Expected Multi-utility Theorems with Topological Continuity Axioms

(Draft, comments are welcomed)

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January 17, 2005

Abstract

By employing continuity axioms with respect to alternative topologies on the set of all lotteries, we present expected multi-utility theorems, in the sense of Dubra et al. [3], for preorders defined on alternative sets of lotteries: the set of all simple probability measures on a non-empty set X of all prizes, the set of all discrete probability measures on X, and the set of all finitely additive probability measures on an algebra of subsets of X.

JEL classification numbers: D11, D81

Keywords: Incomplete preference relations, Expected utility

*I am grateful to Efe A. Ok for calling my attention to the subject and for his valuable suggestions. I am also indebted to Kevin Hasker and Farhad Hüsseinov for their detailed comments and suggestions which improved the presentation substantially. Comments of participants of the Study Group at Bilkent University are also gratefully acknowledged. Of course, the usual disclaimer applies.

1 Introduction

The first attempt toward developing an expected utility theory for incomplete preferences was the paper of Aumann [2]. In this work, the author showed that if a binary relation defined on a finite dimensional mixture set satisfies all axioms other than the completeness axiom of the classical expected utility theory, there is an expected utility function that extends the relation. Kannai [8] gave necessary and sufficient conditions for possibility of such a representation for the case of relations defined on infinite dimensional vector spaces. Two recent contributions are those of Shapley and Baucells [11] and Dubra et al. [3]. In these works, the authors obtain a set of expected utility functions that completely characterize a relation instead of a single function that extends it. In the present paper we shall follow this recent approach.

Before explaining the contribution of this paper, let us briefly discuss representation results of Shapley and Baucells [11] and Dubra et al. [3]. In their second representation theorem, Shapley and Baucells [11] show that a preorder on a mixture set can be represented by a set of linear utility functions provided that it satisfies the standard independence axiom in addition to an algebraic continuity axiom. Another crucial assumption in this representation result is a more technical "properness" condition[1]. Unfortunately, apart from difficulties associated with identification of primitive restrictions on preferences that can guarantee this technical condition, a further problem with the "properness" assumption is that it is a sufficient but not a necessary condition for representability (see Example 3.5). It is clear that if a mixture set contains non-simple lotteries[2], linear utility functions obtained by the theorem Shapley and Baucells [11] do not necessarily posses an expected-utility form. Dubra et al. [3] show that if the domain of a preorder is the set of all countably additive Borel probability measures on a compact metric space, and if a stronger topological continuity axiom is satisfied, these linear utility functions will indeed posses an expected utility form. The topological approach of Dubra et al. [3] also allow the authors to drop the "properness" condition of Shapley and Baucells [11] for a preorder defined on this particular set, so that the representation theorem can be expressed solely in terms of the conventional primitive assumptions of independence and continuity.

The purpose of the present paper is to amplify the scope of the topological approach of Dubra et al. [3], so that several other cases of particular importance within the domain of the classical expected utility theory can be included. To this end, at the first step, we provide a topological characterization of representability of a preorder defined on a convex subset of an arbitrary vector space (see Lemma 2.4). Then, we focus on preorders defined on specific lottery spaces

[1]For a definition of this condition see Subsection 3.2.

[2]By a "lottery" we mean a roulette lottery: a device, say, a course of action, that produces random outcomes (prizes) with known or estimated probabilities. With mathematical terminology, a lottery on a set X is a random variable with values in X. As usual, we assume that all lotteries having the same probability distribution are indistinguishable as far as preferences are concerned, and treat a lottery and its distribution as if they are the same objects. A "simple lottery" refers to a lottery that obtains finitely many different values.

that are outside the scope of the main theorem of Dubra et al. [3], and provide necessary and sufficient primitive conditions for representability of these relations in specific forms.

To put more concretely, let us denote by X an arbitrary non-empty set of all prizes, and let $\mathcal{A}$ be an algebra of subsets of X. It will be shown that any preorder $\succsim$ on the set of all probability measures that belong to a Riesz subspace of the Riesz space of all finitely additive signed measures on $\mathcal{A}$ can be represented by a set of linear utility functions, provided that $\succsim$ satisfies the independence axiom as well as a continuity axiom with respect to (w.r.t.) the total variation norm (see Theorem 3.2). Let us emphasize that the topological continuity axiom used in this result is quite weak and plausible, but naturally still stronger than that of the classical algebraic approach. We leave a formal statement and a discussion of this axiom to Subsection 3.2. The reader should notice that, in this result, the utility functions do not necessarily possess an expected utility form. Hence, in fact, this theorem is a restricted version of the second representation theorem of Shapley and Baucells [11]. The main advantage of the present approach is that it replaces "properness" condition with a plausible continuity condition. It is also worth to note that this "properness" condition becomes a real problem only if X is infinite, and hence, our contribution concerns only this case.

As an immediate consequence of the above theorem, we obtain an expected multi-utility theorem for a preorder defined on the set of all simple lotteries on X. We shall also show that such a representation is still possible for a preorder on the set of all discrete lotteries[3] on X. A comparison of these expected-multi utility theorems with that of Dubra et al. [3] reveals their importance. First, we do not make any assumption on X so that sets of prizes that cannot be compactified in a natural manner are included. For instance, the case of monetary or physical prizes available at different points of an infinite time horizon fall into this category. Another point is that these theorems rely on a weaker continuity axiom than the continuity notion of Dubra et al. [3]. Most importantly, these theorems show that unless inclusion of non-discrete lotteries is necessary, one can dispense with the assumption that preferences are defined on the set of all Borel probability measures on X, which may be an extremely large set even if X is a compact metric space.

A major difficulty with a large set of lotteries as the domain of a preference relation is that it may contain lotteries of extremely complex forms.[4] As Aumann [2] points out, one of the main motivations behind relaxation of completeness axiom is that a decision problem might be too complex for an individual and the decision maker might prefer to stay indecisive in such problems. Allowing incompleteness itself may not solve the problem since continuity properties implied by the intended form of representation may require that a great many of lotteries are comparable with each other, no matter how complex they

[3] By a "discrete lottery" we mean a lottery that obtains countably many different values.

[4] Suppose for example that X is the interval $[0, 1]$, and let μ be the Lebesgue measure. Take any μ-measurable, bounded and non-negative real function f on X that takes positive values on a set of positive μ-measure. Then the set function that assigns to a Borel set E the number $\int_E f d\mu / \int_X f d\mu$ is a countably additive Borel probability measure on X.

are; comparability of lotteries of very simple forms may lead to comparability of lotteries of very complex forms.[5] Of course, the most natural way to avoid such a problem is to assume that preferences are defined on the smallest of all suitable lottery sets. Since many decision making problems involve only simple or discrete lotteries rather than a continuous probability distribution over the set of all outcomes, our contribution in this direction is of importance.[6] For example, in cases where an agent is supposed to make a proposal that might be materialized or rejected, each offer x of the agent defines a simple lottery that yields x with some (known) probability p_x, and the null outcome with probability $1 - p_x$. If the agent is given the opportunity to revise her offer once the previous offer is rejected, in a model with infinite time horizon where the time of materialization matters, a strategy consisting of a sequence of proposals would define a discrete lottery. Another good example of a simple lottery is an insurance contract offered to a consumer that produces two different outcomes: the cost of insurance in case loss does not take place, and the net payment in case of loss. More generally, in any situation that can be expressed as an extensive form game and that involves finitely many uncontrollable random parameters each obtaining finitely many different values, a profile of pure strategies would define a simple lottery determined by the (objective) distribution of parameters.

