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On the Multiplicity of Indecomposable Stable Summands for the Classifying Spaces of Elementary Abelian p-Groups

Dedicated to Professor Shôrô Araki on his 60th birthday

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Abstract

The purpose of the paper is to determine the multiplicity of certain indecomposable summands for the classifying spaces of elementary abelian p-groups.

1. Introduction

Let $B(\boldsymbol{Z}/p)^n$ be the classifying space of an elementary abelian p-group $(\boldsymbol{Z}/p)^n$. There has been much interest recently in the stable wedge decomposition of the space $B(\boldsymbol{Z}/p)^n_+$.

In [2], Harris and Kuhn showed that the homotopy classes of the indecomposable summands of such decomposition are in one to one correspondence with the isomorphism classes of irreducible $\boldsymbol{F}_p[M_n(\boldsymbol{Z}/p)]$-modules, and a given homotopy type appears with the multiplicity equal to the dimension of the corresponding module.

With each summand X of $B(\boldsymbol{Z}/p)^n_+$ we associate the Poincaré series

$$P(X;t)=\sum_{i=0}^{\infty}m(X,i)\,t^i,$$

where $m(X,i)$ is the multiplicity of X in $B(\boldsymbol{Z}/p)^i_+$.

The purpose of the paper is to determine the Poincaré series $P(X;t)$ for certain indecomposable summands of $B(\boldsymbol{Z}/p)^n_+$.

In section 2 and 3, we review general results of stable splitting and multiplicity of stable summands. In section 4, we state the properties and results about $P(X;t)$. The tables in the section are what we obtained by computer calculations.

2. Stable Splitting of $B(\boldsymbol{Z}/p)^n_+$

Let p be a prime number and $M_n(\boldsymbol{Z}/p)$ the semigroup of all $n \times n$ matrices with entries in $\boldsymbol{Z}/p$.

A *p-regular partition*, α, is a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ of nonnegative integers such that $0 \leq \alpha_i - \alpha_{i+1} \leq p-1$ for all $i \geq 1$. The number of positive entries in a p-regular partition α is called the *length* of α, and we write $l(\alpha)$ for it. Then there are p^n such partitions with length $\leq n$. Put $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{l(\alpha)}$.

Let $W^\alpha(n)$ be the Weyl module ($\boldsymbol{F}_p[M_n(\boldsymbol{Z}/p)]$-module) associate to p-regular partition α of length $\leq n$. Let $F^\alpha(n)$ be the top composition factor of $W^\alpha(n)$. Then $\{F^\alpha(n) \mid \alpha$ is a p-regular partition of length $\leq n\}$ is a complete set of irreducible $\boldsymbol{F}_p[M_n(\boldsymbol{Z}/p)]$-modules. For detail, see [2] and [3].

Let X_α be the indecomposable summand of $B(\boldsymbol{Z}/p)^n_+$ associate to $F^\alpha(n)$. Let $m(X_\alpha, n)$ be the multiplicity of X_α in $B(\boldsymbol{Z}/p)^n_+$. Then by [2], $m(X_\alpha, i)$ is the dimension of $F^\alpha(n)$.

We define a Poincaré series of multiplicities $P(X_\alpha;t)$ by

$$P(X_\alpha;t) = \sum_{i=0}^{\infty} m(X_\alpha, i)\, t^i.$$

3. Multiplicity of summands

In order to compute the multiplicity $m(X_\alpha, n)$ of an indecomposable summand X_α in $B(\boldsymbol{Z}/p)^n_+$, we recall the following property of Weyl module $W^\alpha(n)$.

Proposition 3.1. [2, 3]

(1) *There is a basis*

$\{e_t \mid t$ *is a semistandard* α*-tableau with entries in* $\{1, 2, \ldots, n\}\}$ *for* $W^\alpha(n)$.

(2) *There is a bilinear form* Φ^α *on* $W^\alpha(n)$ *such that* $F^\alpha(n) = W^\alpha(n) \cap W^\alpha(n)^\perp$, *where* $W^\alpha(n)^\perp = \{w \in W^\alpha(n) \mid \Phi^\alpha(w, v) = 0$ *for all* $v \in W^\alpha(n)\}$.

