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Dynamic Factor Demands and Technology Measurement under Arbitrary Expectations *

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Abstract
We present a dynamic model of factor demands based on expected discounted costs minimization. While making only very mild assumptions on expectations and technology, we are able to establish a duality relationship between contemporary factor demands and the technology, and we provide formula for easily recovering marginal products, returns to scale, and technological change from estimated factor demands. Parametrization and implementation are illustrated in a detailed example.

Key words: Dynamic duality; Investment; Expectations; Expected future cost function; Factor demands; Returns to scale; Technological change.

Résumé
Nous présentons un modèle dynamique de demande de facteurs de production basé sur un comportement de minimisation de l'espérance des coûts cumulatifs actualisés. Sous des hypothèses peu restrictives sur les anticipations et la technologie, nous établissons une relation de dualité entre les demandes courantes de facteurs et la technologie. Produits marginaux, rendements d'échelle et progrès technologique peuvent se calculer simplement à partir des demandes de facteurs. Nous illustrons à travers un exemple détaillé une façon de paramétriser et d'appliquer le modèle.

Mots-clés: Dualité dynamique; Investissement; Anticipations; Demandes de facteurs de production; Rendements d'échelle; Progrès technique.

1 Introduction

During the 1980's, dynamic firm theory has evolved along three major avenues. Dynamic duality was investigated in a series of papers by McLaren and Cooper (1980a,b); Epstein (1981a); and Epstein and Denny (1983). Despite some important results, it failed to become an encompassing tool of empirical analysis in a similar fashion as static duality did. As an alternative, following Hansen and Sargent (1980), several authors used and explicitly solved expected-utility (or profit) maximization problems of greater theoretical generality, but whose implementability remains limited to linear-quadratic forms and subject to considerable computational difficulties. Finally, other authors traded theoretical completeness for generality and simplicity, by focusing on Euler equations.

In this paper, we present a simple alternative way to formulate dynamic factor demand models, and a simple way to recover the main features of the primal technology. The model is very general, the only restrictions on expectations being restrictions that ensure the existence of an optimum program. Although we present the case of a non-rival firm, this generality implies that our approach can be extended to many situations involving strategic behavior, the main restriction being the existence of a unique, differentiable, value function. We establish the existence of a duality relationship between factor demands and technology. In itself, this does not imply that it is easy to recover the technology from the expenditure system; however, our procedure also yields simple formula for the computation of marginal products, returns to scale, and technological progress from a system of factor demands.

In the rest of this introduction, we present and discuss the main existing dynamic factor demand models in more detail. Then, in Section 2, we introduce our model, define the relevant cost functions and establish the corresponding system of factor demands. The properties of factor demands are given in Section 3. Section 4 establishes the duality results and deals with the recovery of primal technology characteristics. Some expectation formation processes, rational expectations in particular, give rise to specific properties that can be tested; we discuss how this might be done in Section 5. Before concluding, in Section 6, we illustrate how to go about implementation by formulating a second-order factor demand system and deriving the parameter restrictions implied by the theoretical model.

Interest toward duality theory, static or dynamic, arises mainly out of the hope that it can simplify empirical work substantially, at no cost in terms of generality. It is because of its relative lack of generality and simplicity that dynamic duality never became successful as a tool of empirical analysis.

Let us outline some major aspects of the methodology developed by McLaren and Cooper (1980a, b), and Epstein (1981a). The model is formulated under perfect certainty. In the case of producer theory, the objective is the minimization of cumulated discounted expenditures, or the maximization of cumulated discounted net profits, subject to technological constraints defined over each period. Intertemporal links are provided by, say, investment. The assumption of perfect certainty is not sufficient for tractable results, however: since the planning period extends over an infinite horizon, the problem involves an infinite number of parametric prices, unless the latter are assumed to be constant or to evolve according to some simple rule.

As the discussion by Epstein and Denny (1983) makes clear, the relaxation of this assumption is not simply a matter of extra algebra but puts the whole apparatus under question. Finally the theoretical model is developed as a generalization of the flexible accelerator model of investment. This is not necessarily a limitation by itself, but introduces the apparently benign assumption that the model has a steady state. As a result other classic dynamic aspects of the theory of the firm (exhaustible-resource extraction; learning by doing) require non trivial adjustments to the theoretical model if they are to be included in its realm of validity.

Under the list of assumptions sketched above, the theory of dynamic duality is built up from the value function G corresponding to the problem of minimizing cumulative discounted costs. The Hamilton-Jacobi equation for that problem defines a duality relationship between the production function f, and G. Thus the primal technology f can be recovered from the value function G , provided the latter satisfies appropriate regularity conditions, and vice versa. The dynamic factor demand system can be derived from G. This system is rather complex, involving, in particular, third derivatives of G. Thus the dynamic dual approach does not have the same clear edge in terms of simplicity as its static counterpart, under which factor demands are directly derived from an observable, easily characterized, cost function, by use of Shephard's lemma.

Empirical analysts might have overcome these difficulties had the rewards been commensurate with the efforts. However, because the model lacks generality, other alternatives were privileged. Interpreted in the current context, the model of Hansen and Sargent (1980) is based on the minimization of expected cumulated discounted cash flows. In discrete time, for a quadratic technology, linear equations of motion, and linear investment rules, the problem is a special linear-quadratic dynamic game. The solution concept is a decision rule (a contingency plan) rather than a program.

The model can be interpreted as a rational-expectations model if, besides purely exogenous information and variables specific to the firm, the information set on which investment decisions are based only contains information that reflects the aggregate impact of other economic agents' strategies and is considered exogenous by the firm (see Sargent, 1985, for a nice discussion of these issues). Despite the restrictions imposed by "quadratic-linearity", this methodology can be extended beyond the traditional investment decision framework to cases that do not necessarily admit a steady state, such as, e.g., resource extraction (Epple,1985). Despite various attempts to simplify it or to extend its somewhat rigid framework (Epstein and Yatchew, 1985; Kollintzas, 1985), this methodology has proved difficult to apply.

Indeed, it is very difficult to construct, as do Hansen and Sargent and their followers, econometric models based on explicit closed form solutions. Cooper et al. (1989) have made an advance in this line of investigation by finding conditions for the existence of explicit closed form solutions in dynamic stochastic consumer demand models. Other authors (Pindyck and Rotemberg, 1985; Shapiro, 1986; Bernstein and Nadiri, 1989), while using a similar model (without the linear quadratic restrictions), have chosen to stop short of providing a closed-form solution and work from Euler equations.

For investment demand, the Euler equations at date t involve the expected values of the marginal products of quasi-fixed factors, $\partial f/\partial k_{t+1}$, at $t+1$; the latter depend on the arguments of f which are typically, with costs of adjustment, a vector x_{t+1} of n variable factors, a vector k_{t+1} of m quasi-fixed factors, and i_{t+1}, the corresponding vector of gross investment flows. Under rational expectations, this expected marginal-product vector does not differ in any predictable way from the value obtained by substituting into f the values of x_{t+1}, $k_{t+1,}$ and i_{t+1} as observed *ex post*, at $t+1$. It is thus possible to estimate the Euler equations directly, although not without some loss in information, as Euler equations do not exhaust necessary conditions[1]. From an econometric point of view, the presence of k_{t+1}, i_{t+1}, x_{t+1} in the equation for period t causes the error term to be correlated with the dependent variable, which calls (see, e.g. Pindyck and Rotemberg, 1985; Bernstein and Nadiri, 1989) for the use of instrumental methods such as the generalized method of moments (Hansen, 1982) in the estimation.

To sum up, each of the three major approaches to dynamic firm theory has one or several of the following weaknesses: restrictive assumptions on expectations or the technology; complexity; failure to use relevant information. In the analysis presented

[1] One alternative approach is to work out numerical solutions based on the Euler equations and the appropriate transversality conditions. Again this approach appears to be cumbersome and we are not aware that other authors than Prucha and Nadiri (e.g. 1989) have attempted to use it.

below, our first aim is to establish and characterize a dynamic factor demand system which can be used as a basis for empirical work: it must be general; it must be convenient; it must make use of available information; and it must incorporate the restrictions implied by the underlying economic theory.

Our second aim is the measurement of key technological concepts: marginal factor products; returns to scale; technological change. The duality results presented below establish the existence of a technology corresponding to the system of factor demands, and we provide simple formula for key technology measurements. Thus, one of our achievements is to propose a model and a methodology which ensure that expectations do not interfere with the study of factor demands and technology.