The main shortcoming of our expected multi-utility theorems is that, even if X is a compact metric space, in contrast to the main theorem of Dubra et al. [3], Bernoulli utility functions are not necessarily continuous on X. Since continuity of these functions might be important for existence and identification of alternatives that are maximal w.r.t. a preorder, this is a notable disadvantage.[7]

Despite the associated problems indicated above, it is of interest to know whether one can provide expected multi-utility theorems for relations containing non-discrete lotteries in their domain. After all, domain of a relation is a primitive of the theory, and a decision making problem may involve lotteries of non-discrete form, say, in a model with continuous time, a monetary lottery that obtains random values over time. By employing a continuity axiom w.r.t. the topology induced by the set of all bounded $\mathcal{A}$-measurable mappings, we shall present an expected multi-utility theorem for preorders defined on the set of all finitely additive probability measures on $\mathcal{A}$. Since, as far as we know, a topological approach within the context of this theorem has never been used in the standard theory (for a detailed exposition see Fishburn [5, Chapter 10]), we should stress that the mentioned continuity axiom is a necessary condition. A far more interesting case is the case of a binary relation defined on the set of all countably additive probability measures on $\mathcal{A}$, for finitely additive measures are unnatural objects that can hardly arise in economic problems. Unfortunately,

[5] See the weak*-continuity axiom in Subsection 3.3 and a classical monotonicity assumption given in Footnote 16. Even after the completeness axiom is relaxed, these conditions are necessary for representation in expected multi-utility form.

[6] Note, however, our expected-multi utility theorems do not function on an arbitrary convex set of lotteries. Since even the inclusion of all simple lotteries will rarely be necessary, an interesting open question is whether, under a set of plausible primitive restrictions, a generalization in this direction would be possible.

[7] See Footnote 13.

this case turns out to be much more difficult to handle than those mentioned previously. Even without imposing the restriction of continuous Bernoulli utility functions, we could not obtain a direct extension of the main theorem of Dubra et al. [3] to more general spaces of prizes. More specifically, since the case of discrete probability measures is already covered by one of the theorems introduced above, the problem arises when the space of certain prizes is uncountable and is not a compact metric space.

The organization of the paper is as follows. In Section 2, we introduce the axioms to be used in a general format and state a topological characterization of representability. In Section 3, we present our representation theorems and give a general uniqueness theorem in the sense of Dubra et al. [3].

2 Axioms and a Technical Characterization

The purpose of this section is to formulate the axioms and the problem in a general setting where the set of lotteries is a subset of an arbitrary (real) vector space. We also give a topological characterization of representability.

Throughout the section, $\succsim$ denotes an arbitrary binary relation on a convex set P contained in a vector space Y. The sets $\{\gamma\,(p-q) : \gamma > 0, p \succsim q\} \subset Y$ and $\{(p,q) : p \succsim q\} \subset P \times P$ will be denoted by cone($\succsim$) and $Gr(\succsim)$, respectively. Following the terminology of Dubra et al. [3], the term "preference relation" is used instead of the term "preorder" which, as usual, refers to a reflexive and transitive binary relation. The reader should notice that, in contrast to the standard theory, completeness is omitted in this definition of a preference relation.

Given a linear, Hausdorff topology τ on Y, (P,τ) will denote the topological space obtained by endowing P with its topology relative to the topological vector space (t.v.s.) (Y,τ).[8] Given another vector space Y' and a bilinear functional $(y,y') \rightarrow \langle y, y'\rangle$ on $Y \times Y'$, $\langle Y, Y'\rangle$ is said to be a *dual pair* if

$$\begin{aligned} \text{for any } y \text{ in } Y\setminus\{0\} \text{ there exists a } y' \text{ in } Y' \text{ such that } \langle y, y'\rangle &\neq 0, \\ \text{for any } y' \text{ in } Y'\setminus\{0\} \text{ there exists a } y \text{ in } Y \text{ such that } \langle y, y'\rangle &\neq 0. \end{aligned}$$

For any dual pair $\langle Y, Y'\rangle$, $\sigma\,(Y, Y')$ will denote the topology on Y induced by Y'.[9] $(Y,\tau)'$ stands for the topological dual of the t.v.s. (Y,τ). We denote the algebraic dual of Y by Y^*. For a set C contained in a vector space, cone(C) stands for the convex cone generated by C, that is, $\text{cone}(C) := \bigcup_{\gamma \geq 0} \gamma\, \text{co}(C)$, where co($C$) is the convex hull of C.

Next, starting with the standard independence axiom, we introduce the axioms to be used throughout the study.

[8] By a t.v.s. (Y,τ), we always mean a vector space Y endowed with a linear, Hausdorff topology τ.

[9] Under the topology $\sigma\,(Y,Y')$, a net $\{y_\lambda\}_{\lambda \in \Lambda}$ in Y converges to $y \in Y$ if and only if $\lim_\lambda \langle y_\lambda, y'\rangle = \langle y, y'\rangle$ for all $y' \in Y'$.

Independence axiom. For all $p, q, r \in P$ and for all $\alpha \in (0, 1)$,

$$p \succsim q \quad \text{implies} \quad \alpha p + (1 - \alpha) r \succsim \alpha q + (1 - \alpha) r.$$

The following continuity condition plays a central role in [11].

Weak continuity axiom. $\{\alpha \in [0, 1] : \alpha p + (1 - \alpha) q \succsim \alpha r + (1 - \alpha) w\}$ is a closed subset of $[0, 1]$ for all $p, q, r, w \in P$.

In what follows, we shall use the weak continuity axiom only to obtain a technical characterization of representability, and work almost exclusively with the two stronger continuity axioms given below. These axioms are asserted for a linear, Hausdorff topology τ on Y.

Sequential τ-continuity axiom. $Gr(\succsim)$ is sequentially $\tau \times \tau$-closed in $P \times P$. That is, for any pair of convergent sequences $\{p_n\}$ and $\{q_n\}$ in (P, τ),

$$p_n \succsim q_n \text{ for all } n \quad \text{imply} \quad \lim_n p_n \succsim \lim_n q_n.$$

In one of our results we will have to employ the following stronger form of continuity condition.

τ-continuity axiom. $Gr(\succsim)$ is $\tau \times \tau$-closed in $P \times P$. That is, for any pair of convergent nets $\{p_\lambda\}_{\lambda \in \Lambda}$ and $\{q_\lambda\}_{\lambda \in \Lambda}$ in (P, τ),

$$p_\lambda \succsim q_\lambda \text{ for all } \lambda \in \Lambda \quad \text{imply} \quad \lim_\lambda p_\lambda \succsim \lim_\lambda q_\lambda.$$

It should be emphasized that this paper relies purely on exploitation of topological continuity axioms given above. In contrast to the classical theory, though in our expected utility theorems continuity properties of the Bernoulli utility functions are out of consideration, we work with continuity axioms w.r.t. appropriately chosen topologies instead of an Archimedean axiom. We believe that even if a continuity axiom is a necessary condition for a desired form of representation, it is of importance to know whether one can replace it with a weaker continuity axiom. After all, this might be considered as an evidence supporting plausibility of the intended form of representation. Of course, results to be presented below do not constitute an exception, and it is of interest to know whether the same or similar results can be obtained with weaker continuity axioms.

The representation notion to be used throughout the paper is given next.

Definition 2.1 Let $\mathfrak{T}$ be a set of linear functionals on Y. We say that $\mathfrak{T}$ is *a set of utility functions that represents* $\succsim$ if, for all $p, q \in P$,

$$p \succsim q \quad \text{if and only if} \quad T(p) \geq T(q) \text{ for all } T \in \mathfrak{T}. \tag{1}$$

The following lemma is due to Shapley and Baucells [11].

