknown horizons is of very practical interest.

Repeated games with probabilistic horizon are defined as those games where players have a common structure of probability over the length of the game's repetition, T . In particular, they assign a probability p_t to the event that "*the game ends in period t* ". In this framework we analyze Generalized Prisoners' Dilemma games in both finite stage and differentiable stage games.

Papers in this line of research are Bernheim and Dasgupta (1995, BD hereafter) and Jones (1998, 1999, J hereafter). The former authors study a class of repeated games that retains the desirable features of both finite and infinite horizon assumptions, while avoiding their undesirable features. They refer to this class of games as games with "asymptotically finite" horizon: there is always a positive continuation probability but it varies over time. Their results have an application in both differentiable stage and finite stage games. For the first ones BD obtain a sufficient and necessary condition over continuation probabilities for the existence of cooperative equilibria. For finite games they show a sufficient condition for negative results. J (1998, 1999) analyzes cooperation in finite stage games through linear strategies. In his 98 paper the set of cooperation vectors associated with subgame perfect publicly correlated equilibria is examined, meanwhile the 99 paper focuses on the conditions for the existence of cooperative subgame perfect equilibria.

Our construction shows that it is possible to reach cooperative equilibria under some conditions on the distribution of the discrete random variable T even if the expected length of the game is finite. More precisely, we completely characterize the existence of sub-game perfect cooperative equilibrium

in finite stage games by the (first order) convergence speed: the behavior in the limit of the ratio between the ending probabilities of two consecutive periods. The convergence speed, in turn, measures the speed at which p_t converges to zero as t tends to infinity. It allows to completely classify the set of distributions over the length of the game in three subsets. In one of them cooperation is always sustained, in another one it never occurs and finally, in the third one cooperation depends on the payoff matrix of the game.

Our approach to modeling the uncertainty over the length of the game allows us to solve analytically a wider family of problems than those previously analyzed in the literature. "*Leptokurtic distributions*" are defined as those distributions whose convergence speed is zero. This is the class of probability distributions of the length of the game, for which no cooperative equilibrium exists, independently of the finite stage game under consideration. Also, we relate them with both quasi-finite (J, 1999) and asymptotically finite (BD, 1995) continuation probabilities. Moreover, since by changing the structure of the probability over the future, we vary the way in which cooperation may be sustained, our approach allows us to work with a wider family of games of unknown length. Thus, finite stage games with probabilistic horizon unify the analysis for finitely repeated game without discount factor, infinitely repeated games with discount factor and infinitely repeated games with limit average payoffs. Common and wide families of probability distributions over the length of the game as the positive Poisson, the geometric, the harmonic and the negative binomial distributions, among others, are useful to analyze and characterize the existence of cooperation.

We reinterpret BD (1995) results for differentiable games and classify them according to both the convergence speed of the probability distribution over the length of the game and the convergence speed of its logarithm. Also, just a subset of the leptokurtic distributions precludes cooperation in these classes of games.

To conclude our work we present a classification of the distributions over the length of any probabilistic Generalized Prisoners' Dilemma game which allows to ascertain the existence of cooperation in both finite and differentiable stage games. This taxonomy is made according to the behavior of both the first and the second order convergence speeds. We classify discrete distributions in three categories which include five subcases and we analyze in which of them cooperation is attainable.

The paper is organized as follows. Section 2 sets up the probabilistic horizon model and defines the associated statistical concepts. The main results for both finite and differentiable stage games are presented in section 3. In subsections 3.1-3.2, we present the finite stage model and we characterize the existence of cooperation by means of the first order convergence speed. Subsection 3.3, introduces the class of distribution for which cooperation is never attainable in finite stage games and is related with alternative negative results. The analysis of differentiable stage games is undertaken in subsection 3.4, where the existence of cooperation is characterized in terms of the second order convergence speed. Finally, section 4 concludes the paper by offering a summary and a classification of probabilistic horizon games.

2 THE MODEL

2.1 The stage game

Let us consider a simultaneous move game, G , played by N players. Each player i selects an action $s_i \in S_i$, where S_i is a subset of R , which may be either finite, (for the analysis of finite stage games) or compact (for the case of differentiable stage games). Let $S = \times_{i=1}^N S_i$, and let s denote an element of S . The payoff to each player i is given by the function $\pi_i : S \rightarrow R$.

Let $s = (s_i, s_{-i})$ be an action profile where $s_i \in S_i$ and $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$ and define

$$\phi_i(s) = \max_{s'_i \in S_i} \pi_i(s'_i, s_{-i}) - \pi_i(s)$$

as a function that measures the maximum gain that player i can achieve by deviating unilaterally from s . The existence of this function is obviously guaranteed when S_i is a finite set and under continuity of π_i when S_i is a compact set. Trivially, $s \in S$ is a Nash equilibrium of G iff $\phi_i(s) = 0$ for all i . In other words, if any deviation from s is a loss to any player.

We consider the class of the generalized Prisoners' Dilemma games introduced by BD (1995). Thus, assume that in the stage game there is a unique Nash equilibrium, s^* , and normalize payoffs so that $\pi_i(s^*) = 0$ for all i . Further, there exists at least one efficient action s^0 whose payoffs strictly Pareto dominates those of the Nash equilibrium.

2.2 The probabilistic horizon repeated game

Assume that the stage game is repeatedly played by the same players an unknown number of times.

Players have a common probability structure over the length of the repeated game. More precisely, if we define the discrete random variable T as the length of the repeated game, the players will assign a probability p_t to the event $T = t$,

$$Pr(T = t) = p_t, \quad \forall t = 1, 2, \dots, \quad p_t \geq 0, \quad \sum_{t=1}^{\infty} p_t = 1$$

Once the random variable T is defined as above, denote by $E[T]$ the expected length of the game. More precisely,

Definition 1 (Expected length of the game) *If the length of the repeated game, T , is a random variable with distribution $\{p_t\}_{t=1}^{\infty}$, the expected length of the game is defined as the expected value of the random variable T , i.e.,*

$$E[T] = \sum_{t=1}^{\infty} tp_t \quad (1)$$

Notice that the expected length of the game could not exist if the series in (1) diverges. In this case $E[T] = \infty$ and it means that players believe that the stage game is going to be repeated forever.

Players maximize average expected payoffs. For any sequence of action profiles $\bar{s} = \{s_t\}_{t=1}^{\infty}$ and distribution $\{p_t\}_{t=1}^{\infty}$, player i will receive a payoff of $\pi_i(s_1)$ if the game lasts for just one period, an event that has probability p_1 . If the game lasts for two periods, with probability p_2 , player i will receive a payoff of $\pi_i(s_1) + \pi_i(s_2)$. In general, if the game is played during t periods, an event that has probability p_t , player i will receive a payoff of $\pi_i(s_1) + \dots + \pi_i(s_t)$. The payoff function to each player in the repeated game is defined by the limit of average expected payoffs.

Thus, for any sequence of action profiles $\bar{s} = (s_1, s_2, \dots)$, player i 's expected average payoff is given by¹

$$\begin{aligned} \pi_i(\bar{s}) &= \lim_{t \rightarrow \infty} \frac{1}{\sum_{n=1}^t np_n} (p_1\pi_i(s_1) + \dots + p_t(\pi_i(s_1) + \dots + \pi_i(s_t))) = \\ &= \lim_{t \rightarrow \infty} \frac{1}{\sum_{n=1}^t np_n} \sum_{j=1}^t \left(\sum_{n=1}^j \pi_i(s_n) \right) p_j = \\ &= \lim_{t \rightarrow \infty} \frac{1}{\sum_{n=1}^t np_n} \sum_{n=1}^t \left(\sum_{j=n}^t p_j \right) \pi_i(s_n) \end{aligned} \quad (2)$$

if the limit exists.

¹By Fubini's theorem we can change the order of the summations.

When the expected length of the game is finite and the payoffs of the stage game are bounded, the above expression translates to:

$$\frac{1}{E[T]} \sum_{n=1}^{\infty} \left(\sum_{j=n}^{\infty} p_j \right) \pi_i(s_n).$$

On the other hand, $E[T]$ could be ∞ but then either i) player i 's payoffs for all actions in \bar{s} are bigger than a fixed value, $\pi_i(s_n) > \alpha > 0$ for all n , and then the limit in (2) exists and $\pi_i(\bar{s}) > \alpha$, or ii) \bar{s} has a finite set of actions different from the Nash equilibrium and then $\pi_i(\bar{s}) = 0$ for all i .

We refer to the above defined repeated game as a *probabilistic horizon game*. Notice that this model is more general than the usual discount factor model for infinitely repeated games. In the latter, the discount factor is the subjective present value of one payoff unit received after a delay of one period. However, the discount factor also represents the knowledge that players have about the exact length of the game. In other words, players, at present, do not give all the value to future payoffs since they cannot be sure that the game will be infinitely played. Taking $p_t = (1 - \delta)\delta^{t-1}$, $\delta \in (0, 1)$, the payoff function becomes the one of the traditional discounted (infinitely) repeated game.

While the game is played the expected length of the game is updated each period by Bayes' rule, so that if the game reaches period t , then the expected length of the game will be conditioned upon this event. In general,

Definition 2 (Conditional expected length) *The expected length of the game after period t has been played is the conditional expected value of the random variable T , i.e.,*

$$E[T|T \geq t] = \frac{1}{1 - p_1 - \dots - p_{t-1}} \sum_{n=t+1}^{\infty} np_n = \frac{\sum_{n=t}^{\infty} np_n}{\sum_{n=t}^{\infty} p_n} \quad (3)$$

The conditional expected length of the game is the simplest way to sum up the information contained in the distribution of T .

