University of Wisconsin-Madison
Department of Agricultural & Applied Economics

Staff Paper No. 496 March 2006

The Duopolistic Firm with Endogenous Risk Control: Case of Persuasive Advertising and Product Differentiation

By

Shinn-Shyr Wang and Kyle W. Stiegert

AGRICULTURAL & APPLIED ECONOMICS

STAFF PAPER SERIES

The Duopolistic Firm with Endogenous Risk Control

– Case of Persuasive Advertising and Product Differentiation

Shinn-Shyr Wang* Kyle W. Stiegert†

February 2006

Abstract

In this paper, a two-period game is constructed, where duopoly firms choose advertising strategies in the first period and compete in price or quantity in the second period by maximizing the value of firm equity. Using certainty equivalence, we demonstrate the impacts of uncertainty and modes of competition on duopoly firms' optimal pricing, production, and advertising strategies. Equilibrium price and quantity outcomes emerge as significantly different from the standard industrial organization model of profit maximization. It turns out that the common measurement of market power, the Lerner index, is generally mis-stated. In contrast to the literature, we also find that firms will optimally switch from quantity to price competition either when advertising costs are low, demand is high, or if idiosyncratic risk is reduced. A series of simulations confirm these findings.

Keywords: persuasive advertising, product differentiation, quantity (Cournot) competition, price (Bertrand) competition, CAPM.

JEL Codes: D43, G12, L13, L21

*Corresponding author, E-mail: swang17@wisc.edu. Associate Scientist, Food System Research Group, Department of Agricultural and Applied Economics, University of Wisconsin-Madison. Mailing address: 427 Lorch Street, Madison, WI 53706; Phone: (608)263-6383; Fax: (608)262-4376.

†Associate Professor, Director of Food System Research Group, Department of Agricultural and Applied Economics, University of Wisconsin-Madison.

1 Introduction

This paper investigates firm pricing and production decisions building from a model of equity value maximization under imperfect competition. The vast majority of industrial organizational theory is constructed on a rather simplistic premise that a firm's first priority is to maximize profits. In practice, however, firm managers driven by equity-based incentive packages are more apt to focus equally on multiple objectives, including profitability, stability of profits, and creating conditions to foster strong anticipated growth of profits. When viewed through the objective of equity market valuation, different market outcomes, interpretations, and policy-relevant factors begin to emerge.

The basic single-period model of imperfectly competitive markets usually assumes that the demand function is known with certainty. In cases when demand uncertainty is allowed, it is generally assumed to be exogenous.[1] This study considers whether risk can be partially controlled, and if so, the corresponding implications in the equity market for a firm's capital. The justification for such a behavioral assumption rests on the common observations of firm behavior. To elaborate, we note that most firms use various tools to reduce risk exposure, including those available in the product and financial markets. Such tools include: (1) hedging transactions in which firms manage risk exposure in commodities, interest rates, or currency fluctuations, (2) R&D exploratory investment in which firms face technological uncertainty, (3) supply chain management strategies that secure uncertain supplies of raw inputs and manage price and cost risks, and (4) advertising in which firms face demand uncertainty in the product market.

Point 4 is the focus of this paper. In particular, we focus on promotional efforts

[1] For several aspects of decision making with uncertain demand, see, for example, Baron (1971), Sandmo (1971), Leland (1972), and Klemperer and Meyer (1986).

for otherwise identical products by firms facing uncertain demand.[2] We take note of the endogenous control of risk evidenced when firms use persuasive advertising to differentiate their products. Promotion-induced brand identity is modeled to implicitly raise entry barriers and as such to cause more inelastic demand, which in turn promotes more stable market shares and higher margins. As a result, firms may deem a product as contributing to capital value, because said product sales add to profit accumulation and stabilize current and future revenue.

To evaluate cash flows from sales of a retail product, we use a certainty equivalence approach. The risk-adjusted net present value of cash flows, realized at the end of the project, is established by using the single-period Capital Asset Pricing Model (CAPM) due to Sharpe (1964), Lintner (1965), and Mossin (1966). In it, systematic (nondiversifiable) risk measures how rates of return on the security issued by an individual firm relate with that of the overall market portfolio. The underlying market structure and/or the microeconomic determinants of the product market are given scant attention in the context of CAPM. In reality, most firms operate within an environment in which they interact with others continuously, as most market structures are neither purely monopolistic nor perfectly competitive. It is apparent that the strategic interactions among firms may impact financial market variables. In their seminal work, Subrahmanyam and Thomadakis (1980) successfully integrate real and financial views of a firm. We consider the case of two firms and discuss market structure impacts on systematic risk.

Following Dixit (1979), this study introduces persuasive advertising in the duopoly competition. Each firm simultaneously chooses either a quantity to produce or a price to charge, and faces constant marginal costs and no capacity constraints. The

[2]Of course, one might also consider the case that firms invest in R&D or innovation activities to differentiate products intrinsically. Evaluating the trade-off between generating perceived versus intrinsic product differentiations is also a topic worthy of future research.

demand structure is linear and restricted to the case of substitute goods. In addition, product differentiations are endogenous made through advertising. When firms act to reduce risk and maximize profits, significant differences in duopoly price and quantity outcomes are possible.

Several related contributions focus on mode of competition and product differentiation. Singh and Vives (1984) discuss the nature of duopoly competition in Bertrand and Cournot markets. For the case of substitutes, they conclude that, when compared to Bertrand competition Cournot competition always yields higher prices and higher profits, but lower social welfare. Klemperer and Meyer (1986) consider the role that uncertainty plays when firms seek to determine the choice variables in a single-period context. The present study is different from theirs and emphasizes endogenous uncertainty and product differentiation. While Motta (1993) also deals with different modes of competition, his study considers only the case of vertical and endogenous product differentiation.

Several comparative views of persuasive and informative advertising appear in the literature. According to Kaldor (1950) and Bain (1956), persuasive advertising is socially wasteful because it changes tastes and enhances brand loyalty by subjective or perceived product differentiation. Alternatively, Stigler (1961) and Telser (1964) emphasize the informative role of advertising, and hold that advertising primarily affects demand by conveying information, lowering search costs, and increasing competition. Considerable theoretical research has examined the role of informative advertising. For example, see Nelson (1974), Grossman and Shapiro (1984), Milgrom and Roberts (1986), and Bagwell and Ramey (1988).[3] Relatively little theoretical work has investigated the role of purely persuasive advertising. Additionally, this literature generally

[3]Stigler and Becker (1977) proposed the concept of complementary advertising. See Bagwell (2001) for a survey of different views of advertising.

lacks cohesion and fail to provide conclusive evidence about just how advertising may affect consumers' preferences. Becker and Murphy (1993) assume that advertising enters into consumers' utility function and that advertising is complementary to the consumption of the advertised product. More directly, various studies assume that advertising "shifts demand" for the advertised goods, for example, Dixit and Norman (1978, 1979, 1980), Fisher and McGowan (1979), and Shapiro (1980).

This paper explicitly models how advertising fosters product differentiation and then explores the subsequent impacts on product demand and revenue shocks to a firm. One closely related model is offered in von der Fehr and Stevik (1998), where the authors distinguish the effects of persuasive advertising on preferences in three ways: increases willingness to pay, changes ideal product variety, and increases perceived product differences. Our model differs from theirs in at least four aspects. First, this study assumes a firm's main objective to be asset value maximization, while in theirs it is profit maximization. Second, we consider various modes of competition, whereas they consider price competition alone. Third, we additionally investigate the impact of uncertainty on firms' optimal advertising strategies. Finally, we use the linear demand of representative agents as opposed to a linear Hotelling model of differentiated goods.

The remainder of the paper is organized as follows. Section 2 describes a model in which advertising may create subjective product differentiation, compares quantity and price competition, and explores the impacts of advertising on product differentiation and risk reduction, and then presents comparative statics and simulations of the model. Finally, section 3 offers conclusive remarks and suggestions for future research.

Figure 1: The Model Timeline

2 The Model

The model in this study is a two-stage sequential duopoly game. In the first period, each firm differentiates its single product through persuasive advertising. In the second period, each firm produces its product with two factors of production: labor and capital. Both factors are assumed to arrive at the firm from perfectly competitive factor markets. Assume that while each firm faces uncertain demand, a constant marginal cost is known with certainty. Both firms' revenues are subject to a random shock that neither can observe when the strategic variables are chosen.

Figure 1 presents the timeline. Firm i finances the entire project by borrowing from the investment bank, with a risk-free interest rate r.[4] The borrowed cash in the first period is $m_i = K_i + g(A_i)$, where K_i is firm i's capital stock for production, A_i is the advertising level, and $g(A_i)$ is the cost of advertising. Firm i learns $g(A_i)$ before choosing A_i.[5] Note that capital is a numeraire in the model. In the second period, the wealth of firm i is $W_i = \tilde{R}_i - wL_i - (1+r)m_i$, where $\tilde{R}_i$ is the total revenue of firm i, w is the wage rate, and L_i is the input of labor. As a result, the expected

[4] This study does not deal with the conflicts of interest between debt-holders and equity-holders. We assume that the decisions are made by owner-managers or that there exists no agency problem between investors and managers.

[5] The timing of learning $g(A_i)$ is crucial in the model. If firms know the cost structure of advertising after choosing A_i, they must have an expectation over $g(A_i)$. This creates issues of incomplete information.

discounted cash flow for the undertaken project in period 1 is

$$E(\tilde{\pi}_i) = \frac{W_i}{1+r} = \frac{\tilde{R}_i - wL_i}{1+r} - K_i - g(A_i). \tag{1}$$

According to Equation (1), the total cost of production valued in the first period is

$$C_i = \frac{wL_i}{1+r} + K_i.$$

That is, we assume that the firms build up some amount of capital stock[6] in the first period in order to pay for the employment of labor in the second period. In other words, capital is purchased in advance of production, while labor is purchased as production proceeds. Finally, we assume $wL_i = cX_i$, where, for simplicity, c is the constant marginal cost of both firms and X_i is quantity produced.

On the revenue side, firm i's total revenue is given by

$$\tilde{R}_i = p_i X_i(1+\tilde{e}), E(\tilde{e}) = 0, Var(\tilde{e}) = \sigma_e^2, \tag{2}$$

where p_i is the price and the random variable $\tilde{e}$ is an idiosyncratic shock on the revenue of firm i. Without loss of generality, it is assumed to have mean of zero. σ_e is the standard deviation of the shock. It is further assumed for every demand curve that the support of the noise is small enough so that negative revenue never occurs.

Suppose firms face a linear inverse demand function[7] in period 2.

$$p_i = \alpha - (b-\gamma)X_i - \gamma X_j,\ b/2 \geq \gamma \geq 0, i, j = 1, 2, i \neq j. \tag{3}$$

$\gamma \geq 0$ implies directly the case of substitutes and $b/2 \geq \gamma$ implies that the own effect $(b-\gamma)$ is at least as large as the cross effect (γ).

[6] K_i is sufficiently large, so no capacity constraint exists. On the other hand, it cannot be too large for non-negative profits. The fixed capital stock in the first period is assumed. The relevant extensions are discussed in the last section.

[7] See also Dixit (1979), Singh and Vives (1984), and Vives (1999).

