

VCG mechanisms and efficient ex ante investments with externalities

Daniel Krähmer and Roland Strausz*
Free University Berlin

February 8, 2006

Abstract

For mechanism design with independent values, we identify a subclass of Vickrey–Clarke–Groves (VCG) mechanisms that induce efficient ex ante investments even with externalities. The Vickrey second price auction does not belong to this class.

JEL Classification: D82

Keywords: Externalities; ex ante investments; efficient mechanisms; mechanism design; second price auction.

*Corresponding author: Roland Strausz, Boltzmannstrasse 20, 14195 Berlin, Germany. Telephone: +49.30.838.53685 Fax: +49.30.838.54142. Emailaddress: strausz@zedat.fu-berlin.de. Financial support by the DFG (German Science Foundation) under SFB/TR 15 is gratefully acknowledged. Roland Strausz acknowledges financial support under grant STR-991/1.

1 Introduction

For mechanism design with independent values, we identify a subclass of Vickrey–Clarke–Groves (VCG) mechanisms that induce efficient ex ante investments when investments exhibit externalities. Hence, we extend the result that there is no trade-off between ex ante efficiency and ex post efficient implementation in the independent private value setting (e.g. Bergemann and Välimäki 2002, Arozamena and Cantillon 2004). Yet, in contrast to the case without externalities not every VCG induces efficient ex ante investments. In particular, the familiar Vickrey second price auction does not induce efficient investments and therefore does not belong to the identified subclass of VCG mechanisms. Spectral auctions for mobile phones, where entrants benefit indirectly from an incumbent’s investment in fixed-line telephone connections, are typical examples of an implementation setting that exhibits ex ante investments with externalities.

2 Setup

There are n players $i = 1, \dots, n$, a social planner, a set X of possible decisions, and for each player i a set of types $S_i = [\underline{s}_i, \bar{s}_i] \subseteq \mathbb{R}$. Let $S = \times_i S_i$. There are two periods. In period 1, each player i independently and simultaneously chooses an investment $\alpha_i \geq 0$ at a personal cost $c_i(\alpha_i)$. After the players have chosen their investment levels their types $s \in S$ are realized. In particular, player i ’s type s_i is the realization of a random variable \tilde{s}_i with cdf $F_i(s_i; \alpha)$. Since the distribution may depend on the entire investment profile $\alpha = (\alpha_1, \dots, \alpha_n)$, our setup allows for investment externalities. We assume that the family $(\tilde{s}_1, \dots, \tilde{s}_n)$ is stochastically independent for all α so that we consider a framework of independent, private values. Moreover, we assume that \tilde{s}_i has a pdf $f_i(s_i; \alpha)$ that is bounded, strictly positive, and differentiable in α for all s_i and α .

In period 2, a decision $x \in X$ is implemented together with a transfer schedule $t = (t_1, \dots, t_n)$ specifying a transfer t_i of agent i to the social planner. We assume that agents have quasi-linear utility functions that are additively separable in decision, transfer, and cost. That is, given an investment profile α , a type realization s , a decision x , and a

transfer t_i , agent i receives a utility

$$u_i(\alpha, s, x, t) = v_i(x, s_i) - t_i - c_i(\alpha_i).$$

As is standard, we assume that the social planner collects the transfers. Therefore, total social welfare is

$$u(\alpha, s, x, t) = \sum_i u_i(\alpha, s, x, t) + \sum_i t_i = \sum_i \{v_i(x, s_i) - c_i(\alpha_i)\}.$$

3 Efficiency

Given a realization of types $s = (s_1, \dots, s_n)$, a decision $x^* \in X$ is *ex post efficient* if and only if

$$x^* \in \arg \max_{x \in X} \sum_i v_i(x, s_i). \quad (1)$$

We call a *decision rule* $x : S \rightarrow X$ *ex post efficient* if $x^*(s)$ satisfies (1) for every $s \in S$.