Moreover, since $E[T|T \geq t]$ is the total number of periods expected to be played after period t , the next definition expresses the expected extra number of periods to be played after period t and it plays a key role under our approach.

Definition 3 *The remaining expected length of the game is*

$$E[T|T \geq t] - t = \frac{\sum_{n=t}^{\infty} (n-t)p_n}{\sum_{n=t}^{\infty} p_n} \quad (4)$$

The value $E[T|T \geq t] - t$ gives players the expected number of periods they have to punish any deviation in period t .

Throughout the paper, we will focus on the subgame perfect Nash equilibria. Repeating the Nash equilibrium of the stage game, s^* , in every period is clearly a subgame perfect Nash equilibrium; we will refer to it as the degenerate equilibrium.

Definition 4 (Cooperative equilibrium) *A cooperative equilibrium is any subgame perfect equilibrium with actions different from those of s^* for some history of the game. Repeating the Nash equilibrium, s^* , in every period and for any history is a degenerate equilibrium.*

3 MAIN RESULTS

Our construction shows that it is possible to reach cooperative equilibria under some conditions on the distribution of the discrete random variable T even if the expected length of the game is finite, i.e. $E[T] < \infty$.

The conditions for cooperation depend on the class of stage games being analyzed: finite games or differentiable games. For the former both the value of the payoffs in the stage game and the distribution of the length of the game are the key factors. For the latter only the structure of the underlying distribution has to be considered.

The difference between these two classes of games relies on the compactness of the players' action space. This compactness is used to design punishments to support subgame perfect equilibria. The intuition tells us that the longer the expected length of the game the more cooperation is possible. In games with a low probability to be infinitely played, future payoffs have a small probability to be reached and there is no room to punish any deviation. Moreover, if the remaining length of the game decreases as the game goes on, then subgame perfect equilibria can only be supported when the gains of deviations from the equilibrium path decrease as time goes on. This is only possible in general if the game has a compact action space.

In next sections we show our results for both finite stage games and differentiable stage games, both with common knowledge about the probability distribution of the length of the game.

3.1 Finite stage games

We will consider first finite stage games with probabilistic horizon in which the distribution on T is equal for both players and common knowledge. We will show that, in most of the cases, the necessary and sufficient condition for cooperation depends not only on the distribution, $\{p_t\}_{t=1}^{\infty}$, but also on the payoff matrix. This condition has a natural interpretation in terms of the expected value of the length of the players' horizons, and it is translated to the behavior in the limit of the ratio between the ending probabilities of two consecutive periods.

A strategy such that players will play a fixed collusive outcome in any period can be played whenever deviations can be punished. A rough way

to think of it is the following: if player i deviates from action s' in period t he wins $\phi_i(s')$ and in the remaining periods he will be punished by a loss of $\pi_i(s')$. (Recall that we have normalized the payoff of the Nash equilibrium so that $\pi_i(s^*) = 0$ for all i .) Thus, the ratio between these two values gives us the number of times that player i has to be punished. If players estimate that the expected length of the game is equal to or bigger than this ratio any deviation can be punished and hence cooperation could be obtained. Of course, if the expected length of the game is lower than $\frac{\phi_i(s')}{\pi_i(s')}$ no collusion is attainable.

We are concerned with the existence of strategies that can support subgame perfect cooperative equilibria. As in BD (1995) and J(1998, 1999) we select an action that has relatively the least benefit from deviations. Thus, consider the action s^0 such that it verifies the following two conditions:

- i) $\pi_i(s^0) > \pi_i(s^*) \quad \forall i \in N$
- ii) $s^0 \in \arg \min_{s \in S} \left\{ \max_{i \in N} \frac{\phi_i(s)}{\pi_i(s)} \right\}$

That is, s^0 is an efficient action with respect to s^* yet it also minimizes the gains from deviations. By assumption there exists at least one action in the stage game which verifies condition i) and we select among them the one which also verifies condition ii).

We look for the conditions that ensure the existence of cooperative equilibria. To this end we remove two types of distributions which yield degenerate cases.

Consider first a terminating p-function as defined by Carroll (1987): a distribution $\{p_t\}_{t=1}^{\infty}$ is a p-function if $p_t = 0$ for all $t > t_0$, with t_0 a fixed integer. In a probabilistic horizon game in which the distribution is a terminating p-function the only equilibrium is the degenerate one, consisting on the repetition of s^* as long as the game is played. The proof of this result is trivial by using backward induction. As Becker (1990) points out this kind of games is equivalent to the class of finitely repeated games.

Alternatively, we could model any finitely repeated game as a probabilistic horizon repeated game in the following way. For an l -times repeated game, consider the distribution $\{p_t\}_{t=1}^{\infty}$ such that $p_t = 0$ for all $t \neq l$, and $p_l = 1$. Player i 's expected average payoff is $\sum_{n=1}^l \pi_i(s_n)/l$. Now we could analyze the behavior in the limit, when the length of the finitely repeated game grows. Thus, we obtain a degenerate discrete distribution $\{p_t^{\infty}\}_{t=1}^{\infty}$ which represents

the infinitely repeated game, i.e. $p_t^\infty = 0$ for all t . This is a degenerate case in the sense that $\sum p_t$ is zero, but it arises as the limit of a proper distribution and thus player i 's expected average payoff can be calculated as $\lim_{l \rightarrow \infty} \sum_{n=1}^l \pi_i(s_n)/l$, i.e. the usual time-average payoff (Fudenberg and Tirole, 1991).

Given the above discussion, we only consider discrete distributions such that there exists an infinite number of periods with probability of ending different from zero. More precisely,

Condition (C): given $\{p_t\}_{t=1}^\infty$ there exists an infinite and strictly increasing sequence of integer numbers $\{t_l\}_{l=1}^\infty$ such that $p_t > 0$ if and only if $t \in \{t_l\}_{l=1}^\infty$.

The following proposition gives the condition for the existence of cooperative equilibria. Recall that by definition of s^0 if there exists an efficient strategy which plays any cooperative action in all periods, then there exists a strategy which plays s^0 in all periods. Moreover, it is sufficient to consider trigger strategies because any subgame perfect outcome can be supported by a trigger strategy (Abreu, 1988).

Proposition 1 *In any probabilistic horizon Generalized Prisoners' Dilemma game which satisfies condition (C), there exists a cooperative equilibrium which plays s^0 in all periods if the distribution of the length of the games satisfies*

$$\max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0)} \right\} \leq E[T|T \geq t] - t \quad \forall t > 0 \quad (5)$$

Proof: A strategy which plays s^0 in all periods is a subgame perfect equilibrium if any deviation can be punished. Suppose that player i deviates from cooperation in period t , so that he obtains an extra gain of $\phi_i(s^0)$. In the remaining periods both players play s^* , but the length of this phase is unknown. The punishment loss per period is $\pi_i(s^0) - \pi_i(s^*) = \pi_i(s^0)$, and the number of punishment periods is the remaining length of the repeated game. Then, players estimate the expected punishment loss as,

$$(E[T|T \geq t] - t)\pi_i(s^0)$$

Thus, s^0 is a subgame perfect equilibrium if the one stage gain for deviation is smaller than the expected punishment loss, for all $t_0 > 0$ and for all i , i.e

$$\phi_i(s^0) \leq (E[T|T \geq t] - t)\pi_i(s^0) \quad \forall t > 0, \quad \forall i \in N$$

which is (5). □

Note that the right hand side of (5) is, in general, bigger than zero, ($E[T|T \geq t] > t$, for all t), but it does not have to be bigger than $\max_i \{\frac{\phi_i(s^0)}{\pi_i(s^0)}\}$ as t increases. To determine whether or not (5) is verified, we need to know the behavior in the limit of the conditional expected length of the game.

3.2 The remaining expected length of the game

Intuitively it is clear that if the remaining expected length of the game after any period, $E[T|T \geq t] - t$, is infinity there is room to punish properly any deviation and, hence, there exists cooperative equilibria; similarly, if, after any period, it is some constant different from zero then, it is possible to find, under some assumptions, cooperative equilibria in the repeated game. But if, as the game goes on, this remaining expected length goes to zero then players may not have time to punish deviations and only degenerate equilibria will be available as subgame perfect equilibria.

Next results characterize the behavior in the limit of $E[T|T \geq t] - t$. First,

Lemma 1 *Let $\{p_t\}_{t=1}^\infty$ be a distribution, with $\{t_l\}_{l=1}^\infty$ an infinite strictly increasing sequence of integer numbers such that $p_t > 0$ if and only if $t \in \{t_l\}_{l=1}^\infty$. Then, $\forall t, t'$ such that $t_{l-1} < t' < t < t_l$ it is verified that*

$$E[T|T \geq t'] - t' > E[T|T \geq t] - t > E[T|T \geq t_l] - t_l$$

Proof. Recall that $p_t = 0$ if $t_{l-1} < t < t_l$. Then the proof is straightforward since by (3)

$$E[T|T \geq t] = \frac{\sum_{n=t}^\infty np_n}{\sum_{n=t}^\infty p_n} = \frac{\sum_{n=t_l}^\infty np_n}{\sum_{n=t_l}^\infty p_n} = E[T|T \geq t_l] \quad t_{l-1} < t \leq t_l$$

□

Lemma 1 says that if the probabilistic horizon game is moving along non-ending periods the remaining expected length of the game decreases, since the conditional expected length does not change².

