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Tax Smoothing with Stochastic Interest Rates: A Re-assessment of Clinton's Fiscal Legacy *

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Résumé:
Le retour vers une politique fiscale “raisonnable” après les déficits budgétaires élevés des années 1980 et du début des années 1990 a été acclamé par beaucoup comme la réalisation la plus importante de l’administration Clinton. Dans cet article, nous évaluons la politique fiscale des États-Unis d’après-guerre en utilisant une extension du modèle de lissage des impôts de Barro (1979), modèle généralisé pour permettre des variations stochastiques des taux d’intérêt et des taux de croissance. Nous montrons que l’évolution de ratio dette/PIB américain a été remarquablement conséquent avec le paradigme de lissage des impôts, même durant les années 1980. Le seul écart important a eu lieu durant la fin des années 1990, lorsque le ratio dette/PIB est tombé plus rapidement que ce qu’aurait prédit un lissage optimal des impôts.

Abstract:
The return to “sound” fiscal policy after the high budget deficits of the 1980s and early 1990s has been hailed by many as the Clinton administration’s most important achievement. In this article, we evaluate post-war, US fiscal policy using an extension of Barro’s (1979) tax-smoothing model, generalized to allow for stochastic variation in interest rates and growth rates. We show that the evolution of the US debt-GDP ratio has been remarkably consistent with the tax-smoothing paradigm, even during the 1980s. The only major departure occurred during the late 1990s, when the debt-GDP ratio fell more rapidly than predicted by optimal tax smoothing.

Keywords:
Public debt, tax smoothing, stochastic discounting

JEL classification: E6, F3, H6

1 Introduction

In a seminal paper, Barro (1979) develops a positive theory of debt determination which generates the classic tax smoothing result and implications for the evolution of the public debt. He demonstrates that between 1916 and 1976 government debt policy in the UK and the US, was surprisingly consistent with his simple theory. Recently however, many have argued that the debt experiences of the US (and other OECD economies) in the 1980s were seriously at odds with the predictions of the tax smoothing paradigm.[1] The basic theory implies that the budget deficit should only increase temporarily in response to shocks to government spending and growth, whereas the budget deficits in the 1980s and early 1990s were persistently high (see Figure 1). In a recent assessment of US fiscal policy, Alesina (2000) states: "*While the mediocre growth performance in the period 1979-1982 contributes to the increase in deficits, the rest of the 1980s clearly show a radical departure from tax smoothing, as budget deficits accumulated in a period of peace and sustained growth.*" He concludes that "*the fiscal policy of the 1980s was unsound from the point of view of tax smoothing.*"[2]

In 1993, perhaps heeding economists' criticisms, the US congress passed the Omnibus Budget Reconciliation Act that included a variety of tax increases and spending cuts. As Figure 1 illustrates, this policy measure along with strong GDP growth contributed to dramatic reductions in budget deficits and debt in the late 1990s. The reduction in the public debt has been widely hailed in many corners and is viewed as a major achievement of the Clinton administration. In this article, we argue that the high budget deficits and rising public debt in the 1980s were caused mainly by shocks to the interest rate and GDP growth, rather then any significant departure from sound fiscal policy. Taking these shocks into account, we show that US fiscal policy in the 1980s was perfectly consistent with tax smoothing. Rather, we contend that it is the recent budget cuts and the rapid reduction of the US public debt that represents a departure from the principle of tax smoothing.

— FIGURE 1 GOES HERE –

Figure 2 illustrates the primary deficit along with the overall budget deficit. While on average

[1] See, among others, Roubini and Sachs (1989), Alesina and Tabellini (1990), and Alesina and Perotti (1995).
[2] Underlines added by the authors.

the primary deficits in the late 1970s and early 1980s were higher than those in the period before 1975, this was mainly because of two drastic but *temporary* increases in the primary deficit during the two big recessions: 1974–1976 and 1981–1983. The reason that the budget deficits were *persistently* high is that interest payments on the debt as a percentage of GDP increased significantly during those years (see Figure 3). This increase was mainly due to the high interest rates in the 1980s. Figure 4 illustrates how the debt–GDP ratio would have evolved if the growth–adjusted effective interest rate[3] had remained at a constant level equal to the pre–1980 average. It is obvious from this counter factual that the high interest rates and low growth rates of the late 1970s and early 1980s largely account for the rising debt–GDP ratio.[4]

— FIGURE 2 GOES HERE –

Should the Reagan and Bush administrations have significantly raised the tax rate to offset the impact of rising interest rates on the debt? What is the optimal tax response to interest and growth rate shocks implied by the tax smoothing theory? Barro's (1979) model cannot be used to address these questions because it assumes deterministic interest and growth rates. In this paper, we generalize Barro's tax smoothing model to allowing for these sources of variation and use it to assess the usefulness of the tax smoothing theory in accounting for US fiscal policy during the post war period. We characterize the optimal tax policy in this model and show that the optimal marginal response of the tax rate to the debt–GDP ratio is almost the same as the optimal marginal response of the tax rate to a pure transitory government expenditure shock. When we calibrate the parameters of our model to match post–war US data, we find that the optimal marginal response of taxes to both the debt and temporary government spending shocks turn out to be quantitatively small, and that the debt–GDP ratio implied by the tax smoothing theory matches the actual data remarkably well.

If a sustained increase in debt occurs because of a sustained increase in government expenditures, then taxes should be increased. However, the sustained increase in the debt during the 1980s was largely due to a significant and persistent increase in the interest rate rather than a

[3] The growth-adjusted effective interest rate equals the average nominal interest rate the federal government pays on its debt minus the nominal GDP growth rate.

[4] The interest rate and GDP growth rate also played important roles prior to 1975. Despite budget deficits for most of the years between 1955 and 1974, the debt–to–GDP ratio declined sharply because the interest rate on debt was significantly below the GDP growth rate.

significant increase in the permanent component of government expenditures. As we show in this paper, the optimal tax response to an increase in debt due to an interest rate shocks should be very modest — of the same order of magnitude as the response to a transitory expenditure shock. The intuition behind these results follow directly from the basic principles of tax–smoothing. The contribution to the government's liability of a one dollar increase in gross debt (including interest payment) is the same as the contribution of a one dollar temporary increase in government expenditure. Note that this argument does not depend on the persistence of the interest rate.[5]

Throughout our analysis, we (like Barro) take the interest rate faced by the government to be independent of fiscal policy. We justify this assumption on three grounds. First, since the main purpose of this paper is to study the quantitative response to interest rate shocks, we need a model that generates a realistic distribution of such shocks. Standard equilibrium business cycle models have difficulties in generating a realistic interest rate process. To mimic the interest rate movements in the data we would still have to introduce some exogenous interest rate shocks in a general equilibrium model. Second the assumption allows us to analyze optimal taxation without state–contingent debt. Although there exist several general equilibrium analyses of optimal taxation (e.g. Lucas and Stokey 1983, Zhu 1992 and Chari, Christiano and Kehoe 1994), they all assume that the government uses state–contingent debt. This has the unrealistic implication that the debt–GDP ratio increases during periods when government expenditures are temporarily high and decreases when government expenditures are temporarily low, purely because of the state contingency.[6] Moreover, as pointed out by Marcet, Sargent and Seppala (1999), the persistence of the optimal debt–GDP ratio implied by these models are significantly lower than is observed in the data. Imposing the restriction that the government can only issue risk–free debt can generate more realistic debt dynamics. However, analyzing the optimal taxation problem in a general equilibrium model with risk–free borrowing is compositionally very difficult.[7] Finally, empirical studies find at most a small effect of budget deficits on interest rates (Plosser 1982 and Evans 1987). So an exogenous interest rate process may not be a bad assumption empirically.

The rest of the paper is organized as follows: Section 2 develops the model and section 3

[5]However, the distribution and persistency of the interest rate marginally (but only marginally) affects the response to both transitory expenditure shocks and interest rate shocks.

[6]We thank Larry Christiano for pointing this out to us.

[7]See Chari, Christiano and Kehoe (1995, p.366). Sargent, Marcet and Seppala (1999) is the only one that we know of that tackles such a problem. But they do not consider interest rate shocks.

characterizes the optimal tax policy under stochastic interest rates. Section 4 provides several analytically tractable examples to illustrate the main qualitative implications of the model. Section 5 studies the quantitative implications for the debt–GDP ratio that result when the model is calibrated to US data, and section 6 provides some concluding remarks. Technical details are relegated to the Appendix.

2 The Model

We extend Barro's (1979) tax–smoothing model by allowing for stochastic interest and GDP growth rates. In this model, output, interest rate, and government expenditures are taken as exogenous, and the government can finance its expenditures through taxation or by issuing nominally risk-free debt.

