The Emergent Seed: Simplifying the Analysis of Dynamic Evolution.

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Abstract

In a model of Dynamic Evolution—as first popularized by Kandori, Mailath and Rob [14]—there is an underlying structure that helps determine the long run viability of limit sets, called the *emergent seed*. Relative to this structure long run viability is the additive component of the *security level*—the minimal distance out of a limit set's basin of attraction—and the *core attraction rate*—the cost of evolving from one particular limit set to the limit set in question.

The usefulness of this approach is shown by characterizing long run viability in all games with two limit sets, analyzing bargaining and contract games.

JEL codes: C63 C73 C78 C79

1 Introduction

Assuming equilibrium behavior can not be justified by rational learning. This is one of the implications of Kalai and Lehrer's study of rational learning [13]. What is an alternative? One is to assume a specific type of "limitedly rational" behavior; allow players using this behavior to interact in an economy; and then study the resulting long run behavior. If one part of the model of limitedly rational behavior is that players occasionally "experiment" or "mutate" in a suboptimal way, then this is a model of dynamic evolution.

Dynamic evolution—which has been called "Noisy Evolution", and "Evolution with Noise" in various papers[1]—was introduced to the economic community by a pair of seminal papers in 1993: Kandori, Mailath

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[1] There is no general terminology for this model in the literature. Samuelson and coauthors ([2] and [18]) have referred to it as a model of "noisy evolution," and Ellison [8] as one of "evolution with noise." However the definition of *Evolutionarily Stable Strategies* (Maynard Smith [17]) also requires "noise." The essential difference is that in that analysis one always stays in the same state while in this analysis there are multiple state transitions, or this model is dynamic.

and Rob [14, KMR hereafter] and Young [25]. While these models are intuitively appealing they are often complicated to solve, and leave the reader with no clear intuition about what makes a strategy evolutionarily successful.

This intuition can be discovered if the analyst first finds the *emergent seed*. Relative to this core structure being evolutionarily successful depends on having a high *security level* and a low *core attraction rate*. Intuitively the *security level* is the degree that one player's plans do not need to change if other's do; in some interactions it is instead the degree to which they do not need to change in the long run. The core is a critical subset of strategies in the emergent seed, and the *core attraction rate* is the speed at which evolution passes from this subset to the strategy in question.

In contrast, in Peyton Young's analysis [25] finding strategies which were evolutionarily successful (or *stochastically stable*) required finding the least cost way to pass from all other potentially viable strategies to the given strategy—a tree minimization problem. The complexity of this problem limits the applicability of dynamic evolution. The results here provide analytic clarity for all problems and for some simplifies the analysis.

Analysis of the emergent seed is often sufficient to find the evolutionarily successful strategies (or sets of strategies, formally *limit sets*). For example in all games with two limit sets—like any two action interaction—the limit set with maximal security level will survive in the long run. Other games where analyzing the emergent seed is sufficient are bargaining and contract games first studied by Young [26] and [27].

Since the seminal papers of Kandori, Mailath and Rob and Young there have been essentially three branches of the literature: applications, variations of the basic model, and simplification or clarification of the analysis.

Papers analyzing variations of the basic model have been extremely prolific, for example Bergin and Lipman [3], Squintani and Valimaki [24], van Damme and Weibull [5] have shown that the original independent mutations can be replaced with reasonable "bandwagon" mutations without changing the basic results. Another batch of papers analyze the effect of changing the matching rule, these include Ellison [7], [8], Canals and Vega-Redondo [4] and Ely [9]. Some other papers of note are Amir and Berninghaus [1]; Binmore, Samuelson, and Vaughan [2]; Nöeldeke and Samuelson [18]; Robson and Vega-Redondo [22]; and Saez-Marti and Weibull [23].

There have been many applications of the theory as well. Examples are Ellingsen [6]; Johnson, Levine and Pessendorfer [12]; Kandori and Rob [15] and [16]; Nöeldeke and Samuelson [19]; Robles [21]; and Young [26] and [27].

The final category—clarification, to which this paper is contribution—has not been widely developed. Kandori and Rob [15] and [16] both provide some results, but the only paper focusing on this subject is Ellison [8]. That paper has two goals. The primary goal is to characterize how long evolution will take, to do this it develops a sufficient characterization of stochastic stability: the radius and (modified) coradius. A secondary goal is simplification, and while this technique can be applied to many papers in the literature Ellison states that he is not sure that it would simplify their analysis. The reason might be because he does not first find the emergent seed. The radius is the security level, the coradius is similar to (but generally greater than) the core attraction rate, thus his sufficient characterization is not far from the representation theorem in this paper. The difference is that by understanding the emergent seed the underlying architecture of the problem is revealed and sometimes this can simplify analysis. For example, the results in Johnson, Levine and Pessendorfer's [12] analyzes of the evolution of cooperation can be generalized using the emergent seed methodology.

In the next section I will motivate the general model with a specific example: Kandori, Mailath, and Rob's coordination game. I will then present the general model, and explain the model in analysis. I close this section by briefly explaining Young's key results since they are integral to some proofs in this paper. In subsection 3.1 of section 3 I then explain my results by first showing a game where the structure I am looking for is obvious, using this I show how to always find the first level of the emergent seed. For the case where this first level is completely connected I then present the general result. In the following subsection I show how to find the general solution, and characterize it. In section 4 I then show the benefits of my analysis with several applications.

2 The Model.

The model used in the analysis of dynamic evolution is several steps removed from the models that motivate our analysis. Thus this section is broken into several subsections to clarify these connections. First a brief exposition of one of the most popular models is given, the coordination game that was first studied by Kandori, Mailath, and Rob [14]. Next the general model (as first described by Ellison [8]) is presented, and the links between this model and the model used in analysis are explained.

The model actually used in analysis is described in subsection 2.3 and readers either familiar with the general model or interested only in tree minimization problems can begin there.

2.1 A Motivating Example—Two Action Coordination.

Consider the following symmetric coordination game:

	A	B
A	$2, 2$	$-2, 0$
B	$0, -2$	$1, 1$

Kandori, Mailath, and Rob [14] motivate this game as representing the choice of operating system, but it has been applied to describe many other interactions. We will assume that there are N players—an even and large number—who are matched by equal likelihood to play this game an infinite number of periods. Each period with probability τ a player updates their beliefs about other's actions. Their new beliefs will be the current distribution of actions of other players, and they believe it will not change in the future. If their beliefs are updated there is a small probability, ε, that they experiment by choosing an action at random, and they will keep playing this action until their beliefs update again. This probability of experimentation (or mutation) is small, and our analysis focuses on the case where it converges to zero.