Unfortunately, there does not exist a clear relationship between $E[T|T \geq t_{l-1}] - t_{l-1}$ and $E[T|T \geq t_l] - t_l$, but the analytical expression of $E[T|T \geq t] - t$ involves series and succession terms (see (4)), and thus we can use different results concerning both the convergence of series and the limit of successions to obtain its behavior in the limit. In particular, the First Stolz's convergence criterion transforms the limit of partial sums (series) into the easier limit of successions. We state it next for completeness.

Proposition 2 (First Stolz's Criterion) *Let $\{a_n\}$ and $\{b_n\}$ be two successions converging to zero, such that*

$$b_1 > b_2 > \dots > b_n > \dots$$

If the limit of $\frac{a_{n+1} - a_n}{b_{n+1} - b_n}$ is finite or infinite with a proper sign, then

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

Our approach has a simple intuition. From a statistical viewpoint the conditional expected value of a random variable depends on where the probability mass of this variable is concentrated: either around some specific value, or around all the values of the variable (dispersed). To study whether or not this random variable is concentrated we need to know the weights of the tails. In our setting, a measure of how fast the tails go to zero (if we understand them as functions of time) is to evaluate the behavior of $\frac{p_{t+1}}{p_t}$ in the limit.

The above criterion allows us to calculate the behavior in the limit of $E[T|T \geq t] - t = \frac{\sum_{n=t}^{\infty} (n-t)p_n}{\sum_{n=t}^{\infty} p_n}$ when both the numerator and the denominator are considered as successions of t . And, it turns out that this behavior depends on the ratio $\frac{p_{t+1}}{p_t}$.

²Although players update the probability distribution by Bayes' rule period by period, the underlying distribution does not vary.

Moreover, since the behavior in the limit of $\frac{p_{t+1}}{p_t}$ is also a way to measure the speed at which p_t goes to zero, cooperation in repeated games with probabilistic horizon can be easily characterized by the underlying probability distribution over the length of the game. In fact, the order of convergence of p_t depends on the order of convergence of the mass of probability concentrated in either far periods or along all the distribution. Thus, define

Definition 5 (Convergence speed) *If $\{p_t\}_{t=1}^\infty$ is a distribution such that $p_t > 0$ for all t , the measure at which p_t tends to zero, the convergence speed, is defined as $\lim_{n \rightarrow \infty} \frac{p_{t+1}}{p_t}$. The closer to zero this limit the faster p_t decreases.*

Now, D’Alambert’s criterion on convergence of series helps us to bound the possible values of the convergence speed. We also state it next for completeness.

Proposition 3 (D’Alambert’s criterion) *Let $\sum a_n$ be a series and let $\lambda = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \in R \cup \{+\infty\}$, if it exists. Then*

- i) $\lambda < 1 \Rightarrow \sum a_n$ is absolutely convergent $\Rightarrow \sum a_n$ is convergent*
- ii) $\lambda > 1$ (or $\lambda = +\infty$) $\Rightarrow \sum a_n$ is divergent*
- iii) $\lambda = 1$ the criterion does not give a precise answer.*

Thus, given that $\sum_{n=1}^\infty p_n = 1$ and by the above criterion, we have that

Corollary 1 $\lim_{n \rightarrow \infty} \frac{p_{t+1}}{p_t} \in [0, 1]$.

The next Theorem characterizes the behavior in the limit of $E[T|T \geq t] - t$ according to the convergence speed (the proof appears in the Appendix). This result is our first contribution to the analysis of finite stage games with probabilistic horizon and it allows to completely classify the set of distributions over the length of the game in three subsets. In one of them cooperation is always sustained, in another one it never occurs and finally, in the third one cooperation depends on the payoff matrix of the game.

Theorem 1 *Assume that $\lim_{l \rightarrow \infty} \frac{p_{t_l+1}}{p_{t_l}}$ exists then,*

$$\lim_{l \rightarrow \infty} E[T|T \geq t_l] - t_l = \begin{cases} 0 & \Leftrightarrow \lim_{l \rightarrow \infty} \frac{p_{t_l+1}}{p_{t_l}} = 0 \\ \frac{\gamma}{1-\gamma} (0 < \gamma < 1) & \Leftrightarrow \lim_{l \rightarrow \infty} \frac{p_{t_l+1}}{p_{t_l}} = \gamma \\ \infty & \Leftrightarrow \lim_{l \rightarrow \infty} \frac{p_{t_l+1}}{p_{t_l}} = 1 \end{cases}$$

To illustrate our findings let us consider three natural families of discrete distributions each one corresponding to each of the above cases: the positive Poisson distribution, the geometric distribution and the harmonic distribution. In the three cases $p_t > 0$ for all t .

- *The positive Poisson distribution:* $p_t = (e^\lambda - 1)^{-1} \lambda^t / t!$ for $t = 1, 2, \dots$. It is not difficult to see that $\frac{p_{t+1}}{p_t} = \frac{\lambda}{t+1}$, which converges to zero as t tends to infinity. The expected value for T under this distribution is $\frac{\lambda}{1-e^{-\lambda}}$, that goes from 1 as $\lambda \rightarrow 0$, to ∞ as $\lambda \rightarrow \infty$.

- *The geometric distribution:* $p_t = (1-\gamma)\gamma^{t-1}$. The geometric distribution verifies a property which characterizes it: $Pr(T = T_0 + t | T \geq T_0) = Pr(T = t)$ for all T_0 and t . Thus, $E[T | T \geq t] - t$ is constant and equal to $\frac{\gamma}{1-\gamma}$. ($E[T] = \frac{1}{1-\gamma}$, that goes from 1 as $\gamma \rightarrow 0$, to ∞ as $\gamma \rightarrow 1$).

- *The harmonic distribution:* $p_t = K/t^\alpha$, with $\alpha > 1$ and K a normalization parameter equal to $1/\sum \frac{1}{t^\alpha}$. It is not difficult to check that $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 1$ and that $E[T] = \infty$ if $1 < \alpha \leq 2$.

We summarize the remaining expected length of the game behavior for these three distributions in the next Corollary. To prove it just combine the previous Theorem and the definitions of the distributions.

Corollary 2 *The value of $\lim_{t \rightarrow \infty} E[T | T \geq t] - t$ is,*

- i) zero, if p_t is distributed as a positive Poisson distribution.*
- ii) $\frac{\gamma}{1-\gamma}$, if p_t follows a geometric distribution with ratio γ .*
- iii) infinity, if p_t is distributed according to a harmonic distribution.*

Conditions for the existence of cooperative equilibria are given next. They follow directly from Proposition 1, Lemma 1 and Theorem 1.

Proposition 4 *Consider a finite Generalized Prisoners' Dilemma game with probabilistic horizon determined by the distribution $\{p_t\}_{t=1}^\infty$ which satisfies condition (C) and which defines the payoff function for any path $\{s_t\}_{t=1}^\infty$ as,*

$$\frac{1}{E[T]} \sum_{n=1}^{\infty} \left(\sum_{t=n}^{\infty} p_t \pi_i(s_n) \right)$$

then, there exists a cooperative equilibrium which plays s^0 in all periods if

$$\lim_{l \rightarrow \infty} \frac{p_{t_{l+1}}}{p_{t_l}} > \max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} \right\} \quad (6)$$

Proof: Since $\max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} \right\} \in (0, 1)$, we consider the case $\lim_{l \rightarrow \infty} \frac{p_{t_{l+1}}}{p_{t_l}} = \gamma > \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)}$ for all i , and $\gamma \in (0, 1)$. Or, alternatively that $\frac{\phi_i(s^0)}{\pi_i(s^0)} < \frac{\gamma}{1-\gamma}$ for all i . Now, by Theorem 1, $\lim_{l \rightarrow \infty} E[T|T \geq t_l] - t_l = \frac{\gamma}{1-\gamma}$. Thus, there exists a integer l_0 such that $E[T|T \geq t] - t > \frac{\phi_i(s^0)}{\pi_i(s^0)}$ for all i and $t > t_{l_0}$. By proposition 1 there exists a cooperative equilibrium. \square

A clarification is at hand. In case that (6) is fulfilled with strict equality cooperation is still sustained if the ratio $\frac{p_{t_{l+1}}}{p_{t_l}}$ is never lower than $\max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} \right\}$.

The next two corollaries establish the sufficient (and necessary) conditions for the existence and non-existence, respectively, of cooperative equilibria, independently of the payoffs of the finite stage game under analysis.

Corollary 3 *There exists a cooperative equilibrium in pure strategies for any finite Generalized Prisoners' Dilemma game with probabilistic horizon if and only if $\lim_{l \rightarrow \infty} \frac{p_{t_{l+1}}}{p_{t_l}} = 1$*

Proof: If $\lim_{l \rightarrow \infty} \frac{p_{t_{l+1}}}{p_{t_l}} = 1$, then (Theorem 1) $\lim_{l \rightarrow \infty} E[T|T \geq t_l] - t_l = \infty$, and by (5) in Proposition 1, necessity is trivially satisfied.