Let Y_t denote GDP, P_t the price level, G_t government expenditures, and τ_t the tax rate in period t. Let B_t be the stock of public debt at the beginning of period t and r_{t-1} the risk–free nominal interest rate paid on the debt in period t, which is determined in period $t-1$. Normalizing gives us the debt–GDP ratio, $b_t = B_t/P_{t-1}Y_{t-1}$, expenditure–GDP ratio $g_t = G_t/P_tY_t$, and the growth rate of nominal GDP: $v_t = \ln(P_tY_t/P_{t-1}Y_{t-1})$. The government's period–by–period budget constraint can therefore be expressed in GDP units as[8]

$$b_{t+1} = \exp(r_{t-1} - v_t)b_t + g_t - \tau_t. \tag{1}$$

Taxes impose a deadweight loss on the economy in period t that is proportional to GDP and a quadratic function of the tax rate:

$$\gamma(\tau_t, Y_t) = \frac{1}{2}\tau_t^2 Y_t. \tag{2}$$

The government's objective is to choose the optimal tax policy that minimizes the present discounted expected deadweight losses[9]

$$V(b_0) = \max_{\{\tau_t\}_{t\geq 0}} -\frac{1}{M_0}\sum_{t=0}^{\infty} E_0 M_t \frac{1}{2}\tau_t^2 Y_t \tag{3}$$

subject to the dynamic budget constraint (1) and the no–Ponzi game restriction

$$\lim_{j\to\infty} E_t M_{t+j} b_{t+j+1} Y_{t+j} \leq 0. \tag{4}$$

[8] The gross interest on the debt is $1 + r'$, where r' is the ratio of interest payments to debt. However, we define the effective interest rate as $r = \ln(1 + r')$, so that the gross interest is e^r. This transformation is for analytical convenience only.

[9] This is the same assumption used by Barro (1979).

Here, we assume that the government uses the market stochastic discount factor, M_t, to discount future deadweight losses. For the postwar period, the average GDP growth rate exceeded the average one-year interest rate. If we use the average one-year interest rate as the discount rate for the government, the government's objective function would be unbounded and the optimal policy would not be well defined. However, with a stochastic discount factor, this is not a problem provided that the risk–premium associated with GDP growth shocks is sufficiently large.[10] In addition, when tax rates, interest rates and GDP growth rates are deterministic, discounting using the stochastic discount factor is equivalent to discounting using one-year interest rate.

The government's problem can be written as a dynamic programing problem:

$$V(b_t) = \max_{\tau_t}\{-\frac{1}{2}\tau_t^2 Y_t + \frac{1}{M_t}E_t\left[M_{t+1}V(b_{t+1})\right]\} \tag{5}$$

subject to the constraint (1). The first order and envelope conditions for the dynamic programming problem yield the following first–order condition

$$\tau_t Y_t = E_t\left[\frac{M_{t+1}}{M_t}\exp(r_t - \nu_{t+1})\tau_{t+1}Y_{t+1}\right]. \tag{6}$$

If we define the nominal stochastic discount factor as:

$$M_t^P = M_t/P_t, \tag{7}$$

then the first–order condition (6) can be rewritten more succinctly as

$$\tau_t = E_t\left[q_{t+1}\tau_{t+1}\right], \tag{8}$$

where

$$q_{t+1} = \frac{M_{t+1}^P}{M_t^P}\exp(r_t). \tag{9}$$

Since r_t is the risk–free nominal interest rate, the no–arbitrage condition implies that

$$E_t[q_{t+1}] = E_t\left[\frac{M_{t+1}^P}{M_t^P}\exp(r_t)\right] = 1. \tag{10}$$

Let z_t represent a vector of exogenous shocks in period t, which include r_t, v_t, g_t and any shocks to M_t^P, and let $z^{(t)}$ be the history of the shocks up to t. Assume that $z^{(t)}$ has a well defined probability density function $\pi_t(z^{(t)})$. Then, (10) implies that

$$\hat{\pi}_t(z_{t+1}|z^{(t)}) = q_{t+1}\pi_t(z_{t+1}|z^{(t)}) \tag{11}$$

[10] In the literature, authors have side–stepped this problem by using the interest rate on long–term bonds rather than one–year interest rate on debt. But there is no justification for using a long–term interest rate to discount annually.

is also a conditional density function, which we call the risk–adjusted probability density function. Under this risk–adjusted probability density function, (8) can be written as

$$\tau_t = \tilde{E}_t[\tau_{t+1}]. \tag{12}$$

Proposition 1: *The optimal tax rate follows a martingale process under the risk–adjusted probability distribution.*

If both the interest rate r_t and the growth rate v_t are constant, and the government uses the interest rate as its discount rate, then, $q_{t+1} \equiv 1$, and we have Barro's tax smoothing result that the optimal tax rate follows a martingale process under the original probability distribution. Proposition 1 is simply a generalization of Barro's result to the case of a stochastic interest rate and a stochastic GDP growth rate. The key implication of Barro's model, that tax rate follows a martingale process, remains valid in the generalized model under the risk-adjusted probability distribution. In the next section, we turn to characterizing the optimal tax policy in the presence of shocks to interest rate, GDP growth rate, and government expenditures.

3 Characterizing the Optimal Tax Policy

In this Section, we specify more explicitly the shock processes and the stochastic discount factor. This allows us to generate sharper characterizations of the optimal tax policy for use in our quantitative analysis of Section 4. However, as we show in the appendix, the nature of the solution remains the same under much more general conditions. We adopt the following specifications:

The stochastic discount factor: We directly specify a parametric process for the stochastic discount factor:

$$-\ln\left(\frac{M^P_{t+1}}{M^P_t}\right) = r_t + \frac{1}{2}\sigma^2_m + \varepsilon_{m,t+1}, \tag{13}$$

where $\varepsilon_{m,t+1}$ is an i.i.d. variable with distribution $N(0, \sigma^2_m)$. This specification ensures that the no–arbitrage condition (10) for the nominal interest rate is always satisfied. This approach has recently been used by several authors to study the term–structure of interest rates and to analyze the optimal portfolio allocation problem.[11] It has the advantage of being able to generate realistic

[11] See, for example, Campbell, Lo and MacKinlay (1997) and Campbell and Viceira (1998).

distributions of interest rates and asset returns, which is important for our analysis of optimal policy under stochastic interest rates.

For any risky nominal return $r_{i,t+1}$, the following no–arbitrage condition must hold

$$E_t\left[\frac{M^P_{t+1}}{M^P_t}\exp(r_{i,t+1})\right] = 1. \tag{14}$$

If we assume that the unexpected return $\varepsilon_{i,t+1} = r_{i,t+1} - E_t[r_{i,t+1}]$ has a normal conditional distribution, then, substituting (13) into (14) implies that

$$E_t[r_{i,t+1} - r_t] + \frac{1}{2}\mathrm{Var}_t(\varepsilon_{i,t+1}) = \mathrm{Cov}_t(\varepsilon_{i,t+1}, \varepsilon_{m,t+1}). \tag{15}$$

That is, the expected excess return of asset i (after adjusting a variance term for log–returns) equals the conditional covariance between the asset return and the innovation in the stochastic discount factor, which measures the risk-premium on asset i. We assume that $\varepsilon_{m,t+1}$ is proportional to the unexpected return of the market portfolio. So, our model implies that the expected excess return to asset i equals the conditional covariance between the asset return and the unexpected return of the market portfolio, which is the same implication of the standard capital asset pricing model (CAPM).

The shock processes: The interest rate is assumed to follow a first–order Markov process, and the processes for GDP growth rates and government expenditures are given by the following equations:

$$v_{t+1} = \overline{v} + \frac{1}{2}\sigma^2_\nu + \varepsilon_{v,t+1}, \tag{16}$$

$$g_{t+1} = (1-\rho_g)\overline{g} + \rho_g g_t + \varepsilon_{g,t+1}, \tag{17}$$

where $0 < \rho_g < 1$, and $\varepsilon_{v,t+1}$, and $\varepsilon_{g,t+1}$ are independent i.i.d. variables with distributions $N(0, \sigma^2_v)$ and $N(0, \sigma^2_g)$, respectively. We further assume that $\{r_t\}_{t\geq 0}$ is independent of $\{\varepsilon_{v,t}\}_{t\geq 0}$ and $\{\varepsilon_{g,t}\}_{t\geq 0}$.