Lemma 2.2 *Let $\succsim$ be a preference relation on a convex subset P of a vector space Y. Assume further that $\succsim$ satisfies independence and weak continuity axioms. Then for any $p, q \in P$,*

$$p \succsim q \quad \text{if and only if} \quad (p-q) \in \operatorname{cone}(\succsim).$$

The reader should notice that if $\succsim$ is a preference relation that satisfies the independence axiom, then the set $\operatorname{cone}(\succsim)$ is a convex cone. Proofs of representation results of both [11] and [3] are based on applications of well known separation theorems to separate points outside the set $\operatorname{cone}(\succsim)$ from this set. Following the topological approach of [3] in a more general framework, in the next section, we will endow a vector space Y of signed measures with a topology τ and try to obtain a non-empty set of utility functions $\mathfrak{T} \subset (Y,\tau)'$ that satisfies (1). Here, the choice of a particular topology τ will determine the form of the functionals in $\mathfrak{T}$. Dubra et al. [3] show that τ-closedness of $\operatorname{cone}(\succsim)$ is a sufficient condition for this idea to be applicable.[10] The next lemma states that the converse of this observation is also true. Since it is of particular importance, we repeat their sufficiency proof.

Lemma 2.3 *Let P be a non-empty, convex subset of a locally convex t.v.s. (Y,τ), and let $\succsim$ be a preference relation on P. Assume further that $\succsim$ satisfies the weak continuity and independence axioms. Then there exists a non-empty, $\sigma\left((Y,\tau)', Y\right)$-closed and convex set of utility functions $\mathfrak{T} \subset (Y,\tau)'$ that represents $\succsim$ if and only if*

$$\overline{\operatorname{cone}}(\succsim) \cap [P-P] \subset \operatorname{cone}(\succsim), \tag{2}$$

where $\overline{\operatorname{cone}}(\succsim)$ is the τ-closure of $\operatorname{cone}(\succsim)$.

Proof. First assume that (2) holds. Put

$$\mathfrak{T} := \left\{T \in (Y,\tau)' : T(\mu) \geq 0, \forall \mu \in \operatorname{cone}(\succsim)\right\}.$$

Obviously, $\mathfrak{T}$ is a non-empty, $\sigma\left((Y,\tau)', Y\right)$-closed and convex set.

We shall now show that $\mathfrak{T}$ represents $\succsim$. For any pair of points p and q in P with $p \succsim q$, $p-q$ belongs to $\operatorname{cone}(\succsim)$, and so, by definition of $\mathfrak{T}$, $T(p) \geq T(q)$ for all $T \in \mathfrak{T}$.

To see that the converse is also true, suppose to the contrary that for a pair of points p and q in P with $p \not\succsim q$ we have $T(p) \geq T(q)$ for all $T \in \mathfrak{T}$. Then, by (2) and Lemma 2.2, $p-q$ does not belong to $\overline{\operatorname{cone}}(\succsim)$. Since $\succsim$ is a preference relation that satisfies independence axiom and since P is non-empty, clearly, $\overline{\operatorname{cone}}(\succsim)$ is a non-empty, convex and τ-closed set. Now, as τ is locally convex, by Hahn-Banach separation theorem [10, Theorem 3.4], there is a functional $T \in (Y,\tau)'$ and a number δ such that

$$T(p-q) < \delta \leq T(\mu) \quad \text{for all } \mu \in \overline{\operatorname{cone}}(\succsim).$$

[10] They prove this fact for a particular topology on a particular space of measures, yet their proof is applicable in a more general framework.

Since cone($\succsim$) is a cone that contains 0, clearly, this implies that $\delta \leq 0 \leq T(\mu)$ for all $\mu \in$ cone($\succsim$). Thus, T belongs to $\mathfrak{T}$, and in fact, $T(p-q) < 0$, which is a contradiction. So, indeed, the set $\mathfrak{T}$ represents $\succsim$.

Now suppose that there exists a set of utility functions $\mathfrak{T} \subset (Y,\tau)'$ that represents $\succsim$, and let $\mu := p - q$ be an arbitrary point in $\overline{\text{cone}}(\succsim) \cap [P - P]$ where p and q are elements of P. Then there exists a τ-convergent net $\{\gamma_\lambda (p_\lambda - q_\lambda)\}_{\lambda \in \Lambda}$ in cone($\succsim$) such that $\gamma_\lambda > 0$, $p_\lambda \succsim q_\lambda$ for all $\lambda \in \Lambda$, and

$$\lim_\lambda \gamma_\lambda (p_\lambda - q_\lambda) = p - q. \tag{3}$$

Now, as $\mathfrak{T}$ represents $\succsim$, we must have $T(\gamma_\lambda (p_\lambda - q_\lambda)) \geq 0$ for all $\lambda \in \Lambda$, and for all $T \in \mathfrak{T}$. So, since each functional in $\mathfrak{T}$ is τ-continuous, from (3) it follows that $T(p-q) \geq 0$ for all $T \in \mathfrak{T}$. Obviously, this implies that $p \succsim q$, and hence, $\mu \in$ cone($\succsim$). This completes the proof. ■

The dual of the question answered in Lemma 2.3 is the following: given a subspace Y' of Y^*, what are the necessary and sufficient conditions for existence of a non-empty set $\mathfrak{T} \subset Y'$ that represents a relation? We close this section with the following lemma which answers this question provided that Y' is sufficiently large so that $\langle Y, Y' \rangle$ constitutes a dual pair under the duality mapping $(p, T) \rightarrow T(p)$. It is worth to note that $\langle Y, Y^* \rangle$ is always a dual pair.

Lemma 2.4 *Let P be a non-empty, convex subset of a vector space Y, and let Y' be a linear subspace of Y^* such that $\langle Y, Y' \rangle$ is a dual pair. Let furthermore $\succsim$ be a preference relation on P that satisfies the weak continuity and independence axioms. Then there exists a non-empty, $\sigma(Y', Y)$-closed and convex set of utility functions $\mathfrak{T} \subset Y'$ that represents $\succsim$ if and only if*

$$\overline{\text{cone}}(\succsim) \cap [P - P] \subset \text{cone}(\succsim),$$

where $\overline{\text{cone}}(\succsim)$ is the $\sigma(Y, Y')$-closure of cone($\succsim$).

Proof. By a well known duality theorem we have $(Y, \sigma(Y, Y'))' = Y'$ [1, Theorem 4.69], and the proof follows from Lemma 2.3. ■

3 Expected Multi-utility Theorems

3.1 Preliminaries

Throughout the rest of the paper, X denotes a non-empty set of all prizes, and $\mathcal{A}$ stands for an arbitrary algebra of subsets of X. If no additional information is given, a measure on $\mathcal{A}$ should be considered as finitely additive and signed. For a measure p on $\mathcal{A}$, the total variation of p over $E \in \mathcal{A}$ is defined as $v(p \mid E, \mathcal{A}) := \sup \sum_{i=1}^n |p(E_i)|$, where the supremum is taken over all finite sequences $\{E_1, ..., E_n\}$ of disjoint sets in $\mathcal{A}$ such that $E_i \subset E$ for all $i \in \{1, ..., n\}$. It is well known that the set function $v(p \mid \cdot, \mathcal{A})$ defines a measure on $\mathcal{A}$, and the function $v(\cdot \mid X, \mathcal{A})$ defines a norm on the vector space of all measures p

on $\mathcal{A}$ such that $v(p \mid X, \mathcal{A}) < \infty$. We denote this normed space by $ba(X, \mathcal{A})$. Instead of $v(p \mid X, \mathcal{A})$ we will write $\|p\|_{\mathcal{A}}$. Since our analysis will be closely related to order structure of $ba(X, \mathcal{A})$, let us recall the terminology of the theory of ordered vector spaces.