Suppose next that there exists a cooperative equilibrium in pure strategies for any generalized Prisoners' Dilemma but $\lim_{l \rightarrow \infty} \frac{p_{t_{l+1}}}{p_{t_l}} = \gamma < 1$. Then consider the next Prisoners' Dilemma game, where $\alpha > 1$,

	C	D
C	1,1	-1, $\frac{\alpha}{1-\gamma}$
D	$\frac{\alpha}{1-\gamma}, -1$	0,0

The unique efficient strategy is to play $s^0 = (C, C)$ in all periods, and for all i , $1 > \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} = \frac{\frac{\alpha}{1-\gamma} - 1}{1 + \frac{\alpha}{1-\gamma} - 1} = \frac{\alpha - 1 + \gamma}{\alpha} > \gamma$. Then by proposition 2 there does not exist a cooperative equilibrium in this game. But this fact

contradicts our assumption. Hence, $\lim_{l \rightarrow \infty} \frac{p_{t+1}}{p_t} = 1$ is a sufficient condition. \square

Next, notice that by Proposition 1 and by Theorem 1, there does not exist a cooperative equilibrium which plays s^0 in all periods if

$$\lim_{l \rightarrow \infty} \frac{p_{t+1}}{p_t} < \max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} \right\}$$

In particular, we have that

Corollary 4 *There does not exist a cooperative equilibrium strategy in pure strategies for any probabilistic horizon finite Generalized Prisoners' Dilemma game if and only if the convergence speed of the length of the game is zero, i.e.*

$$\lim_{l \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$$

Proof: If $\lim_{l \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$, then by Theorem 1, $\lim_{l \rightarrow \infty} E[T|T \geq t_l] - t_l = 0$, which implies that (5) is not satisfied for any t . By Proposition 1, there does not exist a cooperative equilibrium which plays an efficient action of the stage game in all periods.

Suppose now that there does not exist a cooperative equilibrium strategy in pure strategies for any finite Generalized Prisoners' Dilemma but $\lim_{l \rightarrow \infty} \frac{p_{t+1}}{p_t} = \gamma > 0$. Then consider the Prisoners' Dilemma game of the proof of the above corollary, but where $0 < \alpha < 1$, with $\alpha + \gamma > 1$. The unique efficient strategy is to play $s^0 = (C, C)$ in all periods, and for all i we have that $0 < \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} = \frac{\alpha - 1 + \gamma}{\alpha} < \gamma$. Then by Proposition 2 there exists a cooperative equilibrium in this game. But this fact contradicts our assumption. Hence, $\lim_{l \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ is a sufficient condition. \square

The intuition of the above results is clear by Proposition 1 and by Theorem 1. Let us assume that $p_t > 0$ for all t . When $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$, the remaining expected length of the game, after period $t - 1$ has been played, collapses to zero and (6) is never verified. The probability goes to zero too quickly and the future has no relevance. After period $t - 1$, players estimate that the expected length of the game is close enough to t and hence it is not

possible to punish deviations in period t . Thus, cooperation is not obtained in any subgame perfect equilibrium. The game in this case is equivalent to a finitely repeated game and the unique subgame perfect equilibrium is the repetition of the Nash equilibrium of the stage game, s^* .

If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = \gamma$, then the rate at which probabilities decrease is a constant between zero and one³. Now, the remaining expected length of the game after period $t - 1$ has been played is a constant and close to $\gamma/(1 - \gamma)$. In other words, at any period of the game, players expect to play $\gamma/(1 - \gamma)$ more periods. Thus, if the number of periods needed to punish the deviator, $\max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0)} \right\}$, is lower than $\gamma/(1 - \gamma)$ cooperation is possible, and this is equivalent to (6). In this case the expected length of the game is also finite, but, at any period, the remaining expected periods are constant. In other words, no matter how many periods have been played, players always believe that $\gamma/(1 - \gamma)$ periods remain to be played.

Hence, this repeated game with probabilistic horizon is neither equivalent to a finitely repeated game (since at any period the remaining expected periods stay constant), nor to an infinitely repeated game (since the expected length of the game is finite).

Moreover, recall that if $p_t = (1 - \delta)\delta^{t-1}$ then the game with probabilistic horizon becomes an infinitely repeated game with discount factor δ and player i 's expected average payoffs is $(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_i(s_t)$. In this case $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = \delta$ and it is well known that cooperation is possible if and only if $\max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0)} \right\} \leq \delta/(1 - \delta)$.

Finally, if $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 1$ then we have a degenerate case in the sense that the (conditional) expected length of the game goes to infinity. Thus, from some period on, players estimate that they have a huge number of periods to punish any deviation and cooperation holds independently of the parameters of the game given that $\max_i \left\{ \frac{\phi_i(s^0)}{\pi_i(s^0) + \phi_i(s^0)} \right\} < 1$.

This case is similar to an infinitely repeated game where the average payoff of the repeated game is given by the limit of average payoffs, $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \pi_i(s_t)$.

Thus, finite games with probabilistic horizon make it possible to unify the analysis for finitely repeated games without discount factor, infinitely repeated

³This is the case when there exists an external, arbitrary and independent event which makes the game finish and it has a probability γ to occur.

games with discount factor and infinitely repeated games with limit average payoffs.

3.3 Leptokurtic distributions, continuation probabilities and finite games.

This section considers the class of probability distributions of the length of the game, for which no cooperative equilibrium exists, independently of the finite stage game under consideration, and relates it with alternative approaches in the literature. The above Corollary 4 characterizes the class of discrete distributions for which just degenerate equilibria exist, and we define this class of distributions as **leptokurtic** distributions.

Definition 6 (Leptokurtic distributions) *A distribution $\{p_t\}_{t=1}^{\infty}$ is leptokurtic if and only if its convergence speed is zero, i.e.,*

$$\lim_{l \rightarrow \infty} \frac{p_{t_{l+1}}}{p_{t_l}} = 0$$

where $\{t_l\}_{l=1}^{\infty}$ is a strictly increasing sequence of integer numbers such that $p_t > 0$ if and only if $t \in \{t_l\}_{l=1}^{\infty}$.

Probabilistic horizon games are similar to games with continuation probabilities as defined in BD(1995) and⁴ J(1998, 1999). Expected payoffs for these games weigh future payoffs by both the continuation probability and the discount factor.

Definition 7 *A continuation probability is defined by a sequence $\{\beta_t\}_{t=1}^{\infty}$, where β_t is the probability that the game will continue to the k^{th} period given that the $(k-1)^{\text{th}}$ period occurs.*

Both of the above authors analyze the class of continuation probabilities for which cooperative equilibria do not exist, independently of the underlying

⁴While finishing our paper we got to know the work of J(1998,1999). In this section we relate our approach with that of him.

finite Prisoners' Dilemma stage game. In particular, BD (1995) show that if the continuation probabilities are such that $\lim_{t \rightarrow \infty} \beta_t = 0$ and the stage game is finite, then there does not exist any cooperative pure strategy subgame perfect equilibrium. They define this class of continuation probabilities as **asymptotically finite**.

Definition 8 (Bernheim and Dasgupta, 1995) *The continuation probability $\{\beta_t\}_{t=1}^{\infty}$, where $\beta_t \in (0, 1)$, is asymptotically finite if and only if $\lim_{t \rightarrow \infty} \beta_t = 0$.*

On the other hand, J (1998) considers a model where players use publicly correlated strategies and he shows that there does not exist a publicly correlated subgame perfect equilibria for any Generalized Prisoners' Dilemma game if $\lim_{t \rightarrow \infty} \sup_{p \in N} \left(\prod_{j=p+1}^{t+p} \beta_j \right)^{1/t} = 0$. He defines this class of continuation probabilities as **quasifinite**.

Definition 9 (Jones, 1999) *The continuation probability $\{\beta_t\}_{t=1}^{\infty}$, where $\beta_t \in [0, 1]$, is quasifinite if and only if*

$$\lim_{t \rightarrow \infty} \sup_{p \in N} \left(\prod_{j=p+1}^{t+p} \beta_j \right)^{1/t} = 0.$$

Both classes of continuation probabilities are related, so that if a continuation probability is asymptotically finite then it is quasifinite.

Note that under our approach p_t can be expressed as $p_t = \beta_1 \cdots \beta_t (1 - \beta_{t+1})$ or equivalently that

$$\beta_t = \frac{\sum_{n=t}^{\infty} p_n}{\sum_{n=t-1}^{\infty} p_n}$$

Thus, we can relate leptokurtic distributions with both asymptotically finite and quasifinite continuation probabilities. The next results link all the concepts named above.

Corollary 5 *An asymptotically finite continuation probability has an underlying leptokurtic distribution over the length of the game.*

Proof: If $\lim_{t \rightarrow \infty} \beta_t = 0$ then we can consider, without loss of generality, that $\beta_t < 1$ for all t and thus, $p_t = \beta_1 \cdots \beta_t (1 - \beta_{t+1}) > 0$ for all t . Thus,

$$\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = \lim_{t \rightarrow \infty} \frac{\beta_{t+1}(1 - \beta_{t+2})}{1 - \beta_{t+1}} = 0$$

□

The reverse result is not true as we see in the next example. Consider the distribution,

$$p_t = \begin{cases} 0 & t \text{ is even} \\ \frac{n}{n!(n+1)} & t \text{ is odd.} \end{cases}$$

Clearly the convergence speed is,

$$\lim_{l \rightarrow \infty} \frac{p_{2l+3}}{p_{2l+1}} = \lim_{l \rightarrow \infty} \frac{l+1}{l(l+2)} = 0$$

But the continuation probability defined by the above distribution, is,

$$\beta_t = \begin{cases} \frac{2}{t} & t \text{ is even} \\ 1 & t \text{ is odd.} \end{cases}$$

that is not asymptotically finite.