Growth risk–premium: We assume that innovations to the stochastic discount factor $\{\varepsilon_{m,t}\}_{t\geq 0}$ are independent of $\{r_t\}_{t\geq 0}$ and government expenditure shocks $\{\varepsilon_{g,t}\}_{t\geq 0}$, but are correlated with shocks to GDP growth $\{\varepsilon_{v,t}\}_{t\geq 0}$. We also assume that the random vector $(\varepsilon_{m,t}, \varepsilon_{v,t})$ is i.i.d. and has a joint normal distribution with a constant covariance given by

$$\gamma = Cov(\varepsilon_{v,t}, \varepsilon_{m,t}). \tag{18}$$

From (??) we know that γ may be interpreted as the risk–premium associated with shocks to the GDP growth rate.

Given these assumptions, a fairly straightforward characterization of the optimal tax policy is possible:

Proposition 2: *If there exists a unique function $\phi(.)$ and a constant $\phi^* > 0$ such that $0 < \phi^* \leq \phi(r_t) < 1$ and*

$$\phi(r_t) = \frac{e^{r_t - \overline{v} + \gamma} E_t\left[\phi(r_{t+1})\right]}{1 + e^{r_t - \overline{v} + \gamma} E_t\left[\phi(r_{t+1})\right]}, \tag{19}$$

then, the optimal tax rate is given by

$$\tau_t = \overline{g} + \phi(r_t) e^{r_{t-1} - v_t} b_t + \psi(r_t)(g_t - \overline{g}), \tag{20}$$

where $\psi(r_t)$ is the unique bounded solution to the linear functional equation

$$\psi(r_t) = (1 - \phi(r_t))\rho_g E_t\left[\psi(r_{t+1})\right] + \phi(r_t), \tag{21}$$

and where $\phi(r_t)$ and $\psi(r_t)$ are increasing functions of r_t.

Proof: To verify that this is a solution, first use (20) to substitute for τ_t into (1). This yields

$$b_{t+1} = (1 - \phi(r_t)) e^{r_{t-1} - v_t} b_t + (1 - \psi(r_t))(g_t - \overline{g}). \tag{22}$$

Leading (20) forward one period and using (22) to substitute for b_{t+1} yields

$$\tau_{t+1} = \overline{g} + \phi(r_{t+1}) e^{r_t - v_{t+1}} \left[(1 - \phi(r_t)) e^{r_{t-1} - v_t} b_t + (1 - \psi(r_t))(g_t - \overline{g})\right] + \psi(r_{t+1})(g_{t+1} - \overline{g}) \tag{23}$$

Substituting (20) and (23) into the first–order condition (8) gives

$$\begin{aligned} \phi(r)_t e^{r_{t-1} - v_t} b_t + \psi(r_t)(g_t - \overline{g}) \quad &= \quad E_t[q_{t+1} \phi(r_{t+1}) e^{r_t - v_{t+1}} [(1 - \phi(r_t)) e^{r_{t-1} - v_t} b_t + (1 - \psi(r_t))(g_t - \overline{g})]] \\ &\quad + E_t[q_{t+1} \psi(r_{t+1})(g_{t+1} - \overline{g})] \end{aligned} \tag{24}$$

where we have used the fact that $E_t\left[q_{t+1}\right] = 1$. It follows that for (20) and (21) to be a solution to the problem, it is sufficient that

$$\phi(r_t) = (1 - \phi(r_t)) E_t \left[q_{t+1} \phi(r_{t+1}) e^{r_t - v_{t+1}}\right] \tag{25}$$

and

$$\psi(r_t)(g_t - \overline{g}) = E_t \left[q_{t+1} \phi(r_{t+1}) e^{r_t - v_{t+1}}\right] (1 - \psi(r_t))(g_t - \overline{g}) + E_t\left[q_{t+1} \psi(r_{t+1})(g_{t+1} - \overline{g})\right]. \tag{26}$$

Substituting for v_{t+1} using (16) and noting that $q_{t+1} = e^{-\frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1}}$, $Cov(r_{t+1}, \varepsilon_{m,t+1}) = Cov(r_{t+1}, \varepsilon_{v,t+1}) = 0$ and $Cov(\varepsilon_{\nu,t+1}, \varepsilon_{m,t+1}) = \gamma$, (25) simplifies to

$$\phi(r_t) = (1 - \phi(r_t))e^{r_t - \overline{v} + \gamma} E_t \left[\phi(r_{t+1})\right]. \tag{27}$$

Rearranging (27) yields (19). Notice that this solution is consistent because the function ϕ that solves it depends only on r_t and no other variable. Using (25) to substitute for $E_t\left[q_{t+1}\phi(r_{t+1})e^{r_t - v_{t+1}}\right]$ in (26) and rearranging yields

$$\left[\psi(r_t) - \phi(r_t)\right](g_t - \overline{g}) = (1 - \phi(r_t))E_t\left[q_{t+1}\psi(r_{t+1})(g_{t+1} - \overline{g})\right] \tag{28}$$

From (17), $g_{t+1} - \overline{g} = \rho_g(g_t - \overline{g}) + \varepsilon_{g,t+1}$, and since $Cov(r_{t+1}, \varepsilon_{m,t+1}) = Cov(r_{t+1}, \varepsilon_{g,t+1}) = 0$, it follows that (26) simplifies to the linear functional equation in (21). Observe that if there exists a strictly positive unique solution to (19) then $(1 - \phi(r_t))\rho_g < 1$. It follows that (21) can be solved forward to get the unique function

$$\psi(r_t) = \phi(r_t) + \sum_{i=1}^{\infty} \rho_g^i E_t\left[\phi(r_{t+i}) \prod_{j=1}^{i} (1 - \phi(r_{t+j-1}))\right]. \tag{29}$$

QED

In the appendix we show that a unique function $\phi(\cdot)$ satisfying the conditions in Proposition 2 exists provided that

$$r_t - \overline{v} + \gamma \geq \delta \quad \text{almost surely} \tag{30}$$

for some constant $\delta > 0$. Condition (30) is non–trivial. In the US, the average interest rate during the post–war period was below the average GDP growth rate, so that $\overline{r} - \overline{v} < 0$.[12] For condition (30) to hold, the risk–premium on shocks to the growth rate, γ, must be sufficiently large.[13]

Thus, if the risk premium is sufficiently large, Proposition 2 implies that the optimal tax rate can be decomposed into three parts: the long–term mean of government expenditures, $\overline{g}$, the tax response to debt, $\phi(r_t)\exp(r_{t-1} - v_t)b_t$, and the tax response to government expenditure shocks, $\psi(r_t)(g_t - \overline{g})$. Shocks to the interest rate and the GDP growth rate affect the optimal tax policy through their impacts on the debt–GDP ratio and through their impacts on the marginal responses of the tax rate to debt and government expenditure shocks.

[12] Note that the term σ_v^2 is very small.

[13] Condition (30) is only a sufficient condition. Even for a value of γ such that the condition does not hold, there may still exists a uniformly bounded solution.

Let $\theta_t = \phi(r_t)\exp(r_{t-1} - v_t)$ denote the marginal tax response to the debt, and let $R_t = e^{r_t - \overline{v} + \gamma} - 1$ denote the growth–adjusted interest rate on the debt. Then, the evolution of the debt–GDP ratio implied by the optimal tax policy is described by

$$b_{t+1} - b_t = [1 - \psi(r_t)](g_t - \overline{g}) + (R_t - \theta_t)b_t. \tag{31}$$

Since $\psi(r_t) < 1$, the debt–to–GDP ratio will increase if there is a positive shock to government expenditures. Starting from a positive level, the debt–GDP ratio will also increase in the absence of government expenditure shocks whenever the marginal tax response to debt, θ_t, is less than the growth–adjusted interest rate, R_t.

4 Some Illustrative Examples

In Section 5, we numerically characterize the quantitative implications of our tax–smoothing model calibrated to US data. However, in order to develop some intuition for the nature of our results it is useful to consider a number of special cases that can be solved analytically.

Example 1 – Constant Interest Rate: In this example, we show that whenever the GDP growth rate is far enough below its average level, the optimal debt–GDP ratio rises, *even in the absence of government expenditure shocks* .

Suppose that $r_t = \overline{r}$ for all t, , and that inflation is zero. This case is almost identical to Barro's original model except that the nominal GDP growth rate is stochastic. It follows that the solution to equation (19) is given by

$$\phi(\overline{r}) = 1 - e^{-(\overline{r} - \overline{v} + \gamma)}, \tag{32}$$

and the solution to (29) by

$$\psi(\overline{r}) = \frac{\phi(\overline{r})}{1 - (1 - \phi(\overline{r}))\rho_g}. \tag{33}$$

The optimal tax policy is identical to that in Barro's original model if we replace the interest rate $\overline{r}$ with the risk–adjusted interest rate $\overline{r} + \gamma$. Note that, even this special case permits the average one-year interest rate to be lower than the average GDP growth rate (as is the case for post-war US data), provided that the growth risk-premium, γ, is sufficiently large.