2 A model of expected cost minimization with arbitrary expectations and technology

2.1 The model

As with static firm theory, our analysis can easily be carried out in terms of the profit function, treating output as endogenous. Because supply decisions are often more complex than factor demand choices, analysts have often focused on factor demands, given output. Consequently, the static duality between the cost function and the production function has been exploited empirically more than any other static duality relation. We adopt a similar approach in what follows, by treating output as given. As in static setups, this may be interpreted to mean that output is truly exogenous, or is determined at some other stage of the decision process. Although these two alternative views have different econometric implications, they have little implications on the theoretical analyses which follows.

As far as future output levels are concerned, they must be anticipated, as other future exogenous variables, based on current information. If we consider that output is not truly exogenous but determined, in each period, at a separate stage of the decision making process, expectations of future outputs are conditional on the use of an appropriate profit maximizing rule at the time such output levels will be chosen. But this will not affect our analysis, as we do not model the expectation formation process explicitly; what matters here is that future output anticipations are based on current information.

Although several cost functions will be defined below, our analysis will focus on the duality between a system of current factor demands, including investment demands, on one hand, and the current production function on the other hand. This is the same relationship as the one between the cost function and the production function.

In a static framework, as is well known, one obtains factor demands from the cost function by Shephard's lemma or the cost function from factor demands by summing optimal expenditures. Consequently there is no distinction to be made between the production function - cost function relationship on one hand, and the production function - factor demand system relationship on the other hand.

In the dynamic framework that we are going to introduce, such a distinction is useful, although only as a matter of convenience. In fact, factor demands, which include investment demands, are observable while the corresponding expenditures are not: there is no observable variable reflecting the appropriate shadow rental price applying to capital expenditures. As shown below, that shadow price can be computed, but only using factor demand functions. Consequently, it is algebraically simpler to focus on the duality between factor demands and the production function. Looking for a duality between factor demands and technology might be considered unusual in production economics. However, there is a long tradition of focusing on the duality between demands and preferences in consumer theory. We adopt the convention of referring to the production function as the primal representation of the technology, while factor demands belong to the dual side.

The firm operates in a stochastic environment, attempting to minimize the expected value of cumulated discounted expenditures given its current output (taken as exogenous), its current quasi-fixed inputs, and its current information, under technological constraints. Technology at date t is represented by a production function $f(x_t, k_t, i_t; \gamma_t)$ where x_t, k_t, and i_t respectively are vectors of n variable factors, m quasi-fixed factors, and m gross investment flows corresponding to k_t; w_t and q_t are the prices corresponding to x_t and i_t respectively; γ_t is a vector of parameters which represent the current state of technology (in most empirical applications, γ is simply the date). As in the most general treatment of the cost-of-adjustment model, f may be rising or decreasing in i, depending on the level of i and other arguments.

Decisions at date t are based on current information and involve contemporary control variables x_t and i_t. The choice of $x_{t+\tau}$ and $i_{t+\tau}$ is postponed until the relevant information is revealed so that, formally the problem is to choose decision rules for x and i at all future dates so as to

$$\min_{\{x_\tau(\cdot), i_\tau(\cdot)\}} E_t \left\{ \sum_{\tau=t}^{\infty} \left[\frac{\beta_\tau}{\beta_t} (w_\tau^T x_\tau + q_\tau^T i_\tau) \right] \right\} \tag{1}$$

subject to

$$y_\tau \leq f(x_\tau, k_\tau, i_\tau; \gamma_\tau), \ \tau = t, \ldots, \infty \tag{2}$$

$$k_{\tau+1} = [I - \delta_\tau] \, k_\tau + i_\tau, \quad \tau = t, \ldots, \infty \tag{3}$$

$$y_t \text{ and } k_t \text{ given} \tag{4}$$

where β_τ is the discount coefficient from date τ to zero, based on a sequence of discount rates $\{r_o, \ldots, r_\tau, \ldots, r_\infty\}$, while I and δ_τ are diagonal matrices of dimension m whose

j^{th} diagonal elements are respectively 1 and the rate of depreciation δ_τ^j for the j^{th} fixed factor. We make the following standard assumption

Assumption 1 *The production function $f(x_\tau, k_\tau, i_\tau; \gamma_\tau)$ has the following properties*

1. *twice continuously differentiable: $f \in \mathcal{C}^2$;*

2. *monotonously increasing in x and k: $f_x > 0_{(n\times1)},\ f_k\ > 0_{(m\times1)}$;*

3. *strongly quasi-concave in x and i: $\phi \neq 0_{([n+m]\times1)}$ and $[f_x \quad f_i]\,\phi = 0$*

$$\Rightarrow \phi^T \begin{bmatrix} f_{xx} & f_{xi} \\ f_{ix} & f_{ii} \end{bmatrix} \phi < 0.$$

Although the variables on which future decisions will be based are stochastic, and neither the process by which k will evolve in the future, nor the future states of technology, may be known at t, problem (1) is a standard, non-stochastic, dynamic programming problem as the current motion of k is non stochastic and the current technology is known (Cooper and McLaren (1987) study a situation where current investment and future investments have stochastic effects on capital; when their model is restricted to deterministic current investment, it is a particular case of ours).

We do not attempt to establish conditions for the existence of a solution; however, a solution is known to exist for several expectation formation rules, static expectations and rational expectations being prime examples that have been developed in the literature. Restricting our attention to the set of expectation formation rules under which a solution exists, let $G(\tilde{J}_t, k_t, y_t)$ be the optimized value function, where $\tilde{J}_t$ is the vector of all information which condition the expected value operator E_t in (1).

Under static expectations and a constant technology, $\tilde{J}_t = (w_t, q_t, r_t)$; more generally, $\tilde{J}_t$ may also contain past values of w, q, and r, as well as current and past values of γ and a vector θ_t of other relevant variables. Current and past decisions by other agents are not included in θ_t because of the assumption of perfect competition; however, in a rational expectations competitive equilibrium, the firm may watch current and past values of a vector of aggregate state variables (e.g. G.N.P., the rate of inflation, wage inflation, etc.) that determine the evolution of market prices. θ_t may also include firm specific variables such as past and current values of δ. Thus we may define $\tilde{J}_t$ as

$$\tilde{J}_t = (w_t, q_t, \rho_t, \gamma_t, \sigma_t)$$

where $\rho_t = \frac{\beta_{t+1}}{\beta_t}$ and where $\sigma_t = \left(\{w_\tau, q_\tau, \rho_\tau, \theta_\tau, \gamma_\tau\}_{t-S}^{t-1}, \theta_t\right)^T$ is the column-vector of relevant information, excluding current input prices, the current discount factor, and current technology indicators, S being the number of past periods over which information is relevant for current decisions; the dimension of σ is $\Sigma = S\,[n + m + 1 + \Gamma] + [S + 1]\,\Theta$ where Θ is the dimension of θ_t and Γ is the dimension of γ_t.

It is for future analytical convenience that we keep current prices and technology indicators distinct from other information variables in $\tilde{J}_t$. Unlike other variables, they would be present in the model as decision parameters even if they did not play any informational role in future periods. In what follows, it will also be important to keep the state of technology distinct from other variables. For example, if γ_t is just the date, we will need to express technology in a form that does not depend on expectation variables, but is conditioned on the date; this requires allowing all variables in $\tilde{J}_t$ to vary while γ_t is kept constant. Thus we partition $\tilde{J}$ as

$$\tilde{J}_t = (J_t, \gamma_t) \tag{5}$$

with

$$J_t = (w_t, q_t, \rho_t, \sigma_t) \tag{6}$$

Finally, if we take the view that current output is in fact endogenous, although chosen at some other stage of the decision process, then it does not play any role in the formation of expectations: indeed if current output is actually under the control of the firm, the latter does not change its expectations about future prices and output levels when it considers alternative current levels of y_t. Alternatively, if we take the view that output is truly exogenous, then it may play a role in the formation of anticipations implying that current and past levels of y are part of σ_t. In either interpretation, future levels of y are unknown; they are anticipated from the information vector J_t in the same way as, say, future prices.

Before defining the cost functions that generate variable factor demands and investment demands, we make an assumption on the information set that will greatly simplify the analysis.

Assumption 2 *S is finite and independent of t; the dimension of θ is finite; the composition of θ and γ is time invariant.*

The finite dimension of the information vector is a fairly common assumption in discrete-time models; the constant dimension assumption is not crucial to the analysis but simplifies notation and derivations greatly. Instead of S, we could also have chosen one specific duration for each component of the information vector, at the cost of increasing the notational burden.