(3) *The dimension of* $F^\alpha(n)$ *is the rank of Gram matrix* $(\Phi^\alpha(e_s, e_t))$, *where* s *and* t *run through the semistandard* α*-tableaux with entries in* $\{1, 2, \ldots, n\}$

For a semistandard α–tableau t, let S_t be the set of all entries in t. Let $s(\alpha, n)$ be the rank of the matrix $(\Phi^{\alpha}(e_s,e_t))$ where s and t run through the semistandard α–tableaux with $S_s=S_t=\{1,2,\ldots,n\}$. Then the following proposition is useful for our calculations.

Proposition 3.2. [1] *We have*

(1) $m(X_{\alpha}, n)=\sum_{i=0}^{n}\binom{n}{i}s(\alpha,i)$, *and*

(2) $s(\alpha,n)>0$ *if and only if* $l(\alpha)\leq n\leq|\alpha|$.

4. Poincaré series $P(X;t)$

In this section, we state properties and results about $P(X;t)$.

First we prove the following lemma.

Lemma 4.1. *For* $k\geq 0$, *we have*

(1) $\sum_{i=0}^{\infty}\binom{i}{k}t^{i}=\frac{t^{k}}{(1-t)^{k+1}}$, *and*

(2) $\sum_{i=0}^{\infty}\binom{k+i-1}{k}t^{i}=\frac{t}{(1-t)^{k+1}}$.

Proof. (1) : It is clear for $k=0$. Put $f_k(t)=\sum_{i=0}^{\infty}\binom{i}{k}t^{i}$, and by induction we assume $k>0$.

$$f_k(t)=\sum_{i=0}^{\infty}\binom{i}{k}t^{i}=\sum_{i=0}^{\infty}\binom{i-1}{k-1}t^{i}+\sum_{i=0}^{\infty}\binom{i-1}{k}t^{i}$$
$$=t\sum_{i=0}^{\infty}\binom{i}{k-1}t^{i}+t\sum_{i=0}^{\infty}\binom{i}{k}t^{i}=tf_{k-1}(t)+tf_k(t).$$

Therefore

$$f_k(t)=\frac{t}{1-t}f_{k-1}(t)=\frac{t^{k}}{(1-t)^{k+1}}.$$

(2) : It is clear for $k=0$. Put $f_k(t)=\sum_{i=0}^{\infty}\binom{k+i-1}{k}t^{i}$ and assume $k>0$.

$$f_k(t)=\sum_{i=0}^{\infty}\binom{k+i-1}{k}t^{i}=\sum_{i=0}^{\infty}\binom{k+i-2}{k-1}t^{i}+\sum_{i=0}^{\infty}\binom{k+i-2}{k}t^{i}$$
$$=\sum_{i=0}^{\infty}\binom{k+i-2}{k-1}t^{i}+t\sum_{i=0}^{\infty}\binom{k+i-1}{k}t^{i}=f_{k-1}(t)+tf_k(t).$$

Therefore

$$f_k(t)=\frac{1}{1-t}f_{k-1}(t)=\frac{t}{(1-t)^{k+1}}.$$

Now we have the following theorem.

Theorem 4.2. *There is a polynomial*

$$M(X_\alpha;t)=c_{l(\alpha)}t^{l(\alpha)}+c_{l(\alpha)+1}t^{l(\alpha)+1}+\cdots+c_{|\alpha|}t^{|\alpha|}\ (c_i\in\boldsymbol{Z})$$

such that

$$P(X_\alpha;t)=\frac{M(X_\alpha;t)}{(1-t)^{|\alpha|+1}}$$

Proof. By Proposition 3.2 and Lemma 4.1(1),

$$P(X_\alpha;t)=\sum_{i=0}^{\infty}m(X_\alpha,i)t^i=\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}s(\alpha,k)t^i=\sum_{k=l(\alpha)}^{|\alpha|}\sum_{i=0}^{\infty}s(\alpha,k)\binom{i}{k}t^i$$

$$=\sum_{k=l(\alpha)}^{|\alpha|}s(\alpha,k)\frac{t^k}{(1-t)^{k+1}}=\frac{\sum_{k=l(\alpha)}^{|\alpha|}s(\alpha,k)t^k(1-t)^{|\alpha|-k}}{(1-t)^{|\alpha|+1}}.$$