Note that the above continuation probability is not asymptotically finite since it has a subsequence that does not converge to zero. Also, when $\beta_t = 1$ then $p_{t-1} = 0$, but the leptokurtic behavior of the distribution is driven by the convergence to zero of the ratio of the subsequence of strictly positive probabilities. However, the existence of any subsequence converging to zero, does not guarantee that the distribution is leptokurtic: for instance, consider the distribution,

$$p_t = \begin{cases} \frac{1}{3(e^{0.1}-1)} \frac{0.1^{t/2}}{(t/2)!} & t \text{ is even} \\ \left(\frac{1}{2}\right)^t & t \text{ is odd.} \end{cases}$$

It is easy to check that $\lim_{t \rightarrow \infty} \frac{p_{2t+2}}{p_{2t}} = \lim_{t \rightarrow \infty} \frac{\lambda}{t+1} = 0$ but that $\lim_{t \rightarrow \infty} \frac{p_{2t+1}}{p_{2t-1}} = \frac{1}{4}$. Thus, the distribution is not leptokurtic and there exists a cooperative equilibrium for some stage game when played indefinitely. In fact, since

$E[T|T \geq t] - t \geq 5/9$ for all t , by Proposition 1 we can find a Generalized Prisoners' Dilemma stage game such that there exists a cooperative equilibrium for its repetition.

Thus, the class of asymptotically finite continuation probabilities has underlying distributions on T which are a subset of the class of leptokurtic distributions. The next corollary relates quasifinite continuation probabilities with leptokurtic distributions.

Corollary 6 *The continuation probability $\{\beta_t\}_{t=1}^\infty$ is quasifinite if and only if the associated distribution on T is leptokurtic.*

The proof of this corollary follows directly from definitions 6 and 9 and Stolz's criteria for convergence of successions. In this way, the characterization of quasifinite continuation probabilities has a natural translation to their underlying distributions on T . Thinking of distributions instead of continuation probabilities has some advantages: firstly, it is easier to translate players' beliefs over the length of the game by means of a distribution than by a continuation probability. Secondly, from an analytical point of view, the calculus of the convergence speed entails solving an easier limit than knowing whether or not the continuation probability is quasifinite. In fact and even for very standard distributions $\lim_{t \rightarrow \infty} \sup_{p \in N} \left(\prod_{j=p+1}^{t+p} \beta_j \right)^{1/t}$ can only be calculated if the convergence speed ratio is known since, if we consider, without loss of generality, that $p_t > 0$ for all t , then

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(\prod_{j=2}^{t+1} \beta_j \right)^{1/t} &= \lim_{t \rightarrow \infty} \left(\prod_{j=2}^{t+1} \frac{\sum_{n=j}^{\infty} p_n}{\sum_{n=j-1}^{\infty} p_n} \right)^{1/t} = \\ &= \lim_{t \rightarrow \infty} \left(\sum_{n=t+1}^{\infty} p_n \right)^{1/t} = \lim_{t \rightarrow \infty} \frac{\sum_{n=t+2}^{\infty} p_n}{\sum_{n=t+1}^{\infty} p_n} = \\ &= \lim_{t \rightarrow \infty} \frac{p_{t+2}}{p_{t+1}} \end{aligned}$$

where the second equality holds by the functional relationship between any distribution and their associated continuation probability; the fourth by the

second Stolz's criterion⁵ and the fifth by (the first) Stolz's criterion.

Thus, when $\beta_t < 1$ for all t , the asymptotic geometric average of Jones is equal to the limit of the continuation probabilities which, in turn, can be approximated by the limit of the ratio of the underlying probabilities. In other words, the limit of the geometric average of the continuation probabilities is equal to the limit of this succession (Cauchy's criterion), which, by Bayes' rule, is the convergence speed.

Note that the above result and theorem 4.5 in Jones(1999), allow us to extend directly our approach to cooperative equilibria in publicly correlated strategies: at every round t , players publicly correlate their actions and with probability λ_t agree to all play an efficient action (s^0) and with probability $(1 - \lambda_t)$ all play the Nash equilibrium of the stage game, s^* .

3.4 Differentiable stage games

These classes of games was analyzed by BD (1995) by considering asymptotically finite continuation probabilities. They showed under which conditions cooperation is supported. The most important fact is that given the compactness of players' action spaces, the conditions that give rise to cooperative equilibria do not depend on the stage game under consideration, but just on the continuation probability. Hence, cooperation is a property of the continuation probability sequence. Our main contribution in differentiable stage games with probabilistic horizon is to find the threshold of the speed at which the distributions on the length of the game have to converge to sustain cooperation.

Let us recall first some assumptions for differentiable games (also in BD(1995)). Let S_i , be a compact subset of R for each i and assume,

Assumption 1: π_i is C^2 , $\pi(s) = (\pi_1(s), \dots, \pi_N(s))$.

⁵The second Stolz's Criterion says the following : *Let $\{a_n\}$ be a positive succession and $\{b_n\} \rightarrow \infty$ be increasing. If the limit of $\left(\frac{a_{n+1}}{a_n}\right)^{1/(b_{n+1}-b_n)}$ is finite or infinite with a proper sign, then*

$$\lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{a_n} \right)^{1/(b_{n+1}-b_n)} = \lim_{n \rightarrow \infty} a_n^{1/b_n}$$

Assumption 2: $s^* \in \text{Int}(S)$ and $D\pi(s^*)$ is regular, where $D\pi(s)$ is the $N \times N$ matrix of partial derivatives, so that the element i, j is $\frac{\partial \pi_i(s)}{\partial s_j}$.

Assumption 3: There exist a \underline{b} , and $\epsilon > 0$ such that for all $s \in S$ with $|s - s^*| < \epsilon$, $\sum_{i=1}^N \phi_i(s) \geq \underline{b}|s - s^*|^2$.

The first condition establishes a minimum condition in order to work. Assumption 2 ensures that s^* is locally inefficient, i.e. there exists a vector v such that if we move from s^* in the direction of v all players increase their payoffs. Formally, $\exists v \in R^N, \alpha' > 0$ such that $\forall \alpha < \alpha', s^* + \alpha v \in S$, and

$$\left. \frac{d\pi_i(s^* + \alpha v)}{d\alpha} \right|_{\alpha=0} > 0 \quad \forall i$$

Assumption 3 is important since it guarantees local uniqueness of the Nash equilibrium s^* .

Assume that the stage game is repeatedly played an unknown number of times. Repeating the unique s^* in every period, irrespective of the history, is clearly a subgame perfect equilibrium. As said above, by the local inefficiency of s^* (assumption 2) there exists a vector v such that if we move from s^* in the direction of v all players increase their payoffs. Since payoffs are approximately linear within a neighborhood of s^* , we may consider the set of efficient actions defined by $s^* + \alpha v$. The idea is that players choose a sequence $\{\alpha_t\}_{t=1}^{\infty}$ to play $s^* + \alpha_t v$ in period t . If anybody deviates, then players play s^* in all subsequent periods. As long as the gains from deviating in period t are smaller than following the constructed play, this play will be a cooperative subgame perfect equilibrium.

Thus, cooperation in this framework translates to the existence of a sequence of action profiles $\{s_t\}_{t=1}^{\infty}$ such that it verifies

$$\max_i \left\{ \frac{\phi_i(s_t)}{\pi_i(s_t)} \right\} \leq E[T|T \geq t] - t \quad \forall t > 0$$

We have to analyze under which conditions the above sequence exists in our framework. Without loss of generality we assume that $p_t > 0$ for all t .

If the convergence speed is slow, $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} > 0$ then cooperative equilibria will be trivially obtained. In the differentiable case, in contrast to the finite case, cooperation is attainable independently of the payoffs of the

stage game. The next proposition shows this result, which parallels that of BD (1995).

Proposition 5 *If a probabilistic horizon differentiable Generalized Prisoners' Dilemma game satisfies assumptions 1 and 2 and the distribution of the length of the games verifies that $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} > 0$ then there exists a cooperative equilibrium.*

Proof: If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} > 0$, by lemma 1 and Theorem 1, $E[T|T \geq t] - t > \alpha_1$ for all $t > t_0$ and $\alpha_1 > 0$.

On the other hand, by assumptions 1 and 2, there exists $a > 0$, $\alpha_2 > 0$, and a vector v such that for all $\alpha < \alpha_2$, $\pi_i(s^* + \alpha v) \geq a\alpha$. Moreover, the Envelope theorem implies that there exists $b > 0$ such that for all $\alpha < \alpha_2$, $\phi_i(s^* + \alpha v) \leq b\alpha^2$. Hence, if $s^0 = s^* + \alpha v$, we have that $\frac{\phi_i(s^0)}{\pi_i(s^0)} < \frac{b}{a}\alpha$ for all i .