Let the expected future cost function $\bar{C}(J_t, k_{t+1}, \gamma_t)$ be defined as the minimum, *as expected at* t, and discounted to t, of cumulated expenditures from $t+1$ on

$$\bar{C}(J_t, k_{t+1}, \gamma_t) = \rho_t E_t \min_{\{x_\tau(\cdot), i_\tau(\cdot)\}} \left\{ \sum_{\tau=t+1}^{\infty} \left[\frac{\beta_\tau}{\beta_{t+1}} (w_\tau^T x_\tau + q_\tau^T i_\tau) \right] \mid (2) \text{ and } (3) \right\} \tag{7}$$

We need to strengthen the existence assumption mentioned earlier as follows:

Assumption 3 *Expectations are formed in such a way that, given f, there exists a unique solution to problem (1). The solution is such that the expected future cost function*

$\bar{C}(J_t, k_{t+1}, \gamma_t)$ defined as part of the solution to (1) is unique and twice continuously differentiable in J_t, k_{t+1}, and γ_t.

It is easy to see that $\bar{C}$ is non increasing in k_{t+1}: suppose it is not; then, for some (J_t, k, γ_t), there exists $k' > k$ such that $\bar{C}(J_t, k', \gamma_t) > \bar{C}(J_t, k, \gamma_t)$. However, with $k_{t+1} = k'$, it is possible to use decision rules that will yield the same values of x_τ and i_τ as are obtained, optimally, when $k_{t+1} = k$; doing so would achieve the same expected cumulated discounted costs while exceeding the production constraints (since f is increasing in k); thus $\bar{C}(J_t, k', \gamma_t) > \bar{C}(J_t, k, \gamma_t)$ cannot result from an optimal factor allocation rule, proving the result. Since $\bar{C}_{k_{t+1}}$ gives the reduction in expected future cumulative discounted costs associated with marginal increases in k_{t+1}, acquired at unit cost q_t, $q_t + \bar{C}_{k_{t+1}}(J_t, k_{t+1}, \gamma_t)$ may be interpreted as the vector of quasi-fixed factor shadow prices at t. This shadow prices reflect both expectations and the technology.

Although we do not attempt to characterize the set of expectations under which Assumption 3 holds, it is easy to see that this set is non empty. Indeed, for any $\bar{C}_{k_{t+1}}(J_t, k_{t+1}, \gamma_t) \leq 0_{(1\times m)}$, suppose that expectations at t are such that $t+1$ is expected with certainty to be the last operating period (e.g. $w_\tau = \infty$, $q_\tau = 0$, and $y_\tau = 0$ for $\tau > t+1$) and that y_{t+1}, as well as all relevant parameters at $t+1$, are known with certainty as functions of current information J_t. In that case, $\bar{C}(J_t, k_{t+1}, \gamma_t) = c(w_{t+1}(J_t, \gamma_t), q_{t+1}(J_t, \gamma_t), k_{t+1}, y_{t+1}(J_t, \gamma_t))$ is a static cost function, the (unique, twice continuously differentiable) solution to the minimization of expenditures at $t+1$ subject to (2); it is easily shown that $\frac{\partial \bar{C}(J_t, k_{t+1}, \gamma_t)}{\partial k_{t+1}} = -\frac{\partial c(w_{t+1}, q_{t+1}, k_{t+1}, y_{t+1})}{\partial y_{t+1}} \frac{\partial f(x_{t+1}, k_{t+1}, i_{t+1}; \gamma_{t+1})}{\partial k_{t+1}}$. Since, for any f satisfying Assumption 1, $\frac{\partial c}{\partial y}$ may be chosen to equal any non negative value by choice of w_{t+1} and q_{t+1}, this provides an example of the construction of $\bar{C}$.

Other examples may be obtained by reformulating the models of Epstein (1981a), Epstein and Denny (1983), or Hansen and Sargent (1980) to fit the above framework. Thus if we assume that the discount rate and the input prices are constant and known over the entire period and that the technology is invariant over time, we obtain Epstein's model. On the other hand, if the technology is quadratic and if we assume that firms do not make any systematic forecasting errors, we get Hansen and Sargent's model. Thus we may state:

Lemma 1 *For any technology satisfying Assumption 1, the set of expectations satisfying Assumption 3 is non empty.*

By a standard dynamic programming argument

$$G(J_t, k_t, y_t, \gamma_t) = \min_{x_t, i_t} \left\{ w_t^T x_t + q_t^T i_t + \bar{C}(J_t, k_{t+1}, \gamma_t) \right\} \tag{8}$$

subject to $y_t \leq f(x_t, k_t, i_t; \gamma_t)$, $k_{t+1} = [I - \delta_t] k_t + i_t$, with k_t and y_t given. Let $x_t(J_t, k_t, y_t, \gamma_t)$ and $i_t(J_t, k_t, y_t, \gamma_t)$ be the solutions to problem (8). Then we may define the contemporary cost function as

$$C(J_t, k_t, y_t, \gamma_t) = w_t^T x_t(J_t, k_t, y_t, \gamma_t) + q_t^T i_t(J_t, k_t, y_t, \gamma_t) \tag{9}$$

Now we turn to the properties of the model, which will be presented in the form of propositions.

3 Properties

Unless otherwise mentioned, from here on, variables are evaluated at t, the notation z_1 means that the variable z is measured one period into the future relative to other variables, and expected-value operators are conditioned on information relevant at t. For a function $g(l, z)$, g_l refers to the partial derivative (the line-vector of partial derivatives) of g with respect to the variable (the vector) l; also, if a vector h contains l as well as other elements b which are not arguments of g, i.e. $h^T = \left(b^T, l^T\right)$, then $g_h = \left(0^T, g_l\right)$

where 0^T is a transposed vector of zeros with the same dimension as b. Occasionally, we use the same notation for vectors or matrices of different dimensions; thus 0 may be a number, or a matrix of zeros, whose dimension must ensure compatibility with the terms it is combined with. If we need to specify the dimension of such matrices, we use a subscript between parentheses, as in $I_{(n\times n)}$ the diagonal matrix of ones. Unless otherwise mentioned, x and i will either refer to the optimized level of variables x and i, or to the functions $x(.)$ and $i(.)$.

Dynamic factor demands are known not to possess the same testable properties as their static counterparts. The same is true for our model, as it is more general in several respects than most alternative dynamic factor-demand models. It is also well known that this lack of verifiability does not affect variable-factor demands, conditional on (y, k, i). However, here, variable factor demands are conditional on (y, k) only. As the following proposition indicates, their observable properties include the additivity property, but not homogeneity, nor the symmetry and negative definitiveness of x_w: like investment demands, variable factor demands may not be downward-sloping. (Proofs may be found in the Appendix)

Proposition 1 *Variable factor demands, investment demands, and quasi-fixed factor shadow prices have the following properties*

1.

$$\begin{bmatrix} w^T & q^T + \bar{C}_{k_1} \end{bmatrix} \begin{bmatrix} x_J & x_k & x_y & x_\gamma \\ i_J & i_k & i_y & i_\gamma \end{bmatrix} = \begin{bmatrix} 0^T & -\lambda f_k & \lambda & -\lambda f_\gamma \end{bmatrix}$$

where λf_k is a vector whose elements are all positive and λ is a positive scalar.

2. The system of factor demands satisfies the following symmetry and negativity con-

ditions

$$\begin{bmatrix} x_J^T & i_J^T & 0 \end{bmatrix} + i_J^T \bar{C}_{Jk_1} \quad \textit{is symmetric and negative semi-definite}$$

where 0's dimension is $a \times (a - [m+n])$.

Properties 1 and 2 correspond to the familiar homogeneity, additivity, symmetry, and curvature conditions of static factor-demand theory. In particular, in the absence of quasi-fixed factors, $i_J = 0$ and J reduces to w; thus Property 2 reduces to x_w symmetric and negative semi-definite, while Property 1 becomes $w^T x_w = 0$ and $w^T x_y = \lambda$, where λ is marginal cost. By the symmetry of x_w, it follows that $x_w w = 0$, which is the homogeneity property. Note that Property 2 is possible only if $a \geq m + n$, which is the case since J contains at least all factor prices. In general, $a > m+n$, so that the matrix in Property 2 is negative semi-definite, and not negative definite. As is clear from the proof, the first-order conditions for the maximum of g with respect to J may be viewed as a system of a equations in the $n + m$ variables (x, i). The existence and unicity of (x, i) as functions of J thus require functional relationships between the elements of the a equations, as implied by the proposition.

One convenient way to express Proposition 1 in terms of factor demands only is given in the following corollary.