An *ordered vector space* Y is a vector space equipped with a partial order (antisymmetric preorder) $\geq$ such that, for all $p, q, r \in Y$, and for all real numbers $\alpha \geq 0$,

$$p \geq q \quad \text{implies} \quad \alpha p \geq \alpha q \quad \text{and} \quad p + r \geq q + r.$$

An ordered vector space Y is said to be a *Riesz space* if any pair of points p and q in Y have a supremum. In an ordered vector space Y, an element $r \in Y$ is said to be the *supremum* of $\{p, q\} \subset Y$ if

$$\begin{aligned} r &\geq p, q \quad \text{and,} \\ \text{for any } w \text{ in } Y, \quad w &\geq p, q \quad \text{imply} \quad w \geq r. \end{aligned}$$

For any Riesz space Y and for any pair of elements $p, q \in Y$, the supremum of $\{p, q\}$ will be denoted by $p \vee q$. p^+, p^- and $|p|$ will stand for $p \vee 0$, $(-p) \vee 0$ and $p \vee (-p)$, respectively. It can easily be seen that, for any element p of a Riesz space, we have $p = p^+ - p^-$. A *Riesz subspace* L of a Riesz space Y is a linear subspace of Y that satisfies "$p \vee q \in L$ whenever $p, q \in L$." A Riesz space equipped with a norm $\|\cdot\|$ is said to be a *normed Riesz space* if $|p| \leq |q|$ implies $\|p\| \leq \|q\|$. A norm complete normed Riesz space with the property "$p, q \geq 0$ imply $\|p + q\| = \|p\| + \|q\|$" is known as an *AL-space*.

It is well known that $ba(X, \mathcal{A})$ is an AL-space with the ordering $\geq$ which is defined as, for any $r, q \in ba(X, \mathcal{A})$, $r \geq q$ if and only if $r(E) \geq q(E)$ for all $E \in \mathcal{A}$ [1, Theorem 8.70]. Moreover, for any pair of points $r, q \in ba(X, \mathcal{A})$, the supremum of r and q is given by

$$r \vee q(E) = \sup\{r(B) + q(E \backslash B) : B \subset E, B \in \mathcal{A}\} \quad \text{for all } E \in \mathcal{A}.$$

With this convention, for any measure $r \in ba(X, \mathcal{A})$, r^+ and r^- are nothing but the well known positive and negative variations obtained from the Jordan decomposition.

We denote the *set of all finitely additive probability measures on* $\mathcal{A}$ by $P_a(X, \mathcal{A})$. $P_{ca}(X, \mathcal{A})$ will stand for the *set of all countably additive probability measures on* A. A probability measure p on 2^X is said to be simple if $p(F) = 1$ for some finite set $F \subset X$, where 2^X denotes the σ-algebra of all subsets of X. Similarly, we say that a countably additive probability measure p on 2^X is discrete if $p(F) = 1$ for some countable set $F \subset X$. The point mass of a point $x \in X$ will be denoted by δ_x. Clearly, a probability measure p on 2^X is discrete if and only if $p = \sum_{x \in F} \alpha_x \delta_x$ for a countable set $F \subset X$, and for a set of numbers $\{\alpha_x : x \in F\} \subset [0, 1]$ such that $\sum_{x \in F} \alpha_x = 1$; and similarly for simple probability measures. The *set of all simple probability measures on* X and the *set of all discrete probability measures on* X will be denoted by $P_s(X)$ and $P_d(X)$, respectively. We define $S(X)$ and $D(X)$ to be the span of $P_s(X)$

and of $P_d(X)$, respectively. $ca(X,\mathcal{A})$ will stand for the subspace of $ba(X,\mathcal{A})$ that consists of all countably additive measures on $\mathcal{A}$.

Let Y be a Riesz subspace of $ba(X,\mathcal{A})$, and let us denote *the set of all probability measures that belong to* Y by P_Y, that is, $P_Y := P_a(X,\mathcal{A}) \cap Y$. The reader should notice that the span of the set P_Y equals to Y. Our analysis is restricted to relations defined on a set of the form P_Y. Though this restriction narrows the domain of our results considerably, in many cases of special importance this condition will be satisfied. In particular, with some simple algebra it can be verified that $S(X)$ and $D(X)$ are Riesz subspaces of $ba\left(X, 2^X\right)$. It is also true that $ca(X,\mathcal{A})$ is a Riesz subspace of $ba(X,\mathcal{A})$, but the proof of this fact requires some work (see [1, Theorem 8.73]).

For a set $E \in \mathcal{A}$ we denote the characteristic function of E by $\mathbf{1}_E$. A function u is said to be $\mathcal{A}$-*simple* if it is of the form $u = \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}$, where $\{\alpha_i : i = 1, ..., n\} \subset \mathbb{R}$ and $\{E_i : i = 1, ..., n\} \subset \mathcal{A}$. The integral of an $\mathcal{A}$-simple function $u := \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}$ w.r.t. $p \in ba(X,\mathcal{A})$ is defined by $\int_E udp := \sum_{i=1}^n \alpha_i p(E \cap E_i)$ $(E \in \mathcal{A})$. The Banach-space (normed by the sup-norm) of all uniform limits of $\mathcal{A}$-simple functions will be denoted by $M_b(X,\mathcal{A})$. The reader should notice that if $\mathcal{A}$ is a σ-algebra, then u belongs to $M_b(X,\mathcal{A})$ if and only if u is bounded and $u^{-1}(B) \in \mathcal{A}$ for each Borel subset B of the real line. For $u \in M_b(X,\mathcal{A})$, the integral of u w.r.t. $p \in ba(X,\mathcal{A})$ is given by $\int_E udp := \lim_n \int_E u_n dp$ $(E \in \mathcal{A})$ where u_n is a sequence of $\mathcal{A}$-simple functions that uniformly converge to u (for a proof of the fact that this integral is well defined see [4, Lemma III.2.16]). It is also worth to note that $M_b\left(X, 2^X\right)$ is nothing but the Banach space of all bounded real functions on X.

In our expected multi-utility theorems, the following representation notion of Dubra et al. [3] will be used.

Definition 3.1 Let U be a subset of $M_b(X,\mathcal{A})$, and let $\succsim$ be a binary relation defined on a set $P \subset P_a(X,\mathcal{A})$. We say that U is *a set of Bernoulli utility functions that represents* $\succsim$ if, for all $p, q \in P$,

$$p \succsim q \quad \text{if and only if} \quad \int_X udp \geq \int_X udq \text{ for all } u \in U.$$

The reader should contrast the above definition with Definition 2.1, and notice the distinction between our use of the terms "Bernoulli utility function" and "utility function." Now, we are ready to proceed.

3.2 Representation with Norm-continuity Axiom

Throughout the subsection, $\succsim$ stands for a binary relation defined on a set of the form P_Y for some Riesz subspace Y of $ba(X,\mathcal{A})$. Representation results of this subsection rely on the following sequential continuity axiom.