Agent i 's expected utility from an investment profile α and an *ex post efficient* decision rule $x^*(s)$ is

$$U_i(\alpha) = E[v_i(x^*(\tilde{s}), \tilde{s}_i)] - c_i(\alpha_i). \quad (2)$$

An *ex ante* investment profile $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$ is *ex ante efficient* with respect to an *ex post efficient* decision rule $x^*(s)$ if and only if it maximizes the expected sum of utilities $U(\alpha) = \sum_i U_i(\alpha)$. That is,

$$\alpha^* \in \arg \max_{\alpha} U(\alpha). \quad (3)$$

A combination $(\alpha^*, x^*(s))$ satisfying (1) and (3) is *overall efficient*.

Finally, we define the marginal *social contribution* of investment α_i as

$$\frac{\partial}{\partial \alpha_i} U(\alpha) = \frac{\partial}{\partial \alpha_i} E \left[\sum_j v_j(x^*(\tilde{s}), \tilde{s}_j) \right] - c'_i(\alpha_i). \quad (4)$$

We assume that the *ex ante efficient* level α^* is interior and therefore satisfies the first and second order conditions. Hence, $\partial U(\alpha^*)/\partial \alpha_i = 0$ for all $i = 1, \dots, n$.

4 Private Information

Now assume the players' types are private information. In this case the implementation of an ex post efficient decision rule requires the transmission of private information. We suppose that the social planner may employ some mechanism to elicit such information. Due to the revelation principle, we may restrict attention to *direct mechanisms* $(x, t) : S \rightarrow X \times \mathbb{R}^n$ that require each player to independently submit a report $\hat{s}_i \in S_i$ and selects allocation $(x(\hat{s}), t(\hat{s}))$ conditional on the joint reports $\hat{s} = (\hat{s}_1, \dots, \hat{s}_n)$. The mechanism (x, t) implements the decision rule $\xi : S \rightarrow X$ in dominant strategies if $\xi = x$ and truth-telling is a dominant strategy for each player. We are interested in mechanisms that implement an ex post efficient rule $x^*(s)$ and induce an ex ante efficient investment profile α^* . It is well known that $x^*(s)$ is implemented in dominant strategies by a Vickrey-Clarke-Groves (VCG) mechanism (x, t) (e.g. Fudenberg and Tirole 1991, p.271ff). These direct mechanisms are characterized by

$$x(s) = x^*(s) \text{ and } t_i(s) = - \sum_{j \neq i} v_j(x^*(s), s_j) - h_i(s_{-i}),$$

where h_i is an arbitrary function that depends only on the profile s_{-i} of types excluding s_i .

We assume that players play a Nash equilibrium in period 1 thereby anticipating that a VCG mechanism is used in period 2. Given an investment profile α and a type realization $s \in S$, a VCG mechanism yields player i a utility of

$$\begin{aligned} u_i(s, \alpha_i; \alpha_{-i}) &= v_i(x^*(s), s_i) - t(s) - c_i(\alpha_i) \\ &= \sum_j v_j(x^*(s), s_j) + h_i(s_{-i}) - c_i(\alpha_i). \end{aligned}$$

Thus, in period 1 player i 's expected utility is $U_i(\alpha_i; \alpha_{-i}) = E[u_i(\tilde{s}, \alpha_i; \alpha_{-i})]$, and player i 's *private incentive* to invest is

$$\frac{\partial}{\partial \alpha_i} U_i(\alpha_i; \alpha_{-i}) = \frac{\partial}{\partial \alpha_i} E \left[\sum_j v_j(x^*(\tilde{s}), \tilde{s}_j) \right] + \frac{\partial}{\partial \alpha_i} E[h_i(\tilde{s}_{-i})] - c'_i(\alpha_i). \quad (5)$$

Inspection reveals that the private incentive exceeds the social contribution by

$$\delta_i(\alpha_i; \alpha_{-i}) = \frac{\partial}{\partial \alpha_i} E[h_i(\tilde{s}_{-i})].$$

We refer to $\delta_i(\alpha_i; \alpha_{-i})$ as player i 's *excess investment incentive*, because if

$$\delta_i(\alpha^*) = 0, \tag{6}$$

then the private incentive to invest (5) coincides with the social contribution of investment (4). In this case, the first order necessary conditions for a Nash equilibrium $\partial U_i(\alpha)/\partial \alpha_i = 0$ have a solution at α^* . As a result, the VCG mechanism induces efficient investments α^* as an equilibrium. We call a VCG mechanism with (6) *investment efficient*.