Corollary 1 *If* i_q *is invertible, then, at the optimum*

1. $\bar{C}_{k_1} = -\left[w^T x_q i_q^{-1} + q^T\right] < 0$

2. $w^T \left[x_J - x_q i_q^{-1} i_J\right] = 0$

3. $w^T \left[x_k - x_q i_q^{-1} i_k\right] < 0$

4. $w^T \left[x_y - x_q i_q^{-1} i_y\right] > 0$

5. $w^T\left[x_\gamma - x_q i_q^{-1} i_\gamma\right] < 0$ *in case of technological progress* (≤ 0 *in case of regress)*

6. $\begin{bmatrix} x_J^T & i_J^T & 0 \end{bmatrix} + i_J^T\left[\left[I-\delta+i_k\right]^{-1}\left[\frac{\partial\left[w^T x_q i_q^{-1}\right]}{\partial k}\right]^T i_J - \left[\frac{\partial\left[w^T x_q i_q^{-1}+q^T\right]}{\partial J}\right]^T\right]$ *is symmetric and negative semi-definite.*

If i_q is not invertible, another subset of the matrix equation in Proposition 1.1 may be used to obtain an alternative expression for $\bar{C}_{k_1}$ in terms of x and i. In the rest of the paper, we assume that i_q is invertible. Although Corollary 1 only involves rewriting Proposition 1, we have also replaced equations involving λf_k, λ, and λf_γ with inequalities; this is because, λf_k, λ, and λf_γ being unobservable, the only empirically relevant information that they contain in general is their sign.

In a static model, the curvature condition (Proposition 1.2) would only involve C_{JJ} with J reduced to the vector of current prices, thus collapsing to the familiar curvature property of the cost function (concavity in current prices). Although difficult to impose on an econometric system of factor demands *a priori*, the restrictions implied by Proposition 1.2 would be easy to verify *a posteriori*. It is interesting to mention an important group of special cases.

Corollary 2 *If contemporary factor prices w and q do not affect $\bar{C}(J, k_1, \gamma)$, then* $\begin{bmatrix} x_w & x_q \\ i_w & i_q \end{bmatrix}$ *is symmetric and negative-definite.*

This special case (downward sloping, symmetric, factor demands) includes all situations where contemporary prices do not affect current expectations (as, e.g., when expectations are formed with a lag). In fact, the matrix of price effects may be expressed, in general, as the sum of a symmetric negative-definite matrix and a matrix reflecting both expectations and the technology. The lemma spells out sufficient conditions under which the second matrix vanishes.

Before closing this section of the paper it is useful to draw attention to what distinguishes the contemporary cost function C from some other cost functions. C is a restricted cost function (Diewert, 1974) in the sense that k is one of its arguments, but it is not restricted with respect to i. As an important consequence, C is not homogeneous of degree one in current variable factor prices.

It is also interesting to compare our results with Epstein (1981a, b). In 'Generalized Duality and Integrability', Epstein allows the non decision variables (prices and/or other parameters) faced by an agent to enter the optimization problem in a non linear way. Thus, e.g., the linear budget constraint is replaced with a non linear relationship. Our model, besides the fact that $\bar{C}$ is not known in Problem (8), may be seen as an extension to a situation where the agent faces an additional (capital motion) constraint, and as a restriction to the extent that our expenditure function has some structure (before optimization). Proposition 1 is related to Epstein (1981b)'s Theorem 10.

More importantly, as is clear from the contemporaneous or subsequent literature on dynamic factor demands (McLaren and Cooper, 1980a, 1987; Epstein, 1981a; Epstein and Denny, 1983), the 'Generalized Duality' framework was not used in a dynamic context. There may be two reasons for that.

First, Epstein's paper provides sufficient conditions for strong integrability (Theorem 9 and the discussion following Theorem 10). One of them restricts the negative semi-definite matrix corresponding to the matrix in our Proposition 1.2 to be negative-definite; since this is possible only if the number of parameters is equal to the number of factors, this restriction is devastating in most contexts outside the standard static duality framework. However, since it is not necessary, one wonders why such a restriction, although convenient (easy to check), should be requested in dynamic models.

Second, while Epstein's 'Generalized duality' was formalized in terms of a finite number of parameters (e.g. prices), intertemporal optimization involves an infinity of future prices. This dimensionality issue may be resolved, as we have done, by allowing for uncertainty and focusing on expected value functions, but this was not the way early dynamic factor demands models were formulated. Instead, dynamic duality theory was initially established around some non-stochastic form of the Hamilton-Jacobi equation of dynamic programming.

In our model (8) defines a value function. The corresponding Hamilton-Jacobi equation involves partial derivatives of $\bar{C}$ with respect to J. As they depend on expectations, these derivatives are arbitrary. Consequently, unless restrictive assumptions are made about expectations, the Hamilton-Jacobi equation does not imply any additional restrictions on the system of factor demands. The literature on dynamic duality, on the contrary, is characterized by highly restrictive assumptions on expectations. What is remarkable about the duality result that we are presenting below, is the fact that the technology set may be recovered from a system of factor demands satisfying Proposition 1, without any knowledge about expectations.

4 Dynamic duality

One major appeal of duality is the possibility to characterize the primal technology (the production function) entirely from the analysis of factor demands. There are two main issues. The first one is the existence of a technology set dual to the system of factor demands used as an empirical model. The second one is the actual recovery of the primal technology from the factor-demand system; here, in practice, we are mostly interested in some key characteristics of the primal technology: marginal products, returns to scale, and technological change. We shall deal with both issues in turn, starting with the existence issue.

Let $x^*(J, k, y, \gamma)$ and $i^*(J, k, y, \gamma)$ be twice differentiable functions that satisfy Corollary 1; let $\bar{C}^*(J, k_1, \gamma)$ be some twice differentiable real valued function such that, $\bar{C}^*_{k_1}(J, k_1, \gamma) = -\left[w^T x^*_q + q^T i^*_q\right] i^{*-1}_q$ when k_1 is set at its optimal value $k_1 = [1-\delta]\, k + i^*$; let $G^*(J, k, y, \gamma) \equiv w^T x^*(J, k, y, \gamma) + q^T i^*(J, k, y, \gamma) + \bar{C}^*(J, [1-\delta]\, k + i^*, \gamma)$. As a dual to the cost minimization Problem (8) one may define, at any date, the technology set, for all admissible (k, y, γ), as[2]

$$Z^*(k,y,\gamma) = \big\{(x',i') : w^T x' + q^T i' + \bar{C}^*(J,[I-\delta]\,k + i',\gamma) \geq w^T x^*(J,k,y,\gamma) \\ +q^T i^*(J,k,y,\gamma) + \bar{C}^*(J,[I-\delta]\,k + i^*(J,k,y,\gamma),\gamma)\ \ \forall J\big\} \tag{10}$$

Note that Z^* is defined for all J (not only for all w, q, and ρ), and is conditioned on γ. Unlike previous dynamic duality results characterizing future dimensions of technology together with current ones, Z^* represents only the current state of the technology, and the fact the it is conditional on γ will allow the analyst to study its evolution over time, as data on factor demands become available.

We must pause to address an important issue at this stage. As the reader may have noticed, $\bar{C}^*$ is only partially defined. First it is defined by a condition on one of its partial derivatives, $\bar{C}^*_{k_1}$ only. Second, that partial derivative is only known at $k_1 = [I-\delta]\,k + i^*\,(J,k,y,\gamma)$, so that $\bar{C}^*_{k_1}$ is not known from its definition as an explicit function of k_1 but as a function of (J,k,y,γ) : $\bar{C}^*_{k_1}\,(J,[I-\delta]\,k + i^*\,(J,k,y,\gamma)\,,\gamma)$. Indeed this is reflected in the fact that Z^* is not conditioned on k_1 but on k and y. Thus one might conclude that, although (10) is a proper definition of the technology set, it does not permit its recovery because $\bar{C}^*\,(.)$ is not entirely known. In fact the opposite is true; the following result can be established.