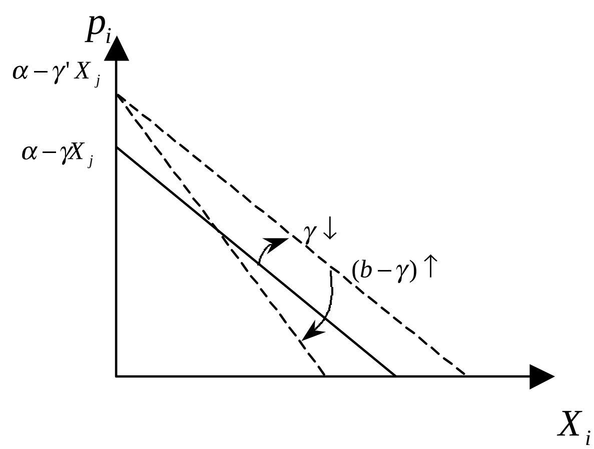

Figure 2: Two Effects of a Decrease in γ on Demand

According to this setting, there exist two effects when γ changes: a shift in demand and a rotation of the demand curve. For example, when products are more differentiated, γ decreases, and the residual demand for firm i is

$$p_i = (\alpha - \gamma X_j) - (b - \gamma)X_i,$$

where both the intercept and the absolute value of slope of demand are increased. These two effects are depicted in Figure 2.

It is worthwhile to note that the aggregate demand in the industry does not change when γ varies. As a result, any activities engaging in changing product differentiation – for instance, advertising in this study – change the substitutability between products, but do not affect the size of the market.

We also define $\delta = \gamma/(b - \gamma)$ to model the degree of (horizontal) product differentiation.[8] The more differentiated the products ($\delta \to 0$), the smaller the effect of change in quantity (price) of brand j on the price (quantity) of brand i. Note that by assumption $0 \leq \delta \leq 1$. Therefore, (3) can be rewritten as

$$p_i = \alpha - \frac{b}{1+\delta}(X_i + \delta X_j),\ 1 \geq \delta \geq 0, i, j = 1, 2, i \neq j. \tag{4}$$

[8]Note that the definition here differs from the common setting seen in, for example, Shy (1995).

Let the degree of differentiation be $\delta = \bar{\delta} - A_i - A_j$, where $\bar{\delta}$ is the initial degree of differentiation, and A_i and A_j are the advertising levels of firm i and j respectively.[9] Because we consider the case where products are ex-ante homogeneous, $\bar{\delta} = 1$.[10] Intuitively, each firm has more market power when its product is more differentiated and as such it is reasonable to expect that both firms will tacitly cooperate in advertising so as to generate higher equity values. This expectation is supported later in the paper. On the cost side, suppose the cost of advertising is the same for both firms and is given by

$$g(A_i) = \frac{\mu A_i^n}{n}, \; i = 1, 2, \tag{5}$$

where $\mu \geq 0$, $n > 1$.[11] Note that a fixed cost of advertising is assumed. That is, this cost is sunk in period 1, and does not vary with the quantity produced in period 2.

Turning now to capital value issues, we assume a financial market characterized by the Sharpe-Lintner equilibrium. That is,

$$E(\tilde{r}_i) = r + \beta_i \left[E(\tilde{r}_m) - r \right], \tag{6}$$

where $E(\tilde{r}_i)$ and $E(\tilde{r}_m)$ are expected rates of return of asset i and market portfolio, respectively, while β_i is systematic risk defined in (9a). Thus, by CAPM, the firm's market value is given by:

$$v_i = \frac{E(\tilde{R}_i) - \lambda Cov(\tilde{R}_i, \tilde{r}_m)}{1+r} - \frac{wL_i}{1+r}, \quad \text{where} \quad \lambda = \frac{E(\tilde{r}_m) - r}{\sigma_m^2},$$

σ_m is the standard deviation of the return of market portfolio, and λ is the market price of risk per unit of variance.[12]

[9] Considering a symmetric equilibrium $A_j = A_i$, $0 \leq A_j = A_i \leq \bar{\delta}/2$.

[10] We can relax this assumption to see the effect of the initial degree of differentiation on the optimal advertising level. Further, each consumer may have a different degree of initial differentiation, which we may characterize by employing a distribution on the initial differentiation. Nevertheless, our qualitative results still hold under these extensive settings.

[11] We use a quadratic cost function in our simulations; that is, $n = 2$.

[12] Dixit and Pindyck (1994) use the New York Stock Exchange Index as the market, $E(\tilde{r}_m) - r \approx$

Because $Cov(\tilde{R}_i, \tilde{r}_m) = Cov(\tilde{e}, \tilde{r}_m)R_i$ and $wL_i = cX_i$,

$$v_i = \frac{R_i(1 - \lambda Cov(\tilde{e}, \tilde{r}_m))}{1+r} - \frac{wL_i}{1+r} = \frac{\phi R_i - wL_i}{1+r} = \frac{X_i(\phi p_i - c)}{1+r} = \frac{\phi X_i(p_i - d)}{1+r}, \quad (7)$$

where certainty equivalent $\phi = 1 - \lambda Cov(\tilde{e}, \tilde{r}_m) = 1 - \lambda \rho \sigma_e \sigma_m$ and ρ is the correlation coefficient between the revenue shock and the return on market portfolio. In general, $\phi \in [0, 1]$[13] and $d = c/\phi$ adjusted marginal cost, provided that $\phi \neq 0$.[14] To keep the model as concise as possible, we focus on the positive ρ and assume $\alpha > d$.

From (7), we see the emergence of a certainty equivalent Lerner index, CEL,

$$CEL_i = \frac{\phi p_i - c}{\phi p_i} = \frac{p_i - c/\phi}{p_i} = \frac{p_i - d}{p_i}. \quad (8)$$

Equation (8) leads to Claim 1:

Claim 1 *Under profit maximization, the common measurement of market power, Lerner index, is generally overstated.*

Claim 1 is straightforward given that $d \geq c$. The equity model carries an added cost associated with the risk profile of the firm. Failure to account for these costs suggests that researchers will either find market power when none is present (Type I error), or will overstate the negative welfare impacts of market power.

By CAPM, systematic risk is defined by

$$\beta_i = \frac{Cov(\tilde{r}_i, \tilde{r}_m)}{Var(\tilde{r}_m)} = \frac{Cov(\tilde{R}_i, \tilde{r}_m)}{v_i Var(\tilde{r}_m)} = \frac{R_i Cov(\tilde{e}, \tilde{r}_m)}{v_i Var(\tilde{r}_m)}. \quad (9a)$$

0.08 and standard deviation $\sigma_m \approx 0.2$, so $\lambda \approx 2$. Note that they measure the market price of risk based on per unit of standard deviation. See also Kaplan and Ruback (1995) and Brealey and Myers (2000).

[13] A parallel literature studies the decisions of firms under uncertainty. The firm is assumed to have a utility function and to maximize expected utility in the sense of von Neumann-Morgenstern, for example, Sandmo (1971) and Leland (1972). This approach is justified in Sandmo (1971). However, it remains difficult to construct a utility function for the firm, see Fama and Miller (1972), pp. 67-68.

[14] We may incorporate cost uncertainty in equation (7) by defining θ as a certainty equivalent parameter on the cost side. For a strictly convex cost function, $\theta > 1$. After redefining $d = c\theta/\phi$, equation (7) still holds true as long as two sources of uncertainty are independent of each other. The new adjusted marginal cost is higher than the original one. However, because we assume the cost function is linear, we need not adjust the cost uncertainty.

We can compute R_i/v_i as follows, using the middle component from (7)

$$\frac{R_i}{v_i} = \frac{(1+r)R_i}{\phi R_i - wL_i} = \frac{(1+r)p_i X_i}{\phi p_i X_i - cX_i} = \frac{(1+r)p_i}{\phi p_i - c} = \frac{1+r}{\phi CEL_i}. \tag{9b}$$

Let $Cov(\tilde{e}, \tilde{r}_m) = \rho\sigma_e\sigma_m$ and $Var(\tilde{r}_m) = \sigma_m^2$. Therefore, we have

$$\beta_i = \rho\frac{\sigma_e}{\sigma_m}\frac{1+r}{\phi CEL_i} = \rho\frac{\sigma_e}{\sigma_m}\frac{1+r}{\phi}\frac{p_i}{p_i - d} = \left\{\frac{1+r}{\frac{\sigma_m}{\rho\sigma_e} - \lambda\sigma_m^2}\right\}\frac{p_i}{p_i - d}. \tag{10a}$$

Or

$$\frac{1}{\beta_i} = \frac{1}{1+r}\frac{\phi\sigma_m}{\rho\sigma_e}CEL_i = \frac{1}{1+r}\frac{\phi\sigma_m}{\rho\sigma_e}\frac{p_i - d}{p_i} = \frac{1}{1+r}\left[\frac{\sigma_m}{\rho\sigma_e} - \lambda\sigma_m^2\right](1 - \frac{d}{p_i}). \tag{10b}$$

As discussed in the introduction, most firms seek to stabilize profits and manage risk. The next section will further explore the role of beta and risk reduction.

Risk Reduction

By CAPM specified in (6), there are two things only we need to know when evaluating an asset's expected return; i.e., the risk premium of the whole market $E(\tilde{r}_m) - r$, and the asset's beta. One important observation is that the portion of the asset return leads to no risk premium and its risk can be diversified as long as the risk is not correlated with the market. As a result, not all of an asset's idiosyncratic risk is taken into account for the expected return, but the part correlated with the market portfolio; that is, $\rho\sigma_e$ illustrated in equation (10a). While the risk-free rate and risk premium of the market portfolio are generally given in the broad market, by (6) beta offers a method to measure an asset's nondiversifiable risk.

Intuitively, it is not difficult to see how changes in ρ, σ_e, or $\rho\sigma_e$ affect β_i. Firms usually differentiate their products so as to maintain stable market shares and revenue streams. In turn, firms bear less risk when the revenue streams are less volatile and/or the revenue shock is less correlated with market portfolio's return generated through product differentiation.

On the other hand, decisions on the production side (quantities to produce or prices to charge) also may alter the nondiversifiable risk. Under the current setting, the systematic risk is inversely related to a firm's market power measure, certainty equivalent Lerner index. To better understand the impacts of production decisions on risk, we highlight the role of production costs, as is summarized in Claim 2.

Claim 2 *To link the financial market with the production market, production costs cannot be ignored.*

In much of the industrial organization literature, production costs are normalized to zero for simplicity and then ignored. However, the linkage between the financial market and the production market cannot be clearly characterized if we neglect the effect of production cost. If $c = 0$, then $d = 0$, implying that β_i is independent of a firm's pricing strategy.

Since risk reduction through ρ and σ_e is straightforward, we will focus mainly on production decisions. That is, we will investigate how firms maximize market values and minimize risks through product differentiation, by holding ρ and σ_e constant in the basic model. In the case with additional benefits from reducing ρ and σ_e, the incentives to differentiate products are enhanced.

Different Modes of Competition in Period Two

This section analyzes the equity valuation model under price and quantity competition. As usual, the two-stage game is solved via backward induction.

Under quantity competition, both firms choose quantity strategies simultaneously, taking each other's strategy as a given. Substituting equation (4) into the value of firm, we get

$$v_i = \frac{\phi X_i(p_i - d)}{1+r} = \frac{\phi}{1+r} X_i \left[\alpha - \frac{b}{1+\delta}(X_i + \delta X_j) - d\right]. \tag{11}$$

Taking a derivative with respect to X_i and rearranging yield

$$(\alpha - d)\frac{1+\delta}{b} = 2X_i + \delta X_j. \tag{12}$$

Equation (12) implicitly defines firm i's reaction function. Solving for optimal quantity and price yields

$$X_i^c = \frac{(\alpha - d)(1+\delta)}{b(2+\delta)}, \quad p_i^c = \frac{\alpha + (1+\delta)d}{2+\delta}. \tag{13}$$

Therefore, the value and inverse of beta of the firm in Cournot equilibrium become

$$v_i^c = \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{1+\delta}{(2+\delta)^2}, \tag{14}$$

$$\frac{1}{\beta_i^c} = \frac{\phi}{1+r} \frac{\sigma_m}{\rho \sigma_e} \left[1 - \frac{(2+\delta)d}{\alpha + (1+\delta)d} \right]. \tag{15}$$

Turning now to competition in price space, and following the procedures used for quantity competition, the equilibrium conditions are:

$$p_i^B = \frac{\alpha(1-\delta) + d}{2-\delta}, \quad X_i^B = \frac{\alpha - d}{b(2-\delta)}, \tag{16}$$

$$v_i^B = \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{1-\delta}{(2-\delta)^2}, \tag{17}$$

$$\frac{1}{\beta_i^B} = \frac{\phi}{1+r} \frac{\sigma_m}{\rho \sigma_e} \left[1 - \frac{(2-\delta)d}{\alpha(1-\delta) + d} \right]. \tag{18}$$

A simple and key finding emerges from equations (13), (14), (16) and (17) and is summarized in the following proposition (see the proof in the appendix).