Norm-continuity axiom. For any pair of convergent sequences $\{p_n\}$ and $\{q_n\}$ in $(P_Y, \|\cdot\|_{\mathcal{A}})$,

$$p_n \succsim q_n \text{ for all } n \quad \text{imply} \quad \lim_n p_n \succsim \lim_n q_n.$$

We would like to stress that since convergence under the norm-topology is a particularly strong form of convergence, norm-continuity axiom is a notably weak form of topological continuity condition. To demonstrate the strength of norm-convergence criterion let us note that, in this topology, a sequence of degenerate lotteries $\{\delta_{x_n}\}$ converges to another degenerate lottery δ_x if and only if $x_n = x$ for all sufficiently large n. Hence, norm-continuity axiom does not impose any restriction to preferences over degenerate lotteries. To illustrate the same point for the case of discrete lotteries, let us consider a norm-convergent sequence of discrete lotteries $\left\{p_n := \sum_{x\in F_n} \alpha_x^n \delta_x\right\}$ with $\lim_n p_n := p := \sum_{x\in F} \alpha_x \delta_x$. It can easily be seen that this can be true if and only if $\lim_n \sum_{x\in F_n\setminus F} \alpha_x^n = 0$ and $\lim_n \sum_{x\in F} |\alpha_x^n - \alpha_x| = 0$. This simply means that, as n becomes large, the lottery p_n yields the certain prizes which are not likely to occur under lottery p with a very small total probability, and yields the prizes offered by p with probabilities so close to those assigned by p that even the sum of their absolute differences is near 0. It is also worth to note that for any sequence $\{p_n\}$ in $ba\,(X, \mathcal{A})$ if $\lim_n \|p_n - p\|_{\mathcal{A}} = 0$ for some $p \in ba\,(X, \mathcal{A})$, then $\lim_n \int_X u dp_n = \int_X u dp$ for all $u \in M_b\,(X, \mathcal{A})$. Thus, norm-continuity axiom is weaker than any other topological continuity condition that could be imposed in terms of the bounded, integrable functions on X.[11] With such a strong convergence criterion, of course, the postulate "$\lim_n p_n \succsim \lim_n q_n$ whenever $p_n \succsim q_n$ for all n" is plausible.

As we noted earlier, since their spans are Riesz subspaces of $ba\,(X, \mathcal{A})$, the sets $P_s\,(X)$, $P_d\,(X)$, $P_{ca}\,(X, \mathcal{A})$, and $P_a\,(X, \mathcal{A})$ are covered by the main theorem of this subsection, which comes next.

Theorem 3.2 *Let $\mathcal{A}$ be an algebra of subsets of a non-empty set X, and let Y be a Riesz subspace of $ba\,(X, \mathcal{A})$ with $P_Y \neq \emptyset$. Then a binary relation $\succsim$ on P_Y is a preference relation that satisfies independence and norm-continuity axioms if and only if there exists a non-empty, $\sigma\left((Y, \|\cdot\|_{\mathcal{A}})', Y\right)$-closed and convex set of utility functions $\mathfrak{T} \subset (Y, \|\cdot\|_{\mathcal{A}})'$ that represents $\succsim$.*

The proof of Theorem 3.2 will be based on the following obvious lemma.

Lemma 3.3 *Let $(Y, \|\cdot\|)$ be an AL-space, and let $\{p_n\}$ be a convergent sequence in $(Y, \|\cdot\|)$ with $\lim_n p_n := p$. Then $\lim_n p_n^+ = p^+$ and $\lim_n p_n^- = p^-$.*

We proceed with a proof of Theorem 3.2.

Proof of Theorem 3.2. Since the "if" part is obvious, we shall prove the "only if" part. Let $\succsim$ be a preference relation on P_Y that satisfies norm-continuity and independence axioms. If we can show that cone($\succsim$) is a norm-closed subset of Y, the proof will follow from Lemma 2.3. To this end, let $\{\mu_n\}$ be a convergent sequence in cone($\succsim$) with $\lim_n \mu_n := \mu \in Y$. Since P_Y

[11] Continuity axioms w.r.t. topology of weak convergence used in [6] and [3] fall into this category. However, since in these papers X is equipped with a topology, and continuity of the Bernoulli utilities is a central issue, this observation does not imply that norm-continuity condition can replace or is superior than these continuity axioms.

is non-empty and since $\succsim$ is reflexive, 0 belongs to cone($\succsim$). Hence, without loss of generality, we may assume that $\mu \neq 0$ and $\mu_n \neq 0$ $(n \in \mathbb{N})$. Note that by definition of cone($\succsim$), we have $\mu_n(X) = 0$ for all $n \in \mathbb{N}$, and therefore, $\mu(X) = 0$. Set $c := \mu^+(X) = \mu^-(X) > 0$ and $c_n := \mu_n^+(X) = \mu_n^-(X) > 0$ $(n \in \mathbb{N})$. Now, as $ba(X, \mathcal{A})$ is an AL-space, by Lemma 3.3, we must have $\lim_n \|\mu_n^+ - \mu^+\|_{\mathcal{A}} = \lim_n \|\mu_n^- - \mu^-\|_{\mathcal{A}} = 0$. From this it follows that $\lim_n c_n = c$, and hence,

$$\lim_n \left\| \frac{\mu_n^+}{c_n} - \frac{\mu^+}{c} \right\|_{\mathcal{A}} = \lim_n \left\| \frac{\mu_n^-}{c_n} - \frac{\mu^-}{c} \right\|_{\mathcal{A}} = 0. \tag{4}$$

Observe that as Y is a Riesz subspace of $ba(X, \mathcal{A})$, the points μ^+/c, μ^-/c, μ_n^+/c_n, μ_n^-/c_n belong to P_Y $(n \in \mathbb{N})$. Thus, by Lemma 2.2, $\mu_n^+/c_n \succsim \mu_n^-/c_n$ for all $n \in \mathbb{N}$. So, from (4) and from norm-continuity of $\succsim$, it follows that $\mu^+/c \succsim \mu^-/c$. Hence, we conclude that the point $\mu = c(\mu^+/c - \mu^-/c)$ belongs to cone($\succsim$). ■

The following expected multi-utility theorem for simple probability measures is a consequence of Theorem 3.2.

Theorem 3.4 (Simple measures) *Let X be a non-empty set and let $\succsim$ be a binary relation on $P_s(X)$. Then the following are equivalent.*

1. *$\succsim$ is a preference relation that satisfies independence and norm-continuity axioms.*

2. *There exists a non-empty set U of bounded Bernoulli utility functions on X that represents $\succsim$.*

Proof. (1$\Longrightarrow$2) Since $S(X)$ is a Riesz subspace of $ba\left(X, 2^X\right)$, by Theorem 3.2, there exists a non-empty set $\mathfrak{T} \subset \left(S(X), \|\cdot\|_{2^X}\right)'$ that represents $\succsim$. For each $T \in \mathfrak{T}$, define the function $u_T : X \to \mathbb{R}$ by $u_T(x) := T(\delta_x)$ for all $x \in X$, and observe that, as the image of a bounded set under a norm-continuous linear functional, the set $\{u_T(x) : x \in X\}$ is a bounded subset of the real line [10, Theorem 1.18]. Finally, note that $\int_X u_T dp = T(p)$ for all $T \in \mathfrak{T}$, and for all $p \in P_s(X)$. Thus, the set $U := \{u_T : T \in \mathfrak{T}\}$ represents $\succsim$.

(2$\Longrightarrow$1) Norm-continuity of $\succsim$ follows from the discussion that precede Theorem 3.2. The remaining implications are obvious. ■

As we noted earlier, if X is a finite set, the above expected multi-utility theorem follows from the second representation theorem of Shapley and Baucells [11]. However, their theorem is based on a properness assumption which, by definition, postulates that the relative algebraic interior[12] of cone($\succsim$) is non-empty. Unfortunately, as Dubra et al. [3] also observe, if X is infinite, it is not easy to see what sort of a primitive axiom on preferences would guarantee this technical condition. Moreover, as we shall show in the following simple example,

[12] The relative algebraic interior of cone($\succsim$) is the set of all points $\mu \in$ cone($\succsim$) that satisfy the following property: for any $\nu \in$ cone($\succsim$), there exists a number $\alpha > 1$ such that the point $\nu + \alpha(\mu - \nu)$ belongs to cone($\succsim$).

properness is not a necessary condition for representability in expected multi-utility form.

