Existence of equilibrium, core and fair division in a heterogeneous divisible commodity exchange economy

(Comments are welcomed)

Farhad Husseinov
Department of Economics
Bilkent University

May 29, 2006

Abstract

Exchange and allocation of a heterogeneous divisible commodity (e.g., land or cake) that is modeled as a measurable space is considered. The exsistence of a competitive equilibrium with additive prices in 'land' trading economy with unordered convex preferences is proved. Also the existence of a weak core and a fair allocation are established.

JEL classification numbers: D51, D63
Keywords: Land trading economy, equilibrium, core, fair division.

1. Introduction

We consider exchange and allocation of a heterogeneous divisible commodity. One notable example of such a commodity is land. This problem is coined in literature as the 'cake division' or 'land division' problem. A heterogeneous divisible commodity is modelled as a measurable space (X, Σ). In theoretical models of land economics, X is assumed to be a Borel measurable subset of Euclidean space R^2 (or more generally R^k) and Σ to be the Borel $\sigma-$algebra $\mathcal{B}(X)$ of subsets of X. It is usual to consider this measurable space with the Lebesgue measure.

Berliant [3] is the first to study a competitive equilibrium in the context of a land trading economy. He shows the existence of a competitive equilibrium in the case when preferences over land plots are represented by utility functions of the form $U(B) = \int_B u(x)dx$, so that U is a measure on $\mathcal{B}(X)$ absolutely continuous with respect to the Lebesgue measure. His proof uses a method that imbeds the land trading economy into an economy with the commodity space $L_\infty(X)$, and then uses Bewley's equilibrium existence results [2] along with the methods of infinite dimensional analysis to establish the existence of an equilibrium.

Dunz [9] studies the existence of the core for substantially more general preferences. In [9] preferences are given by the utility functions that are compositions of quasi-concave functions with a finite number of characteristics of land parcels. Dunz proves that under these assumptions on preferences the weak core of a land trading game is nonempty. These chracteristics are countably-additive over land parcels. Assigning a finite number of additive characteristics to land parcels is a common assumption made in empirical literature on land trading. Dunz [9], based on results of his joint work with Berliant [4], argues that "...if prices are required to be additive ... then an equilibrium might not exist. If no equilibrium with additive prices exists, then it is not clear what the final allocation of the economy will be since there would always be arbitrage opportunities. This suggests that competitive equilibrium might not be the appropriate solution concept for economies with land." However, nonexistence of equilibrium in the example in Berliant-Dunz [4] is of the same nature as one in the classical case of trading divisible commoditiies and is due to nonconvexity of preferences. One of the goals of the present paper is to show that a competitive equilibrium with an additive price exists in land trading economy with rather general unordered 'convex' preferences. In particular, preferences are not assumed to be ordered. In fact this is done in a more abstract context of a measurable space trading economy. We show the existence of an equilibrium with the equilibrium price that is a measure, ν, on (X, Σ), absolutely continuous with respect to the sum of all characteristic measures. For the land trading economy, where all characteristic measures are assumed to be absolutely continuous with respect to the Lebesgue measure λ, we obtain that the equilibrium price ν, is absolutely continuous with respect to the Lebesgue measure λ. Hence the Radon-Nikodim derivative $\frac{d\nu}{d\lambda}$ is an integrable function h on measure space (X, Σ, λ) (see [1, p.]). So $\nu(B) = \int_B h(x) d\lambda(x)$ for all measurable sets B in X. Function h can be interpreted as equilibrium price density on X.

Then, using the standard scheme, we show that a competitive allocation is a weak core allocation. This core existence result generalizes Dunz's [9] core existence theorem in two directions; first, it considers the division problem in the setting of abstract measurable space and does not assume the existence of a reference measure, and the second, preferences are not assumed to be ordered.

The next topic that is dealt with in this paper is the existence of a fair division. Examples of the fair division problem are: division of a heritance fairly among inheritants, designing land reform laws that allows to divide the land owned by a collective farm fairly among members of the collective farm in transition economies. On a deeper level fairness can be regarded as an essential and desirable property of a solution concept in economics (and game theory).

Weller [15] considered a problem of fair division of a measurable space (X, Σ) with a finite number of atomless measures discribing agents' preferences over measurable subsets. He shows the existence of an envy-free and efficient partition in this problem. In a somewhat different setting, namely when X is a measurable subset of the Euclidean space R^k and preference measures are nonatomic and absolutely continuous with respect to the Lebesgue measure, Berliant-Thomson-Dunz [5] shows the existence of a group envy-free and efficient partition. The concept of group envy-free partition is stronger than the concept of envy-free partition. Neither of these results implies the other; Weller's result is concerned with more abstract problem of fair partitioning an abstract measurable space with no reference measure. On the other hand, Berliant-Thomson-Dunz's [5] Theorem 2 states the existence of a fair partition in a stronger sense. Our approach to the fairness problem will be abstract and we will consider much more general preferences over measurable pieces. The result established here implies both of the above discussed results.