Put $M(X_\alpha;t)=\sum_{k=l(\alpha)}^{|\alpha|}s(\alpha,k)t^k(1-t)^{|\alpha|-k}$, then the theorem holds.

The following are examples for $P(X_\alpha;t)$.

Theorem 4.3.

(1) *For* $\alpha=(k)$ *and* $1\leq k\leq p-1$

$$P(X_\alpha;t)=\frac{t}{(1-t)^{k+1}}.$$

(2) *For* $\alpha=(1,1,\ldots,1)$ *of length* l

$$P(X_\alpha;t)=\frac{t^l}{(1-t)^{l+1}}.$$

Proof. To prove (1), recall that the number of semistandard α-tableaux with entries from $\{1,\ldots,i\}$ equals $\binom{k+i-1}{k}$. Since for each content there is at most one such semistandard α-tableau, the Gram matrix is non-singular. Hence $m(X_\alpha,i)=\binom{k+i-1}{k}$. Then Lemma 4.1(1) implies (1). Proof of (2) is similar with $m(X_\alpha,i)=\binom{i}{l}$ and Lemma 4.1(2).

The rest of the paper are tables of $s(\alpha,n)$'s, $m(X_\alpha,n)$'s and coefficients of $M(X_\alpha;t)$'s for $p=2$, 3 obtained by computer calculations. Among them, interesting results for $p=2$ are as follows :

$$P(L(1);t)=\frac{t}{(1-t)^2},$$

$$P(L(2);t)=\frac{2t^2}{(1-t)^4},$$

$$P(L(3);t)=\frac{8t^3(1+t)}{(1-t)^7}, \text{ and}$$

$$P(L(4);t)=\frac{64t^4(1+t)(1+4t+t^2)}{(1-t)^{11}}.$$

For the definition of $L(k)$, see [4].

Table 1.1. $s(\alpha,n)$'s for $p=2$ and $n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
(1)	1									
(11)		1								
(111)			1							
(21)		2	2							
(1111)				1						
(211)			3	2						
(11111)					1					
(2111)				4	4					
(221)			3	8	4					
(111111)						1				
(21111)					5	4				
(2211)				6	10	4				
(321)			8	32	40	16				
(1111111)							1			
(211111)						6	6			
(22111)					10	24	14			
(2221)				4	20	24	8			
(3211)				20	60	60	20			
(11111111)								1		
(2111111)							7	6		
(221111)						15	28	14		
(22211)					10	30	28	8		
(32111)					40	144	168	64		
(3221)				14	90	174	140	40		
(111111111)									1	
(21111111)								8	8	
(2211111)							21	48	26	
(222111)						20	84	112	48	
(321111)						70	217	224	78	
(22221)					5	40	84	64	16	
(32211)					40	160	238	160	40	
(3321)				20	180	580	868	608	160	
(1111111111)										1
(211111111)									9	8
(22111111)								28	54	26
(2221111)							35	112	126	48
(3211111)							112	384	432	160
(222211)						15	70	112	72	16
(322111)						90	483	896	702	198
(32221)					24	240	728	960	576	128
(33211)					74	480	1190	1448	864	200
(4321)				64	704	2880	5824	6272	3456	768

Table 1.2. $m(X_\alpha,n)$'s for $p=2$.