In the absence of spending shocks, the optimal growth in the debt–GDP ratio is then given by

$$R_t - \theta_t = e^{\overline{v} - v_t - \gamma} - 1, \tag{34}$$

so that the average growth is

$$E[R_t - \theta_t] = e^{-\gamma} - 1. \tag{35}$$

Since $\gamma > 0$, the marginal tax response to debt, θ_t, exceeds the effective interest rate, R_t on average and, as a result, the optimal debt–GDP ratio declines on average in the absence of shocks to government expenditure (as a percentage of GDP). However, whenever the realized growth rate is lower than average, so that $v_t < \overline{v} - \gamma$, then $\theta_t < R_t$ and the optimal debt–GDP ratio grows.

Example 2 – Zero Persistence in Government Spending. This example illustrates that the optimal marginal tax response to the gross debt–GDP ratio is of the same order of magnitude as would be the response to transitory spending shocks.

Suppose that $\rho_g = 0$, so that g_t follows an i.i.d. process. It follows from (29) that $\psi(r_t) = \phi(r_t)$, so that the optimal tax policy is given by

$$\tau_t = \overline{g} + \phi(r_t)[\exp(r_{t-1} - v_t)b_t + g_t - \overline{g}]. \tag{36}$$

Thus, in this case, the marginal tax response to the gross debt–GDP ratio, $\exp(r_{t-1} - v_t)b_t$, is *identical to* the marginal response to pure transitory shocks to government expenditures.[14] The optimal tax response to purely transitory expenditure shocks is relatively small — indeed, the key idea of tax–smoothing is that the tax should not fully respond to non–permanent increases in spending. This example therefore implies that we should not expect a significant increase in the optimal tax rate due to a rise in public debt caused by an increase in the growth–adjusted interest rate. The intuition behind this result is as follows: An increase in the growth–adjusted interest rate is like a pure transitory increase in government expenditures in that it increases the stock of government debt as a percentage of GDP with no direct impact on future government expenditures. The optimal tax response, then, is to have a small but permanent increase in the tax rate that will pay off the increase in the stock of debt gradually over time.

Example 3 – Zero growth, no spending shocks, two–state interest rate. This example illustrates that, holding GDP constant, as long as interest rate shocks are not permanent, then the *optimal* debt–GDP ratio rises whenever interest rates are higher than average.

[14] Increasing ρ_g raises the responsiveness of the tax rate to spending shocks, but has no effect on its responsiveness to the gross debt–GDP ratio. Thus, in general, the reponsiveness to the gross debt–GDP ratio is *less than* to spending shocks.

Suppose that the GDP growth rate is zero, that $g_t = \overline{g} \ \ \forall\ t$, and that $\sigma_m^2 = 0$. Then, $\theta_t = \phi(r_t)e^{r_{t-1}}$ and $\tau_t = \overline{g} + \phi(r_t)e^{r_{t-1}}b_t$. Equation (19) becomes

$$\phi(r_t) = \frac{e^{r_t}E_t\left[\phi(r_{t+1})\right]}{1 + e^{r_t}E_t\left[\phi(r_{t+1})\right]}. \tag{37}$$

Assume further that the interest rate r_t follows a two-state Markov process with a state space $\{r_l, r_h\}$, where $r_l < r_h$. Let $\Pr[r_{t+1} = r_l | r_t = r_l] = p_l$ and $\Pr[r_{t+1} = r_h | r_t = r_h] = p_h$ be the transition probabilities, where $0 < p_l < 1$ and $0 < p_h \leq 1$. Then, ϕ can take on two values, $\phi_l = \phi(r_l)$ and $\phi_h = \phi(r_h)$, which are determined by the following equations:

$$\phi_l = \frac{e^{r_l}\left[p_l\phi_l + (1 - p_l)\phi_h\right]}{1 + e^{r_l}\left[p_l\phi_l + (1 - p_l)\phi_h\right]}, \tag{38}$$

and

$$\phi_h = \frac{e^{r_h}\left[p_h\phi_h + (1 - p_h)\phi_l\right]}{1 + e^{r_h}\left[p_h\phi_h + (1 - p_h)\phi_l\right]}. \tag{39}$$

As long as $r_l > 0$, there exist unique solutions to the above two equations and they satisfy the conditions in proposition 2. In addition, let $\phi_s^* = 1 - e^{-r_s}$, $s \in \{l, h\}$. Then, it is straightforward to verify that the solutions to (39) and (40) satisfy

$$\phi_l^* < \phi_l < \phi_h \leq \phi_h^*, \tag{40}$$

and that $\phi_h = \phi_h^*$ if and only if the transition to the high–interest state is permanent, $p_h = 1$.

If $r_t = r_{t-1} = r_l$, then

$$\theta_t = \phi_l \exp(r_l) > \phi_l^* \exp(r_l) = R_t \tag{41}$$

and therefore the debt–to–GDP ratio decreases. On the other hand, if $r_{t-1} = r_t = r_h$, we have

$$\theta_t = \phi_h \exp(r_h) \leq \phi_h^* \exp(r_h) = R_t \tag{42}$$

and the equality holds if and only if $p_h = 1$. Thus, tax–smoothing implies that the *optimal* debt–GDP ratio declines in the low interest rate state and is non-decreasing in the high interest rate state. If the economy stays in the high interest rate state permanently, then the debt–GDP ratio will be a constant. If the economy stays in the high interest rate state only temporarily, then the debt–GDP ratio rises when the interest rate is high. Therefore, the optimal tax response to a positive interest rate shock depends crucially on the persistency of the shock. If the shock is permanent, than the optimal tax response is to fully respond to the shock so that the debt–GDP ratio stays constant. If the shock is temporary, however, the optimal tax response is such that the debt–GDP ratio increases since it is expected that the interest rate will decline and therefore that debt–GDP ratio will decline in the future.

Example 4 – Two–state interest rate with high persistence. This example illustrates that, even if interest rate increases are expected to be permanent, the *optimal* debt–GDP ratio still rises during periods of lower than average GDP growth.

To see this, suppose that everything is the same as in example 3 except v_t is an i.i.d. variable and $p_h = 1$, so that an increase in the interest is permanent. Then, ϕ_h is given by

$$\phi_h = 1 - \exp\left(-(r_h - \overline{v})\right), \tag{43}$$

and ϕ_l is determined by the following equation:

$$\phi_l = \frac{\exp(r_l - \overline{v})\left[p_l \phi_l + (1 - p_l)\phi_h\right]}{1 + \exp(r_l - \overline{v})\left[p_l \phi_l + (1 - p_l)\phi_h\right]}. \tag{44}$$

If $r_{t-1} = r_t = r_h$, then the growth in the debt–GDP ratio is given by

$$R_t - \theta_t = \exp(\overline{v} - v_t) - 1 \tag{45}$$

which is positive if $v_t < \overline{v}$, in which case the debt–to–GDP ratio increases (assuming $b_t > 0$). Thus, even if there is a permanent positive shock to the interest rate, the debt–GDP ratio still increases if the GDP growth rate is temporarily low. Since shocks to the GDP growth rate are generally not persistent, the optimal tax response to negative shocks to GDP growth rate is such that the debt–GDP ratio increases.

The Qualitative Implications of Tax Smoothing for Debt Dynamics:

From these examples, we can see why the tax–smoothing policy implies that the debt–GDP ratio would decline prior to the 1980s and increase in the 1980s. Prior to the 1980s, the real interest rate was low and the GDP growth rate was high so that the effective interest rate was well below its long-term average. In this period, the optimal marginal tax response to debt should be higher than the effective interest rate on debt, which implies that the debt–GDP ratio should decline. In the 1980s, the real interest rate increased significantly and the GDP growth rate dropped. These shocks to the interest rate and the GDP growth rate pushed the effective interest rate above its long-term average and, in this period, the tax response to the debt should optimally be less than the effective interest rate on debt. This, along with the temporary shocks to government expenditure, imply that the debt–GDP ratio should have optimally increased during this period.

So qualitatively, at least, the dynamics of the US debt appear to have been consistent with that predicted by the tax smoothing theory.

Of course, this does not imply that the tax–smoothing model predicts an increase in the debt–GDP ratio of the magnitude that was observed in the 1980s. To address this issue it is necessary to compare the quantitative predictions of the model with the data.

5 Quantitative Implications of Tax Smoothing

In this section we study the dynamics of the US public debt implied by the optimal tax policy characterized above. To do so we estimate the shock processes and calibrate the risk–premium parameter γ.