In proofs of the main results of the paper we use the following scheme. We reduce a problem of trading a heterogeneous divisible commodity to the one of trading a finite number of homogeneous divisible commodities (totality of subjectively attributed characteristics of measurable pieces), where endowments are subsets in the commodity space rather than commodity bundles. We then transform this economy to the general model of economy introduced by Gale and Mas-Colell [10] with the advantage of employing their competitive equilibrium existence theorem.

This introduction is followed by a section devoted to definititions and some preliminary results. Section 3, the central to the paper, studies the existence of a competitive equilibrium and core of the measurable space trading economy. Section 4 studies fairnes criteria for this economy.

2. Preliminaries

We consider a measurable space trading problem set in the following way. Let (X, Σ) be a measurable space (the cake or land plot) and let $\bar{P} = \{A_1, A_2, \ldots, A_n\}$ be a measurable ordered partition of X. Let $\mu_1, \mu_2, \ldots, \mu_n$ be nonatomic finite vector-measures on (X, Σ) of dimensions $s_1, s_2, \ldots, s_n$, respectively. The interpretation is that there are n persons $N = \{1, 2, \ldots, n\}$ each contributing his share A_i $(i \in N)$ and parts of the cake, X, are valued by individuals according to their measures $\mu_1, \mu_2, \ldots, \mu_n$, respectively. The components of vector-measure $\mu_i(B)$ are interpreted as measures of different attributes of a measurable piece B attached to this piece by individual i. We assume that individual i has a preference $\succ_i$ over his subjective attributes profiles $\mu_i(B)$, $B \in \Sigma$ and hence over measurable sets $B \in \Sigma$. We will use the same symbol $\succ_i$ for denoting both of these preferences. No confusion should arise. Every ordered measurable partition $B_1, B_2, \ldots, B_n$ will be interpreted as a feasible allocation of X.

Definition 1. A pair $(P = \{B_1, B_2, \ldots, B_n\},\ \nu)$ consisting of a feasible partition P and a measure μ is said to be a *competitive equilibrium* if for each individual i subset B_i maximizes his preference $\succ_i$ in his *budget set*

$$\mathcal{B}_i(\nu) = \{B \in \Sigma \mid \nu(B) \leq \nu(A_i)\}.$$

In this case $P = \{B_1, B_2, \ldots, B_n\}$ is called an *equilibrium allocation* and measure ν is called an *equilibrium price.*

A *coalition* is an arbitrary nonempty subset of N. The set of all coalitions is denoted as $\mathcal{N}$. All partitions considered further are assumed to be ordered and measurable. Further, the terms partition and division will be used interchangably.

Definition 2. We say a coalition $I \subset N$ *improves (weakly improves) upon* a division $P = \{B_1, B_2, \ldots, B_n\}$ if there exists a partition $Q = \{C_i \mid i \in I\}$ of $A(I) = \cup_{i \in I} A_i$ such that $C_i \succ_i B_i$ for all $i \in I$ (not $B_i \succ_i C_i$ for all $i \in I$ and $C_i \succ_i B_i$ at least for one $i \in I$.)

Definition 3. Partition $P = \{B_1, B_2, \ldots, B_n\}$ is said to be *a weak core allocation (core allocaton)* if there is no coalition that improves (weakly improves) upon allocation P. The set of all weak core allocations is called the *weak core (core)* of the measurable space trading problem.

Next, we introduce two concepts of Pareto efficient partition.

Definition 4. Partition $P = \{B_1, B_2, \ldots, B_n\}$ of X is said to be *weak Pareto efficient (Pareto efficient)* if there is no partition $P' = \{B'_1, B'_2, \ldots, B'_n\}$ of X such that $\mu_i(B'_i) \succ_i \mu_i(B_i)$ for all $i \in N$ (not $B_i \succ_i B'_i$ for all $i \in N$ and $B'_i \succ_i B_i$ for at least one $i \in N$.)

We will identify a vector of vectors (perhaps of different dimensions) as a long vector with scalar coordinates arranged in the lexicographic order. Sometimes we will denote coordinates with double indexes, with the first index being the index of the component vector and the second one the index of a component in that component vector .