n	1	2	3	4	5	6	7	8	9	10
(1)	1	2	3	4	5	6	7	8	9	10
(11)		1	3	6	10	15	21	28	36	45
(111)			1	4	10	20	35	56	84	120
(21)		2	8	20	40	70	112	168	240	330
(1111)				1	5	15	35	70	126	210
(211)			3	14	40	90	175	308	504	780
(11111)					1	6	21	56	126	252
(2111)				4	24	84	224	504	1008	1848
(221)			3	20	74	204	469	952	1764	3048
(111111)						1	7	28	84	210
(21111)					5	34	133	392	966	2100
(2211)				6	40	154	448	1092	2352	4620
(321)			8	64	280	896	2352	5376	11088	21120
(1111111)							1	8	36	120
(211111)						6	48	216	720	1980
(22111)					10	84	392	1344	3780	9240
(2221)				4	40	204	736	2136	5328	11880
(3211)				20	160	720	2400	6600	15840	34320
(11111111)								1	9	45
(2111111)							7	62	306	1110
(221111)						15	133	658	2394	7140
(22211)					10	90	448	1632	4860	12540
(32111)					40	384	2016	7680	23760	63360
(3221)				14	160	924	3738	12052	33120	80760
(111111111)									1	10
(21111111)								8	80	440
(2211111)							21	216	1214	4940
(222111)						20	224	1344	5760	19800
(321111)						70	707	3920	15786	51600
(22221)					5	70	469	2136	7606	22780
(32211)					40	400	2198	8784	28528	79840
(3321)				20	280	1960	9408	35272	110800	304480
(1111111111)										1
(211111111)									9	98
(22111111)								28	306	1826
(2221111)							35	392	2394	10548
(3211111)							112	1280	7920	35200
(222211)						15	175	1092	4860	17326
(322111)						90	1113	7280	33714	124398
(32221)					24	384	2912	14848	58608	192896
(33211)					74	924	6104	28552	106380	336248
(4321)				64	1024	8064	43008	177408	608256	1812096

Table 1.3. Coefficients of $M(X_\alpha;t)$ for $p=2$.

n	1	2	3	4	5	6	7	8	9	10
(1)	1									
(11)		1								
(111)			1							
(21)		2	0							
(1111)				1						
(211)			3	-1						
(11111)					1					
(2111)				4	0					
(221)			3	2	-1					
(111111)						1				
(21111)					5	-1				
(2211)				6	-2	0				
(321)			8	8	0	0				
(1111111)							1			
(211111)						6	0			
(22111)					10	4	0			
(2221)				4	8	-4	0			
(3211)				20	0	0	0			
(11111111)								1		
(2111111)							7	-1		
(221111)						15	-2	1		
(22211)					10	0	-2	0		
(32111)					40	24	0	0		
(3221)				14	34	-12	6	-2		
(111111111)									1	
(21111111)								8	0	
(2211111)							21	6	-1	
(222111)						20	24	4	0	
(321111)						70	7	0	1	
(22221)					5	20	-6	-4	1	
(32211)					40	0	-2	4	-2	
(3321)				20	80	60	8	-8	0	
(1111111111)										1
(211111111)									9	-1
(22111111)								28	-2	0
(2221111)							35	7	7	-1
(3211111)							112	48	0	0
(222211)						15	10	-8	-2	1
(322111)						90	123	-13	-1	-1
(32221)					24	120	8	-24	0	0
(33211)					74	110	10	18	-12	0
(4321)				64	320	320	64	0	0	0

Table 2.1. $s(\alpha,n)$'s for $p=3$ and $n\leq 8$.