5.1 Estimating the Shock Processes:

We assume that the interest rate also follows an AR(1) process:

$$r_{t+1} = (1 - \rho_r)\overline{r} + \rho_r r_t + \varepsilon_{r,t+1}, \tag{46}$$

where $\varepsilon_{r,t}$ is an i.i.d. variable with distribution $N(0, \sigma_r^2)$. We estimate equations (16), (17), and (46) using OLS. The estimated results are reported in Table 1. To solve the functional equations (19) and (21), however, we need to discretize the process for the interest rate r_t. We do so using a 10–state Markov chain to approximate the estimated AR(1) process of r_t specified in (46). The details of the approximation are given in the appendix.

5.2 Calibrating the Risk–Premium Parameter γ:

We allow the market portfolio to consist of both financial and human capital, and approximate the return on human capital by the per capita GDP growth rate. Thus, we have:

$$\varepsilon_{m,t+1} = \beta\left[\lambda\varepsilon_{e,t+1} + (1 - \lambda)\varepsilon_{v,t+1}\right], \tag{47}$$

where β is the ratio of $\varepsilon_{m,t+1}$ to the unexpected return on the market portfolio, $\varepsilon_{e,t+1}$ is the unexpected return on a market index and λ is the weight of financial capital in the market portfolio. We assume that $\varepsilon_{e,t+1}$ is distributed normally: $N(0, \sigma_e^2)$. This specification follows that of Jagannathan and Wang (1996) who show that allowing for human capital to be part of the market portfolio can significantly improve the fit of the CAPM in accounting for the cross–section of expected returns on the NYSE.[15] They argue that aggregate loans against future human capital (e.g. mortgages, consumer credit and personal bank loans) account for as much wealth in the US as equities. Moreover, there are also active insurance markets for hedging the risk to human capital (e.g. life insurance, UI and medical insurance). In the calibration exercise below, we use the traditional CAPM with no human capital in the portfolio (i.e., $\lambda = 1$), as the benchmark, but we also investigate the sensitivity of our results to other choices of λ.

For any given value of λ, we use the no–arbitrage condition to calibrate the value of β. From (15) and (47), we have

$$E_t \left[r_{e,t+1} - r_t \right] + \frac{1}{2}\sigma_e^2 = \beta \left[\lambda\sigma_e^2 + (1-\lambda)\sigma_{ev} \right]. \tag{48}$$

Taking unconditional expectation on both sides of the equation and solving for β yields

$$\beta = \frac{E\left[r_{e,t+1} - r_t \right] + \frac{1}{2}\sigma_e^2}{\lambda\sigma_e^2 + (1-\lambda)\sigma_{ev}}. \tag{49}$$

Since both the variance of the unexpected market return, σ_e^2, and the covariance between the market return and GDP growth, σ_{ve}, can be estimated from the data, we compute β from (49) by replacing the expectation $E\left[r_{e,t+1} - r_t \right]$ with the sample mean, $\overline{r}_e - \overline{r}$. The growth risk–premium is given by

$$\gamma = \beta \left[\lambda\sigma_{ev} + (1-\lambda)\sigma_\nu^2 \right] \tag{50}$$

which can be computed by substituting for the value of β using (49). The calibration results for the benchmark case are reported in Table 1.

Given the estimated shock processes and the calibrated parameter for the stochastic discount factor, we solve the functional equation (19) numerically. Given the solution to (19), $\phi(r_t)$, we then numerically solve the functional equation (21) to get $\psi(r_t)$. Given $\phi(r_t)$, $\psi(r_t)$ and the initial level of the debt–GDP ratio b_0, we calculate the optimal tax rate and the debt–GDP ratio iteratively using to equations (20) and (1).

Table 1 – Benchmark Parameter Values

[15] Jagannathan and Wang (1996) proxy the market return to human capital using the growth in labor income.

Parameter	1955-1999
$\overline{r}$	0.074706
ρ_r	0.953118
$\overline{v}$	0.070729
σ_v^2	0.00042086
σ_{ev}	0.0013757
σ_e^2	0.0201
$\overline{r}_e$	0.11562
$\overline{g}$	0.17594
ρ_g	0.827396
λ	1
β	2.535489
γ	0.003488

5.3 Results

Figure 5 compares the actual taxes and debt and those predicted by the tax smoothing policy for the benchmark case. In the data we follow Barro by computing the actual effective tax rate as tax revenues as a percentage of GDP. The volatility of the predicted tax rate is somewhat less than the volatility of the actual tax rate. However, it is remarkable how well the time–average of optimal tax rate predicted by the model matches that in the data. The average level from the model is determined solely by the government's intertemporal budget constraint, so this implies that on average during the postwar period tax revenues have been quite consistent with intertemporal budget balance.

Despite the relative smoothness of the predicted tax rate, it can be seen that the predicted debt tracks the dynamics of the actual debt well for the period up to 1994 and especially in the 1980s. In other words, the excess volatility of the actual tax rate is neither great enough nor persistent enough to make much difference to the evolution of the debt. Given that the optimal tax is extremely smooth, it is not surprising that the implied–GDP debt is very sensitive to the shocks to government expenditures and the growth–adjusted interest rate. The sharp increase in the US debt–GDP ratio in the 1980s resulted from the fact that adverse interest rate and growth shocks were not offset by tax rate movements. Our results demonstrate that this is perfectly consistent with the tax smoothing theory.

Since 1994, however, the actual debt–GDP ratio declined faster than that predicted by the tax smoothing theory. This rapid reduction in debt has been associated with a significant increase in taxes as a percentage of GDP, partly due to the new tax increases enacted in the 1993 Omnibus Budget Reconciliation Act.

— FIGURE 5 GOES HERE –

We now consider the sensitivity of our results to changes in the model's parameter values.

The Persistence of Interest Rate Shocks, ρ_r:

Example 3 shows that the optimal tax response to debt is sensitive to the persistence of interest rate shocks. In particular, a higher ρ_r implies a larger marginal tax response to debt and therefore a smaller effect of interest rate shocks on debt. Figure 6 shows the tax rates and debt–GDP ratios implied by the tax smoothing policy for $\rho_r = 0$, 0.953118, and 1, respectively. While it is true that the tax rates are higher for higher value of ρ_r, the quantitative difference is fairly small. As example 4 shows, the marginal tax response to debt depends on the persistence of both the interest rate and the GDP growth rate. Since we have assumed that the GDP growth rate is i.i.d., the persistence of the growth–adjusted interest rate is quite low even if the interest rate itself follows a random walk. As a result, the debt dynamics implied by the tax smoothing theory is not very sensitive to our assumptions regarding the persistence of interest rate shocks.

The Composition of the Market Portfolio, λ:

Although Jagannathan and Wang (1996) show that the assuming that wealth consists of human and not just financial wealth improves the fit of the CAPM to US market data, the appropriate value of λ is unknown.[16] We therefore consider the sensitivity our results to changes in the value of λ. This parameter enters the model only via the risk–premium parameter γ. Using (49) and (50), it is straightforward to show that

$$\operatorname{sign}\left(\frac{d\gamma}{d\lambda}\right) = \operatorname{sign}\left[\sigma_{ev}^2 - \sigma_v^2\sigma_e^2\right], \tag{51}$$

and for our benchmark parameters in Table 1 it can be verified that $\frac{d\gamma}{d\lambda} < 0$. Thus, reducing the value of λ increases the growth risk–premium, which implies that the marginal tax response to debt is larger and the debt implied by tax smoothing is lower. Figure 7 shows the tax rates and the debt–GDP ratios implied by tax smoothing for $\lambda = 1$, 0.3, and 0 respectively. The results

[16] Kandel and Stambaugh (1995) argue that even if stocks constitute a small fraction of total wealth, the stock index portfolio return could be a good proxy for the return on the portfolio of aggregate wealth.

are quantitatively very similar for $\lambda = 1$ and 0.3. For $\lambda = 0$, the implied growth risk premium is significantly higher and therefore the optimal tax rates are significantly higher, which implies that the predicted debt is significantly below the actual debt. However, this case represents a very extreme market portfolio consisting of no financial wealth.

6 Concluding Remarks

The movement of the US public debt has been influenced greatly by the variations in the interest rate and GDP growth rate. In this paper we extend Barro's (1979) tax–smoothing theory to allow for stochastic movements in the interest rate and the GDP growth rate. We show how the optimal response of the tax rate to increases in the debt–GDP ratio and to transitory government expenditure shocks depend on movements in the interest rate, the GDP growth rate and the risk–premium associated with GDP growth variability. The optimal tax policy implies that the response to increases in the debt–GDP caused by non–permanent increases in the growth–adjusted interest rate are of the same order of magnitude as the response to transitory spending shocks. As a result, during periods of higher than average interest rates and lower than average growth, an increase in the debt–GDP ratio arises as part of an optimal tax–smoothing policy, even in the absence of spending shocks.