The following theorem is a generalization of a result known as Dubins-Spanier's theorem (see also C. Aliprantis and K. Border [1] page 358) and easily follows from this result. It is to be noted that this theorem was discovered a decade earlier Dubins-Spanier's theorem by Chernoff [7].

Theorem 1. *Let (X, Σ) be a measurable space and let $\mu_1, \mu_2, \ldots, \mu_n$ be nonatomic finite vector measures on (X, Σ) of dimensions $s_1, s_2, \ldots, s_n$, respectively. Then the following set in R^s, where $s = \sum_{j=1}^{n} s_j$,*

$$\mathcal{R} = \{(\mu_i(B_i))_{i=1}^{n} \in R^s \mid P = (B_1, B_2, \ldots, B_n) \text{ a partition of } X\}$$

is compact and convex.

Proof of Theorem 1 based on Dubins-Spanier's theorem. Let $\mu = (\mu_k)_{k=1}^{s}$ be a vector measure $(\mu_1, \mu_2, \ldots, \mu_n)$ of dimension s. With every partition $P = (B_1, B_2, \ldots, B_n) \in \Pi^n(X)$ we associate the $s \times n$ matrix of reals $M(P) = (\mu_k(B_i))$. Denote by $M^{s\times n}$ the space of all $s \times n$ matrices with real entries. By the Theorem 1 in Dubins and Spanier [8] the range $\mathcal{R}' \subset M^{s\times n}$ of matrix-valued function M is compact and convex.

Let $L : M^{s\times n} \to R^s$ be a mapping defined in the following way. The first s_1 components of $L(M)$ are the first s_1 entries in the first column of matrix M, the second s_2 components are the entries in the second column of M with the column indexes $s_1 + 1$ through $s_1 + s_2$, and so on. Clearly L is a linear mapping with $L(\mathcal{R}') = \mathcal{R}$. Since $\mathcal{R}'$ is compact and convex it follows that so is $\mathcal{R}$. ■

3. Existence of a competitive equilibrium and a core

In this section for a preference $\succ_i$ on $R_+^{s_i}$ we denote $P_i(x_i) = \{x'_i \in R_+^{s_i} \mid x'_i \succ_i x_i\}$. Clearly, correspondence P_i defines $\succ_i$ in the unique way. We assume that preferences

$\succ_i$ or P_i $(i \in N)$ are continuos, that is graphs of correspondences P_i are open relative to $R_+^{s_i} \times R_+^{s_i}$, and that they satisfy the following assumption.

Assumption (Weak Monotony). If for $x_i,\ x_i' \in R_+^{s_i}$ and $x_i' \geq x_i$, then $P_i(x_i') \subset P_i(x_i)$ for all $i \in N$.

We assume the following about the initial endowments of individuals.

Assumption (Positive Endowments). For each $i \in N$, set A_i can be divided into n measurable parts A_{ij} $(j \in N)$ so that $\mu_j(A_{ij}) > 0$ for all $j \in N$.

In the case, when for each $i \in N$ there exists a component measure $\mu_{ij}(A_i) > 0$ and all the component measures are mutually absolutely continuous, then it is easily seen that the assumption of positive endowments is satisfied.

The central result of this paper is the following competitive equilibrium existencce theorem.

Theorem 2. *If attribute vector-measures μ_i, $i \in N$ are nonatomic, preferences $\succ_i$, $i \in N$ are irreflexive continuos weakly monotone and convex, then there exsists a competitive equlibrium $(P = \{B_1, B_2, \ldots, B_n\},\ \nu)$ in the measurable space trading economy. Moreover, the equilibrium price measure ν is absolutely continuous with respect to the sum of all component measures of vector-measures μ_i, $i \in N$.*

Proof. We will reduce the above exchange economy to an economy of exchange of a finite number of divisible homogeneous commodities, where endowments of individuals are sets in the consumption spaces, rather than a single commodity bundle, from which the individuals are free to choose.

There are s commodities in this economy. Thus the commodity space is R^s, the $s-$dimensional Euclidean space. R_+^s and R_{++}^s denote the nonnegative and positive cones in this space, respectively. For $i \in N$, $R_+^{s_i}$ will be a consumption space of individual i. It will be considered as a coordinate subspace in R_+^s.

We define the initial endowment set $E_i \subset R^s$ of individual i in the following way:

$$E_i = \{(\mu_1(C_1), \mu_2(C_2), \ldots, \mu_n(C_n)) \mid \{C_1, C_2, \ldots, C_n\} \text{ is a partition of } A_i\}.$$

By Theorem 1 initial endowment sets are compact and convex.