Lemma 2 *The technology set may be expressed in terms of factor demands only:*

$$Z^*(k,y,\gamma) = \big\{(x',i') : w^T x' + q^T i' \geq w^T x^*(J,k,y,\gamma) + q^T i^*(J,k,y,\gamma)$$

$$- \left[w^T x^*_q(J,k,y,\gamma) + q^T i^*_q(J,k,y,\gamma)\right] i^{*-1}_q(J,k,y,\gamma)\,[i' - i^*(J,k,y,\gamma)] + \mathcal{O}(J,k,y,\gamma,\epsilon)\forall J\big\} \tag{11}$$

[2]Equivalently, the technology may be defined by the production function $f^*(x,k,i,\gamma) = \max_y \left\{y\ |G(J,k,y,\gamma) \leq w^T x + q^T i + \bar{C}([I-\delta]\,k + i, J,\gamma)\ \forall\ J\ \right\}$.

where $\epsilon = i' - i^*(J,k,y,\gamma)$*,* $\mathcal{O}(J,k,y,\gamma,\epsilon) = \frac{1}{2}\epsilon^T \bar{C}^*_{k_1 k_1}([1-\delta]\,k + i^*(J,k,y,\gamma), J, \gamma)\epsilon +$ *higher-order terms, and* $\mathcal{O}(J,k,y,\gamma,\epsilon)$ *may be expressed in terms of factor demands by use of the identity* $\bar{C}^*_{k_1} \equiv -\left[w^T x^*_q + q^T i^*_q\right] i^{*-1}_q$*.*

The fact that (11) is a very complex expression is irrelevant, as it is not meant ever to be implemented. Its usefulness resides with the next proposition, which states our main duality result, providing a theoretical foundation for the specification of factor demand equations, and justifying their use as an approach to study the primal technology.

Proposition 2 *If factor demands* $x^*(J,k,y,\gamma)$ *and* $i^*(J,k,y,\gamma)$ *satisfy Corollary 1*, *and if we define* $Z^*(k,y,\gamma)$ *as in (11), then* $x^*(J,k,y,\gamma)$ *and* $i^*(J,k,y,\gamma)$ *solve Problem (8) for a firm with production set* $Z^*(k,y,\gamma)$*.*

Given that expectations and the technology together determine factor demand decisions, it is surprising that technology may be recovered from factor demands without a complete knowledge of expectations. In reality, there is nothing mysterious about this result. Current factor demands give the knowledge of $\bar{C}^*_{k_1}$, which is enough information about expectations to specify the value to the firm of the link between present and future provided by the marginal unit of each type of capital. This in turn allows the recovery of the current technology, while leaving future technologies unknown, and still not differentiated from expectations. Had it been assumed, as in Epstein (1981a), that technology was not changing over time, then of course the recovery would be complete.

5 The recovery of the primal technology

By use of (11), it is possible, in theory, to construct Z^* from the estimated system of factor demands. Fortunately, content with the existence result of Proposition 2, most researchers will not need to carry out such a tedious exercise, but will only possibly be interested in marginal products, returns to scale, and technological change. As we show now, these magnitudes are easily obtained analytically. We assume that $f^*(x, k, i, \gamma)$, as defined in footnote 2, is once continuously differentiable, and we drop the '*' superscript from both f and factor demands to alleviate notation.

5.1 Marginal products

From the first-order conditions for Problem (8) we obtain $f_x = w^T/\lambda$ and $f_i = (q^T + \bar{C}_{k_1})/\lambda$, where λ is the Lagrangian multiplier associated with the technological constraint. By Proposition 1.1, $\lambda = w^T \left[x_y - x_q i_q^{-1} i_y\right]$, and by Corollary 1.1, $\bar{C}_{k_1} = -\left[w^T x_q i_q^{-1} + q^T\right]$. Substituting, we have

$$f_x = \frac{w^T}{w^T \left[x_y - x_q i_q^{-1} i_y\right]} \tag{12}$$

$$f_i = \frac{-w^T x_q i_q^{-1}}{w^T \left[x_y - x_q i_q^{-1} i_y\right]} \tag{13}$$

Also, from the equations involving partial derivatives with respect to k in Proposition 1.1, we derive, after substituting for $\bar{C}_{k_1}$ as above,

$$f_k = -\frac{w^T \left[x_k - x_q i_q^{-1} i_k\right]}{w^T \left[x_y - x_q i_q^{-1} i_y\right]} \tag{14}$$

5.2 Returns to scale

There are competing definitions of returns to scale. Because we deal with a different, dynamic, cost function, it is important to use an undisputable definition of returns to scale, based on the production function. With k fixed at $\bar{k}$, short-term returns to scale are defined as

$$\mu(x, \bar{k}, i; \gamma)^S = \frac{d \ln f(\phi x, \bar{k}, \phi i; \gamma)}{d \ln \phi}, \qquad \text{evaluated at } \phi = 1,$$

$$= \sum_{j=1}^{n} \frac{f_{x^j} x^j}{f} + \sum_{l=1}^{m} \frac{f_{i^l} i^l}{f} \tag{15}$$

where x and i are evaluated at their optimized levels. Substituting (12) and (13) into (15), we have

$$\mu^S = \frac{w^T \left[x - x_q i_q^{-1} i\right]}{y w^T \left[x_y - x_q i_q^{-1} i_y\right]} \tag{16}$$

Consequently, the primal measure μ^S is fully recoverable from the system of factor demands. Note that the formula for short-run returns to scale and marginal variable-factor products reduce to the well-known static expressions $\mu^S = \frac{C}{yC_y}$ and $f_x = \frac{w^T}{C_y}$ when the terms involving i are removed from both the numerator and the denominator, while $w^T x$ and $w^T x_y$ are respectively replaced by C and C_y, using the definition of a static cost function.

Similar results apply to long-run returns to scale to which we turn now. Long-run returns to scale are defined as

$$\begin{aligned} \mu(x,k,i;\gamma)^L &= \frac{d \ln f(\phi x, \phi k, \phi i; \gamma)}{d \ln \phi}, \qquad \text{evaluated at } \phi = 1, \\ &= \mu(x,k,i;\gamma)^S + \sum_{l=1}^{m} \frac{f_{k^l} k^l}{f} \end{aligned} \tag{17}$$

Here again, using Proposition 1 and a similar succession of substitutions as before, μ^L is fully recoverable from the system of factor demands:

$$\mu^L = \frac{w^T \left[x - x_q i_q^{-1} i\right] - w^T \left[x_k - x_q i_q^{-1} i_k\right] k}{y w^T \left[x_y - x_q i_q^{-1} i_y\right]} \tag{18}$$

5.3 Technological change

In a dynamic setup where current costs not only reflect the state of technology but also current expectations, and where current cost changes may have a counterpart in the future, a measure of technological change based on the production function is less subject to interpretation errors than a measure based on a cost function. The primal measure of technological change is defined as the shift in the production function over time, a shift associated with changes in the components of γ over time (in many models, γ is simply defined as t). Expressed as a rate, this is

$$\frac{\dot{A}}{A} = \sum_{v=1}^{\Gamma} \frac{f_{\gamma^v} \gamma^v}{f} \frac{\dot{\gamma}^v}{\gamma^v} \tag{19}$$

This can be computed using the fact that, from the equations involving partial derivatives with respect to γ in Proposition 1.1

$$f_\gamma = -\frac{w^T \left[x_\gamma - x_q i_q^{-1} i_\gamma\right]}{w^T \left[x_y - x_q i_q^{-1} i_y\right]} \tag{20}$$

Equivalently $\frac{\dot{A}}{A}$ may be measured by substracting from $\frac{\dot{y}}{y}$ the contribution of all variables not in γ

$$\frac{\dot{A}}{A} = \frac{\dot{y}}{y} - \sum_j \frac{f_{x^j} x^j}{f} \frac{\dot{x}^j}{x_j} - \sum_l \left[\frac{f_{i^l} i^l}{f} \frac{\dot{i}^l}{i^l} + \frac{f_{k^l} k^l}{f} \frac{\dot{k}^l}{k^l} \right]$$

Using the first-order conditions for Problem (8) as well as Proposition 1, we obtain

$$\begin{aligned} \frac{\dot{A}}{A} = \frac{\dot{y}}{y} - \frac{C}{\left[C_y - \bar{C}_{k_1} i_y\right] y} \left\{ \sum_j \frac{w^j x^j}{C} \frac{\dot{x}^j}{x^j} + \sum_l \left[\frac{q^l i^l}{C} \frac{\dot{i}^l}{i^l} - E_{ck^l} \frac{\dot{k}^l}{k^l} \right] \right\} \\ - \frac{C}{\left[C_y - \bar{C}_{k_1} i_y\right] y} \sum_l \left[\frac{\bar{C}_{k_1^l} i^l}{C} \frac{\dot{i}^l}{i^l} - \sum_e \frac{\bar{C}_{k_1^l} i^e_{k^l} k^l}{C} \frac{\dot{k}^l}{k^l} \right] \end{aligned} \tag{21}$$

where $C = w^T x + q^T i$, $C_y = \frac{\partial C}{\partial y}$, $\bar{C}_{k_1}$ is defined by Corollary 1.1, $\bar{C}_{k_1^l}$ being its l^{th} element, and $E_{ck^l} = \partial \ln C / \partial \ln k^l$.