When we calibrate our model to post–war US data, we find that the optimal tax rate and debt dynamics predicted by our model closely resemble those of the actual debt. In particular, we find that the sharp increases in the US debt–GDP ratio in the 1980s, with no large increase in tax rates, was quite consistent with the tax smoothing paradigm. Indeed the only significant departure from the principle of tax–smoothing occurred during the Clinton administration when the debt–GDP ratio fell more rapidly than predicted by the model.

It should be recognized that the tax–smoothing paradigm is about the optimal method of financing (i.e. taxation or debt) taking as given the process for government expenditures. Our estimated process for spending is based on past US experience. The fact that the recent debt–GDP ratio has fallen more rapidly than predicted by the model implies that taxes were too high, *given the estimated process for spending.* It does not necessarily imply that taxes should be cut if spending is anticipated to be persistently high in the near future. In particular, if it is anticipated that the cost of social security payments will rise substantially and that this increase will be unusually persistent, then the current level of taxes may be warranted. This caveat does not, however, affect the main message of this paper: US fiscal policy during the 1980s was *not* unsound from the point of view of tax–smoothing.

Although, our analysis demonstrates that our generalization of Barro's (1979) model provides a reasonable characterization of post war US policy, this need not be the case for other countries. In particular, some countries (e.g. Belgium, Canada and Italy) experienced much larger increases to their debt–GDP levels during the 1980s than did the US, and these increases may well reflect the political constraints suggested by Alesina and Tabellini (1990) and Alesina and Perotti (1995). In a related paper we assess the extent to which the fiscal policies of other OECD economies conform to our extended tax–smoothing model.

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Appendix A

In this appendix we derive general conditions that ensure that an optimal tax policy like the one described by Proposition 2 exists and is unique, and characterize the general nature of the tax–smoothing policy. Instead of the specific processes given in the text, we assume only that the state can be fully described by the vector $\{z_t\}_{t>0}$ which contains r , g, and v and follows a first–order Markov process.

Proposition 3: *Define the mapping T as follows:*

$$(T\phi)(z) = \frac{\exp(r)E_z[q' \exp(-v')\phi(z')]}{1 + \exp(r)E_z[q' \exp(-v')\phi(z')]}, \tag{52}$$

where T is a monotone operator on the space of positive measurable functions. If there exists an $\omega > 0$ such that

$$\exp(r)E_z[q' \exp(-v')] \geq \omega, \tag{53}$$

then T is a contraction on the space of measurable functions of z, with a unique fixed point ϕ.

Proof: Let $\mu = 1 - \omega^{-1} > 0$, and let D be the space of measurable functions of z such that $1 \geq \phi(z) \geq \mu$ for all z. Then D is a complete norm space with the sup–norm. For any $\phi \in D$, we have, from the condition in the proposition,

$$(T\phi)(z) \geq \frac{\exp(r)E_z[q' \exp(-v')]\mu}{1 + \exp(r)E_z[q' \exp(-v')]\mu} \geq \frac{\omega\mu}{1 + \omega\mu} = \mu. \tag{54}$$

So $T(D) \subseteq D$. We now prove that T is a contraction mapping on D by verifying Blackwell's discounting condition. Using the intermediate value theorem, for any $a > 0$, and $\phi \in D$, we have

$$(T(\phi + a))(z) = (T\phi)(z) + \frac{\exp(r)E_z[q' \exp(-v')]}{(1 + \exp(r)E_z[q' \exp(-v')] + \epsilon \exp(r)E_z[q' \exp(-v')])^2} a \tag{55}$$

for some $\epsilon \in [0, a]$. Again, from the condition in the proposition, we have

$$\begin{aligned} & \frac{\exp(r)E_z[q' \exp(-v')]}{(1 + \exp(r)E_z[q' \exp(-v')] + \epsilon \exp(r)E_z[q' \exp(-v')])^2} \\ \leq \; & \frac{\exp(r)E_z[q' \exp(-v')]}{(1 + \exp(r)E_z[q' \exp(-v')])^2} \leq \frac{1}{1 + \exp(r)E_z[q' \exp(-v')]} \\ \leq \; & \frac{1}{1 + \omega} < 1. \end{aligned} \tag{56}$$

So, we have

$$(T(\phi + a))(z) \leq (T\phi)(z) + \frac{1}{1 + \omega} a. \tag{57}$$

That is, the discounting condition is satisfied. Thus, T is a contraction, which implies that it has a unique fixed point ϕ in D. By definition we know that $\phi < 1$. Q.E.D.

Remark: In the special case described in the text the sufficient condition in Proposition 3 is

$$e^{r_t} E_t[e^{-\frac{1}{2}\sigma_m^2 - \varepsilon_{m,t+1}} e^{-\bar{v} - \frac{1}{2}\sigma_v^2 - \varepsilon_{v,t+1}}] = e^{r_t - \bar{v} + \gamma} \geq \omega \tag{58}$$

Letting $\delta = \ln \omega$, yields (30).

Proposition 4: *The fixed point ϕ of T is strictly increasing in r_t and γ.*

Proof: From Proposition 3 we know that the mapping T is a contraction. Let ϕ be the unique fixed point. We know that it is the limit of $T^n \phi_0$ for an arbitrary function $\phi_0 \in D$. Let ϕ_0 be an increasing function of r_t and γ, then, it can be easily shown that so are $T^n \phi_0$ for any $n \geq 1$. So is ϕ. Finally, since ϕ is increasing in r_t and γ, from the functional equation we know that it must be strictly increasing in r_t and γ. The properties of ψ_t can be proved using standard the dynamic programming argument. Q.E.D.

Appendix B: The Data

Figure 1: US Budget Deficit and Debt (% of GDP)

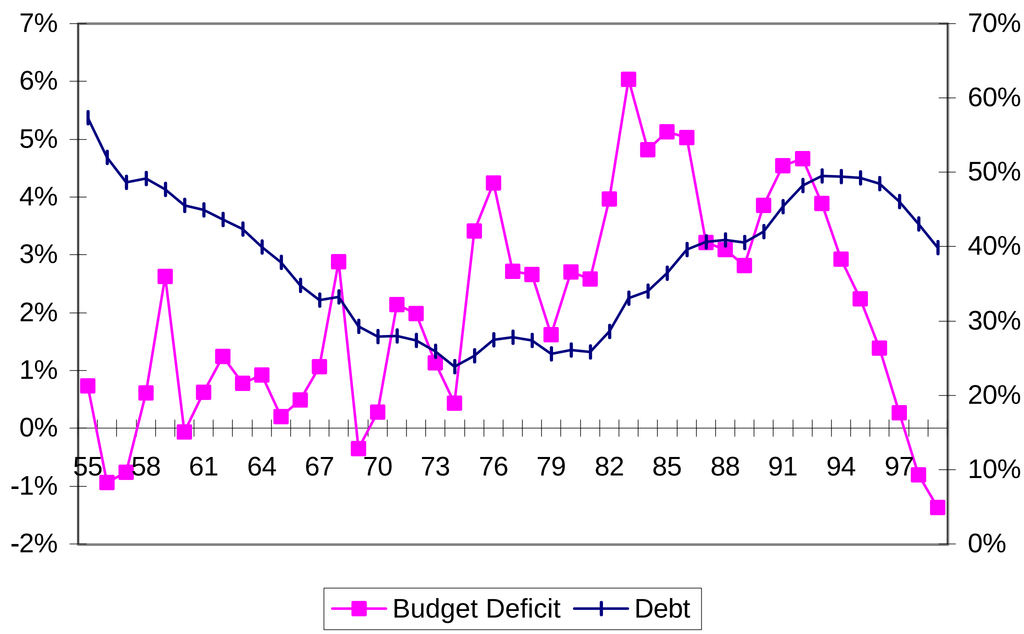


Figure 2: Budget Deficit and Primary Deficit (% of GDP)