Denote Δ the unit simplex in R^s. A price p will be an element of Δ. Wealth of individual i is defined as

$$\alpha_i(p) = \max\{p \cdot x \mid x \in E_i\} \text{ for all } i \in N.$$

Budget set of i is defined as

$$\mathcal{B}_i(p) = \{x \in R^{s_i} \mid p^{s_i} \cdot x \leq \alpha_i(p)\}.$$

Preferences of individual i are defined through mapping $P_i : R^{s_i}_+ \to R^{s_i}_+$ that is irreflexive, that is $x_i \notin P_i(x_i)$, has an open graph in $R^{s_i}_+ \times R^{s_i}_+$, and its values are nonempty convex sets.

Define the set of aggregate endowment vectors as the algebraic sum of individual endowment sets, that is $E = \sum_{i \in N} E_i$, and the technology as $Y = E + R^s_-$.

Fact 1. *Y is closed, has a nonempty bounded intersection with the nonnegative cone R^s_+.*

Proof is routine.

Let $E_0 \subset E$ be the Pareto frontier of Y, otherwise the smallest set with $Y = E_0 + R^s_-$.

Definition 4. A competitive equilibrium in the above described economy $\mathcal{E}$ is defined as an $(2N+1)-$ tuple $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{y_1}, \bar{y_2}, \ldots, \bar{y_n}, \bar{p}) \in ((\Pi_{i \in N} R^{s_i}_+) \times \Pi_{i \in N} E_i) \times \Delta$ such that

$$\bar{x_0} = \sum_{i \in N} \bar{x_i} = \sum_{i \in N} \bar{y_i} = \bar{y_0} \in E_0, \tag{1}$$

$$\bar{p} \cdot \bar{x_i} = \bar{p} \cdot \bar{y_i} = \alpha_i(\bar{p}) \ \text{ for } \ i \in N, \tag{2}$$

and

$$P(\bar{x_i}) \cap \mathcal{B}_i(\bar{p}) = \emptyset \ \text{ for } \ i \in N. \tag{3}$$

Define

$$\Pi(p) = \sup \ p \cdot Y \text{ for } p \in \Delta. \tag{4}$$

Obviously sup in (4) is attained for each $p \in \Delta$.

Fact 2. $\Pi : \Delta \to R_+$ *is a nonnegative continuous function.*
Proof is routine.
By the definition of Y and E we have

$$\Pi(p) = \sum_{i \in N} \alpha_i(p) \text{ for } p \in \Delta. \tag{5}$$

(This is known as 'aggregation' in Microeconomics, see Mas-Colell et al [12] Proposition 5.E.1.) Following Gale and Mas-Colell [10], observe that

$$\bar{p} \cdot \bar{y_0} = \sum_{i \in N} \bar{p} \cdot \bar{y_i} = \sum_{i \in N} \alpha_i(\bar{p}) = \max \bar{p} \cdot Y.$$

By the Positive Endowments assumption E_i contains a strictly positive vector. It follows that

$$\alpha_i(p) > 0 \text{ for all } p \in \Delta,\ i \in N. \tag{6}$$

Now we have the following economy

$$\mathcal{E}_0 = \{Y,\ \{R^{s_i}, P_i, \alpha_i\}_{i \in N}\}$$

satisfying all of assumptions of Gale-Mas-Colell existence theorem [10]. So there exists an $(N+1)-$ tuple $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{p})$ such that conditions

$$\sum_{i \in N} \bar{x_i} = \bar{x_0} \in Y, \tag{7}$$

$$\bar{p} \cdot \bar{x_i} = \alpha_i(\bar{p}) \ \text{ for } \ i \in N, \tag{8}$$

and

$$P(\bar{x_i}) \cap \mathcal{B}_i(\bar{p}) = \emptyset \ \text{ for } \ i \in N \tag{9}$$

are satisfied.

Definition 5. Call an $(N+1)-$ tuple $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{p})$ satisfying conditions (7)-(9) a competitive pseudoequilibrium in $\mathcal{E}$.

So a competitive equilibrium in $\mathcal{E}_0$ is a competitive pseudoequilibrium in $\mathcal{E}$. Next we will show that for every competitive pseudoequilibrium $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{p})$ in $\mathcal{E}$ there exists $\bar{y_i}$ $i \in N$ such that $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{y_1}, \bar{y_2}, \ldots, \bar{y_n}, \bar{p})$ is a competitive equilibrium in $\mathcal{E}$.

In this step we make use of the Weak Monotony assumption. So let $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{p})$ be a competitive equilibrium in $\mathcal{E}_0$. Then $\sum_{i \in N} \bar{x_i} = \bar{x_0} \in Y$.