Then, taking $\alpha < \min\{\frac{a}{b}\alpha_1, \alpha_2\}$ the inequality $\frac{\phi_i(s^0)}{\pi_i(s^0)} < E[T|T \geq t] - t$ holds for all $t > t_0$ and player i . Now, without loss of generality, we can rescale time so that period $t_0 + 1$ is period 1, and assume that (5) is satisfied for all t . By proposition 1 there exists a cooperative equilibrium. \square

According to Theorem 1, the above result tells us that players consider that the remaining expected length of the game after any period is bounded from below by a fixed value bigger than 0. Players just have to play an efficient action s_0 (near the Nash equilibrium s^*) in any period such that its payoff, $\pi_i(s_0)$, is bigger than zero, and the gain from a deviation, $\phi_i(s_0)$, is low enough. If some player deviates then s^* is played in all subsequent periods. Clearly, this is an equilibrium since $\phi_i(s_0) \leq (E[T|T \geq t] - t)\pi_i(s_0)$, is satisfied for all i and for all t .

However, when $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ cooperation is still possible under more demanding requirements as the next proposition shows. As we already saw in the above section if p_t goes to zero too quickly the only possible equilibrium in Generalized finite Prisoners' Dilemma games is the repetition of the stage game Nash equilibrium. In contrast, the next Proposition tells us that in differentiable games cooperation is reached even if we assign a low probability to future payoffs, but this probability has a lower bound.

Proposition 6 *Suppose that $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ and assumptions 1 to 3 are satisfied. Then the necessary and sufficient condition for the existence of a cooperative equilibrium is*

$$\lim_{t \rightarrow \infty} \sum_{k=2}^t 2^{-k} \ln(p_k) > -\infty \quad (7)$$

The way to prove Proposition 6 (see the Appendix for a sketch of the proof) is not remarkably different from that used by BD(1995), once we show first the equivalence between the convergence behavior of condition (7) with the one of BD in terms of the underlying distribution⁶ (see step 1 in the proof of the Proposition).

We can understand (7) as a lower bound for the convergence speed. It restricts the distributions $\{p_t\}_{t=1}^{\infty}$ to those that verify $p_t > \lambda^{2^t}$ for some positive⁷ $\lambda < 1$, i.e. it gives us the threshold of the speed at which the distributions on the length of the game have to converge to sustain cooperation. In fact, it classifies the leptokurtic distributions (see definition 6) over T with respect to the distribution λ^{2^t} .

Moreover, $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} > 0$ implies condition (7). If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = \alpha > 0$ then $p_t > \alpha^{t-t_0} p_{t_0}$ for all $t < t_0$. Thus,

$$\lim_{t \rightarrow \infty} \sum_{k=t_0}^t 2^{-k} \ln(p_k) > \lim_{t \rightarrow \infty} \sum_{k=t_0}^t 2^{-k} \ln(\alpha^{k-t_0} p_{t_0}) =$$

⁶The translation of the necessary and sufficient condition given by BD(1995) in terms of the underlying distribution is,

$$\sum_{k=2}^{\infty} 2^{-k} \ln \left(\frac{\sum_{n=k}^{\infty} p_n}{\sum_{n=k-1}^{\infty} p_n} \right) > -\infty$$

⁷If $\{p_t\}$ has a convergence speed equal to zero and verifies (7), then there exists t_0 , a positive number c and two real numbers $\varepsilon, \lambda \in (0, 1)$, such that

$$c\varepsilon^t > p_t > \lambda^{2^t} \quad \forall t > t_0$$

$$\begin{aligned}
&= \sum_{k=t_0}^{\infty} 2^{-k} \ln(\alpha^{k-t_0}) + \sum_{k=t_0}^{\infty} 2^{-k} \ln(p_{t_0}) = \\
&= (\ln(\alpha)(t_0 + 2) + \ln(p_{t_0})) \left(\frac{1}{2}\right)^{t_0-1} > -\infty
\end{aligned}$$

Clearly, by adding the first t_0 terms in the sum, (7) is satisfied.

Condition (7) can be translated to a more operative expression. Notice that, when $\frac{p_{t+1}}{p_t}$ goes to zero and by Theorem 1 also $E[T|T \geq t] - t$ does it, we need a finer analysis of the convergence process, characterized by the above Proposition. This finer analysis proceeds by considering the limit of $\frac{\ln(p_{t+1})}{\ln(p_t)}$. We can consider the convergence speed as a first order behavior of $\{p_t\}_{t=1}^{\infty}$. Thus, the second order convergence speed is defined by,

Definition 10 *If $\{p_t\}_{t=1}^{\infty}$ is a distribution such that $p_t > 0$ for all t , the second order convergence speed is defined as $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)}$, that is always equal to or greater than one if the (first order) convergence speed is zero.*

The next Corollary characterizes cooperation in terms of the second order convergence speed and gives a more useful condition than (7) to ascertain whether or not cooperation is attainable. It is a direct consequence of D’Alambert’s criterion for series.

Corollary 7 *Suppose that $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ and assumptions 1 to 3 are satisfied, Then,*

- i) if $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} < 2$ cooperation is attainable,*
- ii) if $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} > 2$ cooperation is not attainable.*

In differentiable stage games, the knowledge of the first and second order convergence speeds allow players to know whether or not cooperation is possible. However, if the first order convergence speed is zero and the second one is two, there is no way to know whether or not cooperation is attainable. For instance, consider a distribution $\{p_t\}_{t=1}^{\infty}$ such that $p_t = \lambda^{\frac{1}{\alpha} 2^t}$, where $\alpha > 0$

and $\lambda \in (0, 1)$ is such that $\sum p_t = 1$. It is easy to check that $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ and $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} = 2$ but (7) is verified depending on α , since

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{k=2}^t 2^{-k} \ln(p_k) &= \lim_{t \rightarrow \infty} \sum_{k=2}^t 2^{-k} \ln(\lambda^{\frac{1}{k^\alpha} 2^k}) = \lim_{t \rightarrow \infty} \sum_{k=2}^t 2^{-k} 2^k \frac{1}{k^\alpha} \ln(\lambda) = \\ &= \ln(\lambda) \sum_{k=2}^{\infty} \frac{1}{k^\alpha} \end{aligned}$$

And the harmonic series $\sum_{k=2}^{\infty} \frac{1}{k^\alpha}$ converges or equivalently (7) is verified if and only if $\alpha > 1$.

We saw that leptokurtic distributions over T preclude cooperation in probabilistic horizon games with finite stage games. However this is not a sufficient condition on the probability distribution to preclude cooperation in differentiable stage games. In fact, Corollary 7 characterizes the subset of the leptokurtic distributions for which cooperation is not possible. We define them as,

Definition 11 *A distribution $\{p_t\}_{t=1}^{\infty}$ is ultra-leptokurtic if and only if*

$$\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} > 2$$

Thus, cooperation is never attained in differentiable stage games with probabilistic horizon whenever the distribution over the length of the games is ultra-leptokurtic.

4 CONCLUDING REMARKS

To conclude our work we present a classification of the distributions over the length of any probabilistic Generalized Prisoners' Dilemma game which allows us to ascertain the existence of cooperation in both finite and differentiable stage games. This taxonomy is made according to the behavior of both the first and the second order convergence speeds.

First, the next result is needed.

Lemma 2 *The relationship between the first and second order convergence speed is given by,*

- i) $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} > 1 \Rightarrow \lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$
- ii) $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0 \Rightarrow \lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} \geq 1$

Proof: i) If $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} > 1$ then there exists a $\alpha > 1$ such that $\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} > \alpha > 1$. Then, there exists a t_0 such that $\ln(p_{t+1}) < \alpha \ln(p_t)$ for all $t > t_0$. But this expression is equivalent to $p_{t+1} < p_t^\alpha$ which implies $\frac{p_{t+1}}{p_t} < p_t^{\alpha-1}$ for all $t > t_0$. Thus,

$$0 \leq \lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} \leq \lim_{t \rightarrow \infty} p_t^{\alpha-1} = 0$$

where the last equality holds given that $\alpha - 1 > 0$.

ii) If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ then for any $0 < \varepsilon < 1$ there exists a t_0 such that $p_{t+1} < \varepsilon p_t$ for all $t > t_0$. Then, taking logarithms on both sides we have that $\ln(p_{t+1}) < \ln(\varepsilon) + \ln(p_t)$ for all $t > t_0$. But this expression is equivalent to $\frac{\ln(p_{t+1})}{\ln(p_t)} > \frac{\ln(\varepsilon)}{\ln(p_t)} + 1$ for all $t > t_0$, where $\frac{\ln(\varepsilon)}{\ln(p_t)} > 0$. Thus,

$$\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)} \geq 1 + \lim_{t \rightarrow \infty} \frac{\ln(\varepsilon)}{\ln(p_t)} = 1$$

□

The above Lemma is useful to classify discrete distributions in three categories which include five subcases. This unifies the analysis of both finite and differentiable stage games and it summarizes our main contributions to the study of repeated games with unknown horizon.

The first category consists of these distributions whose second order convergence speed is one but the first order convergence behavior varies among 0, γ , and 1, for $\gamma \in (0, 1)$. In all of these three subcases cooperation is always attainable in probabilistic horizon Generalized Prisoners' Dilemma games with differentiable stage games, but this is not the case for finite stage games. The former subset, when the convergence speed is zero, stands for leptokurtic distributions, under which cooperation is never possible in finite stage games; when the first convergence speed is γ , cooperation in finite stage games depends on this value; and finally, cooperation is always sustained in the third case. The positive Poisson distribution, the geometric distribution and the harmonic distribution defined above are examples of this first category.