Example 3.5 Let X be an infinite set and let X' be an arbitrary infinite subset of X with $X' \neq X$. Define the binary relation $\succsim$ on $P_s(X)$ as, for any pair of points $p, q \in P_s(X)$, $p \succsim q$ if and only if $p(\{x\}) \geq q(\{x\})$ for all $x \in X'$. Note that, by definition, $\succsim$ is represented by the set of bounded Bernoulli utility functions $\left\{\mathbf{1}_{\{x\}} : x \in X'\right\}$. However, relative algebraic interior of the set cone($\succsim$) is empty. To illustrate this point, let us take an arbitrary element μ of cone($\succsim$). Now, obviously, the set $Z := \{x \in X' : \mu(\{x\}) \neq 0\}$ is finite. So, as X' is infinite, we can pick a point $x_0 \in X' \backslash Z$. Now, take a point $y_0 \in X \backslash X'$, and observe that $\delta_{x_0} - \delta_{y_0} \in$ cone($\succsim$). Let $\alpha > 1$, and set $\mu_\alpha := (\delta_{x_0} - \delta_{y_0}) + \alpha(\mu - (\delta_{x_0} - \delta_{y_0}))$. Then, $\mu_\alpha(\{x_0\}) = 1 - \alpha < 0$, and hence, $\mu_\alpha \notin$ cone($\succsim$). Thus, indeed, μ does not belong to relative algebraic interior of the set cone($\succsim$).

Next, we give an expected multi-utility theorem for discrete probability measures.

Theorem 3.6 (Discrete measures) *Let X be a non-empty set and let $\succsim$ be a binary relation on $P_d(X)$. Then the following are equivalent.*

1. *$\succsim$ is a preference relation that satisfies independence and norm-continuity axioms.*

2. *There exists a non-empty set U of bounded Bernoulli utility functions on X that represents $\succsim$.*

Proof. Since the other implication is obvious, we shall prove that "1" implies "2". Assume that "1" holds. Since $D(X)$ is a Riesz subspace of $ba\left(X, 2^X\right)$, by Theorem 3.2, there exists a non-empty set $\mathfrak{T} \subset (D(X), \|\cdot\|_{2^X})'$ that represents $\succsim$. For each $T \in \mathfrak{T}$, define the bounded real function u_T on X as in the proof of Theorem 3.4. We will complete the proof by showing that $\int_X u_T dp = T(p)$ for all $T \in \mathfrak{T}$, and for all $p \in P_d(X)$. To this end, let T be an element of $\mathfrak{T}$, and let $p := \sum_{i=1}^{\infty} \alpha_i \delta_{x_i}$ be a discrete probability measure, where $\{x_i : i \in \mathbb{N}\}$ is a countable subset of X, and the set $\{\alpha_i : i \in \mathbb{N}\} \subset [0,1]$ is such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, the sequence $\{p_n := \sum_{i=1}^{n} \alpha_i \delta_{x_i}\}$ converges in norm-topology to p. Thus, from norm-continuity of T it follows that $\lim_n \sum_{i=1}^{n} \alpha_i u_T(x_i) = \lim_n T(p_n) = T(p)$. Moreover, as we noted earlier, we have $\lim_n \int_X u dp_n = \int_X u dp$ for any bounded real function u on X. So, in particular, $\lim_n \sum_{i=1}^{n} \alpha_i u_T(x_i) = \lim_n \int_X u_T dp_n = \int_X u_T dp$. Hence, $\int_X u_T dp = T(p)$ as we claimed. ∎

The reader should notice that if X is a countable set, the set of all discrete probability measures on X coincides with the set of all countably additive probability measures on 2^X. Hence, for this particular case, Theorem 3.6 provides an extension of the main theorem of [3].

As we mentioned in Section 1, in contrast to the main theorem of Dubra et al. [3], our approach does not yield a set of continuous Bernoulli utility functions.[13] Even if X is a compact metric space, neither Theorem 3.4 nor Theorem 3.6 can be strengthened to read as, under the same assumptions, there is a set of Bernoulli utility functions $U \subset C_b(X)$ that represents $\succsim$, where $C_b(X)$ is the set of bounded continuous functions on X. The problem is fairly clear: the norm-continuity axiom together with additional standard assumptions do not guarantee the continuity condition of [3], and failure of this condition prevents existence of continuous Bernoulli utilities.[14] Indeed, in Example 3.5 if we set $X := [0,2]$ and $X' := [1,2]$, continuity condition of [3] would fail since the sequence $\{\delta_{1-1/n}\}$ converges to δ_1 in the topology of weak convergence and the lotteries δ_2 and δ_1 are incomparable, but we have $\delta_2 \succsim \delta_{1-1/n}$ for each $n \in \mathbb{N}$.

We should emphasize that, at least without imposing some further conditions, it is not possible to drive expected multi-utility theorems from Theorem 3.2 for probability spaces that contain non-discrete measures. Here, the problem is that the norm-dual of $ba(X, \mathcal{A})$ (or $ca(X, \mathcal{A})$) is, in general, too rich and contain functionals other than those of the form $p \to \int_X u dp$ (for a description of the norm-dual of $ca(X, \mathcal{A})$ see [7]). Therefore, richness of the norm-topology, which enabled us to prove that under the hypotheses of Theorem 3.2 cone $(\succsim)$ is a closed set, is at the same time a problematic factor that leads to continuous linear functionals of undesirable forms. Though it is possible to give a concrete example which shows that linear utility functions in Theorem 3.2 do not necessarily take an expected utility form, we shall not do so here, and simply stress that hypotheses of Theorem 3.2 are not strong enough to guarantee even the well known mild monotonicity conditions discussed in [5, Chapter 10] (see also Footnote 16). Hence, in the next subsection, we focus on a stronger continuity axiom to obtain an expected multi-utility theorem for non-discrete probability measures.

3.3 Representation with a Stronger Continuity Axiom

The purpose of this subsection is to provide an expected multi-utility theorem for preference relations defined on the set $P_a(X, \mathcal{A})$. To obtain this result, we will follow the main idea of the proof of the expected multi-utility theorem of [3] in a different framework, and exploit the well known fact that the norm-dual of $M_b(X, \mathcal{A})$ is isometrically isomorphic to $ba(X, A)$ [4, Theorem IV.5.1].

[13] One useful implication of the continuity properties of the Bernoulli utilities is related to existence of maximal alternatives. Let X be a compact metric space and let $\mathcal{A}$ be the Borel σ-algebra of X. Dubra et al. [3] show that if $\succsim$ is a binary relation on a set $P \subset P_{ca}(X, \mathcal{A})$ that admits a set of continuous Bernoulli utilities, then there is a continuous function $u : X \to \mathbb{R}$ such that the functional $p \to \int_X u dp$ represents $\succsim$ in the sense of Aumann [2]. That is, for all $p, q \in P$, we will have $p \succ (\sim) q \Longrightarrow \int_X u dp > (=) \int_X u dq$. Since u is continuous and X is compact, some $\overline{x}$ maximizes u over X. Now, it is easy to check that $\delta_{\overline{x}}$ is maximal w.r.t. $\succsim$.

[14] In [3], the authors focus on a preorder defined on the set $P_{ca}(X, \mathcal{A})$, where X and $\mathcal{A}$ are as in Footnote 13. In our terminology, the continuity axiom they employ is the (sequential) τ-continuity axiom, where τ is the topology of weak convergence. Under this topology, a sequence $\{p_n\}$ in $ca(X, \mathcal{A})$ converges to a point p in $ca(X, \mathcal{A})$ if and only if $\int_X u dp_n \to \int_X u dp$ for each continuous real function u on X.