Again, once the appropriate substitutions are made, the primal measure of technological change can be recovered from the system of factor demands. It is also interesting to verify that this formula simplifies to a more familiar form in the absence of adjustment costs. Indeed, if i is not an argument of f, $\bar{C}_{k_1} = -\left[w^T x_q + q^T i_q\right] i_q^{-1} = -q$, where the last equality is one of the first-order conditions for Problem (8). Defining $CV \equiv w'x$, $\epsilon_{cy} \equiv \frac{CV_y y}{CV}$, and $\epsilon_{ck^l} \equiv \frac{CV_{k^l} k^l}{CV}$ the formula simplifies to

$$\frac{\dot{A}}{A} = \frac{\dot{y}}{y} - \epsilon_{cy}^{-1} \left\{ \sum_j \frac{w^j x^j}{CV} \frac{\dot{x}^j}{x^j} - \sum_l \epsilon_{ck^l} \frac{\dot{k}^l}{k^l} \right\}$$

This is the standard static measure of technological change.

6 Testing expectations

The previous developments have focused on factor demands, costs, and the technology under the minimal assumptions on expectations spelled out in Assumption 3. It was noted in Corollary 2 that, for expectations under which current factor prices do not affect $\bar{C}(J, k_1, \gamma)$, the factor demand system exhibits the symmetry and negativity properties of static factor demands. We will show now that the model yields additional predictions, if we constrain it to the family of rational expectations. Using the definitions of $\bar{C}$ and G, (7) and (8), we may write

$$\bar{C}(J, k_1, \gamma) = \beta E \{G(J_1, k_1, y_1, \gamma_1)\}$$

Since $k_1 = [I - \delta] k + i$ and since the expectation on the right-hand side is a function of k

$$\frac{\partial \bar{C}(J, k_1, \gamma)}{\partial k} = \beta \frac{\partial E \{G(J_1, k_1, y_1, \gamma_1)\}}{\partial k}$$

Taking a second-order expansion of the right-hand side around (J, k, y, γ) and carrying out the differentiation on the left-hand side, while supposing for simplicity that there is only one quasi-fixed input, we obtain

$$[1-\delta]\,\bar{C}_{k_1} = \beta\frac{\partial}{\partial k}\Big(G + G_J E\{dJ\} + G_k dk + G_y E\{dy\} + G_\gamma E\{d\gamma\}$$
$$+\tfrac{1}{2}\sum_l\sum_j G_{J^l J^j} E\left\{dJ^l dJ^j\right\} + \tfrac{1}{2}G_{kk}dk^2 + \tfrac{1}{2}G_{yy}E\{dy^2\} + \tfrac{1}{2}\sum_l\sum_j G_{\gamma^l\gamma^j}E\left\{d\gamma^j d\gamma^l\right\}$$
$$+\sum_j G_{kJ^j}dkE\{dJ^j\} + G_{ky}dkE\{dy\} + \sum_j G_{y\gamma^j}E\{dyd\gamma^j\}$$
$$+\sum_j G_{yJ^j}E\{dydJ^j\} + \sum_l\sum_j G_{\gamma^j J^l}E\left\{dJ^l d\gamma^j\right\} + \sum_j G_{k\gamma^j}dkE\{d\gamma^j\} + hot2\Big)$$

where we used the fact that dk, defined as $k_1 - k$, is not stochastic; for any vector z, dz means $z_1 - z$, z^j being the j^{th} element of z; $hot2$ represents the residual term of the Taylor expansion. Since k is the result of past decisions, and decisions do not affect expectations, all terms of type $\frac{\partial E\{z\}}{\partial k}$ vanish when we carry out the partial differentiation with respect to k on the right-hand side of the above expression. Thus, using the fact that $\frac{\partial}{\partial k}dk = i_k - \delta$ and that $\frac{\partial}{\partial k}\left(dk^2\right) = 2\left[i_k - \delta\right]dk$

$$[1-\delta]\,\bar{C}_{k_1} = \beta\Big(G_k + G_{Jk}E\{dJ\} + G_{kk}dk + G_k\left[i_k-\delta\right] + G_{yk}E\{dy\} + G_{\gamma k}E\{d\gamma\}$$
$$+\tfrac{1}{2}\sum_l\sum_j G_{J^l J^j k}E\left\{dJ^l dJ^j\right\} + \tfrac{1}{2}G_{kkk}dk^2 + G_{kk}\left[i_k-\delta\right]dk$$
$$+\tfrac{1}{2}G_{yyk}E\{dy^2\} + \tfrac{1}{2}\sum_l\sum_j G_{\gamma^l\gamma^j k}E\left\{d\gamma^j d\gamma^l\right\} + \sum_j G_{kJ^j}\left[i_k-\delta\right]E\{dJ^j\}$$
$$+\sum_j G_{kJ^i k}dkE\{dJ^j\} + G_{ky}\left[i_k-\delta\right]E\{dy\} + G_{kyk}dkE\{dy\}$$
$$+\sum_j G_{y\gamma^j k}E\{dyd\gamma^j\} + \sum_j G_{yJ^j k}E\{dydJ\} + \sum_l\sum_j G_{\gamma^j J^l k}E\left\{dJ^l d\gamma^j\right\}$$
$$+\sum_j G_{k\gamma^j}\left[i_k-\delta\right]E\{d\gamma^j\} + \sum_j G_{k\gamma^j k}dkE\{d\gamma^j\} + \frac{\partial hot2}{\partial k}\Big) \tag{22}$$

All partial derivatives on the right-hand side may be seen to be derivatives of elements of G_k. Since, by (8), $G_k = w^T x_k + q i_k + \bar{C}_{k_1}\left[1-\delta+i_k\right]$, where $\bar{C}_{k_1}$ may be expressed in terms of factor demands using Corollary 1.1, all partial derivatives in (22) may be expressed in terms of factor demands.

Let us assume rational expectations. Then the expectations on the right-hand side may be evaluated using actual observations. Thus, for example, $EdJ = J_1 - J + \epsilon$, where ϵ's expected value is zero. As a result, (22), including $\frac{\partial hot2}{\partial k}$, may be entirely expressed in terms of factor demand functions and observed variables. The rational expectation hypothesis may be tested by including (22) into the factor demand model, and testing whether this additional m equation system significantly constrains the estimation.

7 An example

There are two ways to go about the parametrization of the factor demand model. One way consists in using a set of functions satisfying the required properties (Proposition 1) over their whole domain. Econometric estimation then provides a test of the joint hypothesis that the theory is true and that the true technology may be represented by the functions used to parametrize the model. Given that chances are slight that the true parametric representation be selected from within a possibly infinite-dimensional set of admissible parametric representations, the only reason why the joint hypothesis would not be rejected is the lack of power of the statistical tests.

The alternative way is to interpret the parametrization as an approximation of the true technology. If the candidate set of functions are flexible forms, they will represent the true technology at one point but deviate from the true technology outside this point. Under that approach, the theoretical restrictions should hold exactly at one point; at all other points, departures from the theoretical properties may reflect both approximation errors and departures from the theoretical model, making it difficult to test the theory.

The former approach has been privileged in the empirical literature, and has been interpreted to mean that the required properties should hold either over the whole domain, or at least at each observation of a data sample. In practice, the properties which take the form of equalities are imposed *a priori* on the econometric model, while inequality restrictions are usually verified *a posteriori*.[3]

[3]See Diewert and Wales (1987) for a discussion about procedures imposing curvature properties *a priori*.

Let us illustrate model specification and the global imposition of equality restrictions in an example. Rather than starting from a cost function and estimate it together with (all but one) demand or share equations as is most commonly done in static models, the procedure consists in specifying all variable-factor, and investment, demand functions and estimate them simultaneously. This approach has the additional advantage of avoiding the tricky issue, discussed by McElroy (1987) of ensuring the compatibility of error terms in share, and cost, equations.

We choose a system of quadratic factor demands. Although such a model involves a number of parameters that may be too large in many practical situations, it may be specialized to a linear model. But unlike the linear model for which any equality restrictions that holds locally also holds globally (as it does not involve any variable), the quadratic model illustrates the global imposition of the equality restrictions in Corollary 1. It should be noted that the linear specialization of our model **does not** correspond to the so-called linear quadratic model of investment. Here, specifying linear demand functions does not imply that the expected cumulative future cost function $\bar{C}$ is linear or quadratic (only that $\bar{C}_{k_1}$ is linear), nor does it imply that the value function G is known. Note also that prices should not be normalized, as the demand functions are not homogenous of degree zero.