From this behavioral model we will develop two analytic tools. The current state of the system will be $Z = \left\{\frac{\alpha}{N}\right\}_{\alpha=0}^{N}$ where α is the number of players using action A. We also use a cost function:

$$C\left(z'|z\right) = \begin{cases} \text{The number of experiments needed} \\ \text{to transition from } z \text{ in period } t \\ \text{to } z' \text{ in period } t+1 \end{cases}$$

Notice that this cost function is increasing in N, the population size, and our results will be for all N "large enough." This dependence is inconvenient but can not always be overcome. If any given state z appears infinitely often as N goes to infinity then we can normalize these costs. Let $Z(N)$ be the states of the system given N, and

$$c(z'|z) = \lim_{N\to\infty} \left\{ \frac{C(z'|z)}{h(N)} | \{z, z'\} \in Z(N) \right\}$$

where $h(\cdot)$ depends on the matching rule and characteristics of the stage game, here $h(N) = N$. Throughout the paper a lower case cost function is always independent of N and capitalized cost functions are not.

There are many other formulations of this model. In general there is a large population of players, being matched to play some stage game. There is a well specified model of how they gather and use information, and then some noise is added. This noise is sometimes called experimentation, sometimes mutations, sometimes just mistakes but it is integral that this noise be maintained over time. The system is then allowed to percolate for a very large number of periods and analysis finds the strategies most likely to survive in the long run.

2.2 The Formal Model.

The formal model is general enough to characterize all of the games that have been analyzed in the literature and more. This model is a triplet $\{Z, M, M(\varepsilon)\}$ where:

1. Z is a finite set that characterizes the various states of the system.
2. M is a Markov transition matrix on Z.
3. $M(\varepsilon)$ is a family of Markov transition matrixes on Z indexed by $\varepsilon \in [0, \bar{\varepsilon})$ such that:
   (a) $M(\varepsilon)$ is ergodic for $\varepsilon > 0$.
   (b) $M(\varepsilon)$ is continuous in ε with $M(0) = M$.
   (c) There is a cost function $C : Z \times Z \to [0, \infty) \cup \infty$ such that for all $z, z' \in Z$ if $C(z'|z) < \infty$ $\lim_{\varepsilon\to 0} \frac{M_{z,z'}(\varepsilon)}{\varepsilon^{C(z,z')}} > 0$ and the limit exists, if $C(z'|z) = \infty$ $\lim_{\varepsilon\to 0} \frac{M_{z,z'}(\varepsilon)}{\varepsilon} = 0$.

In the example above M is given by the best response dynamic, or all players who update their action play a best response. $M(\varepsilon)$ is the transition matrix when players experiment with probability ε. It is ergodic since it is strictly positive—any state transition has a positive probability—and it is continuous in ε. The cost function $C(z'|z)$ is derived from $M_{z,z'}(\varepsilon)$. For example if B is a best response to $z = \frac{\sigma}{N}$, and $z' = \frac{\alpha'}{N}, \alpha' \geq \alpha$ then:

$$\begin{aligned} M_{z,z'}(\varepsilon) &= \sum_{k=(\alpha'-\alpha)}^{N} \binom{N}{k} (1-\tau)^{N-k} \tau^k * \\ & \binom{k}{\alpha'-\alpha} \left(1-\frac{\varepsilon}{2}\right)^{k-(\alpha'-\alpha)} \left(\frac{\varepsilon}{2}\right)^{(\alpha'-\alpha)} \end{aligned}$$

and $C(z'|z) = \alpha' - \alpha$. The reason for using this cost function is that as ε goes to zero $M_{z,z'}(\varepsilon)$ is proportional to $\left(\frac{\varepsilon}{2}\right)^{(\alpha'-\alpha)}$, thus the likelihood of this transition is proportional to $\alpha' - \alpha$.

In this model we try to find the limiting distribution of states. This will be independent of initial condition since $M(\varepsilon)$ is ergodic; or for any μ_0 $\mu(\varepsilon) = \lim_{t\to\infty} M(\varepsilon)^t \mu_0$ is independent of μ_0. The fact that we do not know ε—the experimentation rate—is not problematic since $\mu(\varepsilon)$ is continuous in ε and we are interested in the case where ε is small. Thus we analyze $\lim_{\varepsilon\to 0}\mu(\varepsilon) = \mu^*$. Note that:

$$\mu_z(\varepsilon) = \frac{P(\text{transitioning to } z \text{ from all } z'' \in Z\backslash z)}{\Sigma_{z'\in Z} P(\text{transitioning to } z' \text{ from all } z'' \in Z\backslash z')}$$

and $P(\text{transitioning to } z \text{ from all } z'' \in Z\backslash z)$ is essentially a collection of "absorption trees." In other words if there are three states in Z then it is the sum over the probability of transitioning from z''' to z'' to z plus the probability of transitioning from z'' to z''' to z plus the probability of transitioning from z'' to z and z''' to z. However since each of these transitions is proportional to $\varepsilon^{C(\cdot|\cdot)}$ as ε gets small only the least costly absorption tree matters. Thus the likelihood of z in the limiting distribution is proportional to the cost of it's least costly absorption tree—we call this z's *stochastic potential*—and z will dominate the long run distribution if it has the least costly stochastic potential—in this case z is *stochastically stable.*

2.3 The Model in Analysis.

We now have the key elements which we use in analysis, Z and the cost function $C(\cdot|\cdot)$. Our objective will be to solve the minimal cost absorption tree problem. The problem in the rest of our analysis is essentially the same as the graph theoretic problem of the "optimal convention center." You have conventioneers spread across the globe and want to minimize their cost of coming to a conference—what is the optimal location? This analysis can also be applied to other problems in this family.

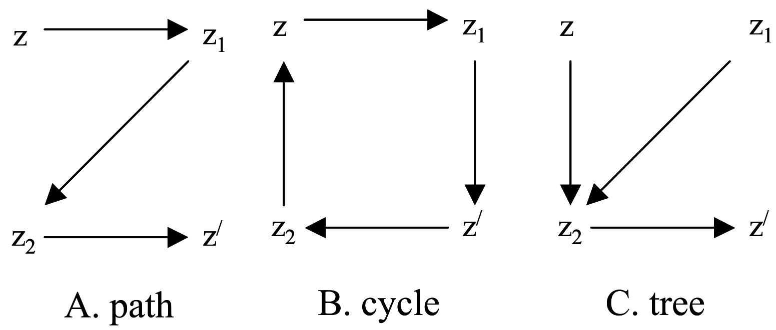
The main object of our analysis will be directed graphs over subsets of Z. Such a graph (g) is a set of vertices $(\tilde{Z} \subseteq Z)$ and an ordering over the vertices, denoted $z_g : \tilde{Z} \to \tilde{Z} \cup \emptyset$, and let the direct resistance of going to the empty set be zero, or if $z_g(z) = \emptyset$ then $C(z_g(z)|z) = 0$, thus $g = \{Z, z_g\}$.[2] The connection with the Markov transition matrix above is that if $z' = z_g(z)$ then in period t we are in state z and in period $t+1$ we are in state z'. We call z' a *direct successor* of z in g if $z' = z_g(z)$ and we also call z a *direct predecessor* of z' in this case. If z' is in the transitive closure of the $z_g(\cdot)$ ordering from a given z then we say that z' is a *successor* of z, and we also say that z is a *predecessor* of z' in this situation. We will denote the cost of such a graph $C(g) = \Sigma_{z\in Z} C(z_g(z)|z)$, and will use the same notation for other cost functions.