Proposition 1 *Firms facing uncertain demand behave less competitively than those facing no uncertainty, ceteris paribus. Value-maximizing firms produce less and charge higher prices, but earn lower cash flows.*

Value-maximizing firms behave less competitively no matter which type of competition (Cournot or Bertrand) is adopted because those firms are concerned with

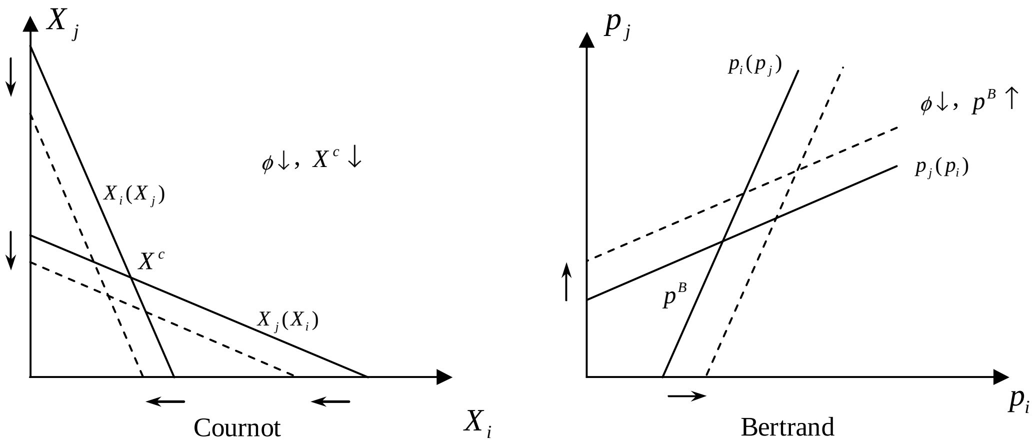

Figure 3: Effects of a Decrease in Certainty Equivalent

profits as well as risks. After accounting for the risk, *ceteris paribus*, the marginal revenue resulting from producing more for value maximization is less than that for profit maximization. Figure 3 demonstrates how changes in certainty equivalent affect equilibrium quantity and price, for Cournot and Bertrand competitions alike, provided that the degree of product differentiation is exogenous. The results of proposition 1 are supported.

Different Degrees of Product Differentiation

There exist three essential but standard results derived from the range of advertising costs. First, when advertising costs are prohibitive ($\mu \to \infty$ in (5), $\delta = 1$), the competition will drive prices down until said prices equal either the marginal costs under the price competition senario[15] or typical Cournot results under the quantity competition senario.

Second, when firms are able to differentiate their products without incurring any

[15]Under price competition, from (17) and (18) each firm has zero value and incurs infinite systematic risk. This result is not likely the case. Economists usually ignore the financial variables as well as the interplay between financial and product markets. This example suggests that the standard Bertrand duopoly model with homogeneous products is not appropriate if we incorporate concerns about financial variables.

costs, i.e., $\mu = 0$, they will differentiate goods such that their products are independent of each other ($\delta = 0$), which obviously generates the monopoly outcome for price and quantity competition. The results of price competition and quantity competition coincide and represent the joint value maximization. Each firm performs just like a monopolist with its product.[16]

Finally, for intermediate cases where $0 < \mu < 1$, (13), (14), (16), and (17) imply that both Bertrand and Cournot firms prefer to produce more differentiated goods. The more differentiated the products, the less δ, and consequently the more v^c and v^B.

Therefore, comparing Cournot competition with Bertrand competition, the following proposition emerges (see the proof in the appendix).

Proposition 2 *Suppose firms face the same demand, and that the degree of product differentiation is exogenous,*
(a) We arrive at the standard industrial organization results. That is, Cournot firms produce less quantity but charge higher prices than Bertrand firms, and therefore obtain higher profits. The differences between Bertrand and Cournot in terms of prices, quantities, and profits are widest when products are homogeneous ($\delta = 1$) and smallest when products are totally differentiated ($\delta = 0$).
(b) When firms maximize capital value, the differences in prices, quantities, and profits are less divergent than the standard industrial organization results of duopolistic competition.

[16]It is also interesting to compare our model with the spatial competition of a "linear city." In terms of spatial competition (e.g., Hotelling model), the equilibrium has the two firms locating at the two extremes of the city (maximal differentiation). Since each firm incurs no costs when choosing the location, each firm locates as far as possible from its rival in order to avoid triggering a low price, and thus price competition is not so harsh. Assuming representative consumers in the markets allows us to analyze the price competition as well as the quantity competition, and then to study their differences without concern for market coverage.

The Cournot firms produce less, charge higher prices, and enjoy more firm value than the Bertrand firms. While similar to findings by Singh and Vives (1984), these results may not hold if firms are able to choose degrees of differentiation for their products. That is, as shown below, firms may choose different levels of differentiation under different modes of competition, and therefore may charge higher prices and garner more revenue under the price competition in some ranges of parameters.

The Advertising Game - Optimal Degree of Differentiation

Up to this point, we have evaluated only the way uncertainty affects quantity and price setting outcomes in a capital value model. Now we move backwards to period 1, in which optimal advertising and thus the degree of differentiation for each mode of competition are solved.

In period 1, firm i chooses A_i, given its rival j's strategy A_j. The objective function is to maximize

$$NPV_i = v_i - K_i - g(A_i). \tag{19}$$

Substituting (14) and (5) into (19) yields the net present discounted value of the undertaken project under quantity competition:

$$NPV_i^c = \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{2 - A_i - A_j}{(3 - A_i - A_j)^2} - K_i - \mu \frac{A_i^n}{n}, \quad i, j = 1, 2, i \neq j.$$

The first order condition for firm i is

$$FOC_i^c = \frac{\partial NPV_i^c}{\partial A_i} = \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{1 - A_i - A_j}{(3 - A_i - A_j)^3} - \mu A_i^{n-1} = 0, \quad i, j = 1, 2, i \neq j. \tag{20}$$

From equation (17), the net present discounted value of firm i under price competition is

$$NPV_i^B = \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{A_i + A_j}{(1 + A_i + A_j)^2} - K_i - \mu \frac{A_i^n}{n}, \quad i, j = 1, 2, i \neq j.$$

The first order condition is

$$FOC_i^B = \frac{\partial NPV_i^B}{\partial A_i} = \frac{\phi(\alpha-d)^2}{b(1+r)}\frac{1-A_i-A_j}{(1+A_i+A_j)^3} - \mu A_i^{n-1} = 0, \quad i,j=1,2, i\neq j. \quad (21)$$

In a symmetric equilibrium, $A_j = A_i$. Solving (20) for A_i and A_j, we obtain optimal advertising, associated price, quantity, value of firm, and systematic risk under quantity competition. Similarly, we can solve (21) for optimal advertising and other variables under price competition. Basically, the solution for A_i involves a quartic equation in each first order condition if a quadratic advertising cost function ($n = 2$) is assumed. It is not possible to obtain a closed-form solution from equations (20) or (21). Therefore, we will proceed toward a simple simulation approach. Before doing so, first we examine the comparative statics of changes in the exogenous parameters on endogenous variables in the system.

Comparative Statics and Simulations

This section explores the effects of changes in the exogenous parameters on equilibrium prices, quantities, and advertising levels as well as on firm's value and systematic risk. These effects are most easily derived by differentiating equations (20) and (21) entirely. We may simplify our computations by rearranging (20) and (21) respectively as:

$$\frac{\phi(\alpha-d)^2}{\mu b(1+r)} = \frac{A_i(3-A_i-A_j)^3}{1-A_i-A_j}, \quad (22)$$

$$\frac{\phi(\alpha-d)^2}{\mu b(1+r)} = \frac{A_i(1+A_i+A_j)^3}{1-A_i-A_j}. \quad (23)$$

Note that $n = 2$ and $d = c/\phi$. Because $\lambda = \frac{E(\tilde{r}_m)-r}{\sigma_m^2}$,

$$\phi = 1 - \lambda\rho\sigma_e\sigma_m = 1 - [E(\tilde{r}_m) - r]\frac{\rho\sigma_e}{\sigma_m}. \quad (24)$$

We want to see how the change in $\alpha, b, \rho\sigma_e, \sigma_m$, $E(\tilde{r}_m) - r$, c, or μ impacts A_i. Because in the symmetric equilibrium $A_j = A_i$, $\frac{\partial A_j}{\partial z} = \frac{\partial A_i}{\partial z}$, where $z = \alpha$, b, $\rho\sigma_e$, σ_m,

$E(\tilde{r}_m) - r$, c, and μ, the following lemma is necessary to obtain a monotonic impact on the optimal advertising level (see the proof in the appendix).

Lemma 1 The right sides of equations (22) and (23) are monotonically increasing if $A_i \in [0, 0.5)$.

The unique role of advertising is worth considering here. In our model, advertising serves as a vehicle to shift and rotate the demand curve. Because of this specific setting, the income effect never dominates the substitution effect even when the slope of the demand curve is sufficiently high. As such, advertising performs like a *normal* good. The above lemma indicates that, subject to the costs, as a rule, advertising is always beneficial to the value of a firm.

Impacts on $\boldsymbol{A_i}$

With lemma 1, we can examine the impacts of z on A_i (see the proof in the appendix).

Proposition 3 *Assuming that $\alpha > d$, an increase in α or σ_m will increase the equilibrium A_i, while an increase in b, $\rho\sigma_e$, $E(\tilde{r}_m) - r$, c, or μ will decrease the equilibrium A_i.*

The results of proposition 3 are presented in column (1) of Table 1. Proposition 3 shows that the equilibrium advertising level increases as its marginal benefit raises, whereas the converse is true if the marginal benefit of advertising declines.

Taking α as an example, an increase in consumers' willingness to pay α leads to an increase in the equilibrium advertising level. This is because a shift out in demand tends to raise the yield on a given firm's advertising, as marginal benefits of advertising are more likely to increase. In addition, because $\delta = 1 - A_i - A_j$, δ decreases as A_i or A_j increases. In fact, $\frac{\partial \delta}{\partial z} = -\frac{2\partial A_i}{\partial z}$ in the symmetric equilibrium, where z is any exogenous variable. With these two facts in hand, we are able to examine the changes of the variables noted in equations (13)-(15) and (16)-(18).

Table 1 Comparative Statics

		(1) A	(2) p	(3) X	(4) v	(5) β
α	Bertrand	$+$	$+$	?	$+$	$-$
	Cournot	$+$	$+$	?	$+$	$-$
b	Bertrand	$-$	$-$	?	$-$	$+$
	Cournot	$-$	$-$	?	$-$	$+$
$\rho\sigma_e$	Bertrand	$-$	?($-$)	?($+$)	$-$	$+$
	Cournot	$-$	?($-$)	?($+$)	$-$	$+$
σ_m	Bertrand	$+$	?($+$)	?($-$)	$+$	$-$
	Cournot	$+$	?($+$)	?($-$)	$+$	$-$
$E(\tilde{r}_m) - r$	Bertrand	$-$	?($-$)	?($+$)	$-$	$+$
	Cournot	$-$	?($-$)	?($+$)	$-$	$+$
μ	Bertrand	$-$	$-$	?	$-$	$+$
	Cournot	$-$	$-$	?	$-$	$+$
c	Bertrand	$-$	?	?	$-$	$+$
	Cournot	$-$	?	?	$-$	$+$

* "?($-$)" and "?($+$)" denote that the impact is generally undetermined, but more likely to be negative or positive, respectively.