If $\bar{x_0} \in E_0$, then since $E_0 \subset E$

$$\bar{x_0} = \sum_{i \in N} \bar{x_i} = \sum_{i \in N} \bar{y_i}$$

for some $\bar{y_i} \in E_i$ $(i \in N)$ and such that equations (1) are satisfied. Since $\bar{p} \cdot \bar{x_0} = \max \bar{p} \cdot Y$ it follows that equations (2) are satisfied. Thus $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{y_1}, \bar{y_2}, \ldots, \bar{y_n}, \bar{p})$ is a competitive equilibrium in $\mathcal{E}$. Assume $\bar{x_0} \notin E_0$. It follows from the definition of Y and E_0 that there exists $\hat{x_0} \in E_0$ such that $\hat{x_0} \geq \bar{x_0}$. Set $\hat{x_i} = \bar{x_i} + (\hat{x_0} - \bar{x_0})^{s_i}$ for $i \in N$. Then

$$\sum_{i \in N} \hat{x_i} = \sum_{i \in N} \bar{x_i} + \sum_{i \in N} (\hat{x_0} - \bar{x_0})^{s_i} = \hat{x_0}.$$

So $(\hat{x_1}, \hat{x_2}, \ldots, \hat{x_n})$ is feasible.

We have

$$\bar{p} \cdot \hat{x}_i = \bar{p} \cdot \bar{x}_i = \alpha_i(\bar{p}) \text{ for } i \in N.$$

Otherwise, $\bar{x}_i$ would not be a profit maximizing consumption at price $\bar{p}$.

Since $\hat{x}_i \geqq \bar{x}_i$, by the weak monotony assumption it follows that $P_i(\hat{x}_i) \subset P_i(\bar{x}_i)$. This inclusion together with equation (9) imply that

$$P_i(\hat{x}_i) \cap B_i(\bar{p}) = \emptyset \text{ for } i \in N.$$

So, we have constructed a new competitive equilibrium $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \bar{p})$ in $\mathcal{E}_0$ such that

$$\sum_{i \in N} \hat{x}_i = \hat{x}_0 \in E_0. \tag{10}$$

We have shown above how to construct a competitive equilibrium in $\mathcal{E}$ from one of $\mathcal{E}_0$ with the property (10). Thus we have proven the existence of a competitive equilibrium in economy $\mathcal{E}$.

Let $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{p})$ be a competitive equilibrium in economy $\mathcal{E}$. By the definition of sets E_i $(i \in N)$ there are partitions $P_i = \{A_i^1, A_i^2, \ldots, A_i^n\}$ of sets A_i such that $(\mu_1(A_i^1), \mu_2(A_i^2), \ldots, \mu_n(A_i^n)) = \bar{y}_i$ for each $i \in N$. Set $B_j = \cup_{i \in N} A_i^j$ $(j \in N)$. Clearly $\{B_1, B_2, \ldots, B_n\}$ is a partition of X. Define a measure ν on Σ by setting

$$\nu(D) = \sum_{i \in N} \bar{p}^{s_i} \cdot \mu_i(D \cap B_i) \text{ for } D \in \Sigma.$$

Obviously ν is a measure on Σ absolutely continuous with respect to $\theta = \sum_{i \in N} \sum_{j=1}^{s_i} \mu_i^j$. We will show that the pair $(\{B_1, B_2, \ldots, B_n\}, \nu)$ is a competitive equilibrium in the measurable space exchange economy. Show that B_i is $\succ_i$ $-$maximal in the budget set of individual i, $\mathcal{B}_i(\nu)$ for $i \in N$. Assume on the contrary, for some i there exists $B \in \mathcal{B}_i(\nu)$ such that $B \succ_i B_i$. Thus

$$\nu(B) \leq \nu(A_i) = \nu(B_i) = \alpha_i(\bar{p})$$

and $\mu_i(B) \in P_i(\mu_i(B_i))$. This preference implies that

$$\bar{p}^{s_i} \cdot \mu_i(B) > \bar{p}^{s_i} \cdot \mu_i(B_i) = \nu(B_i) = \alpha_i(\bar{p}) \geq \nu(B) = \sum_{j \in N} \bar{p}^{s_j} \cdot \mu_j(B \cap B_j).$$

Otherwise

$$\bar{p}^{s_i} \cdot \mu_i(B \setminus B_i) > \sum_{j \in N \setminus \{i\}} \bar{p}^{s_j} \cdot \mu_j(B \cap B_j).$$

This inequality would mean that selling the piece $B \setminus B_i$ at price $\bar{p}^{s_i}$ is more profitable for agents possessing this piece. This contradicts to the profit maximization property, that is to the optimality of divisions $\{A_j^1, A_j^2, \ldots, A_j^n\}$ for $i \in N$. ■