Three types of graph will be mentioned in our analysis. The simplest is a *path*, $g(z'|z) = \left\{\widetilde{Z}, z_g\right\}$ is a path if z is the predecessor of every $\widetilde{z} \in \widetilde{Z}$ and z' is the successor of every such $\widetilde{z}$.[3] In other words a path is just a sequence of states, one occurring after the other as is illustrated in graph A in the figure below. Another simple example is a *cycle*, $g = \left\{\widetilde{Z}, z_g\right\}$ is a cycle if every $z \in \widetilde{Z}$ is the successor of every other $z' \in \widetilde{Z}$. This is graph B in the figure below. The type of graphs we will be most interested in will be trees with base z, denoted t_z^χ where χ indicates what subset of Z t_z^χ is over. For example t_z^0 will be graphs with

[2] Notice that we impose that for each z there is a unique $z_g(z)$.

[3] Notice that since $z_g(\cdot)$ is a function—not a correspondence—we do not need to make the further restriction that there are no $\left\{\widetilde{z}, \widetilde{\widetilde{z}}\right\}$ such that $\widetilde{z}$ is neither the sucesssor or predecessor of $\widetilde{\widetilde{z}}$

vertices in Z. In such a graph the only restriction is that z has no successors and is the successor of every other $\widetilde{z} \in \widetilde{Z}$. A tree looks like an inverted extensive form game; the decision nodes are now states—with z as the initial "decision node." The dynamics are reversed as well, instead of starting with z we start from the ends of the branches and travel back to z—like we are solving the extensive form game by backwards induction. An example is graph C in the figure below.



An arrow from $\hat{z}$ to $\tilde{z}$ means that $\tilde{z}$ is the direct successor of $\hat{z}$ in the illustrated graphs.

Let T_z^0 be the set of t_z^0 then the *stochastic potential of* z is $C_z = \min_{t_z^0 \in T_z^0} C\left(t_z^0\right)$, and z is *stochastically stable* if and only if $z \in \arg\min_{z \in Z} C_z$.

The first difficulty we face is that the solution depends on the cardinality of Z. Z increases with N and therefore the problem is effectively intractable. The key result in Young [25] is that this excess of complexity can be simplified to the analysis of limit sets. To find these limit sets it is convenient to find the *optimal cost function*, if $G(z'|z)$ is the set of paths from z to z' then $C^*(z'|z) = \min_{g \in G(z'|z)} C(g)$ is the best way to transition from z to z'.

A *limit set* is an $\omega_0 \subseteq Z$ such that

1. For every $z \in Z \backslash \omega_0$ $C^*(z|\omega_0) > 0$.

2. For all $g = \{\omega_0, z_g\}$, $C^*(g) = 0$.

Let the collection of these limit sets be Ω_0. In principle there could be a large number of these limit sets, but in general it is a much smaller set than Z and does not increase with the population size.[4] Note that if $z \notin \Omega_0$ then there is some $\omega_0 \in \Omega_0$ such that $C^*(\omega_0|z) = 0$—otherwise z itself would be a limit set. We call the states z for which $C^*(\omega_0|z) = 0$ ω_0's *basin of attraction*, denoted $B(\omega_0)$.

Now consider the trees $t_{\omega_0}^1$ over Ω_0, let

$$C_{\omega_0}^* = \min_{t_{\omega_0}^1} C^*\left(t_{\omega_0}^1\right)$$

then Young's primary results can be summarized in the following lemma.

Lemma 1 *Assume* $C(\cdot|\cdot) \geq 0$*, and derive* C^* *and* Ω_0 *as above, then:*

[4] As a counter example see the analysis of evolution with matching on a grid in Ellison [8].

1. *For all* $z \in \omega_0$ *for some* $\omega_0 \in \Omega_0$

$$C_z = C^*_{\omega_0}$$

2. *For all* $z \notin \Omega_0$

$$C_z = \min_{\omega_0 \in \Omega_0} \left\{ C^* \left(z | \omega_0 \right) + C^*_{\omega_0} \right\}$$

The implications of this result are significant. The first claim states that for all limit sets you only have to consider trees over other limit sets, the second claim implies that you can ignore all other states. There are currently two proofs of the first part of this claim in the literature. The first in Young [25] uses a "cut and paste" technique, the second in Kandori and Rob [15] is based on reduced chains. For completeness we include a third, which also establishes the second part of the claim.

Proof. Let ω_0' be such that $z' \in B(\omega_0')$, $\hat{t}^1_{\omega_0}$ be the tree that has cost $C^*_{\omega_0}$ and $\hat{t}^0_z$ be the tree that has cost C_z. First for $z \in \omega_0$ $C^*_{\omega_0} \geq C_z$ since we can always add a path from z' to ω_0' at zero cost. Second, $C^*_{\omega_0} \leq C_z$ since we can represent $\hat{t}^0_z$ as a graph over Ω_0 without losing any vital information. The only difficulty will be z''s that are *junctures*. A juncture is a point where the tree branches, in other words there are an ω_0'' and ω_0''' such that neither is the successor of the other, and z' is the first successor of both of them. If this is the case add a path from z' to ω_0' at zero cost, assign at least one of ω_0'' and ω_0''' to this path and ignore z'. The second claim is proven by noting that otherwise z would be a juncture. ■

3 The Emergent Seed.

We now define a fundamental underlying graph that—unless there is a good reason not to—the minimal cost trees will follow. This graph is the *emergent seed*, or formally:

Definition 1 *The* emergent seed—E —*is a least cost graph such that:*

1. *Every* $\omega_0 \in \Omega_0$ *has a successor.*

2. *There exists some* $\omega_0^* \in \Omega_0$ *that are the successors of all other* $\omega_0 \in \Omega_0$.

Note that there will always be more than one ω_0^* that satisfy the second part of the definition, and these ω_0^* will form a cycle; we call this sub-graph of the emergent seed the *core*. Sometimes there will be multiple graphs that have the same cost, in this case the analyst can choose the most convenient.