Changes in $\boldsymbol{\alpha}$, $\boldsymbol{b}$, and $\boldsymbol{\mu}$

Next, the effects of changes in α under Cournot and Bertrand competition are explored. We summarize the results in the following proposition (see the proof in the appendix).

Proposition 4 *If consumers' willingness to pay (α) increases, under either Cournot or Bertrand competition:*

(a) the equilibrium advertising level, price, and firm's value increase, but the systematic risk decreases, and

(b) the impact on the equilibrium quantity ($\frac{\partial X_i^j}{\partial \alpha}, j = c, B$) depends on the impact on

the equilibrium advertising level ($\frac{\partial A_i}{\partial \alpha}$). These conditions are provided in equations (25) and (26).

$$\frac{\partial X_i^c}{\partial \alpha} > 0, \quad \text{if} \quad \frac{\partial A_i}{\partial \alpha} < \frac{(1+\delta)(2+\delta)}{2(\alpha - d)}. \tag{25}$$

$$\frac{\partial X_i^B}{\partial \alpha} > 0, \quad \text{if} \quad \frac{\partial A_i}{\partial \alpha} < \frac{2-\delta}{2(\alpha - d)}. \tag{26}$$

An increase in consumers' willingness to pay generates additional firm profits, and also reduces systematic risk. Firms engage in more advertising and produce more differentiated goods. However, the influence on quantity is ambiguous because it depends on changes in levels of advertising. Changes in advertising create two effects: shifts in demand and rotation of the demand curve. For an outward shift, the demand curve will generate an increase in price and quantity. Meanwhile, a less elastic demand curve will increase price but decrease quantity. These two effects offset each other for equilibrium quantity. Obviously, when the rotation effect is small, the price and quantity will increase.

By applying similar logic, we employ the following proposition.

Proposition 5 *If the slope of demand curve (b) or the cost parameter of advertising (μ) increases, under either Cournot or Bertrand competition:*
(a) the equilibrium advertising level, price, and firm's value decrease, but the systematic risk increases, and
(b) the impacts on the equilibrium quantity ($\frac{\partial X_i^j}{\partial z}$, $z = b, \mu$; $j = c, B$) depend on the impacts on the equilibrium advertising level ($\frac{\partial A_i}{\partial z}$). These conditions are given in equations (27) and (28).

From equations (22) and (23), it is easy to see that b and μ play the same role when we compute the impacts of their changes. Similar to proposition 4, the conditions in proposition 5 (b) are

$$\frac{\partial X_i^c}{\partial z} < 0, \quad \text{if} \quad \frac{\partial A_i}{\partial z} > -\frac{(1+\delta)(2+\delta)}{2z}, \tag{27}$$

$$\frac{\partial X_i^B}{\partial z} < 0, \quad \text{if} \quad \frac{\partial A_i}{\partial z} > -\frac{2-\delta}{2z}, \tag{28}$$

where $z = b, \mu$.

If we define the slope elasticity of quantity η_{xb} and the slope elasticity of advertising η_{Ab}, respectively, as

$$\eta_{xb} = \frac{b}{X}\frac{\partial X}{\partial b}, \quad \eta_{Ab} = \frac{b}{A}\frac{\partial A}{\partial b},$$

and $\eta_{x\mu}$ and $\eta_{A\mu}$ are defined in a similar way, then together with $\delta = 1 - 2A_i$ in equilibrium, conditions (27) and (28) have alternative expressions as follows:

$$\eta_{xz}^c < 0, \quad \text{if} \quad \eta_{Az} > -\frac{(2-2A_i)(3-2A_i)}{2A_i}, \tag{29}$$

$$\eta_{xz}^B < 0, \quad \text{if} \quad \eta_{Az} > -\frac{1+2A_i}{2A_i}. \tag{30}$$

where $z = b, \mu$. Of course, readers may find similar formulas for (25) and (26).

Changes in $\rho\sigma_e$, $E(\tilde{r}_m) - r$, and σ_m

From equation (24), we know $\rho\sigma_e$, $E(\tilde{r}_m) - r$, and σ_m serve similar roles, and this changes firms' certainty equivalence, ϕ, as we derive the comparative statics.[17] Therefore, we define the risk elasticity of quantity $\eta_{x\phi}$, the risk elasticity of price $\eta_{p\phi}$, and the risk elasticity of product differentiation $\eta_{\delta\phi}$ as

$$\eta_{x\phi} = \frac{\phi}{X}\frac{\partial X}{\partial \phi}, \quad \eta_{p\phi} = \frac{\phi}{p}\frac{\partial p}{\partial \phi}, \quad \eta_{\delta\phi} = \frac{\phi}{\delta}\frac{\partial \delta}{\partial \phi}.$$

From $\frac{\partial \delta}{\partial z} = -\frac{2\partial A_i}{\partial z}$, equation (24), and proposition 3, $\eta_{\delta\phi} < 0$. Thus, we have lemma 2 (see the proof in the appendix).

[17] The only difference is from the computation of the effect on β. However, the conclusion is the same.

Lemma 2 Under either Cournot or Bertrand competition,

(a) $\eta_{x\phi}$ is bounded above, while $\eta_{p\phi}$ is bounded below.

$\eta_{x\phi} < 0$ and $\eta_{p\phi} > 0$ if $\eta_{\delta\phi} \ll 0$.

(b) $\frac{\partial v_i}{\partial \phi} > 0$.

Lemma 2 indicates that effects on X_i and p_i from a change in the risk attitude – or, equivalently, from certainty equivalence – cannot be extremely positive or negative, respectively. It is quite intuitive that $\eta_{x\phi}$ ($\eta_{p\phi}$) is bounded above (below). This implies that the less averse the risk attitude, the less the quantity produced and the higher the price charged. If an increase in ϕ can induce a large increase in optimal advertising, i.e., $\eta_{\delta\phi}$ or $\frac{\partial \delta}{\partial \phi}$ is significantly negative, then $\eta_{x\phi}$ is negative while $\eta_{p\phi}$ is positive. This assumption will dramatically simplify our analysis. The facts that $\eta_{x\phi}$ is bounded above and that $\eta_{p\phi}$ is bounded below are also supported by the result that $\frac{\partial v_i}{\partial \phi} > 0$.

We summarize the impacts of $\rho\sigma_e$ and $E(\tilde{r}_m) - r$ as well as σ_m in the following proposition (see the proof in the appendix).

Proposition 6 *Under either Cournot or Bertrand competition:*

(a) an increase in $\rho\sigma_e$ (i.e. the correlation coefficient between the project and the market portfolio, multiplied by the volatility of the project), or an increase in the excess rate of return on the market portfolio ($E(\tilde{r}_m) - r$), decreases the equilibrium advertising level as well as firm's value, but increases the systematic risk. Meanwhile, changes in prices (quantities) are generally undetermined but more likely to be negative (positive) with significantly positive $\frac{\partial \delta}{\partial(\rho\sigma_e)}$ or $\frac{\partial \delta}{\partial(E(\tilde{r}_m)-r)}$.

(b) an increase in the volatility of market portfolio (σ_m) increases the equilibrium advertising level and firm's value, but decreases the systematic risk, while changes in prices (quantities) are generally undetermined but more likely to be positive (negative),

with significantly negative $\frac{\partial \delta}{\partial \sigma_m}$.

The above proposition provides the link from the risk attitude ϕ to other related variables. When the certainty equivalence rises as the project risk decreases, the excess rate of return on the market portfolio decreases, or the volatility of market portfolio increases, while the relative benefits of the project are increasing but the systematic risk is decreasing.

Finally, let us deal with the case of changes in the marginal cost of production c.

Changes in c

The effects of a change in the marginal cost of production c can be presented in the following proposition.

Proposition 7 *Under either Cournot or Bertrand competition, when the marginal cost of production c increases:*

(a) the equilibrium advertising level and firm's value decrease but the systematic risk increases, and

(b) the impacts on the equilibrium quantity and price $(\frac{\partial X_i^j}{\partial c}, \frac{\partial p_i^j}{\partial c}, j = c, B)$ *depend on the impact on the equilibrium advertising level* $(\frac{\partial A_i}{\partial c})$*. The conditions are shown in equations (31) and (32).*

Proof. (a) We omit the proof of part (a), as it is similar to the above example.

(b) Similar to proposition 4, we report only the new conditions here.

$$\frac{\partial X_i^c}{\partial c} < 0, \frac{\partial p_i^c}{\partial c} > 0, \quad \text{if} \quad \frac{\partial A_i}{\partial c} > -\frac{(1+\delta)(2+\delta)}{2\phi(\alpha - d)}. \tag{31}$$

$$\frac{\partial X_i^B}{\partial c} < 0, \frac{\partial p_i^B}{\partial c} > 0, \quad \text{if} \quad \frac{\partial A_i}{\partial c} > -\frac{2-\delta}{2\phi(\alpha - d)}. \tag{32}$$

Table 1 presents the results of comparative statics. The analysis shows that exogenous variables have clear relations with the optimal advertising level, value of firm, and systematic risk, whereas impacts on the price and quantity are varied and some effects are undetermined. However, the above discussions permits us to narrow down the results; for example, with significantly negative $\frac{\partial \delta}{\partial \phi}$, the effects on the price (quantity) are negative (positive) for $\rho\sigma_e$ and $E(\tilde{r}_m) - r$, but positive (negative) for σ_m.

In the next section we investigate some simulations to determine what the symmetric pure-strategy equilibrium looks like, as the analytical results are not available.

Simulations

As discussed in the comparative statics, b and μ serve a similar role, as do σ_m, $\rho\sigma_e$, and $E(\tilde{r}_m) - r$. We conduct the simulations in which changes in α, μ, ρ, and σ_e are examined.

In our simulations, the parameters of the base case include $b = 1$, $r = 0.05$, $c = 0.1$, $\sigma_m = 0.2$, $\lambda = 2$, $\sigma_e = 0.2$, $\rho = 0.5$, $\alpha = 1$, $\mu = 0.1$, and $n = 2$. Therefore, $\phi = 1 - \lambda\rho\sigma_e\sigma_m = 1 - 2 * 0.5 * 0.2 * 0.2 = 0.96$. Moreover,

$$\frac{\phi(\alpha - d)^2}{b(1+r)} = \frac{0.96(1 - 0.1/0.96)^2}{1 + 0.05} = 0.7337,$$

which is less than the case of certain demand (i.e. $(1 - 0.1)^2/(1 + 0.05) = 0.7714$).

The simulation results are presented in Table 2, where (a) μ is from 0 to 0.2, (b) α is from 0.5 to 1.5, (c) ρ is from 0 to 1, (d) σ_e is from 0 to 0.4, and (e) c is from 0 to 0.2. The change in μ captures the effect of changes in advertising costs, the change in α captures the effect of shift in demand, the change in ρ or σ_e captures the impact of changes in the revenue shock, and finally, the change in c captures the effect of changes in the production cost. The results confirm the previous comparative statics, and offer some general observations in the following claim.

Claim 3 From a perspective that considers the comparisons between Bertrand competition and Cournot competition, the simulation results also indicate the following observations:

(1) Bertrand firms engage in more advertising activities.