Corollary 3. *Under the conditions of Theorem 2 the weak core in the measurable space trading economy is nonempty.*

Proof. By Theorem 2 there exists an equilibrium $(\{B_1, B_2, \ldots, B_n\}, \nu)$. We show that $\{B_1, B_2, \ldots, B_n\}$ belongs to the weak core. Assume on the contrary, there exists a coalition I that improve upon partition $\{B_1, B_2, \ldots, B_n\}$. Thus there exists a partition $\{C_i \mid i \in I\}$ of $A(I) = \cup_{i\in I} A_i$ such that $C_i \succ_i B_i$ for all $i \in I$. Then since $(\{B_1, B_2, \ldots, B_n\}, \nu)$ is an equilibrium we have $\nu(C_i) > \nu(B_i)$ for all $i \in I$. Adding these inequalities we will get $\nu(C(I)) > \nu(A(I))$. This contradicts to $C(I) = A(I)$. ■

It is obvious that every (weak) core division is (weak) Pareto efficient. For the coincidence of the weak core (the weak Pareto set) and the core (the Pareto set) some assumptions are required.

Proposition 4. *If preferences $\succ_i$ are the strict parts of rational continuous preferences $\succcurlyeq_i$, monotone (for $x_i, x_i' \in R_+^{s_i}$, $x_i' \geq x_i$ implies $x_i' \succ_i x_i$,) and if measures $\eta_i = \sum_{j=1}^{s_i} \mu_i^j$ $(i \in N)$ are absoluely continuous with respect to each other, then the weak core (the weak Pareto set) and the core (the Pareto set) coincide.*

Proof. We now show that the weak core and core coincide. Let a coalition I weakly improve upon a division $P = \{B_1, B_2, \ldots, B_n\}$ via division $Q = \{C_i \mid i \in I\}$ of $A(I) = \cup_{i\in I} A_i$. So we have not $B_i \succ_i C_i$ for all $i \in I$ and $C_i \succ_i B_i$ for at least one $i \in I$.) Since preferences are assumed to be the strict parts of rational preferences [not $B_i \succ_i C_i$] is equivalent to [$C_i \succcurlyeq_i B_i$]. Let $C_{i_0} \succ_{i_0} B_{i_0}$ for $i_0 \in I$. Then by the weak monotony assumption we have $\mu_{i_0}(C_{i_0}) \geq 0$. By the mutual absolute continuity assumption $\mu_i(C_{i_0}) \geq 0$ for all $i \in N$.

By continuity of $\succ_{i_0}$ there exists $d > 0$ such that for $D \subset C_{i_0}$, $\mu_{i_0}(D) < d$ and $C_{i_0} \setminus D \succ_{i_0} B_{i_0}$. By nonatomicity of measure μ_{i_0} such subset D exists. By Theorem 1 there exists a partition D_i, $i \in I)$ for D such that $\mu_i(D_i) = \frac{1}{|I|}\mu_i(D)$ for all $i \in I$. Define $F_{i_0} = C_{i_0} \setminus D$, and $F_i = C_i \cup D_i$ for $i \in I \setminus \{i_0\}$. Then $\{F_i \mid i \in I\}$ is a partition of $A(I)$ such that $F_i \succ_i C_i$ for all $i \in I$. Thus I improves upon division P.

Corollary 3 and Proposition 4 imply

Corollary 5. *If in addition to the assumptions of Propsition 4 preferences are convex, then the core in the measurable space trading economy is nonempty.*

As it is noted in the Introduction the main result of Dunz [9] follows from Corollary 5. Notice that in [9] continuity of utility functions is not explicitly assumed. As the following example shows without this assumption the weak core may be empty. Although in [9] this result is formulated as the existence of the core (rather than the weak core), the method of the proof is based on Scarf theorem on nontransferable utility games which asserts only the existence of the weak core.

Example. Consider $X = [0,2] \times [0,2] \subset R^2$ with the Lebesgue measure μ. Let there be two agents with endowments $A_1 = [0,1] \times [0,2]$ and $A_2 = (1,2] \times [0,2]$ and preferensces are defined in the following way. Let both agents have the common characteristics μ_1, μ_2 defined as $\mu_i(B) = \int_B g_i(x) d\mu(x)$ $(i = 1,2)$ for measurable $B \subset X$, where g_i $(i \in N)$ is the characteristic function of A_i $(i \in N)$. Let $u_1, u_2 : R^2_+ \to R$ be defined as

$$u_1(x_1, x_2) = \begin{cases} x_1 & \text{for } 0 \leq x_1, x_2 \leq 1, \\ 2 - x_1 & \text{for } 1 < x_1 \leq 2,\ 0 \leq x_2 \leq 1, \\ 0 & \text{elsewhere.} \end{cases}$$

and $u_2(x_1, x_2) = u_1(x_2, x_1)$.