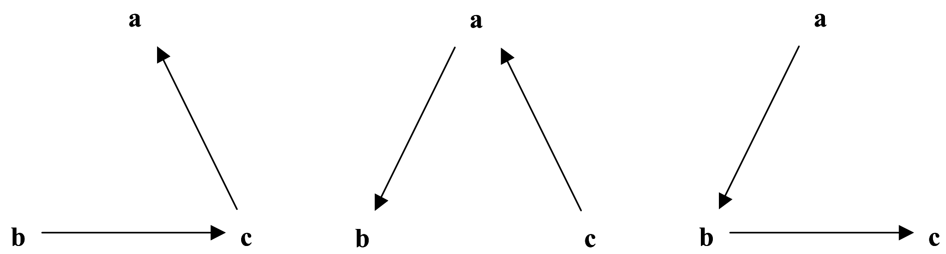
Relative to this fundamental graph there is a clear representation of stochastic potential. This point will be explained in two steps. In the first subsection the first level of the emergent seed will be constructed using two examples. This suffices to explain all of the key elements of the representation, and the relationship between this representation and the radius and modified coradius from Ellison will be discussed. Unfortunately finding the first level does not always complete the emergent seed. In the subsection 3.2 the methodology to complete it is presented culminating in the general representation theorem.

3.1 The First Level, Two Examples.

I will show how to find the first level of the emergent seed by two examples. First consider the stage game:

	a	b	c
a	6,6	0,5	0,0
b	5,0	5,5	0,4
c	0,0	4,0	4,4

I call this the "step" game due to the structure of the row player's payoffs. Using the convention of writing "k," $k \in \{a, b, c\}$, for the state in which all players play k, then in this game the minimum cost trees are as follows:



An arrow from ω to ω' means that ω' is the direct successor of ω in the illustrated graphs.

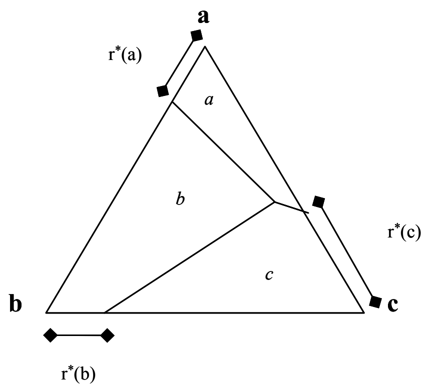
In this case the emergent seed is obvious, it is the union of all three graphs. Furthermore if we write each strategies stochastic potential in terms of the emergent seed:

$$\begin{aligned} c_a^* &= c^*(E) - c^*(b|a) = c^*(E) - \frac{1}{6} \\ c_b^* &= c^*(E) - c^*(c|b) = c^*(E) - \frac{1}{5} \\ c_c^* &= c^*(E) - c^*(a|c) = c^*(E) - \frac{2}{5} \end{aligned}$$

one can immediately see that c is stochastically stable because it is the maximum of $c^*(b|a)$,$c^*(c|b)$, and $c^*(a|c)$. Thus for this problem being stochastically stable is equivalent to having the most costly link in the emergent seed. Finally $c^*(b|a)$,$c^*(c|b)$, and $c^*(a|c)$ are the *security level* of a, b, and c; respectively. This is:

$$R^*(\omega_0) = \min_{\omega_0' \in \Omega_0 \backslash \omega_0} C^*(\omega_0'|\omega_0)$$

. The security level can also be easily found from a graph of the basins of attraction:



A graph of the probability space over the strategies $\{a, b, c\}$. The corner labeled $k \in \{a, b, c\}$ has probability one that all players play action k. The region labeled k is the strategy k's basin of attraction.

As you can see in this example $r^* (\omega_0)$ is the shortest distance out of the best response region of ω_0. In general $B(\omega_0)$ might be larger than the best response region of ω_0, but the security level is always easy to find. Ellison [8] analyzes the same concept but called it the *radius*. I prefer the term security level because for many limit sets it has a clear intuitive meaning. In these case the security level is the minimal fraction of people that must change their actions before other players need to worry about responding. If less than that fraction changes their plans people can feel "secure" about not changing theirs. If the basin of attraction is larger than the best response region (like in the Cournot game) it is possible that at a lower level players might want to change their plans, but in the long run their plans will return to ω_0.

Regardless of the terminology used, the security level is all that is needed to find the first level of the emergent seed.

Definition 2 *The* first level of the emergent seed—E_1—*is a graph with vertices* Ω_0 *and*

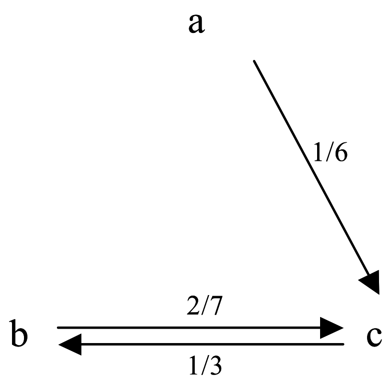
$$z_1 (\omega_0) = \omega_0^* \equiv \arg \min_{\omega_0' \in \Omega_0 \setminus \omega_0} C^* (\omega_0' | \omega_0)$$

.

Now clearly having a maximal security level is not sufficient for every problem. Consider the following game:

	a	b	c
a	$5, 5$	$-3, -1$	$0, 4$
b	$-1, -3$	$4, 4$	$1, 2$
c	$4, 0$	$2, 1$	$2, 2$

In this game the emergent seed is:



An arrow from ω to ω' means that ω' is the direct successor of ω in the illustrated graphs. Beside each arrow is the (normalized) cost of transition.

Now clearly for b and c all that matters is the maximal security level, but what about a? By analyzing the emergent seed we know that we must connect either b or c to a (and not both), but which one? The proper question is what is the relevant cost for connecting b to a instead of c. If we do this we are increasing the total cost by $c^*(a|b)$ but *reducing* the cost by $r^*(b)$, or the new cost is:

$$\Delta^1 c(a|b) = c^*(a|b) - r^*(b).$$

Notice that $\{b, c\}$ form a cycle, we will label all cycles in E_1 as ω_1's, with the set being Ω_1. The optimal cost function of going from ω_1 to a is:

$$\Delta^1 c^*(a|\omega_1) = \min\left\{\Delta^1 c(a|b), \Delta^1 c(a|c)\right\}$$

and the stochastic potential of a then is

$$c_a^* = c^*(E) - r^*(a) + \Delta^1 c^*(a|\omega_1)$$

. This representation can be made very general, define the first difference cost function as:

$$\Delta^1 C(z|z') = \begin{cases} C^*(z|z') - R^*(\omega_0) & \text{if } z' \in \omega_0 \text{ and } z \in \Omega_0 \backslash \omega_0 \\ C^*(z|z') & \text{else} \end{cases}$$

and $\Delta^1 C^*(\cdot|\cdot)$ like before. Then if Ω_1 has a unique element $\Delta^1 C^*(\omega_0|\omega_1)$ is the core attraction rate, or $Ca(\omega_0) = \Delta^1 C^*(\omega_0|\omega_1)$ and:

Theorem 1 *If Ω_1 has a unique element then:*

$$C_{\omega_0}^* = C^*(E) - R^*(\omega_0) + Ca(\omega_0)$$

This is the general representation but here we only prove it assuming Ω_1 has a unique element. Notice that it is much more common for Ω_1 to have a unique element than for there to be one limit set. It is necessary that there are more than three limit sets for this to happen. While it might be impossible to provide a general characterization of all the games where Ω_1 has a unique element, it is certainly a large class.