(2) In general, Cournot firms enjoy more net present value. However, this relation is reversed when the marginal cost of advertising is small, consumers' willingness to pay is large, production cost is small, and/or risk is small. For example, the case occurs when $\mu \leq 0.08$ in Table 2(a), $\alpha \geq 1.1$ in Table 2(b), and $c \leq 0.03$ in Table 2(e).

(3) An increase in α or a decrease in $\rho\sigma_e$, c, or μ tend to benefit Bertrand firms more than Cournot firms. This can be shown by taking the difference between any two corresponding pairs of entries in Table 2.

The Extensive Models

In this section, we extend the model to the extent that advertising can also reduce the risk parameter of revenue shock σ_e and enhance consumers' willingness to pay α. Specifically, advertising reduces the risk of revenue shock inversely to that illustrated in a mean-preserving spread while it shifts outwards the demand curve by enhancing the willingness to pay. First, we explore the case of the risk reduction.

Reduction of $\boldsymbol{\sigma_e}$

With the setting that advertising reduces revenue shock parameter σ_e or changes the correlation coefficient between the undertaken project and the market portfolio ρ, we may interpret advertising activities in a more general way. That is, in addition to changes in product differentiation, A_i can be advertising or any investment in R&D or innovation activities that can either predict the revenue shock more accurately or insulate the undertaken project from market shocks. Both scenarios act as hedging activities. Therefore, this implies that A_i may reduce σ_e defined in equation (2) and/or the correlation coefficient between the undertaken project and the market portfolio ρ.

The following discussion addresses the effect on the reduction of σ_e only by taking ρ as given. We assume the effect is characterized by equation (33) to facilitate the illustration.

$$\sigma_e = \overline{\sigma}_e \exp(\delta - 1) = \overline{\sigma}_e \exp\left[-(A_i + A_j)\right]. \tag{33}$$

Under this setting, the advertising is assumed to reduce the impact of revenue shock inversely to that seen in a mean-preserving spread. That is, the advertising decreases the volatility of the project and the expected value of the project is unchanged. Note that by defining the reduction of σ_e as $\Delta\sigma_e = \sigma_e - \overline{\sigma}_e$, we have $\partial\Delta\sigma_e/\partial\delta > 0$ and $\partial^2\Delta\sigma_e/\partial\delta^2 < 0$. This captures the idea of positive but diminishing marginal benefits of σ_e reduction.

As discussed earlier in this paper, we expect that the incentives to differentiate products will be enhanced when additional benefits of advertising result from reducing ρ and σ_e. This expectation can be examined as follows. The first order conditions for firm i under the new setting are

$$\begin{aligned} FOC_i^{cr} &= \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{1 - A_i - A_j}{(3 - A_i - A_j)^3} - \mu A_i^{n-1} \qquad (34) \\ &\quad + \left\{ \frac{1}{b(1+r)} \frac{2 - A_i - A_j}{(3 - A_i - A_j)^2} (\alpha^2 - d^2)\lambda\rho\overline{\sigma}_e\sigma_m \exp\left[-(A_i + A_j)\right] \right\} = 0, \\ FOC_i^{Br} &= \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{1 - A_i - A_j}{(1 + A_i + A_j)^3} - \mu A_i^{n-1} \qquad (35) \\ &\quad + \left\{ \frac{1}{b(1+r)} \frac{A_i + A_j}{(1 + A_i + A_j)^2} (\alpha^2 - d^2)\lambda\rho\overline{\sigma}_e\sigma_m \exp\left[-(A_i + A_j)\right] \right\} = 0, \end{aligned}$$

where $i, j = 1, 2, i \neq j$.

The positive additional terms within the brackets of (34) and (35) indicate the marginal benefits from σ_e reduction through advertising. Suppose that A^c, A^B, A^{cr} and $A^{Br}$are solutions to equations (20), (21), (34) and (35), respectively. It is easy to examine that $A^c < A^{cr}$ and $A^B < A^{Br}$.

Increase of Willingness to Pay $\boldsymbol{\alpha}$

In the basic model, equation (3) shows that changes in substitutability between products caused by advertising do not change the aggregate demand in the industry. We analyze the model for incorporating the change in willingness to pay by shifting the demand curve outwards in addition to the change in perceived product differentiation. As a result, advertising has impacts on both horizontal and vertical differentiations. This implies that advertising may generate more benefits by expanding the market size or boosting consumers' willingness to pay, which is characterized by the intercept of demand function α. Suppose that

$$\alpha_i = \overline{\alpha}_i + \psi_i(A_i) + \xi\psi_j(A_j),$$

where $\overline{\alpha}_i$ is ex ante willingness to pay. The ex post willingness to pay is affected by both firms' advertising levels. We assume $\psi_i'(A_i) > 0$, $\psi_i''(A_i) < 0$, $\psi_j'(A_j) > 0$, and $\psi_j''(A_j) < 0$. However, the marginal effect of firm j's advertising (characterized by $\xi\psi_j'(A_j)$) is ambiguous because it may combine somewhat like the predatory and spillover effects of advertising introduced by Roberts and Samuelson (1988). It turns out that we adopt a particular specification for the purpose of tractability in the numerical simulations.[18]

$$\alpha_i = \overline{\alpha}_i + \sqrt{A_i} - \delta\sqrt{A_j} \tag{36}$$

Under this setting, the cross-effect of advertising is negative as we try to model that the advertising of firm j may offset the impact of firm i's advertising. Therefore, in addition to cooperating in the horizontal differentiation, firms' advertising strategies compete with each other in the vertical differentiation. The advertisement may take a form, such as "*our product is very different from theirs, their product is bad, and the quality of ours is way better than theirs.*" Furthermore, the ratio of

[18] See Gasmi et al. (1992) for this specification and for more discussions about predatory and spillover effects.

cross-effect of advertising to the own-effect is assumed to be the degree of horizontal product differentiation. This assumption indicates that the cross-effect in (36) is the same as the effect on the horizontal differentiation. The effects of firms' advertising strategies may offset each other, but the negative impact from the rival diminishes as the products become more differentiated.

As in equations (34) and (35), Cournot and Bertrand firms's additional marginal benefits from increasing consumers' willingness to pay are

$$\frac{2\phi}{b(1+r)}\frac{2-A_i-A_j}{(3-A_i-A_j)^2}(\alpha-d)\left[1/(2\sqrt{A_i})+\sqrt{A_j}\right] \text{ and}$$
$$\frac{2\phi}{b(1+r)}\frac{A_i+A_j}{(1+A_i+A_j)^2}(\alpha-d)\left[1/(2\sqrt{A_i})+\sqrt{A_j}\right], \text{ respectively.}$$

This leads to conclusions similar to those from (34) and (35). The higher advertising levels are induced under either Bertrand or Cournot competition because advertising generates additional benefits in the extensive models, i.e., advertising reduces idiosyncratic risk and increases consumers' willingness to pay. As a result, firms enjoy increased market values, but incur less systematic risks. While not reported here, the simulation results of the extensive models confirm these assessments.

Based on this discussion, it is straightforward to extend the model to combine the effects of an increase in α with those of a decrease in σ_e. This tends to benefit firms more by increasing their values and decreasing their systematic risks.

3 Conclusions and Extensions

In this study, we developed a model of equity value maximization that allows persuasive advertising to influence the risk facing firms and thereby affecting the competition between duopolists. This study makes six major findings. First, we showed that the traditional Lerner index is generally overstated when systematic financial market risk is ignored. This result may have policy implications for antitrust authorities. The main question raised here is whether conventional antitrust analysis under profit maximization overstates welfare losses due to market power and/or results in Type I error regarding the condition of the industry. The empirical implications of this finding are the focus of a companion study of the U.S. margarine and butter markets (Wang, Stiegert, and Dhar, 2005). Second, the model reveals that the common practice of ignoring production cost acts to disengage key linkages between financial and product markets. Third, when firms can reduce their risk by differentiating products, the marginal benefits of advertising will rise. Consequently, firms advertise more, enjoy an increased cash flow, and incur fewer systematic risks. These results apply under Bertrand or Cournot competition alike, but are generally ignored when profits are assumed to be the sole objective of the firm.

Fourth, the conventional results from pure profit maximization suggest that Bertrand firms always engage in a higher degree of competition than Cournot firms, and as such that they earn fewer profits and incur higher systematic risks. Our model suggests that this scenario may not hold true when advertising costs are low, demand is high, and/or when idiosyncratic risk is reduced. These results may have empirical implications and at minimum suggest the need to test the mode of competition in industries of branded or highly differentiated products.

Our model presented a sequence of lemmas and propositions in section 2. We first proved that both Bertrand and Cournot firms prefer to produce more differentiated goods in Proposition 1. Proposition 2 showed that when the degree of product differentiation is exogenous, firms prefer Cournot competition to Bertrand competition, and the differences of prices, quantities, and firm values are less divergent when firms maximize capital value.

Because advertising is assumed to be a vehicle to shift and rotate the demand curve, Lemma 1 showed that the income effect of advertising never dominates the substitution effect. That is, advertising performs like a normal good, and as such subject to the associated costs, advertising is beneficial to the value of firm. With lemma 1, the comparative statics are easy to manage. Propositions 3 to 7 explored how changes in exogenous variables can impact equilibrium advertising, price, quantity, systematic risk, and firm value. The exogenous variables include consumers' willingness to pay (α), the slope of the demand curve (b), the cost parameter of advertising (μ), the correlation between the undertaken project and the market portfolio multiplied by the volatility of the project ($\rho\sigma_e$), the excess rate of return on the market portfolio ($E(\tilde{r}_m)-r$), the volatility of the market portfolio (σ_m), and the marginal cost of production (c). Table 1 summarized the results of comparative statics.

Fifth, the numerical simulations confirmed the finding that Bertrand firms engage in more advertising activities than Cournot firms, and that the changes in exogenous variables affect Bertrand firms more. In addition, Table 2 showed that Bertrand firms enjoy more net present value when the marginal cost of advertising is small, consumers' willingness to pay is large, production cost is small, and/or risk is small.

Finally, we extend the model to investigate the cases that advertising can also reduce revenue shock and enhance consumers' willingness to pay. The higher advertising levels realized under either Cournot or Bertrand competition occur because advertising generates additional benefits by reducing idiosyncratic risk and increasing consumers' willingness to pay. The theoretical models and simulations confirm these conclusions.

The results of this study are derived under the assumptions that the sources of uncertainty come from the demand side, that any shock is proportional to revenue, that demand is linear, and so on. These assumptions can be somewhat restrictive. Reasonable generalizations should include supply shocks, the general impacts of shock, and the general format of demand. In addition, this paper does not address the role of capital stock, K_i. Indeed, relaxing the assumption of fixed capital stock may raise some relevant issues regarding risk management and the interplay of production and capital structure. Froot, Scharfstein, and Stein (1993) argue that risk management is important as an appropriate strategy to avoid unnecessary fluctuations in either funds raised from outside investors or investment spending if the external sources of finance (for example, borrowing) are more costly than internally generated funds (such as existing cash flows). Second, while a financial variable like systematic risk is affected by real variables of production, financial variables like financial structure or the debt-equity ratio may have impacts on the product side as well. The influential work is pioneered by Brander and Lewis (1986) who examine the strategic role of debt with limited liability. It is worthwhile to incorporate additional financial concerns to expand the current framework.

As mentioned above, one of most important issues addressed here is the role of risk reduction. In addition to the advertising issues that we have addressed, our observations further suggest that vehicles for risk reduction may include contracting, R&D, growth options, and so on. Let us take "contracting" as an example. Imagine that several downstream and upstream firms exist in the industry. To reduce risk, upstream firms may seek contracts with downstream firms so as to ensure a certain level of sales before the uncertainty of demand is realized. Likewise, downstream firms may have incentives to sign contracts with upstream firms so as to ensure a portion of their needed supply.[19] In addition, those downstream or upstream firms may choose to merge horizontally or integrate vertically so as to achieve their common goal: reducing risk. Risk reduction provides a different perspective for analyzing the boundary of a firm as the transaction cost approach is familiar to most economists. This research avenue is worthy of future study.