It is easily verified that functions $u_1,\ u_2$ are quasiconcave. It is easy to calculate $V(1) = \{(U_1, U_2) \mid U_1 \leq 0\}$, $V(2) = \{(U_1, U_2) \mid U_2 \leq 0\}$ and $V(1,2) = \{U_1, U_2) \mid U_1 < 0,\ U_2 < 0\}$. Thus $V(1,2)$ is open and therefore the weak core and hence the core is empty.

4. Existence of fair divisions

Let (X, Σ) be a measurable space. Let as above preferences of agents over measurable subsets in X are defined in the following way. Agent $i \in N$ attributes s_i quantifiable characteristics to these subsets modeled as positive finite measures $\mu_1^1, \mu_i^2, \ldots \mu_i^{s_i}$ on Σ. We denote $\mu_i = (\mu_1^1, \mu_i^2, \ldots \mu_i^{s_i})$. Then preferences of agent i are given through a preference mapping $P_i : R_+^{s_i} \longmapsto R_+^{s_i}$ that is nonempty-valued, irreflexive and has an open graph in $R_+^{s_i} \times R_+^{s_i}$.

Definition 6. A division $P = \{A_1, A_2, \ldots, A_n\}$ of X is said to be *fair* if it is
(a) *Pareto optimal,* that is if there is no other division $Q = \{C_1, C_2, \ldots, C_n)\}$ such that $\mu_i(C_i) \in P_i(\mu_i(A_i))$ for $i \in N$,
and
(b) *envy-free,* that is if $\mu_i(A_j) \notin P_i(\mu_i(A_i))$ [otherwise, not $A_j \succ_i A_i$] for $i, j \in N$.

Definition 7. A division $\{A_1, A_2, \ldots, A_n\}$ is *weak group envy-free* if for every pair of coalitions N_1, N_2 with $|N_1| = |N_2|$ there is no division $\{C_i\}_{i \in N_1}$ of $\cup_{j \in N_2} A_j$ such that $C_i \in P_i(A_i)$ for all $i \in N_1$.

This definition is adapted from Berliant-Thomson-Dunz [5]. Obviously if an allocation is weak group envy-free then it is envy-free and weak Pareto efficient.

Definition 7'. A division $\{A_1, A_2, \ldots, A_n\}$ is *group envy-free* if for every pair of coalitions N_1, N_2 with $|N_1| = |N_2|$ there is no division $\{C_i\}_{i\in N_1}$ of $\cup_{j\in N_2} A_j$ such that $A_i \notin P_i(C_i)$ for all $i \in N_1$ and $C_i \in P_i(A_i)$ at least for one $i \in N_1$.

Of course when preferences P_i are derived from rational preferences $\succcurlyeq_i$ then the last part of Definition 7' will be read as "$C_i \succcurlyeq_i (A_i)$ for all $i \in N_1$ and $C_i \succ_i A_i$ at least for one $i \in N_1$."

As in the proof of Proposition 4 it can be shown that under the assumptions of Proposition 4 every weak group envy-free division is group envy-free, that is two concepts coincide.

Theorem 6. *Under the assumptions of Theorem 2 there exists a group envy-free and Pareto efficient allocation.*

Let $\mu_i^j,\ i \in N$ be nonatomic measures on (X, Σ). If there existed a partition of X into n parts, say $C_j,\ j \in N$, so that restrictions of vector-measure $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ into $C_j,\ j \in N$ have identical ranges, then one could exploit the standard scheme of a proof of the existence of a fair division by assigning each individual j the piece C_j. The author does not know whether such a partition exists. Therefore we are not able to derive the existence of a fair division from the existence of an equilibrium division.

For proving the existence of a fair (or more generally, a group envy-free and efficient) allocation, we will use the method exploited above for establishing the existence of a competitive equilibrium. More explicitly, we will first construct an economy with the aggregate endowment set, and then generate from it an economy of the type as in Gale-Mas-Colell [10], in which individuals are given equal profits. Further, we will use a competitive equilibrium of the latter economy for constructing a division in the measrable space division problem that is group envy-free and Pareto efficient.