Proof. Beginning with an arbitrary tree, t^1_ω, we will reduce it's cost until it is in the above form. First, for every ω_0' that is not a successor of ω_1 in t^1_ω make ω_0''s direct successor it's direct successor in the emergent seed. This must reduce the cost of the entire tree. This leaves several unoptimized paths from ω_1 to ω_0. Choose one at random and move all others into the emergent seed graph, again this reduces the total cost. Finally, for the path from ω_1 to ω_0 note that including a ω_0' in this path instead of having it in the (otherwise optimal) emergent seed graph will increase costs by exactly $\Delta^1 C^*(\omega_0''|\omega_0')$ if ω_0'' is it's direct successor in this path, thus minimizing the cost of the path between ω_1 and ω_0 with respect to the $\Delta^1 C^*(\cdot|\cdot)$ cost function minimizes the total cost of t^1_ω, the resulting tree has minimal cost and is as above. ■

At this level the representation in Theorem 1 begs comparison with Ellison [8]. The modified coradius in that paper is $\widetilde{CR}(\omega_0) = \max_{\omega_0' \in \Omega_0 \setminus \omega_0} \{\Delta^1 C^*(\omega_0|\omega_0') + R^*(\omega_0')\}$ and if $\widetilde{CR}(\omega_0) < R^*(\omega_0)$ then ω_0 is stochastically stable. If the emergent seed has one level then there are bounds on $\widetilde{CR}(\omega_0)$ in terms of $Ca(\omega_0)$.

Corollary 1 *If Ω_1 has one element then:*

$$Ca(\omega_0) + R^*(\omega_1) \leq \widetilde{CR}(\omega_0) \leq Ca(\omega_0) + \max_{\omega_0' \in \Omega_0 \setminus \omega_0} R^*(\omega_0')$$

and if ω_0 is in the core then:

$$\widetilde{CR}(\omega_0) = \max_{\omega_0' \in \Omega_0 \setminus \omega_0} R^*(\omega_0')$$

This corollary shows the relationship between stochastic potential and the speed of evolution. Ellison shows that the expected time to reach the long run is no more than $\varepsilon^{-\widetilde{CR}(\omega_0)}$ and this result bounds that function. Thus for games with one level in the emergent seed finding the stochastically stable strategy also bounds how quickly evolution occurs.

3.2 The Representation Theorem.

It is now possible to establish the general representation with little further work, and this section is the proof. The first issue is what to do when there are multiple elements in Ω_1. This is immediate from Theorem 1 and Lemma 1. Like we have done before, define $t^2_{\omega_1}$ as a tree with base ω_1 and vertices in Ω_1 space, and $\Delta^1 C^*_{\omega_1}$ as the minimal cost of such a tree.

Lemma 2 *If Ω_1 has more than one element then:*

$$C^*_{\omega_0} = C^*(E_1) - R^*(\omega_0) + \min_{\omega_1 \in \Omega_1} \{\Delta^1 C^*(\omega_0|\omega_1) + \Delta^1 C^*_{\omega_1}\}$$

Proof. Since $\Delta^1 C(\cdot|\cdot) \geq 0$ by construction, from Lemma 1 we know that relative to $\Delta^1 C$ $\min_{\omega_1 \in \Omega_1} \{\Delta^1 C^*(\omega_1|\omega_0) + \Delta$ is correct. From Theorem 1 we know that if ω_0' is not used in either $\Delta^1 C^*(\omega_1|\omega_0)$ or $\Delta^1 C^*_{\omega_1}$ it is optimal to have it in the E_1 graph, and thus it's cost is correct. Finally $\Delta^1 C^*(\cdot|\cdot)$ is the appropriate cost metric for taking a limit set out of the E_1 graph. ■

There is now a new problem to simplify, the fundamental structure of the trees that give the $\Delta^1 C^*_{\omega_1}$'s. This problem seems like it is the same as the original problem, but it is not exactly. We still need to define:

$$\begin{aligned}\Delta^1 R^*(\omega_1) &= \min_{\omega_1' \in \Omega_1 \backslash \omega_1} \Delta^1 C^*(\omega_1'|\omega_1) \\ \omega_1^* &= \arg\min_{\omega_1' \in \Omega_1 \backslash \omega_1} \Delta^1 C^*(\omega_1'|\omega_1)\end{aligned}$$

But we will not want the successor of every ω_1 to be ω_1^*. The reason is that we are still trying to find a graph over Ω_0, every time we have $z(\omega_1) = \omega_1^*$ the cost of the graph over Ω_0 increases. Thus we will drop connections that cause cycles in Ω_1, and the second level of the emergent seed is defined as follows:

Definition 3 *The* second level of the emergent seed*—E_2—is a graph over Ω_1 found by the following algorithm:*

Let $L^0 = \Omega_1$ and $\forall \omega_1 \in \Omega_1$, $b^0(\omega_1) = \emptyset$

Initial Step Find $\omega_1 \in \arg\min_{\omega_1 \in L^0} \Delta^1 R^*(\omega_1)$. Then $z_2(\omega_1) = \omega_1^*$, $b^1(\omega_1^*) = \omega_1$, and $L^1 = L^0 \backslash \omega_1$

Iterative Step If $L^t \neq \emptyset$, find $\omega_1 \in \arg\min_{\omega_1 \in L^t} \Delta^1 R^*(\omega_1)$. Then

(a) If $\omega_1^* \notin b^t(\omega_1)$, $z_2(\omega_1) = \omega_1^*$, $b^{t+1}(\omega_1^*) = \omega_1 \cup b^t(\omega_1^*)$, and $L^{t+1} = L^t \backslash \omega_1$.

(b) Else $L^{t+1} = L^0 \backslash \omega_1$.

The ω_1's that have no successors in E_2 are labeled ω_2, elements of Ω_2. The new cost function relative to E_2 only changes on elements of Ω_1 that are not elements of Ω_2.

$$\Delta^2 C(z'|z) = \begin{cases} \Delta^1 C^*(z'|z) - \Delta^1 R^*(\omega_0) & \text{if } z \in \omega_1 \in \Omega_1 \backslash \Omega_2 \text{ and } z' \in (\Omega_1 \backslash \Omega_2) \backslash \omega_1 \\ \Delta^1 C^*(z'|z) & \text{else} \end{cases}$$