For simplicity we assume that γ is reduced to t and that i has dimension one. Then the model is

$$\begin{aligned} x^j \;=\;& \alpha^j + \sum_{l=1}^{n} \alpha_l^j w_l + \alpha_q^j q + \sum_{m=1}^{a-n-1} \alpha_m^j \phi_m + \alpha_k^j k + \alpha_y^j y + \alpha_t^j t \qquad (23) \\ & + \frac{1}{2} \sum_{l=1}^{n} \sum_{l'=1}^{n} \beta_{ll'}^j w_l w_{l'} + \sum_{l=1}^{n} \beta_{lq}^j w_l q + \sum_{l=1}^{n} \sum_{m=1}^{a-n-1} \beta_{lm}^j w_l \phi_m + \sum_{l=1}^{n} \beta_{lk}^j w_l k \\ & + \sum_{l=1}^{n} \beta_{ly}^j w_l y + \sum_{l=1}^{n} \beta_{lt}^j w_l t + \frac{1}{2} \beta_{qq}^j q^2 + \sum_{m=1}^{a-n-1} \beta_{qm}^j q \phi_m + \beta_{qk}^j q k + \beta_{qy}^j q y \end{aligned}$$

$$+\beta^j_{qt}qt+\frac{1}{2}\sum_{m=1}^{a-n-1}\sum_{m'=1}^{a-n-1}\beta^j_{mm'}\phi_m\phi_{m'}+\sum_{m=1}^{a-n-1}\beta^j_{mk}\phi_m k+\sum_{m=1}^{a-n-1}\beta^j_{my}\phi_m y$$
$$+\sum_{m=1}^{a-n-1}\beta^j_{mt}\phi_m t+\frac{1}{2}\beta^j_{kk}k^2+\beta^j_{ky}ky+\beta^j_{kt}kt+\frac{1}{2}\beta^j_{yy}y^2+\beta^j_{yt}yt+\frac{1}{2}\beta^j_{tt}t^2$$

$\forall\, j=1,\ldots,n$ and

$$\begin{aligned}
i \;=\; & \nu+\sum_{l=1}^{n}\nu_l w_l+\nu_q q+\sum_{m=1}^{a-n-1}\nu_m\phi_m+\nu_k k+\nu_y y+\nu_t t \\
&+\frac{1}{2}\sum_{l=1}^{n}\sum_{l'=1}^{n}\mu_{ll'}w_l w_{l'}+\sum_{l=1}^{n}\mu_{lq}w_l q+\sum_{l=1}^{n}\sum_{m}^{a-n-1}\mu_{lm}w_l\phi_m+\sum_{l=1}^{n}\mu_{lk}w_l k \\
&+\sum_{l=1}^{n}\mu_{ly}w_l y+\sum_{l=1}^{n}\mu_{lt}w_l t+\frac{1}{2}\mu_{qq}q^2+\sum_{m=1}^{a-n-1}\mu_{qm}q\phi_m+\mu_{qk}qk+\mu_{qy}qy \\
&+\mu_{qt}qt+\frac{1}{2}\sum_{m=1}^{a-n-1}\sum_{m'=1}^{a-n-1}\mu_{mm'}\phi_m\phi_{m'}+\sum_{m=1}^{a-n-1}\mu_{mk}\phi_m k+\sum_{m=1}^{a-n-1}\mu_{my}\phi_m y \\
&+\sum_{m=1}^{a-n-1}\mu_{mt}\phi_m t+\frac{1}{2}\mu_{kk}k^2+\mu_{ky}ky+\mu_{kt}kt+\frac{1}{2}\mu_{yy}y^2+\mu_{yt}yt+\frac{1}{2}\mu_{tt}t^2
\end{aligned} \tag{24}$$

where, by Young's theorem, $\beta^j_{ll'}=\beta^j_{l'l}\ \forall\, j$; $\beta^j_{mm'}=\beta^j_{m'm}\ \forall\, j$; $\mu_{ll'}=\mu_{l'l}$; and $\mu_{mm'}=\mu_{m'm}$. In each equation, the first line gives the linear specialization of the model.

The equality restrictions implied by Corollary 1 consist of item #2: $w^T\left[x_J-x_q i_q^{-1}i_J\right]=0$ and the symmetry of $\begin{bmatrix}x_J^T & i_J^T & 0\end{bmatrix}+i_J^T\left[[I-\delta+i_k]^{-1}\left[\frac{\partial\left[w^T x_q i_q^{-1}\right]}{\partial k}\right]^T i_J-\left[\frac{\partial\left[w^T x_q i_q^{-1}+q^T\right]}{\partial J}\right]^T\right]$ in item #6. It is simple, although tedious, to substitute the appropriate expressions derived from (23) and (24) into these restrictions to obtain a series of equations in model parameters and variables. Since we want to impose the restrictions globally, they must be valid for any value of the variables. This yields the following set of parameter restrictions

$$\begin{aligned}
\alpha^j_l &= \frac{\alpha^j_q\nu_l}{\nu_q}\ \forall\,(j,l)\quad \alpha^j_m=\frac{\alpha^j_q\nu_m}{\nu_q}\ \forall\,(j,m)\\
\beta^j_{ll'} &= \frac{\alpha^j_q\mu_{ll'}}{\nu_q}\ \forall\,(j,l,l')\quad \beta^j_{lq}=\frac{\alpha^j_q\mu_{lq}}{\nu_q}\ \forall\,(j,l)\quad \beta^j_{lm}=\frac{\alpha^j_q\mu_{lm}}{\nu_q}\ \forall\,(j,l,m)
\end{aligned}$$

$$
\begin{aligned}
\beta_{qq}^{j} &= \frac{\alpha_q^j \mu_{qq}}{\nu_q} \;\forall\, j \quad \beta_{qm}^{j} = \frac{\alpha_q^j \mu_{qm}}{\nu_q} \;\forall\, (j,m) \quad \beta_{mm'}^{j} = \frac{\alpha_q^j \mu_{mm'}}{\nu_q} \;\forall\, (j,m,m') \\
\beta_{lk}^{j} &= \frac{\alpha_q^j \mu_{lk}}{\nu_q} \;\forall\, (j,l) \quad \beta_{qk}^{j} = \frac{\alpha_q^j \mu_{qk}}{\nu_q} \;\forall\, j \quad \beta_{mk}^{j} = \frac{\alpha_q^j \mu_{mk}}{\nu_q} \;\forall\, (j,m) \\
\beta_{ly}^{j} &= \frac{\alpha_q^j \mu_{ly}}{\nu_q} \;\forall\, (j,l) \quad \beta_{qy}^{j} = \frac{\alpha_q^j \mu_{qy}}{\nu_q} \;\forall\, j \quad \beta_{my}^{j} = \frac{\alpha_q^j \mu_{my}}{\nu_q} \;\forall\, (j,m) \\
\beta_{lt}^{j} &= \frac{\alpha_q^j \mu_{lt}}{\nu_q} \;\forall\, (j,l) \quad \beta_{qt}^{j} = \frac{\alpha_q^j \mu_{qt}}{\nu_q} \;\forall\, j \quad \beta_{mt}^{j} = \frac{\alpha_q^j \mu_{mt}}{\nu_q} \;\forall\, (j,m)
\end{aligned}
$$

where the first line corresponds to the linear model.

Once model (23)-(24) has been estimated subject to the above restrictions, one may verify if the items #1, #3, #4, and #5, as well as the negative semi-definitiveness in #6, are satisfied. All key measurement characterizing the primal technology may be computed from the estimated model: marginal products, using (12), (13), and (14); short-run and long-run returns to scale, using (16) and (18); technological change, using (19). If the analyst so wishes, it is then possible to test rational expectations by including (22) as an additional equation into model (23)-(24). Similarly, it is possible to test the expectation structure in Corollary 2 by adding the symmetry of $\begin{bmatrix} x_w & x_q \\ i_w & i_q \end{bmatrix}$ to the restrictions implied by Proposition 1.6; if this additional restriction is not rejected, then one must also check the negative-definitiveness of $\begin{bmatrix} x_w & x_q \\ i_w & i_q \end{bmatrix}$.

8 Conclusion

We have outlined a theory of dynamic factor demands which admits standard expectation formation assumptions as special cases, but is not limited to them. Even when very little is known about expectations, factor demands exhibit restrictive, testable, properties that reflect the cost minimization behavior of the firm and assumptions on the technology. Our results show that the empirical analyst may be fairly agnostic about expectations while obtaining unambiguous measurements of such important dimensions of technology as marginal products, scale economies and productivity.