Due to independence, the pdf of the multivariate random variable \tilde{s}_{-i} is $f_{-i}(s_{-i}; \alpha) = \prod_{j \neq i} f_j(s_j; \alpha)$ with support $S_{-i} = \times_{j \neq i} S_j$ and cdf $F_{-i}(s_{-i}; \alpha) = \prod_{j \neq i} F_j(s_j; \alpha)$. Expanding δ_i yields:

$$\begin{aligned} \delta_i(\alpha^*) &= \frac{\partial}{\partial \alpha_i} \int_{S_{-i}} h_i(s_{-i}) f_{-i}(s_{-i}; \alpha_i, \alpha_{-i}^*) ds_{-i} \Big|_{\alpha_i = \alpha_i^*} \\ &= \int_{S_{-i}} h_i(s_{-i}) \frac{\partial}{\partial \alpha_i} f_{-i}(s_{-i}; \alpha^*) ds_{-i}. \end{aligned}$$

Proposition 1 *A VCG mechanism is investment efficient if*

- (i) $\frac{\partial}{\partial \alpha_i} f_{-i}(s_{-i}; \alpha) = 0$ for all α , or
- (ii) h_i is a constant.

Proof (i) trivial. (ii) We have

$$\int_{S_{-i}} h_i \frac{\partial}{\partial \alpha_i} f_{-i}(s_{-i}; \alpha^*) ds_{-i} = h_i \frac{\partial}{\partial \alpha_i} [F_{-i}(\bar{s}_{-i}, \alpha) - F_{-i}(\underline{s}_{-i}, \alpha)].$$

The right hand side is zero, since $F_{-i}(\bar{s}_{-i}, \alpha) = 1$ and $F_{-i}(\underline{s}_{-i}, \alpha) = 0$ for all α . \square

Condition (i) says that player j 's investment has no effect on player i 's type distribution and describes the case without investment externalities. Thus, in the absence of externalities, any VCG mechanism is investment efficient. This is the observation made in Bergemann and Välimäki (2002), Theorem 1.

Condition (ii) says that even in the presence of externalities, *ex ante* efficiency can be achieved by simply setting h_i equal to a constant. By the appropriate choice of the constants h_i , any division of the *ex ante* surplus among the players can be achieved. In particular, the social planner can achieve an *ex ante* balanced budget, i.e. the sum of expected transfers to the planner is 0. Note that there are other, non-constant functions h_i that satisfy (6), but these depend on the specifics of the densities f_i . Hence, following the Wilson doctrine of “detail-free” implementation (e.g. Dasgupta and Maskin, 2000) condition (ii) defines a subclass of VCG mechanisms with the advantage of being independent of any distributional details.

The significance of Proposition 1 is partially owed to a prominent VCG mechanism, the second price auction, which is not *ex ante* efficient. This is illustrated in the following example.

5 Vickrey auction is not *ex ante* efficient

There are two players and a single object. Let $X = \{x_1, x_2\}$, where x_i means that player i gets the object. Let $S_i = [0, 1]$ and interpret s_i as player i 's valuation of the object. Suppose only player 1 can invest. Let $\alpha_1 \in [0, 1]$ and consider

$$\begin{aligned} F_1(s_1; \alpha_1) &= s_1^{\alpha_1}, \\ F_2(s_2; \alpha_1) &= s_2^{k+\gamma\alpha_1}, \end{aligned}$$

where $k > 0$ and γ are constants such that $k + \gamma\alpha_1 \geq 0$. That is, player 1's investment decreases ($\gamma < 0$) or increases ($\gamma > 0$) player 2's valuation in the sense of first order stochastic dominance. In other words, there is a negative or positive spillover from player 1 to player 2.

Recall that the Vickrey auction is a specific VCG with

$$h_i(s_{-i}) = -s_{-i}.$$

Hence, player 1's private excess incentive is

$$\delta_1(\alpha_1) = \frac{-\gamma}{(k + \gamma\alpha_1 + 1)^2}.$$

Therefore, $\delta_1(\alpha_1) \leq 0 \Leftrightarrow \gamma \geq 0$, and we conclude that there is too much (little) equilibrium investment when there is a negative (positive) spillover.

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