Every ω_0's cost can be represented as:

$$\begin{aligned}C^*_{\omega_0} &= C^*(E_1) - R^*(\omega_0) \\ &+ \min_{\omega_1 \in \Omega_1} \left\{ \Delta^1 C^*(\omega_0|\omega_1) + \Delta^1 C^*(E_2) - \Delta^1 R^*(\omega_1) + \min_{\omega_2 \in \Omega_2} \left\{ \Delta^2 C^*(\omega_1|\omega_2) + \Delta^2 C^*_{\omega_2} \right\} \right\}\end{aligned}$$

where $\Delta^2 C^*_{\omega_2}$ is defined like $\Delta^1 C^*_{\omega_1}$ before, which can be simplified to:

$$\begin{aligned}C^*_{\omega_0} &= C^*(E_1) + \Delta^1 C^*(E_2) - R^*(\omega_0) \\ &+ \min_{\omega_1 \in \Omega_1, \omega_2 \in \Omega_2} \left\{ \Delta^1 C^*(\omega_0|\omega_1) - \Delta^1 R^*(\omega_1) + \Delta^2 C^*(\omega_1|\omega_2) + \Delta^2 C^*_{\omega_2} \right\}\end{aligned}$$

. At this point the reader can iteratively define Ω_k, $\Delta^k C(\cdot|\cdot)$, $\Delta^k R^*(\cdot)$, and E_k and from the definitions for Ω_2, $\Delta^2 C(\cdot|\cdot)$, $\Delta^2 R^*(\cdot)$, and E_2 . Since there are a finite number of elements in Ω_0 there is a finite $K+1$ such that Ω_{K+1} has a single element, call this element ω^*. Then the emergent seed is:

Definition 4 *The emergent seed—E—is found by projecting* $\{E_k\}_{k=1}^{K}$ *onto graphs on subsets of* Ω_0*. If* ω_0 *has a direct successor in* E_{k^*} *but not any* $k > k^*$ *then that direct successor is* ω_0*'s direct successor in* E*.*

The general definition of the core attraction rate is:

$$Ca(\omega_0) = \min_{\omega_k \in \Omega_k, k=1,2,3,...K} \left\{ \sum_{k=1}^{K} \left[\Delta^k C^* (\omega_{k-1}|\omega_k) - \Delta^k R^* (\omega_k) \right] + \Delta^K C^* (\omega_K|\omega^*) \right\}$$

and the representation theorem can now be stated.

Theorem 2 *For* $\omega_0 \in \Omega_0$:

$$C^*_{\omega_0} = C^*(E) - R^*(\omega_0) + Ca(\omega_0)$$

With proof by construction above. While this representation adds clarity to the analysis of dynamic evolution, algorithimically it is not better (or worse) than the best alternative in the graph theory literature. However this approach also generates information at every stage. One useful fact is:

Corollary 2 *For* $k \in \{0, 1, 2, ..., K-1\}$, $\Omega_{k+1} \subseteq \Omega_k$

This can be sufficient to find the solution to some problems, because:

Corollary 3 (Predecessor Dominance) *If* ω_0 *is a successor of* ω_0' *in some* E_k *and* $R^*(\omega_0) > R^*(\omega_0')$ *then* ω_0' *is not stochastically stable.*

Proof. Since ω_0 is a successor of ω_0' it is less costly to go from ω_0' to ω_0 than vice versa, thus $Ca(\omega_0') \leq Ca(\omega_0)$. Since $R^*(\omega_0) > R^*(\omega_0')$ ω_0' is not stochastically stable. ■

This is especially useful if a strategy in the core has a high security level. This technique works in bargaining games (below), and in the second example above we did not actually have to calculate $\Delta^1 c^*(a|\omega_1)$ since $r^*(c) > r^*(a)$.

4 Applications.

In this paper I will limit myself to providing a characterization of games with two limit sets and explaining the results in Young [26] and Young [27]. The author has also applied this technique to generalizing the results in Johnson, Levine, and Pessendorfer [12] and analyzing the evolution of social norms ([10] and [11]) but will present those results elsewhere.

4.1 Games with Two Limit Sets.

A disappointing problem in dynamic evolution is our inability to characterize the solution to broad classes of problems. Instead for each game it seems like the stochastically stable strategy appears mysteriously from the analysis, with no clear relationship to other analyses. With the techniques in this paper this problem can finally be overcome at least for one class of games, those with two limit sets.

Corollary 4 *If $|\Omega_0| = 2$ and ω_0 has maximal security level then ω is stochastically stable.*

The proof is given by Corollary 3. Many games have only two equilibria, if these games are also *acyclic* (Young [25]) then they have two limit sets and stochastic stability is easy to characterize.

4.2 Bargaining and Contracts.

It is noteworthy that Young [26] implicitly used the emergent seed to find the stochastically stable strategies—and published before the seminal papers of Kandori, Mailath, and Rob [14] and Young [25]. There is also a simple emergent seed in contract games (Young [27]).

The similarity of these games and their transparent emergent seeds makes them excellent examples of how the emergent seed underlies much evolutionary analysis. In a *surplus sharing game* there are two players who must decide how to divide one unit of a good. They will have strictly increasing, concave utility functions with $u_i(1) = 1$ and $u_i(0) = 0$ for $i \in \{1, 2\}$. They do this by simultaneously declaring a share for themselves and the other person $a_i = \left\{s_1^i, s_2^i\right\}$, $s_j^i \in [0, 1]$ for $\{i, j\} \in \{1, 2\}^2$, we will require that $s_j^i \in \left\{\frac{d}{D}\right\}_{d=0}^{D}$ where D is some large finite number. The game is a *contract game* if the surplus sharing rule is:

$$h_c(a_1, a_2) = \begin{cases} \left\{s_1^1, s_2^2\right\} & \text{if } s_1^1 = s_1^2, s_2^2 = s_2^1, s_1^1 + s_2^2 \leq 1 \\ \{0, 0\} & \text{else} \end{cases}$$

the game is a *bargaining game* if the surplus sharing rule is:

$$h_b(a_1, a_2) = \begin{cases} \left\{s_1^1, s_2^2\right\} & \text{if } s_1^1 + s_1^2 \leq 1 \\ \{0, 0\} & \text{else} \end{cases}$$

In our analysis of both games the results are slightly peculiar if players can make extreme demands, so first I will analyze the game where $s_j^i \notin \{0, 1\}$.