The second and third categories describe the beliefs of more pessimistic players in the sense that they concentrate all the probability mass around a fixed period. These situations are equivalent to a second order convergence speed equal to $\alpha > 1$ or infinity. An example for the former is $p_t = K\lambda^{\alpha^{t-1}}$ where $\lambda \in (0, 1)$ and $\alpha > 1$ and for the latter $p_t = K\lambda^{t^\alpha}$ where $\lambda \in (0, 1)$ (in both cases K is a normalization parameter such that $\sum p_t = 1$). Under both categories cooperation is never possible in probabilistic horizon finite stage games. However, in differentiable stage games, it depends on whether or not the distribution is ultra-leptokurtic. This will be the case for the third category and for the second one when $\alpha > 2$ where cooperation is never sustained.

Table 1 summarizes the five possible subcases according to the first and second order convergence speed and it gives an example of distributions in them.

		$\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t}$		
		0	$\gamma \in (0, 1)$	1
$\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)}$	1	$p_t = \frac{\lambda^t}{(e^\lambda - 1)t!}$	$p_t = (1 - \gamma)\gamma^{t-1}$	$p_t \propto t^{-\alpha}, \alpha > 1$
	$\alpha > 1$	$p_t \propto \lambda^{\alpha t - 1}, \lambda \in (0, 1), \alpha > 1$	<i>none</i>	<i>none</i>
	∞	$p_t \propto \lambda^{t^t}, \lambda \in (0, 1)$	<i>none</i>	<i>none</i>

Table 1: Probability distribution samples according to the first and second order convergence speed

The next table summarizes the existence of cooperation in probabilistic horizon games in the above five possible subcases.

		$\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t}$					
		0		$\gamma \in (0, 1)$		1	
<i>F = finite, D = Differentiable</i>		<i>F</i>	<i>D</i>	<i>F</i>	<i>D</i>	<i>F</i>	<i>D</i>
$\lim_{t \rightarrow \infty} \frac{\ln(p_{t+1})}{\ln(p_t)}$	1	<i>NC</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>	<i>C</i>
	$1 < \alpha < 2$	<i>NC</i>	<i>C</i>	<i>none</i>		<i>none</i>	
	$\alpha > 2$	<i>NC</i>	<i>NC</i>	<i>none</i>		<i>none</i>	

Table 2: Cooperation (C) and Non Cooperation (NC) according to the first and second order convergence speed

5 APPENDIX

PROOF OF THEOREM 1: Consider a distribution such that $p_t > 0$ for all t . By definition,

$$\lim_{t_0 \rightarrow \infty} E[T|T \geq t_0] - t_0 = \lim_{t_0 \rightarrow \infty} \frac{\sum_{t=t_0}^{\infty} (t - t_0)p_t}{\sum_{t=t_0}^{\infty} p_t} \quad (8)$$

Now, if both, the numerator and the denominator of (8) converge to zero when t_0 tends to infinite, we have by Stolz's Criterion that,

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} E[T|T \geq t_0] - t_0 &= \lim_{t_0 \rightarrow \infty} \frac{\sum_{t=t_0+1}^{\infty} (t - t_0 - 1)p_t - \sum_{t=t_0}^{\infty} (t - t_0)p_t}{\sum_{t=t_0+1}^{\infty} p_t - \sum_{t=t_0}^{\infty} p_t} = \\ &= \lim_{t_0 \rightarrow \infty} \frac{\sum_{t=t_0+1}^{\infty} p_t}{p_{t_0}} = \\ &= \lim_{t_0 \rightarrow \infty} \sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} \end{aligned} \quad (9)$$

Notice that the denominator of (8) tends to zero for all distributions, so that the proper use of Stolz's Criterion depends on the numerator.

Next we consider the three different cases in the statement of the Theorem,

$$\text{Case 1: } \lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0 \Rightarrow \lim_{t \rightarrow \infty} E[T|T \geq t] - t = 0$$

First we show that (9) is satisfied. It suffices to prove that the numerator of (8) converges to zero. If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ then we have that for all $\epsilon > 0$ there exists an integer K such that $\frac{p_{t+1}}{p_t} < \epsilon$ and hence $\frac{p_t}{p_K} < \epsilon^{t-K}$ for all $t \geq K$.

Then,

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} \sum_{t=t_0}^{\infty} (t - t_0)p_t &= \lim_{t_0 \rightarrow \infty} p_K \sum_{t=t_0}^{\infty} (t - t_0) \frac{p_t}{p_K} \leq \\ &\leq \lim_{t_0 \rightarrow \infty} p_K \sum_{t=t_0}^{\infty} (t - t_0) \epsilon^{t-K} = \\ &= \lim_{t_0 \rightarrow \infty} p_K \frac{\epsilon^{t_0-K+1}}{(1 - \epsilon)^2} = 0 \end{aligned}$$

Hence (9) is satisfied. Now, taking $t_0 > K$ we have,

$$\sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} < \sum_{t=t_0+1}^{\infty} \epsilon^{t-t_0} = \frac{\epsilon}{1-\epsilon}$$

and then $\lim_{t_0 \rightarrow \infty} \sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} = 0$ and by Stolz's Criterion $\lim_{t_0 \rightarrow \infty} E[T|T \geq t_0] - t_0 = 0$.

$$\text{Case 2: } \lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = \gamma \Rightarrow \lim_{t \rightarrow \infty} E[T|T \geq t] - t = \frac{\gamma}{1-\gamma}$$

Here, we also show that (9) is satisfied by proving that the numerator of (8) goes to zero. If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = \gamma$ we have that for any small $\epsilon > 0$ there exists an integer K such that for all $t \geq K$, $0 < \gamma - \epsilon < \frac{p_{t+1}}{p_t} < \gamma + \epsilon < 1$. Hence $(\gamma - \epsilon)^{t-K} < \frac{p_t}{p_K} < (\gamma + \epsilon)^{t-K}$

Then,

$$\begin{aligned} 0 \leq \lim_{t_0 \rightarrow \infty} \sum_{t=t_0}^{\infty} (t - t_0)p_t &= \lim_{t_0 \rightarrow \infty} p_K \sum_{t=t_0}^{\infty} (t - t_0) \frac{p_t}{p_K} \leq \\ &\leq \lim_{t_0 \rightarrow \infty} p_K \sum_{t=t_0}^{\infty} (t - t_0)(\gamma + \epsilon)^{t-K} = \\ &= \lim_{t_0 \rightarrow \infty} p_K \frac{(\gamma + \epsilon)^{t_0-K+1}}{(1 - \gamma - \epsilon)^2} = 0 \end{aligned}$$

Hence (9) is satisfied. Now, taking $t_0 > K$ we have,

$$\begin{aligned} \frac{\gamma - \epsilon}{1 - \gamma + \epsilon} &= \sum_{t=t_0+1}^{\infty} (\gamma - \epsilon)^{t-t_0} < \sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} < \\ &< \sum_{t=t_0+1}^{\infty} (\gamma + \epsilon)^{t-t_0} = \frac{\gamma + \epsilon}{1 - \gamma - \epsilon} \end{aligned}$$

The above expression implies that $\lim_{t_0 \rightarrow \infty} \sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} = \frac{\gamma}{1-\gamma}$ and by Stolz's Criterion that $\lim_{t_0 \rightarrow \infty} E[T|T \geq t_0] - t_0 = \frac{\gamma}{1-\gamma}$

$$\text{Case 3: } \lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 1 \Rightarrow \lim_{t \rightarrow \infty} E[T|T \geq t] - t = \infty$$

The limit of the numerator of (8) may be here different from zero. If it is greater than zero (or infinity) then trivially $\lim_{t_0 \rightarrow \infty} E[T|T \geq t_0] - t_0 = \infty$.

On the other hand, if this limit goes to zero we use the Stolz's Criterion as in the previous cases and (9) is verified. Now, if $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 1$ we have that for all $\epsilon > 0$ there exists a integer K such that $\frac{p_{t+1}}{p_t} > 1 - \epsilon$ and hence $\frac{p_t}{p_K} > (1 - \epsilon)^{t-K}$ for all $t \geq K$.

Then, if $t_0 > K$

$$\sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} > \sum_{t=t_0+1}^{\infty} (1 - \epsilon)^{t-t_0} = \frac{1 - \epsilon}{\epsilon}$$

This implies that $\lim_{t_0 \rightarrow \infty} \sum_{t=t_0+1}^{\infty} \frac{p_t}{p_{t_0}} = \infty$ and then by (9) that $\lim_{t_0 \rightarrow \infty} E[T|T \geq t_0] - t_0 = \infty$.

Now, the reverse implications are trivial since we have analyzed above all the possible values of $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t}$. □

PROOF OF PROPOSITION 6: To prove this result we will proceed in several steps and to simplify notation we denote $\delta_t = \sum_{n=t}^{\infty} p_n$.