[19]Note that hold-up problems may impose additional costs on the risk reduction.

References

[1] Bagwell, Kyle (2001) *The Economics of Advertising*, Cheltenham, UK; Northampton, MA: Edward Elgar.

[2] Bagwell, Kyle and Ramey, Garey (1988) Advertising and Limit Pricing, *Rand Journal of Economics; 19(1), Spring*, 59-71.

[3] Bain, Joe Staten (1956) *Barriers to New Competition, Their Character and Consequences in Manufacturing Industries*, Cambridge: Harvard University Press.

[4] Baron, David P. (1971) Demand Uncertainty in Imperfect Competition, *International Economic Review; 12(2), June*, 196-208.

[5] Becker, Gary S. and Murphy, Kevin M. (1993) A Simple Theory of Advertising as a Good or Bad, *Quarterly Journal of Economics; 108(4), November*, 941-64.

[6] Brander, James A. and Lewis, Tracy R. (1986) Oligopoly and Financial Structure: The Limited Liability Effect, *American Economic Review; 76(5), December*, 956-70.

[7] Brealey, Richard A. and Myers, Stewart C. (2000) *Principles of Corporate Finance*, Sixth Edition, McGraw-Hill Series in Finance. New York; London; Toronto and Sydney: McGraw-Hill.

[8] Dixit, Avinash (1979) A Model of Duopoly Suggesting a Theory of Entry Barriers, *Bell Journal of Economics; 10(1), Spring*, 20-32.

[9] Dixit, Avinash and Norman, Victor (1978) Advertising and Welfare, *Bell Journal of Economics; 9(1), Spring*, 1-17.

[10] Dixit, Avinash and Norman, Victor (1979) Advertising and Welfare: Reply, *Bell Journal of Economics; 10(2), Autumn*, 728-29.

[11] Dixit, Avinash and Norman, Victor (1980) Advertising and Welfare: Another Reply, *Bell Journal of Economics; 11(2), Autumn*, 753-54.

[12] Dixit, Avinash K. and Pindyck, Robert S. (1994) *Investment under uncertainty*, Princeton: Princeton University Press.

[13] Fama, Eugene F. and Miller, Merton H. (1972) *The Theory of Finance*, New York: Holt, Rinehart and Winston.

[14] Fisher, Franklin M. and McGowan, John J. (1979) Advertising and Welfare: Comment, *Bell Journal of Economics; 10(2), Autumn*, 726-27.

[15] Froot, Kenneth A, Scharfstein, David S, and Stein, Jeremy C (1993) Risk Management: Coordinating Corporate Investment and Financing Policies, *Journal of Finance; 48(5), December*, 1629-58.

[16] Gasmi, F., Laffont, J. J., and Vuong, Q. (1992) Econometric Analysis of Collusive Behavior in a Soft-Drink Market, *Journal of Economics and Management Strategy; 1(2), Summer*, 277-311.

[17] Grossman, Gene M. and Shapiro, Carl (1984) Informative Advertising with Differentiated Products, *Review of Economic Studies; 51(1), January*, 63-81.

[18] Kaldor, Nicholas (1950) The Economic Aspects of Advertising, *Review of Economic Studies; 18(1)*, 1-27.

[19] Kaplan, Steven N. and Ruback, Richard S. (1995) The Valuation of Cash Flow Forecasts: An Empirical Analysis, *Journal of Finance; 50(4), September*, 1059-93.

[20] Klemperer, Paul D. and Meyer, Margaret A. (1986) Price Competition vs. Quantity Competition: The Role of Uncertainty, *Rand Journal of Economics; 17(4), Winter*, 618-638.

[21] Leland, Hayne E. (1972) Theory of the Firm Facing Uncertain Demand, *American Economic Review; 62(3), June*, 278-91.

[22] Lintner, John (1965) The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, *Review of Economics and Statistics; 47(1), Feb.*, 13-37.

[23] Milgrom, Paul and Roberts, John (1986) Price and Advertising Signals of Product Quality, *Journal of Political Economy; 94(4), August*, 796-821.

[24] Mossin, Jan (1966) Equilibrium in a Capital Asset Market, *Econometrica; 34(4), Oct.*, 768-783.

[25] Motta, Massimo (1993) Endogenous Quality Choice: Price vs. Quantity Competition, *Journal of Industrial Economics; 41(2), June*, 113-31.

[26] Nelson, Philip (1974) Advertising as Information, *Journal of Political Economy; 82(4), July/Aug.*, 729-54.

[27] Roberts, Mark J. and Samuelson, Larry (1988) An Empirical Analysis of Dynamic, Nonprice Competition in an Oligopolistic Industry, *Rand Journal of Economics; 19(2), Summer*, 200-220.

[28] Sandmo, Agnar (1971) On the Theory of the Competitive Firm under Price Uncertainty, *American Economic Review; 61(1), March*, 65-73.

[29] Shapiro, Carl (1980) Advertising and Welfare: Comment, *Bell Journal of Economics; 11(2), Autumn*, 749-52.

[30] Sharpe, William F. (1964) Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, *Journal of Finance; 19(3), Sep.*, 425-442.

[31] Shy, Oz (1995) *Industrial organization: Theory and applications*, Cambridge and London: MIT Press.

[32] Singh, Nirvikar and Vives, Xavier (1984) Price and Quantity Competition in a Differentiated Duopoly, *Rand Journal of Economics; 15(4), Winter*, 546-54.

[33] Stigler, George J. (1961) The Economics of Information, *Journal of Political Economy; 69(3), June*, 213-225.

[34] Stigler, George J. and Becker, Gary S. (1977) De Gustibus Non Est Disputandum, *American Economic Review; 67(2), March*, 76-90.

[35] Subrahmanyam, Marti G. and Thomadakis, Stavros B. (1980) Systematic Risk and the Theory of the Firm, *Quarterly Journal of Economics; 94(3), May*, 437-51.

[36] Telser, Lester G. (1964) Advertising and Competition, *Journal of Political Economy; 72(6), Dec.*, 537-562.

[37] Vives, Xavier (1999) *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge, Mass.: MIT Press.

[38] von der Fehr, Nils Henrik M. and Stevik, Kristin (1998) Persuasive Advertising and Product Differentiation, *Southern Economic Journal; 65(1), July*, 113-26.

[39] Wang, Shinn-Shyr, Kyle W. Stiegert, and Tirtha P. Dhar (2005) Strategic Pricing Behavior under Asset Value Maximization: An Empirical Assessment of the U.S. Retail Margarine and Butter Markets, *Working Paper*, University of Wisconsin-Madison.

Table 2(a) Simulation Results – Changes in μ

	Bertrand					Cournot				
μ	A	p	X	v	β	A	p	X	v	β
0.00	0.5000	0.5521	0.4479	0.1834	0.6741	0.5000	0.5521	0.4479	0.1834	0.6741
0.01	0.4759	0.5410	0.4590	**0.1822**	0.6773	0.4720	0.5399	0.4601	**0.1822**	0.6776
0.02	0.4565	0.5317	0.4683	**0.1810**	0.6801	0.4430	0.5279	0.4721	**0.1809**	0.6813
0.03	0.4402	0.5236	0.4764	**0.1798**	0.6827	0.4133	0.5163	0.4837	**0.1797**	0.6851
0.04	0.4262	0.5164	0.4836	**0.1786**	0.6851	0.3836	0.5054	0.4946	**0.1785**	0.6889
0.05	0.4139	0.5099	0.4901	**0.1775**	0.6873	0.3547	0.4953	0.5047	**0.1773**	0.6925
0.06	0.4030	0.5039	0.4961	**0.1764**	0.6894	0.3273	0.4861	0.5139	**0.1762**	0.6960
0.07	0.3931	0.4985	0.5015	**0.1754**	0.6913	0.3019	0.4780	0.5220	**0.1752**	0.6993
0.08	0.3842	0.4934	0.5066	**0.1744**	0.6932	0.2786	0.4709	0.5291	**0.1743**	0.7022
0.09	0.3760	0.4887	0.5113	0.1734	0.6950	0.2576	0.4647	0.5353	0.1735	0.7049
0.10	0.3684	0.4842	0.5158	0.1724	0.6968	0.2388	0.4593	0.5407	0.1727	0.7073
0.11	0.3614	0.4800	0.5200	0.1715	0.6984	0.2221	0.4547	0.5453	0.1720	0.7094
0.12	0.3549	0.4761	0.5239	0.1706	0.7000	0.2072	0.4506	0.5494	0.1714	0.7113
0.13	0.3488	0.4723	0.5277	0.1697	0.7016	0.1939	0.4471	0.5529	0.1709	0.7130
0.14	0.3431	0.4687	0.5313	0.1688	0.7031	0.1820	0.4440	0.5560	0.1704	0.7145
0.15	0.3377	0.4653	0.5347	0.1680	0.7046	0.1714	0.4413	0.5587	0.1700	0.7159
0.16	0.3326	0.4620	0.5380	0.1672	0.7061	0.1618	0.4389	0.5611	0.1696	0.7171
0.17	0.3277	0.4589	0.5411	0.1664	0.7075	0.1532	0.4367	0.5633	0.1693	0.7182
0.18	0.3231	0.4558	0.5442	0.1656	0.7089	0.1454	0.4348	0.5652	0.1690	0.7192
0.19	0.3188	0.4529	0.5471	0.1648	0.7102	0.1383	0.4331	0.5669	0.1687	0.7201
0.20	0.3146	0.4501	0.5499	0.1640	0.7115	0.1318	0.4316	0.5684	0.1684	0.7209

[1] $b = 1$, $r = 0.05$, $c = 0.1$, $\sigma_m = 0.2$, $\lambda = 2$, $\sigma_e = 0.2$, $\rho = 0.5$, and $\alpha = 1$.
[2] The numbers with bold type indicate Bertrand firms enjoy more firm value than Cournot firms.

Table 2(b) Simulation Results – Changes in α

	Bertrand					Cournot				
α	A	p	X	v	β	A	p	X	v	β
0.5	0.2364	0.2312	0.2688	0.0284	0.9952	0.0528	0.2409	0.2591	0.0323	0.9634
0.6	0.2740	0.2797	0.3203	0.0476	0.8714	0.0824	0.2790	0.3210	0.0510	0.8726
0.7	0.3045	0.3297	0.3703	0.0717	0.7994	0.1175	0.3197	0.3803	0.0742	0.8112
0.8	0.3298	0.3807	0.4193	0.1006	0.7529	0.1567	0.3632	0.4368	0.1022	0.7668
0.9	0.3508	0.4323	0.4677	0.1342	0.7205	0.1981	0.4098	0.4902	0.1350	0.7333
1.0	0.3684	0.4842	0.5158	0.1724	0.6968	0.2388	0.4593	0.5407	0.1727	0.7073
1.1	0.3834	0.5364	0.5636	**0.2154**	0.6787	0.2766	0.5112	0.5888	**0.2153**	0.6868
1.2	0.3961	0.5886	0.6114	**0.2629**	0.6645	0.3097	0.5645	0.6355	**0.2627**	0.6706
1.3	0.4071	0.6408	0.6592	**0.3151**	0.6530	0.3378	0.6186	0.6814	**0.3148**	0.6576
1.4	0.4165	0.6930	0.7070	**0.3720**	0.6436	0.3610	0.6730	0.7270	**0.3716**	0.6470
1.5	0.4246	0.7452	0.7548	**0.4334**	0.6357	0.3801	0.7274	0.7726	**0.4330**	0.6383

[1] $b = 1$, $r = 0.05$, $c = 0.1$, $\sigma_m = 0.2$, $\lambda = 2$, $\sigma_e = 0.2$, $\rho = 0.5$, and $\mu = 0.1$.
[2] The numbers with bold type indicate Bertrand firms enjoy more firm value than Cournot firms.