4.2.1 The Contract Game.

In a contract game a player's payoff from anyone playing a different strategy is zero. Thus the security level is found by having a small group make the most attractive alternative offer possible. Notice that this group is also requesting the least possible amount for themselves, thus call them "*weak invaders*." Because of their importance denote $\omega^{+1} = \left\{\frac{D-1}{D}, \frac{1}{D}\right\}$ and $\omega^{+2} = \left\{\frac{1}{D}, \frac{D-1}{D}\right\}$. For arbitrary ω the cost of the transition for group i then is the minimal p such that:

$$\begin{aligned} (1-p)\, u_i(\omega_i) &\leq p u_i\left(\omega^{+i}\right) \\ \frac{u_i(\omega_i)}{u_i(\omega^{+i}) + u_i(\omega_i)} &= p \end{aligned}$$

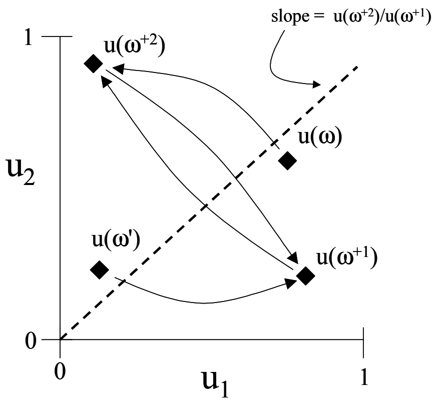
and the security level of ω then is:

$$r^*(\omega) = \min_{i \in \{1,2\}} \left\{ \frac{u_i(\omega)}{u_i(\omega^{+i}) + u_i(\omega)} \right\}$$

and ω^*'s direct successor in the absorbent seed is ω^{+2} if:

$$u_2\left(\omega_2^*\right) \leq \frac{u_2\left(\omega^{+2}\right)}{u_1\left(\omega^{+1}\right)} u_1\left(\omega_1^*\right)$$

One can immediately see the emergent seed from this analysis. The ratio $\frac{u_2\left(\omega^{+2}\right)}{u_1\left(\omega^{+1}\right)}$ defines a line in utility space, all ω with payoffs below this line go to ω^{+2} all ω' with payoffs above go to ω^{+1}, as is depicted in the graph below.



A graph of the payoff space of the contract game. An arrow from $u\left(\hat{z}\right)$ to $u\left(\tilde{z}\right)$ means that $\tilde{z}$ is the direct successor of $\hat{z}$ in the emergent seed.

At this point in the analysis in Young [27] finds a sufficient condition to characterize the stochastically stable strategy. With our new methodology the characterization can be directly stated.

Lemma 3 *For all* $\omega \in \Omega_0$

$$ca\left(\omega\right) = \min_{i \in \{1,2\}, -i = \{1,2\} \backslash i} \left\{ \frac{u_i\left(\omega^{+(-i)}\right)}{u_i\left(\omega\right) + u_i\left(\omega^{+(-i)}\right)} \frac{u_i\left(\omega^{+i}\right) - u_i\left(\omega\right)}{u_i\left(\omega^{+i}\right) + u_i\left(\omega^{+(-i)}\right)} \right\}$$

Proof. It will be shown that the direct predecessor of ω in $ca\left(\omega\right)$ is ω^{+i} for $i \in \{1,2\}$. Let ω^o be ω's direct predecessor in $ca\left(\omega\right)$. Then the first difference cost is:

$$\min_{i \in 1,2} \left\{ \frac{u_i\left(\omega^o\right)}{u_i\left(\omega\right) + u_i\left(\omega^o\right)} \right\} - \min_{i \in 1,2} \left\{ \frac{u_i\left(\omega^o\right)}{u_i\left(\omega^{+i}\right) + u_i\left(\omega^o\right)} \right\}$$

Now if the solution to the first term is i and the solution to the second is $-i = \{1,2\} \backslash i$ then since $\frac{u_i(\omega^o)}{u_i(\omega)+u_i(\omega^o)}$ is an increasing function of $u_i\left(\omega^o\right)$ we want to minimize over $u_i\left(\omega^o\right)$, but that will lead us into the area where the solution to both problems is i. In this case the first difference cost is as above, and it is an increasing function until $\sqrt{u_i\left(\omega^{+i}\right) u_i\left(\omega\right)} \leq u_i\left(\omega^o\right)$, thereafter decreasing. Thus the second function is concave and we should compare it's endpoints, at $u_i\left(\omega^0\right) = u_i\left(\omega^{+(-i)}\right)$ it is less than at $u_i\left(\omega^0\right) = u_i\left(\omega^{+i}\right)$ and we are done. ■

Thus the stochastically stable strategy is:

$$\omega^* \in \arg\min_{\omega\in\Omega_0} \left\{ \begin{array}{l} -\min_{i\in 1,2}\left\{\frac{u_i(\omega)}{u_i(\omega^{+i})+u_i(\omega)}\right\} \\ +\min_{i\in\{1,2\},-i=\{1,2\}\setminus i}\left\{\frac{u_i\left(\omega^{+(-i)}\right)}{u_i(\omega)+u_i\left(\omega^{+(-i)}\right)} \frac{u_i\left(\omega^{+i}\right)-u_i(\omega)}{u_i(\omega^{+i})+u_i\left(\omega^{+(-i)}\right)}\right\} \end{array} \right\}$$

Note that both terms are decreasing in ω thus ω^* is Pareto Efficient. Further characterization is frustrated by the complexity of the second term. Young [27] also has this problem, but notice that as the grid gets finer $u_i\left(\omega^{+(-i)}\right)$ goes to zero, and thus the second term becomes unimportant. As he showed in the limit the solution becomes the Kalai-Smordinsky solution.

This can also be done by allowing $\{0,1\}$ and $\{1,0\}$. In this case each strategy's security level is:

$$r^*(\omega) = \min_{i\in 1,2}\left\{\frac{u_i(\omega)}{1+u_i(\omega)}\right\}$$

but $\{1,0\}$ and $\{0,1\}$ are not limit sets. From the state $\{0,1\}$ the smallest mutation by type 2 players will cause the system to drift, and vice a versa for $\{1,0\}$. Thus the successor in the emergent seed can be any state, and this effectively means that all of the states are in the core. Therefore the stochastically stable strategy is the maximal security level or:

$$\begin{aligned} \omega^* &\in \arg\min_{\omega\in\Omega_0}\left\{-\min_{i\in 1,2}\left\{\frac{u_i(\omega)}{1+u_i(\omega)}\right\}\right\} \\ &\approx \arg\max_{\omega\in\Omega_0}\min_{i\in 1,2} u_i(\omega) \end{aligned}$$

which is the Kalai-Smordinsky solution or maximin welfare.

4.2.2 The Bargaining Game.

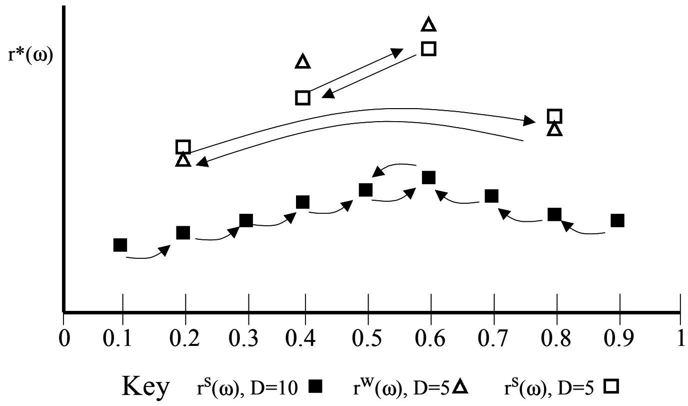
In the bargaining game the limit sets are the Nash equilibria where both parties receive a strictly positive amount—the have the form $\{s, 1-s\}$, $s \in (0,1)$. The security level of a limit set may still be determined by weak invaders, with the same logic as above. But now there is another alternative. Remember that in a bargaining game a player can lower the amount they demand and only loose a tiny amount from everyone. Experimenters can take advantage of this to "push around" their opponents. *Strong invaders* of population $-i$ $(-i = \{1,2\}\setminus i)$ demand $\frac{1}{D}$ more than the current offer (if possible), we denote this payoff $\omega^{\downarrow i} = \left\{\omega_i - \frac{1}{D}, \omega_{-i} + \frac{1}{D}\right\}$. It is a best response to go along with this group if:

$$\begin{aligned} (1-p)\, u_i(\omega) &\le u_i\left(\omega^{\downarrow i}\right) \\ 1 - \frac{u_i\left(\omega^{\downarrow i}\right)}{u_i(\omega)} &= p \end{aligned}$$