Step 1: The next five inequalities are equivalent

- i) $\sum_{k=1}^{\infty} 2^{-k} \ln(p_k) > -\infty$
- ii) $\sum_{k=2}^{\infty} 2^{-k} \ln(\sum_{n=k}^{\infty} p_n) > -\infty$
- iii) $\sum_{k=2}^{\infty} 2^{-k} \ln\left(\frac{\sum_{n=k}^{\infty} p_n}{\sum_{n=k-1}^{\infty} p_n}\right) > -\infty$
- iv) $\exists c > 0, \lambda > 0$ such that $\prod_{k=2}^t (\sum_{n=k}^{\infty} p_n)^{2^{t-k}} \geq c\lambda^{2^t}$, for all t
- v) $\exists \lambda_0 > 0$ such that $\prod_{k=2}^t (\sum_{n=k}^{\infty} p_n)^{2^{t-k}} \geq \lambda_0^{2^t}$, for all t .

i) \Rightarrow ii): This implication follows from $p_k \leq \sum_{n=k}^{\infty} p_n$.

ii) \Rightarrow i): If $\lim_{t \rightarrow \infty} \frac{p_{t+1}}{p_t} = 0$ then we have that for all $\epsilon > 0$ there exists an integer t_0 such that $p_t < \epsilon^{t-t_0} p_{t_0}$ for all $t \geq t_0$. Then, for all $k \geq t_0$,

$$\sum_{n=k}^{\infty} p_n = p_k \sum_{n=k}^{\infty} \frac{p_n}{p_k} < p_k \sum_{n=k}^{\infty} \epsilon^{t-k} = p_k \frac{1}{1 - \epsilon}$$

which implies,

$$\ln \left(\sum_{n=k}^{\infty} p_n \right) < \ln \left(p_k \frac{1}{1-\varepsilon} \right) \Rightarrow 2^{-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) < 2^{-k} \ln \left(p_k \frac{1}{1-\varepsilon} \right)$$

but then

$$\begin{aligned} \sum_{k=t_0}^{\infty} 2^{-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) &< \sum_{k=t_0}^{\infty} 2^{-k} \ln \left(p_k \frac{1}{1-\varepsilon} \right) = \\ &= \sum_{k=t_0}^{\infty} 2^{-k} \ln(p_k) + \sum_{k=t_0}^{\infty} 2^{-k} \ln \left(\frac{1}{1-\varepsilon} \right) < \\ &< \sum_{k=t_0}^{\infty} 2^{-k} \ln(p_k) + \ln \left(\frac{1}{1-\varepsilon} \right) \end{aligned}$$

Then, if ii) is verified,

$$\begin{aligned} -\infty &< \sum_{k=2}^{\infty} 2^{-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) < \sum_{k=t_0}^{\infty} 2^{-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) < \\ &< \sum_{k=t_0}^{\infty} 2^{-k} \ln(p_k) + \ln \left(\frac{1}{1-\varepsilon} \right) \end{aligned}$$

Now, if we add the first $t_0 - 1$ terms to the last term of the r.h.s. of the above expression, we have i).

ii) \Leftrightarrow iii): We will see that ii) is the double of iii). (Recall that $\delta_t = \sum_{n=t}^{\infty} p_n$ and $\delta_1 = 1$):

$$\begin{aligned} \sum_{k=2}^t 2^{-k} \ln \left(\frac{\delta_k}{\delta_{k-1}} \right) &= \sum_{k=2}^t 2^{-k} \ln(\delta_k) - \sum_{k=2}^t 2^{-k} \ln(\delta_{k-1}) = \\ &= \sum_{k=2}^t 2^{-k} \ln(\delta_k) - 2^{-2} \ln(\delta_1) - \sum_{k=3}^t 2^{-k} \ln(\delta_{k-1}) = \\ &= \sum_{k=2}^t 2^{-k} \ln(\delta_k) - \sum_{k=2}^t 2^{-k-1} \ln(\delta_k) = \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=2}^t 2^{-k} \ln(\delta_k) - 2^{-1} \sum_{k=2}^t 2^{-k} \ln(\delta_k) = \\
&= \frac{1}{2} \sum_{k=2}^t 2^{-k} \ln(\delta_k)
\end{aligned}$$

Then, when t goes to infinity, the behavior in the limit of both terms is the same.

iv) \Rightarrow v): We can consider that $c < 1$. Let λ_0 be such that $\lambda_0^2 < c\lambda^2$. Then $\lambda_0^2 < \lambda^2$, that implies $\lambda_0^{2^{t-1}} < \lambda^{2^{t-1}}$. Thus, $\lambda_0^{2^t} < c\lambda^{2^t}$, and the implication follows.

v) \Rightarrow iv): Trivial.

ii) \Leftrightarrow v): v) is equivalent to $\exists \lambda_0 > 0$ such that

$$\sum_{k=2}^t 2^{t-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) \geq 2^t \ln(\lambda_0),$$

for all t , that is equivalent to $\exists \lambda_0 > 0$ such that

$$\sum_{k=2}^t 2^{-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) \geq \ln(\lambda_0),$$

for all t . But if every partial sum exceeds $\ln(\lambda_0)$, then the limit must also exceed it, i.e. $\sum_{k=2}^{\infty} 2^{-k} \ln \left(\sum_{n=k}^{\infty} p_n \right) \geq \ln(\lambda_0) > -\infty$. On the other hand, if the sum is finite, we select λ_0 such that $\ln(\lambda_0)$ be equal to his sum. Since all the terms of the sum are negative, each partial sum will exceed $\ln(\lambda_0)$.

Step 2: *Sufficiency of proposition 6.* (Based on BD(1995)' proof). As in the proof of proposition 5, by assumptions 1 and 2 there exists $a > 0$, $b > 0$, $\alpha_0 > 0$, and a vector v such that for all $\alpha < \alpha_2$, $\pi_i(s^* + \alpha v) \geq a\alpha$ and $\phi_i(s^* + \alpha v) \leq b\alpha^2$.

Assume that $\{\alpha_t\}_{t=1}^{\infty}$ is a sequence of scalars such that the next properties are verified: $0 < \alpha_t < \alpha_0$, and

$$b\alpha_t^2 = a\delta_{t+1}\alpha_{t+1}$$

for all t . Then, for all i ,

$$\begin{aligned}
\phi_i(s^* + \alpha_t v) &\leq b\alpha_t^2 = a\delta_{t+1}\alpha_{t+1} \leq \delta_{t+1}\pi_i(s^* + \alpha_{t+1}v) \leq \\
&\leq \left(\sum_{n=t}^{\infty} (n-t)p_n \right) \pi_i(s^* + \alpha_{t+1}v) \leq \\
&\leq \frac{\sum_{n=t}^{\infty} (n-t)p_n}{\sum_{n=t}^{\infty} p_n} \pi_i(s^* + \alpha_{t+1}v) = \\
&= (E[T|T \geq t] - t)\pi_i(s^* + \alpha_{t+1}v)
\end{aligned}$$

which ensure that the action profile $\{s^* + \alpha_t v\}_{t=1}^{\infty}$ is a subgame perfect equilibrium and there exists a cooperative equilibrium. Thus, we have to prove that there exists a sequence of scalars verifying the above two properties.

Recursive substitution in $b\alpha_t^2 = a\delta_{t+1}\alpha_{t+1}$ allows us to find a general expression of α_t given α_1 :

$$\alpha_t = \left(\frac{b}{a} \right)^{2^{t-1}-1} \alpha_1^{2^{t-1}} \frac{1}{\prod_{n=2}^t \delta_n^{t-n}}$$

Given that condition (7) is satisfied, statement iii) is also verified. Then,

$$\alpha_t \leq \frac{a}{bc} \left(\frac{b\alpha_1}{a\lambda} \right)^{2^{t-1}}$$

Then, just taking α_1 sufficiently small we can guarantee that $0 < \alpha_t < \alpha_0$. \square

Step 3: *Necessity of proposition 6.* The proof is similar to the one of BD(1995) if we replace β_t by $\frac{\delta_t}{\delta_{t-1}}$. (Recall that any distribution $\{p_t\}_{t=1}^{\infty}$ and their continuation probability $\{\beta_t\}_{t=1}^{\infty}$ verifies that $\beta_t = \frac{\sum_{n=t}^{\infty} p_n}{\sum_{n=t-1}^{\infty} p_n}$). \square

6 REFERENCES

Abreu, D. (1988): "On the theory of infinitely repeated games with discounting", *Econometrica*, 56, pp. 383-396.

Becker, N. and Cudd, A. (1990): "Indefinitely repeated games: a response to Carroll", *Theory and Decision*, 28, pp. 189-195.

Benoit, J.P. and Krishna V. (1985): "Finitely Repeated Games", *Econometrica*, 53, pp. 905-922.

Bernheim, B. and Dasgupta, A. (1995): "Repeated games with asymptotically finite horizons", *Journal of Economic Theory*, 67 pp. 129-152.

Carroll, J.W. (1987): "Indefinite terminating points and the iterated prisoner's dilemma", *Theory and Decision*, 22, pp. 247-256.

Fudenberg, D. and Maskin, E. (1986): "The folk theorem in repeated games with discounting or with incomplete information", *Econometrica*, 54, pp. 533-554.

Fudenberg, D. and Tirole, J. (1991): "Game theory", *MIT Press*, Cambridge, MA.

Johnson, N. L. and Kotz, S. (1983): "Discrete distributions", *John Wiley & Sons*, New York.

Jones, M.A. (1998): "Cones of cooperation, Perron-Fröbenius theory and the indefinitely repeated Prisoners' Dilemma", *Journal of Mathematical Economics*, 30, pp. 187-206.

Jones M.A. (1999): "The effect of punishment duration of trigger strategies and quasifinite continuation probabilities for Prisoners' Dilemmas", *International Journal of Games Theory*, 28, pp.533-546.