Table 2(c) Simulation Results – Changes in ρ

	Bertrand					Cournot				
ρ	A	p	X	v	β	A	p	X	v	β
0.0	0.3721	0.4840	0.5160	0.1818	0.0000	0.2477	0.4594	0.5406	0.1820	0.0000
0.1	0.3714	0.4840	0.5160	0.1799	0.1337	0.2460	0.4593	0.5407	0.1801	0.1356
0.2	0.3706	0.4841	0.5159	0.1780	0.2701	0.2442	0.4593	0.5407	0.1783	0.2740
0.3	0.3699	0.4841	0.5159	0.1762	0.4094	0.2424	0.4593	0.5407	0.1764	0.4154
0.4	0.3692	0.4842	0.5158	0.1743	0.5516	0.2406	0.4593	0.5407	0.1746	0.5598
0.5	0.3684	0.4842	0.5158	0.1724	0.6968	0.2388	0.4593	0.5407	0.1727	0.7073
0.6	0.3677	0.4843	0.5157	0.1706	0.8451	0.2370	0.4593	0.5407	0.1709	0.8580
0.7	0.3669	0.4843	0.5157	0.1687	0.9966	0.2352	0.4594	0.5406	0.1690	1.0120
0.8	0.3662	0.4844	0.5156	0.1668	1.1514	0.2333	0.4594	0.5406	0.1672	1.1694
0.9	0.3654	0.4845	0.5155	0.1650	1.3096	0.2314	0.4594	0.5406	0.1653	1.3304
1.0	0.3646	0.4846	0.5154	0.1631	1.4714	0.2295	0.4595	0.5405	0.1635	1.4950

[1] $b = 1$, $r = 0.05$, $c = 0.1$, $\sigma_m = 0.2$, $\lambda = 2$, $\sigma_e = 0.2$, $\alpha = 1$, and $\mu = 0.1$.

Table 2(d) Simulation Results – Changes in σ_e

	Bertrand					Cournot				
σ_e	A	p	X	v	β	A	p	X	v	β
0.00	0.3721	0.4840	0.5160	0.1818	0.0000	0.2477	0.4594	0.5406	0.1820	0.0000
0.02	0.3717	0.4840	0.5160	0.1808	0.0665	0.2469	0.4593	0.5407	0.1810	0.0675
0.04	0.3714	0.4840	0.5160	0.1799	0.1337	0.2460	0.4593	0.5407	0.1801	0.1356
0.06	0.3710	0.4840	0.5160	0.1790	0.2016	0.2451	0.4593	0.5407	0.1792	0.2045
0.08	0.3706	0.4841	0.5159	0.1780	0.2701	0.2442	0.4593	0.5407	0.1783	0.2740
0.10	0.3703	0.4841	0.5159	0.1771	0.3394	0.2433	0.4593	0.5407	0.1773	0.3444
0.12	0.3699	0.4841	0.5159	0.1762	0.4094	0.2424	0.4593	0.5407	0.1764	0.4154
0.14	0.3696	0.4841	0.5159	0.1752	0.4801	0.2415	0.4593	0.5407	0.1755	0.4872
0.16	0.3692	0.4842	0.5158	0.1743	0.5516	0.2406	0.4593	0.5407	0.1746	0.5598
0.18	0.3688	0.4842	0.5158	0.1734	0.6238	0.2397	0.4593	0.5407	0.1736	0.6331
0.20	0.3684	0.4842	0.5158	0.1724	0.6968	0.2388	0.4593	0.5407	0.1727	0.7073
0.22	0.3681	0.4843	0.5157	0.1715	0.7705	0.2379	0.4593	0.5407	0.1718	0.7822
0.24	0.3677	0.4843	0.5157	0.1706	0.8451	0.2370	0.4593	0.5407	0.1709	0.8580
0.26	0.3673	0.4843	0.5157	0.1696	0.9204	0.2361	0.4593	0.5407	0.1699	0.9345
0.28	0.3669	0.4843	0.5157	0.1687	0.9966	0.2352	0.4594	0.5406	0.1690	1.0120
0.30	0.3665	0.4844	0.5156	0.1678	1.0735	0.2342	0.4594	0.5406	0.1681	1.0902
0.32	0.3662	0.4844	0.5156	0.1668	1.1514	0.2333	0.4594	0.5406	0.1672	1.1694
0.34	0.3658	0.4844	0.5156	0.1659	1.2301	0.2324	0.4594	0.5406	0.1663	1.2494
0.36	0.3654	0.4845	0.5155	0.1650	1.3096	0.2314	0.4594	0.5406	0.1653	1.3304
0.38	0.3650	0.4845	0.5155	0.1640	1.3900	0.2305	0.4594	0.5406	0.1644	1.4122
0.40	0.3646	0.4846	0.5154	0.1631	1.4714	0.2295	0.4595	0.5405	0.1635	1.4950

[1] $b = 1$, $r = 0.05$, $c = 0.1$, $\sigma_m = 0.2$, $\lambda = 2$, $\rho = 0.5$, $\alpha = 1$, and $\mu = 0.1$.

Table 2(e) Simulation Results – Changes in c

	Bertrand					Cournot				
c	A	p	X	v	β	A	p	X	v	β
0.00	0.3840	0.4344	0.5656	**0.2173**	0.5469	0.2780	0.4092	0.5908	**0.2172**	0.5469
0.01	0.3825	0.4393	0.5607	**0.2126**	0.5602	0.2743	0.4141	0.5859	**0.2125**	0.5610
0.02	0.3811	0.4443	0.5557	**0.2079**	0.5738	0.2706	0.4191	0.5809	**0.2079**	0.5755
0.03	0.3796	0.4493	0.5507	**0.2033**	0.5878	0.2668	0.4240	0.5760	**0.2033**	0.5904
0.04	0.3781	0.4543	0.5457	0.1987	0.6021	0.2629	0.4290	0.5710	0.1988	0.6057
0.05	0.3765	0.4593	0.5407	0.1942	0.6168	0.2590	0.4340	0.5660	0.1943	0.6215
0.06	0.3750	0.4643	0.5357	0.1898	0.6319	0.2550	0.4390	0.5610	0.1899	0.6377
0.07	0.3734	0.4693	0.5307	0.1854	0.6475	0.2510	0.4441	0.5559	0.1855	0.6543
0.08	0.3718	0.4742	0.5258	0.1810	0.6635	0.2470	0.4491	0.5509	0.1812	0.6715
0.09	0.3701	0.4792	0.5208	0.1767	0.6799	0.2429	0.4542	0.5458	0.1769	0.6891
0.10	0.3684	0.4842	0.5158	0.1724	0.6968	0.2388	0.4593	0.5407	0.1727	0.7073
0.11	0.3667	0.4892	0.5108	0.1682	0.7141	0.2347	0.4645	0.5355	0.1686	0.7260
0.12	0.3650	0.4942	0.5058	0.1641	0.7320	0.2305	0.4696	0.5304	0.1645	0.7452
0.13	0.3632	0.4992	0.5008	0.1600	0.7504	0.2263	0.4748	0.5252	0.1604	0.7651
0.14	0.3614	0.5042	0.4958	0.1559	0.7694	0.2221	0.4800	0.5200	0.1564	0.7855
0.15	0.3596	0.5092	0.4908	0.1519	0.7890	0.2178	0.4853	0.5147	0.1525	0.8066
0.16	0.3578	0.5142	0.4858	0.1480	0.8091	0.2136	0.4906	0.5094	0.1486	0.8283
0.17	0.3559	0.5192	0.4808	0.1441	0.8299	0.2093	0.4959	0.5041	0.1447	0.8507
0.18	0.3539	0.5243	0.4757	0.1402	0.8514	0.2050	0.5012	0.4988	0.1410	0.8737
0.19	0.3520	0.5293	0.4707	0.1364	0.8735	0.2007	0.5066	0.4934	0.1372	0.8975
0.20	0.3500	0.5343	0.4657	0.1327	0.8964	0.1963	0.5120	0.4880	0.1336	0.9221

[1] $b = 1$, $r = 0.05$, $\sigma_m = 0.2$, $\lambda = 2$, $\sigma_e = 0.2$, $\rho = 0.5$, $\alpha = 1$, and $\mu = 0.1$.
[2] The numbers with bold type indicate Bertrand firms enjoy more firm value than Cournot firms.

Appendix

Proof of Proposition 1

From equations (13) and (14), consider the case with no uncertainty, in which we have

$$X_*^c = \frac{(\alpha - c)(1+\delta)}{b(2+\delta)}, \quad p_*^c = \frac{\alpha + (1+\delta)c}{2+\delta}, \quad v_*^c = \frac{(\alpha - c)^2}{b(1+r)} \frac{1+\delta}{(2+\delta)^2}. \tag{37}$$

$$\begin{aligned}
\Delta X_r^c &= X_i^c - X_*^c = -\frac{(d-c)(1+\delta)}{b(2+\delta)} \leq 0, \\
\Delta p_r^c &= p_i^c - p_*^c = \frac{(d-c)(1+\delta)}{2+\delta} \geq 0, \\
\Delta v_r^c &= v_i^c - v_*^c \leq \frac{1}{b(1+r)} \frac{1+\delta}{(2+\delta)^2} \left[\phi(\alpha - d)^2 - \phi(\alpha - c)^2\right] \\
&= -\frac{\phi}{b(1+r)} \frac{1+\delta}{(2+\delta)^2} \left[(\alpha - c) + (\alpha - d)\right](d-c) \leq 0.
\end{aligned} \tag{38}$$

On the other hand, by equations (16) and (17),

$$p_*^B = \frac{\alpha(1-\delta) + c}{2-\delta}, \quad X_*^B = \frac{\alpha - c}{b(2-\delta)}, \quad v_*^B = \frac{(\alpha - c)^2}{b(1+r)} \frac{1-\delta}{(2-\delta)^2}. \tag{39}$$

$$\begin{aligned}
\Delta X_r^B &= X_i^B - X_*^B = -\frac{d-c}{b(2-\delta)} \leq 0, \\
\Delta p_r^B &= p_i^B - p_*^B = \frac{d-c}{2-\delta} \geq 0, \\
\Delta v_r^B &= v_i^B - v_*^B \leq \frac{1}{b(1+r)} \frac{1-\delta}{(2-\delta)^2} \left[\phi(\alpha - d)^2 - \phi(\alpha - c)^2\right] \\
&= -\frac{\phi}{b(1+r)} \frac{1-\delta}{(2-\delta)^2} \left[(\alpha - c) + (\alpha - d)\right](d-c) \leq 0.
\end{aligned} \tag{40}$$

Proof of Proposition 2

(a) From equations (13) and (16), and $0 \leq \delta \leq 1$, we have

$$\Delta p = p_i^c - p_i^B = \frac{\alpha + (1+\delta)d}{2+\delta} - \frac{\alpha(1-\delta) + d}{2-\delta} = \frac{\delta^2(\alpha - d)}{4-\delta^2} \geq 0, \tag{41}$$

$$\begin{aligned}
\Delta X &= X_i^c - X_i^B \\
&= \frac{(\alpha - d)(1+\delta)}{b(2+\delta)} - \frac{\alpha - d}{b(2-\delta)} = -\frac{(\alpha - d)\delta^2}{b(2+\delta)(2-\delta)} \leq 0.
\end{aligned} \tag{42}$$

Moreover, by equations (14) and (17),

$$\Delta v = v_i^c - v_i^B = \frac{\phi(\alpha - d)^2}{b(1+r)} \frac{2\delta^3}{(2+\delta)^2(2-\delta)^2} \geq 0. \tag{43}$$

Together with

$$\frac{\partial \triangle p}{\partial \delta} > 0, \quad \frac{\partial \triangle X}{\partial \delta} < 0, \quad \frac{\partial \triangle v}{\partial \delta} > 0, \quad \forall \delta \in (0,1].$$

The results of part (a) follow.