And as Young [26] shows,

$$
\begin{aligned}
r^*(\omega) &= \min\{r^w(\omega), r^s(\omega)\} \\
r^w(\omega) &= \min_{i\in 1,2}\left\{\frac{u_i(\omega)}{u_i(\omega^{+i}) + u_i(\omega)}\right\} \\
r^s(\omega) &= \min_{i\in 1,2}\left\{1 - \frac{u_i\left(\omega^{\downarrow i}\right)}{u_i(\omega)}\right\}
\end{aligned}
$$

note that if $\omega_i = \frac{1}{D}$ then $\omega^{\downarrow i}$ is not possible, and $r^s(\omega)$ must be modified accordingly. One can easily verify that both of these functions are concave, as well notice that for any ω $r^s(\omega) < r^w(\omega)$ for large enough D. In fact for large enough D by concavity $r^s(\omega) \leq r^w(\omega)$ for all ω in the grid. Below this point is illustrated for $u_1(\omega) = \omega_1$ and $u_2(\omega) = \omega_2^{\frac{4}{5}}$.



A graph of the security level for $D \in \{5, 10\}$. An arrow from $r^*(\hat{z})$ to $r^*(\tilde{z})$ means that $\tilde{z}$ is the direct successor of $\hat{z}$ in the emergent seed.

When $D = 5$ you see that at the endpoints $r^w(\omega) < r^s(\omega)$, but when $D = 6$ this is no longer true. When D is large the case on the bottom dominates, $r^s(\omega)$ determines the security levels. The arrows represent E_1 in each case. Notice that when $D = 5$ we have two ω_1's and thus there is an E_2. But this happens only when D is small, the case when $D = 10$ is more common, the core is in the interior and has the highest security levels, therefore the game is solved by predecessor dominance. Furthermore, the stochastically stable strategy is also the Nash Bargaining Solution for the finite grid. Call the strategy in the core that has the highest payoff for player 1 $\bar{\omega}$ and the other one $\underline{\omega}$, and for clarity if ω^N is the Nash bargaining solution then for the finite grid we know that $u_1\left(\omega^N\right) u_2\left(1-\omega^N\right) \geq u_1\left(s_1^1\right) u_2\left(s_2^2\right)$ for all $s_1^1 + s_2^2 \leq 1$, $\left\{s_1^1, s_2^2\right\} \in \left[\left\{\frac{d}{D}\right\}_{d=0}^{D}\right]^2$

Proposition 1 *Assume D is large enough that $r^s(\omega) \leq r^w(\omega)$ for all relevant ω, then if $\bar{\omega}$ is stochastically stable then $\bar{\omega}$ is the Nash Bargaining solution.*

Proof. Since $\bar{\omega}$ is stochastically stable we know that $r^*(\bar{\omega}) \geq r^*(\underline{\omega})$ since $\bar{\omega}$ is higher than $\underline{\omega}$ for player 1 this means

$$1 - \frac{u_1\left(\bar{\omega}^{\downarrow 1}\right)}{u_1(\bar{\omega})} \geq 1 - \frac{u_2\left(\underline{\omega}^{\downarrow 2}\right)}{u_2(\underline{\omega})}$$

but note that $\underline{\omega}^{\downarrow 2} = \bar{\omega}$ and $\bar{\omega}^{\downarrow 1} = \underline{\omega}$. This means that $u_1(\underline{\omega})\,u_2(\underline{\omega}) \leq u_1(\bar{\omega})\,u_2(\bar{\omega})$. Now consider $\tilde{\omega} = \underline{\omega} - \frac{1}{D_1}$, since the security levels are decreasing we know:

$$1 - \frac{u_1\left(\underline{\omega}^{\downarrow 1}\right)}{u_1(\underline{\omega})} \geq 1 - \frac{u_2\left(\underline{\omega}^{\downarrow 2}\right)}{u_2(\underline{\omega})} \geq 1 - \frac{u_2\left(\left(\underline{\omega}^{\downarrow 2}\right)^{\downarrow 2}\right)}{u_2\left(\underline{\omega}^{\downarrow 2}\right)}$$

and thus $u_2(\underline{\omega})\,u_1(\underline{\omega}) \geq u_1\left(\underline{\omega}^{\downarrow 2}\right) u_2\left(\underline{\omega}^{\downarrow 2}\right)$. Repeating this process for all $\omega \leq \underline{\omega}$ and reversing the argument for all $\omega \geq \bar{\omega}$ proves the Proposition. ■

Notice the speed at which evolution takes place in this model, the modified coradius is $r^*(\underline{\omega}) = 1 - \frac{u_2\left(\underline{\omega}^{\downarrow 2}\right)}{u_2(\underline{\omega})}$, or the expected waiting time converges to zero as D gets large. Allowing for demands of $\{0,1\}$ and $\{1,0\}$ means that if $r^w(\omega) \leq r^s(\omega)$ then ω can be in the core. The only effect this has is being able to allow for log-concave utility functions.

5 Conclusion.

This paper analyzes a fundamental underlying structure in dynamic evolution—the emergent seed. When the stochastic potential of a strategy is written relative to this structure the potential is seen to be an additive combination of the security level and the core attraction rate. The result is a representation theorem that clarifies what makes a strategy evolutionarily successful.

Several examples are presented to show the usefulness of this result, for example stochastically stability in games with two limit sets is completely characterized. In bargaining and contract games the underlying dynamics are clarified by analyzing the emergent seed.

It is far from clear that this methodology would always be easier than Ellison [8]. Examples can be found where it is clear that the emergent seed methodology would give little aid. For example in Kandori and Rob [16] the "bandwagon effect" does not impose enough structure on the underlying game to make the emergent seed useful.

Dynamic evolution is a structural and viable alternative to equilibrium analysis. While still in it's infancy it has shown great promise and insight as a methodology. It is hoped that analysts continue exploring various models of "limitedly rational" behavior to see what types of changes the results are sensitive too. At the same time it is hoped that the emergent seed methodology provides a new window of opportunity for applications of the model, allowing us to see the implications of dynamic evolution in more settings.

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