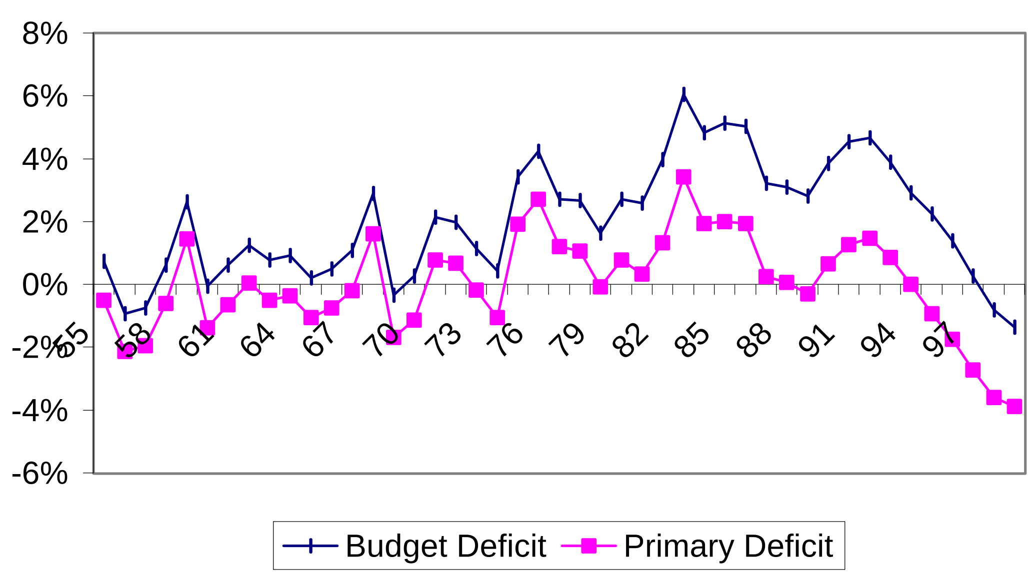


Figure 3: Effective Interest Rate on Debt

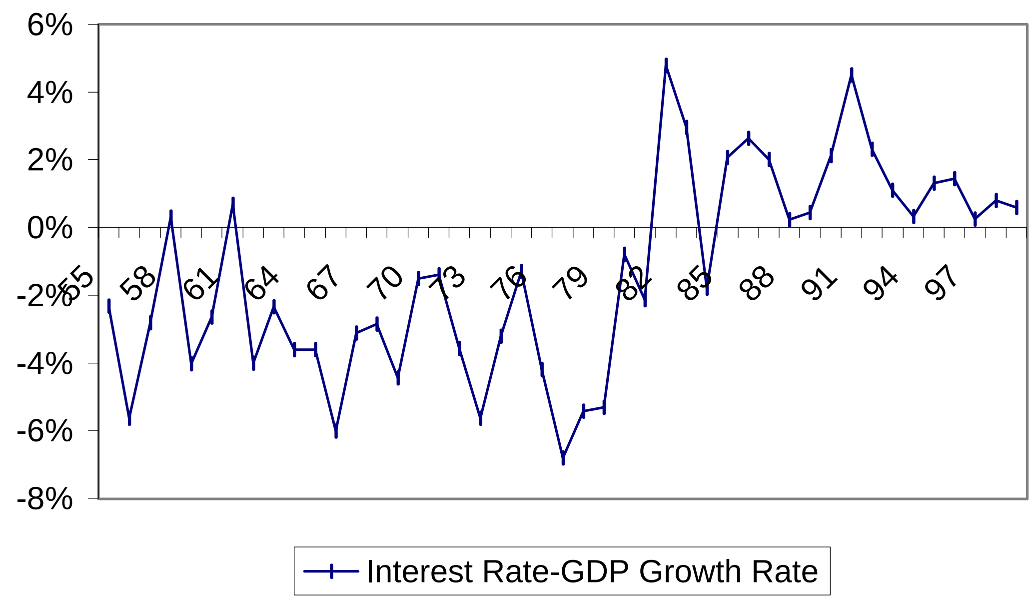
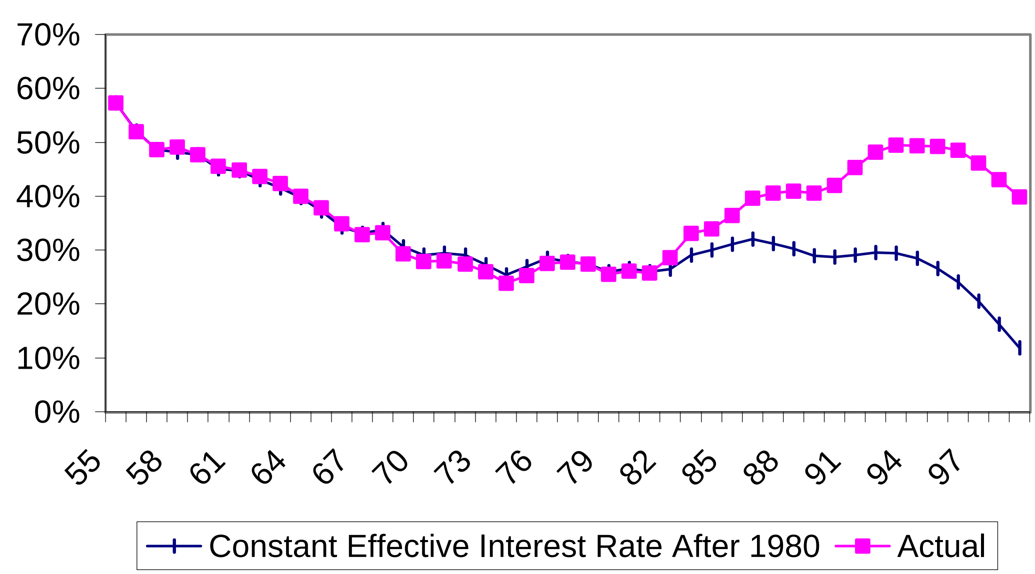


Figure 4: Interest Shocks and Debt GDP Ratio



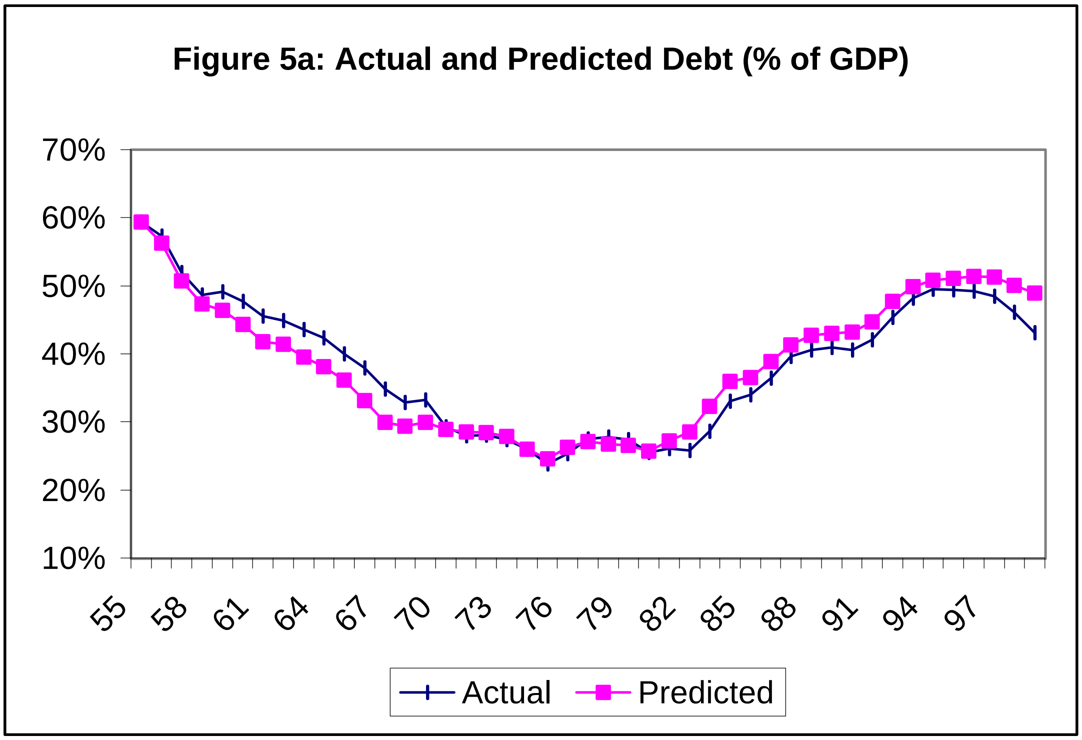


Figure 5a: Actual and Predicted Debt (% of GDP)
70%
60%
50%
40%
30%
20%
10%
55
58
61
64
67
70
73
76
79
82
85
88
91
94
97
Actual
Predicted