(b) Without uncertainty, the differences in (41)-(43) emerge as

$$\begin{aligned}
\Delta p^* &= \frac{\delta^2(\alpha - c)}{4 - \delta^2}, \\
\Delta X^* &= -\frac{(\alpha - c)\delta^2}{b(2+\delta)(2-\delta)}, \\
\Delta v^* &= \frac{(\alpha - c)^2}{b(1+r)} \frac{2\delta^3}{(2+\delta)^2(2-\delta)^2}.
\end{aligned}$$

Because $\phi < 1$ in the uncertain case, we know that $\alpha - d = \alpha - c/\phi < \alpha - c$ and $\phi(\alpha - d)^2 < (\alpha - d)^2 < (\alpha - c)^2$. Thus, $|\Delta p| < |\Delta p^*|$, $|\Delta X| < |\Delta X^*|$, and $|\Delta v| < |\Delta v^*|$.

Proof of Lemma 1

(a) Using the fact that $A_j = A_i$ in equilibrium and taking a derivative with respect to A_i on the right side of (22) yield

$$\frac{\mathrm{d}}{\mathrm{d}A_i} \frac{A_i(3 - 2A_i)^3}{1 - 2A_i} = \frac{(3 - 2A_i)^2(3 - 8A_i + 12A_i^2)}{(1 - 2A_i)^2}. \tag{44}$$

It is easy to check that $3 - 8A_i + 12A_i^2 > 1/3$. Therefore, equation (44) is strictly greater than 0 for all $A_i \in [0, 0.5)$. This completes the proof for Cournot competition.

(b) From equation (23), we have

$$\frac{\mathrm{d}}{\mathrm{d}A_i} \frac{A_i(1 + 2A_i)^3}{1 - 2A_i} = \frac{(1 + 2A_i)^2(1 + 8A_i - 12A_i^2)}{(1 - 2A_i)^2}, \tag{45}$$

$1 + 8A_i - 12A_i^2 \geq 1$, $\forall A_i \in [0, 0.5)$. As a result, equation (45) is strictly greater than 0 for all $A_i \in [0, 0.5)$. Part (b) completes the proof for Bertrand competition.

Proof of Proposition 3

(a) Using the assumption that $A_j = A_i$ and differentiating equation (22) with respect to α, we have

$$\frac{2\phi(\alpha - d)}{\mu b(1+r)} = \frac{(3 - 2A_i)^2(3 - 8A_i + 12A_i^2)}{(1 - 2A_i)^2} \frac{\partial A_i}{\partial \alpha}.$$

Using lemma 1 and $\alpha > d$, we have $\frac{\partial A_i^c}{\partial \alpha} > 0$. The parallel logic can be applied to equation (23), which yields $\frac{\partial A_i^B}{\partial \alpha} > 0$.
(b) Now, applying the relations in equation (24), we can easily obtain the results that the left sides of equations (22) and (23) are increasing in σ_m but decreasing in b, $\rho\sigma_e$, $E(\tilde{r}_m) - r$, c, and μ.
(c) Following the similar steps in part (a) of the proof, it is straightforward to obtain the remaining conclusions.

Proof of Lemma 2
(a) Under Cournot competition, differentiating $b(2+\delta)\phi X_i^c = (\phi\alpha - c)(1+\delta)$ with respect to ϕ and rearranging yield

$$\underset{(+)}{(2+\delta)bX_i^c}(1+\eta_{x\phi}^c) = \underset{(+)}{\alpha(1+\delta)} + \underset{(+)}{(p_i^c - d)\delta}\ \underset{(-)}{\eta_{\delta\phi}}\ .$$

As a result, $\eta_{x\phi}^c$ is bounded above. For the price, $(2+\delta)\phi p_i^c = \alpha\phi + (1+\delta)c$, we get

$$\underset{(+)}{(2+\delta)p_i^c}(1+\eta_{p\phi}^c) = \alpha - \underset{(+)}{(p_i^c - d)\delta}\ \underset{(-)}{\eta_{\delta\phi}}\ .$$

Unlike $\eta_{x\phi}^c$, $\eta_{p\phi}^c$ is bounded below. Moreover, $\eta_{x\phi}^c < 0$ (or $\frac{\partial X_i^c}{\partial \phi} < 0$) and $\eta_{p\phi}^c > 0$ (or $\frac{\partial p_i^c}{\partial \phi} > 0$) if $\eta_{\delta\phi}^c \ll 0$ (or $\frac{\partial \delta}{\partial \phi} \ll 0$).
(b) From equation (47), taking a derivative with respect to ϕ, we get

$$\underset{(+)}{\frac{(\alpha+d)(\alpha-d)}{b(1+r)}} = \underset{(+)}{(2+\delta)^2}\frac{\partial v_i^c}{\partial \phi} + \underset{(+)}{[2(2+\delta)v_i^c]}\ \underset{(-)}{\frac{\partial \delta}{\partial \phi}},$$

which shows that $\frac{\partial v_i^c}{\partial \phi} > 0$. We omit the proof for Bertrand competition because the logic is the same.

Proof of Proposition 4
Rewrite equation (13) as $b(2+\delta)X_i^c = (\alpha - d)(1+\delta)$ and $(2+\delta)p_i^c = \alpha + (1+\delta)d$, then take differentiation with respect to α and rearrange to obtain

$$(2+\delta)\frac{\partial p_i^c}{\partial \alpha} = 1 - (p_i^c - d)\frac{\partial \delta}{\partial \alpha} = 1 + 2(p_i^c - d)\frac{\partial A_i}{\partial \alpha},$$

$$\begin{aligned} b(2+\delta)\frac{\partial X_i^c}{\partial \alpha} &= (1+\delta) + (\alpha - d - bX_i^c)\frac{\partial \delta}{\partial \alpha} \\ &= (1+\delta) - 2(\alpha - d - bX_i^c)\frac{\partial A_i}{\partial \alpha}. \end{aligned} \tag{46}$$

It is easy to check that $\frac{\partial p_i^c}{\partial \alpha} > 0$ because $\frac{\partial A_i}{\partial \alpha} > 0$ and $p_i^c \geq d$. The second term on the right side of equation (46) is weakly greater than 0 because in equilibrium $X_i = X_j$. Hence equation (3) becomes $p_i^c = \alpha - bX_i^c$. Again, $p_i^c - d \geq 0$. As a result, the sign of $\frac{\partial X_i^c}{\partial \alpha}$ depends on $\frac{\partial A_i}{\partial \alpha}$. That is,

$$\frac{\partial X_i^c}{\partial \alpha} > 0, \quad \text{if} \quad \frac{\partial A_i}{\partial \alpha} < \frac{(1+\delta)(2+\delta)}{2(\alpha - d)}.$$

From equation (14), we have

$$\frac{\phi(\alpha - d)^2}{b(1+r)} = \frac{(2+\delta)^2}{1+\delta} v_i^c. \tag{47}$$

Differentiating (47) with respect to α yields

$$2\underset{(+)}{\frac{\phi(\alpha - d)}{b(1+r)}} = \underset{(+)}{\frac{(2+\delta)^2}{1+\delta}} \frac{\partial v_i^c}{\partial \alpha} + \underset{(+)}{\frac{\delta(2+\delta)}{(1+\delta)^2} v_i^c} \underset{(-)}{\frac{\partial \delta}{\partial \alpha}}.$$

Therefore, $\frac{\partial v_i^c}{\partial \alpha} > 0$. Last, from equation (10b),

$$\frac{\partial \beta_i}{\partial \alpha} = -\left(\frac{1}{1+r}\left[\frac{1}{\rho\sigma_e} - \lambda\sigma_m^2\right]\right)^{-1} \frac{d}{(p_i - d)^2} \frac{\partial p_i}{\partial \alpha},$$

$\frac{\partial \beta_i^c}{\partial \alpha} < 0$ as $\frac{\partial p_i^c}{\partial \alpha} > 0$.

Under Bertrand competition, the results are the same as those for Cournot competition except for the change of equilibrium quantity. From equation (16),

$$b(2-\delta)X_i^B = \alpha - d.$$

Taking differentiation with respect to α and rearranging, we get

$$\underset{(+)}{b(2-\delta)^2} \frac{\partial X_i^B}{\partial \alpha} = (2-\delta) + (\alpha - d)\frac{\partial \delta}{\partial \alpha} = \underset{(+)}{(2-\delta)} - 2\underset{(+)}{(\alpha - d)} \underset{(+)}{\frac{\partial A_i}{\partial \alpha}}.$$

This shows us that the impact of change in α on X_i^B depends on $\frac{\partial A_i}{\partial \alpha}$.

$$\frac{\partial X_i^B}{\partial \alpha} > 0, \quad \text{if} \quad \frac{\partial A_i}{\partial \alpha} < \frac{2-\delta}{2(\alpha - d)}.$$

Proof of Proposition 6

By equation (24), we may establish the following relations:

$$\frac{\partial \phi}{\partial(\rho\sigma_e)} = -\frac{E(\tilde{r}_m) - r}{\sigma_m} < 0, \tag{48}$$

$$\frac{\partial \phi}{\partial(E(\tilde{r}_m) - r)} = -\frac{(\rho\sigma_e)}{\sigma_m} < 0, \tag{49}$$

$$\frac{\partial \phi}{\partial \sigma_m} = \frac{(E(\tilde{r}_m) - r)\rho\sigma_e}{\sigma_m^2} > 0. \tag{50}$$

Taking $\rho\sigma_e$ as an example, from lemma 2, we get that $\frac{\partial X_i}{\partial(\rho\sigma_e)}$ is bounded below and $\frac{\partial p_i}{\partial(\rho\sigma_e)}$ is bounded above. Moreover, $\frac{\partial X_i}{\partial(\rho\sigma_e)} > 0$ and $\frac{\partial p_i}{\partial(\rho\sigma_e)} < 0$ if $\frac{\partial \delta}{\partial(\rho\sigma_e)} \gg 0$. In addition, $\frac{\partial v_i}{\partial(\rho\sigma_e)} < 0$.

Let us examine how β can be changed. By equation (15), differentiating with respect to $\rho\sigma_e$ yields

$$\begin{aligned} & \frac{(1+r)}{\beta_i^{c2}} \frac{\partial \beta^c}{\partial(\rho\sigma_e)} \\ = \; & \frac{\sigma_m}{(\rho\sigma_e)^2}[1 - \frac{(2+\delta)d}{\alpha + (1+\delta)d}] + [\frac{\sigma_m}{\rho\sigma_e} - \lambda\sigma_m^2][\frac{\alpha(2+\delta)d' + d(\alpha - d)\delta'}{(\alpha + (1+\delta)d)^2}] \end{aligned}$$

where

$$\begin{aligned} d' &= \frac{\partial d}{\partial(\rho\sigma_e)} = \frac{\partial d}{\partial \phi}\frac{\partial \phi}{\partial(\rho\sigma_e)} = (-\frac{c}{\phi^2})(-[\frac{E(\tilde{r}_m) - r}{\sigma_m}]) > 0, \\ \delta' &= \frac{\partial \delta}{\partial(\rho\sigma_e)} = -\frac{2\partial A_i}{\partial(\rho\sigma_e)} > 0. \end{aligned}$$

It turns out that $\frac{\partial \beta^c}{\partial(\rho\sigma_e)} > 0$.

For Bertrand competition, we omit the computations as the qualitative results are the same.