Proof of Theorem 6. Define

$$E = \{(\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n)) \mid \{A_1, A_2, \ldots, A_n\} \text{ is a partition of } X\}.$$

By Theorem 1, $E \subset R^s$ is a nonempty compact convex set. Set $Y = E + R^s_-$ and $X_i = R^{s_i}_+$ for $i \in N$ as in Section 3. As before define

$$\alpha(p) = \max\ p \cdot Y.$$

Define individual wealth functions by setting $\alpha_i(p) = \frac{\alpha(p)}{n}$, for $i \in N$. It is easily seen that $\alpha(p) > 0$ and hence

$$\alpha_i(p) > 0 \text{ for all } p \in \Delta.$$

Budget sets are defined as

$$B_i(p) = \{x_i \in X_i \mid p^{s_i} \cdot x_i \le \alpha_i(p)\}.$$

So, we have an economy $\mathcal{E}$ for which a competitive economy is defined in the following way.

Definition 8. An $(n+1)-$tuple $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{p})$, where $\bar{x_i} \in X_i$ and $\bar{p} \in \Delta$ is said to be a *competitive equilibrium* in economy $\mathcal{E}$, if $\bar{x_i}$ is a P_i-maximal element in the budget set $B_i(p)$ for all $i \in N$.

For economy $\mathcal{E}$ all conditions of Gale-Mas-Colell equilibrium theorem [10] are satisfied. So, there exists a competitive equilibrium $(\bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}, \bar{p})$ in economy $\mathcal{E}$. We have

$$\bar{x_0} = \sum_{i \in N} \bar{x_i} \in Y \text{ and } \bar{p} \cdot \bar{x_0} = \max \ \bar{p} \cdot Y.$$

If $\bar{x_0} \in PF(Y)$, then $\bar{x_0} \in E$, and hence there exists a division $\{B_1, B_2, \ldots, B_n\}$ of X such that

$$\mu_i(B_i) = \bar{x_i} \text{ for } i \in N.$$

As in Section 3 define a measure ν on Σ by setting

$$\nu(D) = \sum_{i \in N} \bar{p}^{s_i} \cdot \mu_i(D \cap B_i) \text{ for } D \in \Sigma. \tag{11}$$

We have

$$\nu(B_j) = \bar{p}^{s_j} \cdot \mu_j(B_j) = \alpha_j(p) = \frac{\alpha(\bar{p})}{n} \text{ for all } j \in N.$$

We assert that division $B = \{B_1, B_2, \ldots, B_n\}$ is group envy-free and Pareto efficient. Assume it is not group envy-free. Then there exist $N_1, N_2 \subset N$ such that $|N_1| = |N_2|$ and there is a division $\cup_{i \in N_1} C_i$ of $\cup_{j \in N_2} B_j$ such that $C_i \in P_i(B_i)$ for all $i \in N_1$. It follows then $\nu(C_i) > \nu(B_i)$ for all $i \in N_1$. Summing up these inequalities we will have $\nu(\cup_{i \in N_1} C_i) > \nu(\cup_{i \in N_1} B_i)$. But from (11) we have $\nu(\cup_{i \in N_1} C_i) = \nu(\cup_{j \in N_2} B_j) = \frac{|N_2|}{n} \alpha(\bar{p}) = \nu(\cup_{i \in N_1} B_i)$. Assume now that division B is not Pareto efficient. Then there exists a division $C = \{C_1, C_2, \ldots, C_n\}$ of X such that $C_i \in P_i(B_i)$ for all $i \in N$. Then $\nu(C_i) > \mu_i(B_i)$ for all $i \in N$. Summing these inequalities we will have $\sum_{i \in N} \nu(C_i) > \sum_{i \in N} \nu(B_i)$ that is $\nu(X) > \nu(X)$, a contradiction.

If $\bar{x}_0 \notin PF(Y)$, where $PF(Y)$ is the Pareto frontier of Y, then there exists $\hat{x}_0 \geq \bar{x}_0$ such that $\hat{x}_0 \in PF(Y)$, and hence $\hat{x}_0 \in E$. Using the weak monotony assumption as in Section 3 we reduce the situation to the case of $\hat{x}_0 \in N$. ■

In the case when X is a subset of Euclidean space R^k and preferences $\succ_i$ are given by scalar measures on the Borel $\sigma-$algebra of sets in X absolutely continuous with respect to the Lebesgue measure we obtain Theorem 2 of Berliant-Dunz-Thomson [5]. Notice that in their approach there is a reference measure (the Lebesgue measure) while our approach does not involve any such measure.

Corollary 7. *Under the assumptions of Theorem 2 there exists a fair division of a measurable space* (X, Σ).

When each agent i has a single atribute formalized as a finite positive measure μ_i on Σ and preferences $\succ_i$ are defined simply as strictly greater relation on R, in other words, if preferences are given by a scalar measure on Σ, we obtain Weller's fairness result [15].

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