Only standard assumptions are made on the technology, and the sole restriction imposed on expectations is a condition ensuring the existence of a differentiable solution to the intertemporal cost minimization problem faced by the firm. There is a dynamic duality relationship between optimized current expenditures and the technology. As a result, it is possible to recover the underlying technology from factor demand functions, without any further information on expectations. While the analysis presented here is based on a dynamic model of expected expenditure minimization, it can also be formulated in terms of expected profit maximization.

Although our model can also be extended to situations involving strategic behavior on the part of several firms, we have not addressed such issues in this paper, as the dynamic theory of the non-rival firm provided a field which is both in need of a more complete exploration, and whose boundaries are perfectly well defined. We did, however, investigate the case of rational expectations and showed how that assumption can be tested in our framework.

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PROOF APPENDIX[4]

A Proposition 1

1. At the optimum, the Lagrangian for problem (8) is equal to G

$$L = w^T x + q^T i + \bar{C}\left(J, k\left[I-\delta\right]+i, \gamma\right) + \lambda\left[y - f\left(x, k, i; \gamma\right)\right]$$
$$= w^T x\left(J, k, y, \gamma\right) + q^T i\left(J, k, y, \gamma\right) + \bar{C}\left(J, k\left[I-\delta\right]+i\left(J, k, y, \gamma\right), \gamma\right) = G\left(J, k, y, \gamma\right)$$

Differentiating with respect to $J,\ k,\ y,\ \gamma$, using the Envelope theorem, gives the result.

2. Define the function $g\left(J, k, y, \gamma; x, i\right) \equiv G\left(J, k, y, \gamma\right) - \left[w^T x + q^T i + \bar{C}\left(J, k\left[I-\delta\right]+i, \gamma\right)\right]$. This function reaches a maximum of zero with respect to J, subject to $y \leq f(x, k, i; \gamma)$, when x and i are set at the values which solve (8). The first-order conditions for the maximum of g with respect to J imply

$$g_J = G_J - \begin{bmatrix} x^T & i^T & 0 \end{bmatrix} - \bar{C}_J = 0$$

and the second-order condition is

$$g_{JJ} = G_{JJ} - \bar{C}_{JJ} \text{ negative semi-definite}$$

Substitute the demand functions $x\left(.\right)$ and $i\left(.\right)$ into the first-order conditions: the latter become identities, which may be differentiated with respect to J to obtain,

[4] **Corollary 2 and Lemma 1 follow directly from the main text.**

upon rearranging

$$G_{JJ} - \bar{C}_{JJ} \equiv \begin{bmatrix} x_J^T & i_J^T & 0 \end{bmatrix} + i_J^T \bar{C}_{Jk_1}$$

Since the left-hand side is a symmetric matrix, negative semi-definite by the second-order condition, the result is established. ■

B Corollary 1

#1. From Proposition 1.1, $\left[w^T x_J + \left[q^T + \bar{C}_{k_1}\right] i_J\right] = 0$. Thus $\left[w^T x_q + \left[q^T + \bar{C}_{k_1}\right] i_q\right] = 0$ as q is part of J; the result follows if i_q is invertible.

#2, #3, #4, and #5. Substitute the above result into the lines respectively corresponding to $J,\ k,\ y$, and γ in Proposition 1.1.

#6. From Proposition 1.2, $\left[\begin{bmatrix} x_J^T & i_J^T & 0 \end{bmatrix} + i_J^T \bar{C}_{Jk_1}\right]$ is symmetric and negative semi-definite. We want to express $i_J^T \bar{C}_{Jk_1}$ in terms of the factor demand functions. Since $k_1 = [I - \delta]\, k + i$, $\frac{\partial}{\partial k} \bar{C}_{k_1} = [I - \delta + i_k]\, \bar{C}_{k_1 k_1}$; also, $\frac{\partial}{\partial J} \bar{C}_{k_1} = i_J^T \bar{C}_{k_1 k_1} + \bar{C}_{k_1 J}$. It follows that

$$i_J^T \bar{C}_{Jk_1} = i_J^T \left(\frac{\partial}{\partial J} \bar{C}_{k_1}\right)^T - i_J^T \left[I - \delta + i_k\right]^{-1} \left(\frac{\partial}{\partial k} \bar{C}_{k_1}\right) i_J$$

From Corollary 1.1, $\bar{C}_{k_1} = -\left[w^T x_q + q^T i_q\right] i_q^{-1}$. Differentiating with respect to J and k, and substituting the results for the appropriate terms in the above expression, yields the result. ■

C Lemma 2

Replace $\bar{C}^*(J, [I - \delta]\, k + i', \gamma)$ in (10) by a Taylor expansion around $(J, [I - \delta]\, k + i^*(J, k, y, \gamma), \gamma)$. (11) follows after cancellations and rearrangements. Although the identity $\bar{C}^*_{k_1} \equiv -\left[w^T x^*_q + q^T i^*_q\right] i^{*-1}_q$ gives $\bar{C}^*_{k_1}$ as a function of (J,k,y,γ) rather than (J,k_1,γ), higher order derivatives such as $\bar{C}^*_{k_1 k_1}$ are obtained by using the fact that $k_1 = [I-\delta]\,k + i^*(J,k,y,\gamma)$:

$\frac{\partial}{\partial k_1}\bar{C}_{k_1}(J,k_1,\gamma)\,|_{k_1=[1-\delta]k+i^*(J,k,y,\gamma)} = (I-\delta+i^*_k)^{-1}\frac{\partial}{\partial k}\bar{C}_{k_1}(J,[I-\delta]\,k+i^*(J,k,y,\gamma),\gamma)$. ■

D Proposition 2

We want to show that, $\forall J'$, $x' = x^*(J',k,y,\gamma)$ and $i' = i^*(J',k,y,\gamma)$ solve Problem (8) when the choice is out of $Z^*(k,y,\gamma)$. If x' and i' indeed belong to $Z^*(k,y,\gamma)$, then they do solve the problem since this is precisely the way $Z^*(k,y,\gamma)$ is defined. Thus we have to show that, $\forall\ J$, $(x',i') \in Z^*(k,y,\gamma)$; i.e. we have to show that, $\forall\ J^T = (w^T,q^T,\rho,\sigma^T)$

$$\begin{aligned} w^T x' + q^T i' + \bar{C}^*(J,[I-\delta]\,k+i',\gamma) \;\geq\; & w^T x^*(J,k,y,\gamma) + q^T i^*(J,k,y,\gamma) \\ & + \bar{C}^*(J,[I-\delta]\,k+i^*(J,k,y,\gamma),\gamma) \qquad (25) \end{aligned}$$

or, taking a Taylor expansion of the right-hand side around (J',k,y,γ), replacing $x^*(J',k,y,\gamma)$ and $i^*(J',k,y,\gamma)$ by x' and i' respectively, and eliminating the terms that vanish by Proposition 1.1

$$\begin{aligned} w^T x' + q^T i' + \bar{C}^*(J,[I-\delta]\,k+i',\gamma) \geq\; & w'^T x' + q'^T i' + \bar{C}^*(J',[I-\delta]\,k+i',\gamma) \\ & + \left[\begin{bmatrix} x'^T & i'^T & 0 \end{bmatrix} + \bar{C}^*_J(J',k,\gamma)\right][J-J'] + hot \end{aligned}$$

where *hot* represents all terms of order two and higher in the expansion of the right-hand side of (25) around (J',k,y,γ). Simplifying, we obtain

$$\begin{aligned} \bar{C}^*(J,[I-\delta]\,k+i',\gamma) - \bar{C}^*(J',[I-\delta]\,k+i',\gamma) \geq\; & \begin{bmatrix} -x'^T & -i'^T & 0 \end{bmatrix}[J-J'] \\ & + \left[\begin{bmatrix} x'^T & i'^T & 0 \end{bmatrix} + \bar{C}^*_J(J',k,\gamma)\right][J-J'] + hot \end{aligned}$$

Taking a Taylor expansion of $\bar{C}^*(J, [I-\delta]\, k + i', \gamma)$ around $(J', [I-\delta]\, k + i', \gamma)$, and simplifying, the inequality becomes

$$0 \geq hot - hot1$$

where $hot1$ represents all terms of order two and higher in the expansion of $\bar{C}^*(J, [I-\delta]\, k + i', \gamma)$ around $(J', [I-\delta]\, k + i', \gamma)$. By definition, remembering that the right-hand side of (25) is G^*, $hot - hot1$ represents the terms of order two and higher in a Taylor expansion of $G^* - \bar{C}^*$ around J'. By Proposition 1.2, this term is non positive, proving that the inequality holds. ■