n	1	2	3	4	5	6	7	8
(1)	1							
(11)		1						
(2)	1	1						
(111)			1					
(21)		2	1					
(1111)				1				
(211)			3	3				
(22)		1	3	1				
(31)		3	6	3				
(11111)					1			
(2111)				4	4			
(221)			3	4	1			
(311)			6	12	6			
(32)		2	9	12	4			
(111111)						1		
(21111)					5	4		
(2211)				6	15	9		
(3111)				10	15	6		
(222)			1	6	5	1		
(321)			7	16	15	4		
(42)		3	18	36	30	9		
(1111111)							1	
(211111)						6	6	
(22111)					10	24	13	
(31111)					15	30	15	
(2221)				4	10	6	1	
(3211)				16	55	60	20	
(322)			3	24	45	30	6	
(331)			6	36	75	60	15	
(421)			15	56	80	54	13	
(11111111)								1
(2111111)							7	7
(221111)						15	28	13
(311111)						21	42	21
(22211)					10	45	63	28
(32111)					30	99	105	35
(2222)				1	10	15	7	1
(3221)				15	60	81	42	7
(3311)				19	100	186	140	35
(4211)				45	225	405	315	90
(332)			3	33	120	174	105	21
(422)			6	45	105	114	63	13
(431)			15	96	225	249	133	28

Table 2.2. $m(X_\alpha,n)$'s for $p=3$.

n	1	2	3	4	5	6	7	8
(1)	1	2	3	4	5	6	7	8
(11)		1	3	6	10	15	21	28
(2)	1	3	6	10	15	21	28	36
(111)			1	4	10	20	35	56
(21)		2	7	16	30	50	77	112
(1111)				1	5	15	35	70
(211)			3	15	45	105	210	378
(22)		1	6	19	45	90	161	266
(31)		3	15	45	105	210	378	630
(11111)					1	6	21	56
(2111)				4	24	84	224	504
(221)			3	16	51	126	266	504
(311)			6	36	126	336	756	1512
(32)		2	15	60	174	414	861	1624
(111111)						1	7	28
(21111)					5	34	133	392
(2211)				6	45	189	588	1512
(3111)				10	65	246	707	1708
(222)			1	10	45	141	357	784
(321)			7	44	165	474	1148	2464
(42)		3	27	126	420	1134	2646	5544
(1111111)							1	8
(211111)						6	48	216
(22111)					10	84	391	1336
(31111)					15	120	540	1800
(2221)				4	30	126	393	1016
(3211)				16	135	630	2155	6040
(322)			3	36	195	720	2106	5256
(331)			6	60	315	1170	3480	8856
(421)			15	116	510	1674	4556	10856
(11111111)								1
(2111111)							7	63
(221111)						15	133	657
(311111)						21	189	945
(22211)					10	105	588	2352
(32111)					30	279	1428	5327
(2222)				1	15	90	357	1107
(3221)				15	135	666	2394	7021
(3311)				19	195	1071	4207	13293
(4211)				45	450	2430	9450	29700
(332)			3	45	315	1449	5103	14931
(422)			6	69	390	1539	4851	13075
(431)			15	156	855	3339	10486	28224

Table 2.3. Coefficients of $M(X_\alpha;t)$ for $p=3$.

n	1	2	3	4	5	6	7	8
(1)	1							
(11)		1						
(2)	1	0						
(111)			1					
(21)		2	-1					
(1111)				1				
(211)			3	0				
(22)		1	1	-1				
(31)		3	0	0				
(11111)					1			
(2111)				4	0			
(221)			3	-2	0			
(311)			6	0	0			
(32)		2	3	0	-1			
(111111)						1		
(21111)					5	-1		
(2211)				6	3	0		
(3111)				10	-5	1		
(222)			1	3	-4	1		
(321)			7	-5	4	-2		
(42)		3	6	0	0	0		
(1111111)							1	
(211111)						6	0	
(22111)					10	4	-1	
(31111)					15	0	0	
(2221)				4	-2	-2	1	
(3211)				16	7	-2	-1	
(322)			3	12	-9	0	0	
(331)			6	12	3	-6	0	
(421)			15	-4	2	2	-2	
(11111111)								1
(2111111)							7	0
(221111)						15	-2	0
(311111)						21	0	0
(22211)					10	15	3	0
(32111)					30	9	-3	-1
(2222)				1	6	-9	3	0
(3221)				15	0	-9	0	1
(3311)				19	24	0	-8	0
(4211)				45	45	0	0	0
(332)			3	18	18	-18	0	0
(422)			6	15	-15	9	0	-2
(431)			15	21	-9	0	1	0

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