Compared to Archimedean axiom of classical theory, the continuity axiom employed in this result is extremely strong, yet it is still a necessary condition. The topology we shall work with is the weak*-topology[15] of $ba(X, A)$. Under this topology, a net $\{p_\lambda\}_{\lambda\in\Lambda}$ in $ba(X, A)$ converges to a point $p \in ba(X, A)$ if and only if $\lim_\lambda \int_X u dp_\lambda = \int_X u dp$ for all $u \in M_b(X, \mathcal{A})$. The following continuity axiom is asserted for a binary relation $\succsim$ on a set $P \subset P_a(X, \mathcal{A})$ that is endowed with (relative) weak*-topology.

weak*-continuity axiom. [16,17]The set $Gr(\succsim)$ is weak*$\times$weak* closed in the set $P \times P$. That is, for any pair of weak*-convergent nets $\{p_\lambda\}_{\lambda\in\Lambda}$ and $\{q_\lambda\}_{\lambda\in\Lambda}$ in P,

$$p_\lambda \succsim q_\lambda \text{ for all } \lambda \in \Lambda \quad \text{imply} \quad \lim_\lambda p_\lambda \succsim \lim_\lambda q_\lambda.$$

We are now ready to give the promised expected multi-utility theorem. The proof we provide consists of straightforward generalizations of sequential arguments of the main theorem of [3] to arbitrary nets, and applications of some well known theorems.

Theorem 3.7 (Finitely add. measures) *Let X be a non-empty set and let $\mathcal{A}$ be an algebra of subsets of X. A binary relation $\succsim$ on $P_a(X, \mathcal{A})$ is a preference relation that satisfies the weak*-continuity and independence axioms if and only if there exists a closed, convex and non-empty set $U \subset M_b(X, \mathcal{A})$ that represents $\succsim$.*

Proof. The "if" part is obvious. For the "only if" part, by Lemma 2.4, it suffices to show that cone($\succsim$) is weak*-closed in $ba(X, \mathcal{A})$. Since $ba(X, \mathcal{A})$ is the norm dual of the Banach space $M_b(X, \mathcal{A})$, by Krein-Šmulian theorem, the convex set cone($\succsim$) is weak*-closed if and only if for each $k > 0$ the set cone($\succsim$) $\cap\, kB$ is weak*-closed, where $B := \{\mu \in ba(X, \mathcal{A}) : \|\mu\|_{\mathcal{A}} \leq 1\}$ (see [9, Theorem 2.7.11]).

First, notice that $P_a(X, \mathcal{A})$ is a weak*-closed subset of B. Hence, by Banach-Alaoglu theorem, both $P_a(X, \mathcal{A})$ and B are weak*-compact (see [9, Theorem 2.6.18]).

[15] To avoid the awkward notation $\sigma(ba(X, A), M_b(X, \mathcal{A}))$, we depart from our previous notations.

[16] If $\{x\} \in \mathcal{A}$ for each x in X, it can be shown that the set of all simple probability measures is weak*-dense in $P_a(X, \mathcal{A})$. Hence, this continuity axiom together with the independence axiom also imply the following strong form of a classical monotonicity assumption:

$(p(E) = 1, \delta_x \succsim q \text{ for all } x \in E) \implies p \succsim q$, $(p(E) = 1, q \succsim \delta_x \text{ for all } x \in E) \implies q \succsim p$.
In the classical theory, this assumption together with additional standard axioms turn out to be sufficient for representation of complete preference relations in expected utility form. So, a natural question is whether one can dispense with the weak*-continuity axiom by coupling the norm-continuity condition with this mild monotonicity assumption. At the moment we do not know the answer of this question.

[17] Another notable point is that, under the weak*-topology, a net $\{p_\lambda\}_{\lambda\in\Lambda}$ in $P_a(X, \mathcal{A})$ converges to a point p if and only if $p_\lambda(E) = p(E)$ for all $E \in \mathcal{A}$. Hence, this continuity condition can be expressed entirely in terms of convergence of the probabilities of the events. This fact follows from norm-density of the set of all $\mathcal{A}$-simple functions in $M_b(X, \mathcal{A})$, and norm-boundedness of $P_a(X, \mathcal{A})$.

Now let $k > 0$, and let $\{\gamma_\lambda (p_\lambda - q_\lambda)\}_{\lambda\in\Lambda}$ be a convergent net in cone($\succsim$) such that $\gamma_\lambda > 0$, $p_\lambda \succsim q_\lambda$, and

$$\|\gamma_\lambda (p_\lambda - q_\lambda)\|_{\mathcal{A}} \leq k \quad \text{for all } \lambda \in \Lambda. \tag{5}$$

Put $\mu := \lim_\lambda \gamma_\lambda (p_\lambda - q_\lambda)$. Clearly, it suffices to show that $\mu \in$ cone($\succsim$). If there is an index $\lambda_0 \in \Lambda$ such that $\gamma_\lambda (p_\lambda - q_\lambda) = 0$ for all $\lambda \geq \lambda_0$, then $\mu = 0$, and we are done by reflexivity. Therefore, assume that for each $\lambda \in \Lambda$ there exists a $\varphi(\lambda) \in \Lambda$ such that $\varphi(\lambda) \geq \lambda$ and $\gamma_{\varphi(\lambda)} \left(p_{\varphi(\lambda)} - q_{\varphi(\lambda)}\right) \neq 0$. Set $\mu_\lambda := \gamma_{\varphi(\lambda)} \left(p_{\varphi(\lambda)} - q_{\varphi(\lambda)}\right)$ for each $\lambda \in \Lambda$. Then $\{\mu_\lambda\}_{\lambda\in\Lambda}$ is a subnet[18] of the net $\{\gamma_\lambda (p_\lambda - q_\lambda)\}_{\lambda\in\Lambda}$.

For each $\lambda \in \Lambda$, set $w_\lambda := \mu_\lambda^+/c_\lambda$, and $r_\lambda := \mu_\lambda^-/c_\lambda$, where $c_\lambda > 0$ is the common value of $\mu_\lambda^-(X)$ and $\mu_\lambda^+(X)$. Then $\mu_\lambda = c_\lambda (w_\lambda - r_\lambda)$ and $\|\mu_\lambda\|_{\mathcal{A}} = c_\lambda (w_\lambda (X) + r_\lambda (X)) = 2c_\lambda$ $(\lambda \in \Lambda)$. Hence, by (5), $c_\lambda \leq k/2$ for each $\lambda \in \Lambda$. Note that the set $[0, k/2] \times P_a(X, \mathcal{A}) \times P_a(X, \mathcal{A})$ is compact with its product topology. So, the net $\{(c_\lambda, w_\lambda, r_\lambda)\}_{\lambda\in\Lambda}$ has a convergent subnet [1, Theorem 2.29]. Let (c, w, r) be the corresponding limit point of the net $\{(c_\lambda, w_\lambda, r_\lambda)\}_{\lambda\in\Lambda}$. Then, clearly, $c(w - r)$ is a limit point the net $\{\mu_\lambda\}_{\lambda\in\Lambda}$. A net in a topological space converges to a point if and only if every subnet converges to that same point [1, Lemma 2.14]. So, a subnet of the net $\{\mu_\lambda\}_{\lambda\in\Lambda}$ converges to both μ and $c(w - r)$. Thus, as the weak*-topology is Hausdorff, we must have $\mu = c(w - r)$. Now note that, by Lemma 2.2, we have $w_\lambda \succsim r_\lambda$ for all $\lambda \in \Lambda$. So, by weak*-continuity axiom, $w \succsim r$, and this completes the proof. ■

Unfortunately, due to lack of duality relations that enabled us to prove the above theorem, even by employing the weak*-continuity axiom, we could not obtain an expected multi-utility theorem for a relation defined on $P_{ca}(X, \mathcal{A})$. It should be noted that for a relation $\succsim$ satisfying the hypotheses of Lemma 2.4, such a representation is possible if and only if the set cone($\succsim$) is weak*-closed in $ca(X, \mathcal{A})$. However, apart from the particular cases covered by the main theorem of [3] and Theorem 3.6, we do not know under what circumstances this condition can be satisfied.