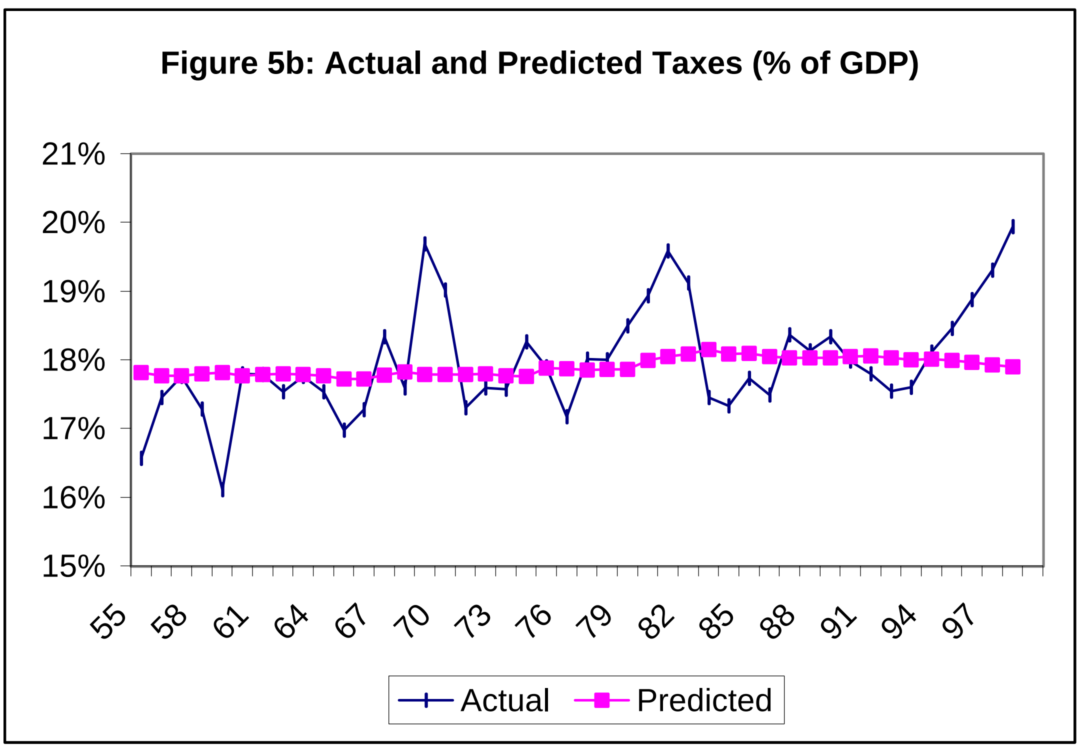


Figure 5b: Actual and Predicted Taxes (% of GDP)
21%
20%
19%
18%
17%
16%
15%
55
58
61
64
67
70
73
76
79
82
85
88
91
94
97
Actual
Predicted

Figure 6a: Persistency of Interest Rate Shocks and Debt

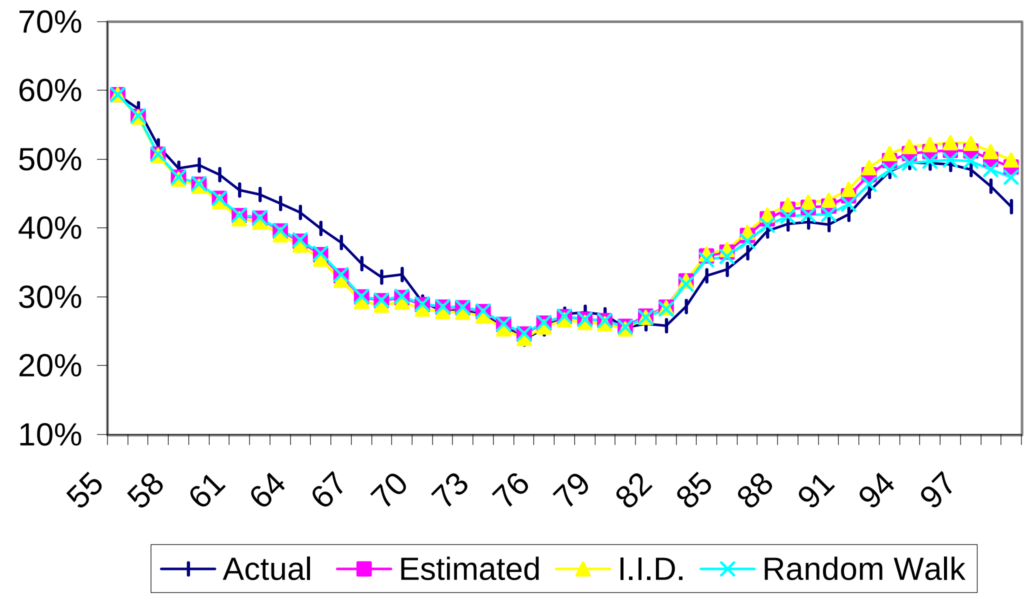


Figure 6b: Persistency of Interest Rate Shocks and Taxes

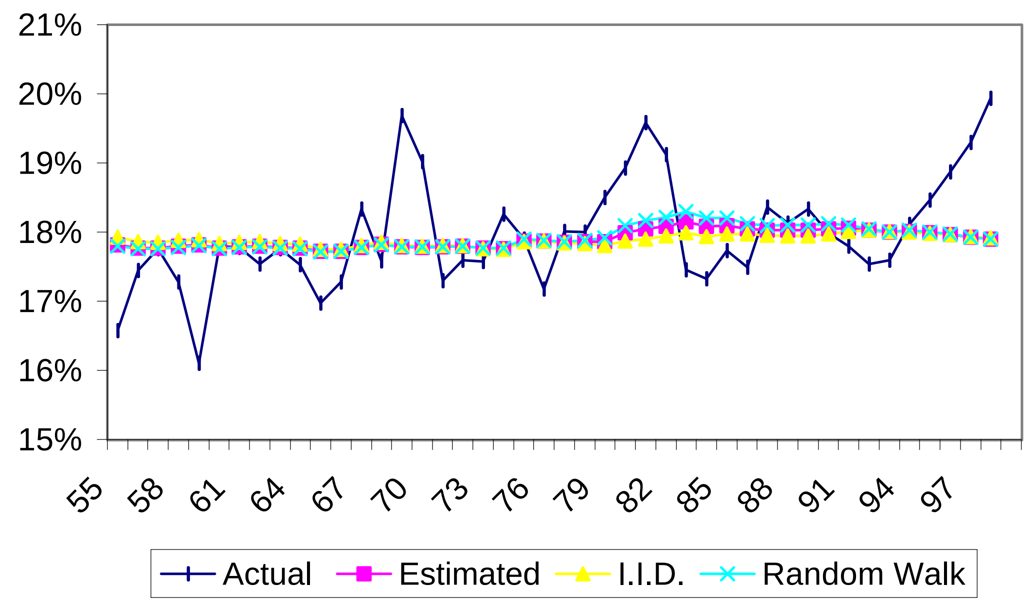


Figure 7a: Growth Risk Premiums and Debt

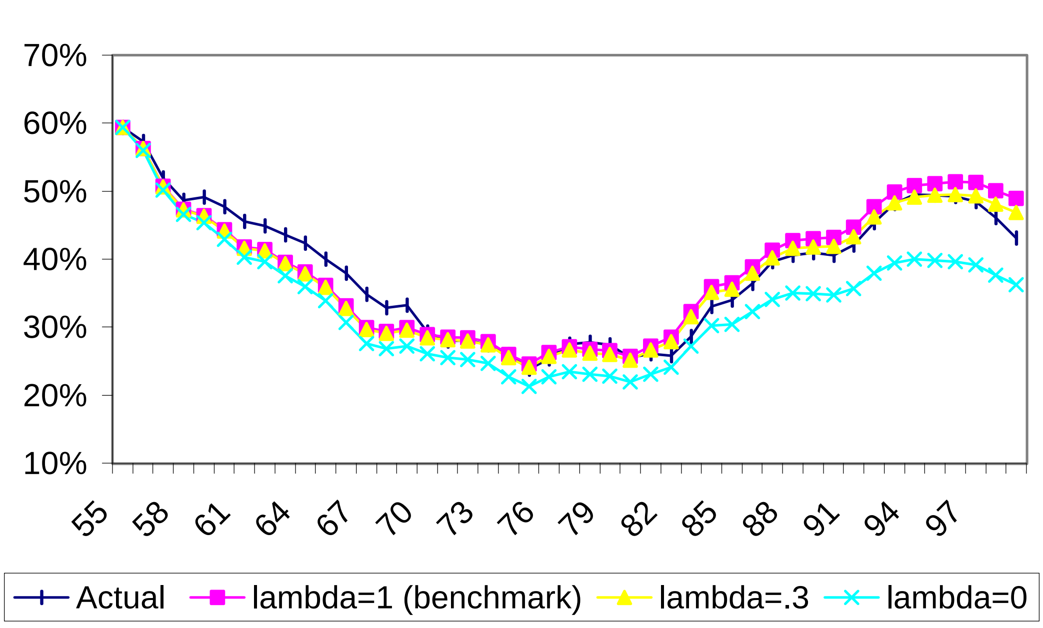


Figure 7b: Growth Risk Premiums and Taxes