3.4 Uniqueness

In this subsection, we give a straightforward generalization of the uniqueness theorem of Dubra et al. [3]. This general form will cover all of our representation theorems.

Let Y be a Riesz subspace of $ba(X, \mathcal{A})$ and let Y' be a subspace of the algebraic dual of Y such that $\langle Y', Y\rangle$ is a dual pair (under the duality mapping $(T, p) \to T(p)$). We denote the linear functional $p \to p(X) = \int_X \mathbf{1}_X dp$ by $\widetilde{\mathbf{1}}_X$. Now, if the functional $\widetilde{\mathbf{1}}_X$ belongs to Y' and if $\mathfrak{T}$ is a subset of Y', the $\sigma(Y', Y)$-closure of the convex cone $\left\{\theta\widetilde{\mathbf{1}}_X : \theta \in \mathbb{R}\right\} + \text{cone}(\mathfrak{T})$ will be denoted by $\langle\mathfrak{T}\rangle_{\langle Y', Y\rangle}$.

[18] $\{y_\alpha\}_{\alpha\in\mathfrak{A}}$ is said to be a subnet of a net $\{z_\lambda\}_{\lambda\in\Lambda}$ if there exists a function $\varphi : \mathfrak{A} \to \Lambda$ such that: $y_\alpha = z_{\varphi(\alpha)}$ for each $\alpha \in \mathfrak{A}$; for each $\lambda_0 \in \Lambda$ there exists an $\alpha_0 \in \mathfrak{A}$ such that $\alpha \geq \alpha_0$ implies $\varphi(\alpha) \geq \lambda_0$.

Similarly, let M' be a subspace of $M_b(X, \mathcal{A})$ and suppose that $\langle M', Y\rangle$ is a dual pair (under the duality mapping $(u, p) \to \int_X udp$). Assume further that $\mathbf{1}_X \in M'$. Then, for any $U \subset M'$, we define $\langle U\rangle_{\langle M', Y\rangle}$ to be the $\sigma(M', Y)$-closure of the convex cone $\{\theta \mathbf{1}_X : \theta \in \mathbb{R}\} + \text{cone}(U)$.

As a slight difference from the approach of Dubra et al. [3], the following uniqueness theorem focuses on the algebraic dual of the span of the domain of the relation $\succsim$ instead of focusing on the space $M_b(X, \mathcal{A})$. We present this version since it covers Theorem 3.2 as well. The proof we provide is a straightforward generalization of the uniqueness theorem of [3].

Theorem 3.8 (Uniqueness of utility functions) *Let Y be a Riesz subspace of $ba(X, \mathcal{A})$, where $\mathcal{A}$ is an algebra of subsets of a non-empty set X. Let furthermore, Y' be a linear subspace of Y^* such that $\widetilde{\mathbf{1}}_X$ belongs to Y' and $\langle Y', Y\rangle$ is a dual pair. Then two non-empty sets $\mathfrak{T}$ and $\mathcal{K}$ in Y' satisfy*

$$T(p) \geq T(q) \;\; \text{for all } T \in \mathfrak{T} \quad \Longleftrightarrow \quad K(p) \geq K(q) \;\; \text{for all } K \in \mathcal{K},$$

for each $p, q \in P_Y$, if and only if $\langle \mathfrak{T}\rangle_{\langle Y', Y\rangle} = \langle \mathcal{K}\rangle_{\langle Y', Y\rangle}$.

Proof. Since the "if" part is obvious, we omit it. To prove the "only if" part, suppose to the contrary that the set $\langle \mathfrak{T}\rangle_{\langle Y', Y\rangle} \setminus \langle \mathcal{K}\rangle_{\langle Y', Y\rangle}$ is non-empty. Then, clearly, we can pick a point T from the set $\mathfrak{T} \setminus \langle \mathcal{K}\rangle_{\langle Y', Y\rangle}$. As dual pairs are weakly dual [1, Theorem 4.69], we have $(Y', \sigma(Y', Y))' = Y$. So, since $\langle \mathcal{K}\rangle_{\langle Y', Y\rangle}$ is a $\sigma(Y', Y)$-closed, non-empty convex cone, as in the proof of Lemma 2.3, we can apply a separating hyperplane theorem to obtain a point $\mu \in Y \setminus \{0\}$ such that

$$T(\mu) > 0 \geq K(\mu) \quad \text{for all } K \in \langle \mathcal{K}\rangle_{\langle Y', Y\rangle}. \tag{6}$$

Now, as $\left\{\theta \widetilde{\mathbf{1}}_X : \theta \in \mathbb{R}\right\} \subset \langle \mathcal{K}\rangle_{\langle Y', Y\rangle}$, we must have $0 \geq \theta \mu(X)$ for all $\theta \in \mathbb{R}$. Thus, $\mu(X) = 0$, and hence, $\mu^+(X) = \mu^-(X) > 0$. Since Y is a Riesz subspace of $ba(X, \mathcal{A})$, it follows that the points μ^+/c and μ^-/c belong to P_Y, where c is the common value of $\mu^+(X)$ and $\mu^-(X)$. So, by (6), we see that $T(\mu^+/c) > T(\mu^-/c)$ and $K(\mu^+/c) \leq K(\mu^-/c)$ for all $K \in \mathcal{K}$, which gives the desired contradiction. ∎

We conclude the paper with a direct analog of the uniqueness theorem of Dubra et al. [3].

Theorem 3.9 (Uniqueness of Bernoulli utility functions) *Let Y be a Riesz subspace of $ba(X, \mathcal{A})$, where $\mathcal{A}$ is an algebra of subsets of a non-empty set X. Let furthermore, M' be a linear subspace of $M_b(X, \mathcal{A})$ such that $\mathbf{1}_X$ belongs to M' and $\langle M', Y\rangle$ is a dual pair. Then two non-empty sets U and V in M' satisfy*

$$\int_X udp \geq \int_X udq \quad \forall u \in U \quad \Longleftrightarrow \quad \int_X vdp \geq \int_X vdq \quad \forall v \in V,$$

for each $p, q \in P_Y$, if and only if $\langle U\rangle_{\langle M', Y\rangle} = \langle V\rangle_{\langle M', Y\rangle}$.

The proof of Theorem 3.9 is obvious: we simply define the functional $\widetilde{u} \in Y^*$ as $\widetilde{u}(p) := \int_X u dp \ (p \in Y, u \in M')$, and then use the linear operator $u \to \widetilde{u}$ to embed the space M' into Y^*, so that Theorem 3.8 can be applied.

The reader should notice that all the expected multi-utility theorems presented so far are included within the scope of Theorem 3.9 since, in each of these results, points of $M_b(X, \mathcal{A})$ can be separated by the domain of the relation $\succsim$ under the duality mapping $(u, p) \to \int_X u dp$. In this theorem, we took subspaces of $M_b(X, \mathcal{A})$ into consideration so that the expected multi-utility theorem of Dubra et al. [3], which functions on the space of all bounded continuous mappings of a metric space X, is also included.

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