University of Wisconsin-Madison
Department of Agricultural & Applied Economics

October 2002 **Staff Paper No. 454**

The Dynamic Behavior of Efficient Timber Prices

By

Bruce McGough
Andrew J. Plantinga
William Provencher

AGRICULTURAL &
APPLIED ECONOMICS

STAFF PAPER SERIES

The Dynamic Behavior of Efficient Timber Prices

Bruce McGough
Department of Economics
Oregon State University

Andrew J. Plantinga
Department of Agricultural and Resource Economics
Oregon State University

William Provencher
Department of Agricultural and Applied Economics
University of Wisconsin

The authors acknowledge helpful comments from seminar participants at the University of California-Davis and the 2001 American Agricultural Economics Association meeting, where earlier versions of this paper were presented.

The Dynamic Behavior of Efficient Timber Prices

Abstract: The problem of when to optimally harvest trees when timber prices evolve according to an exogenous stochastic process has been studied extensively in recent decades. However, little attention has been given to the appropriate form of the stochastic process for timber prices, despite the fact that the choice of a process has important effects on optimal harvesting decisions. We develop a simple theoretical model of a timber market and show that there exists a rational expectations equilibrium in which prices evolve according to a stationary ARMA(1,1) process. Simulations are used to analyze a model with a more general representation of timber stock dynamics and to demonstrate that the unconditional distribution for rational timber prices is asymmetric. Implications for the optimal harvesting literature are: 1) market efficiency provides little justification for random walk prices, 2) unit root tests, used to analyze the informational efficiency of timber markets, do not distinguish between efficient and inefficient markets, and 3) failure to recognize asymmetric disturbances in time-series analyses of historical timber prices can lead to sub-optimal harvesting rules.

The Dynamic Behavior of Efficient Timber Prices

Introduction

In recent decades, the problem of when to optimally harvest trees when timber prices (and other components of forestry profits) are uncertain has received a great deal of attention.[1] In these studies, price is assumed to evolve according to an exogenous stochastic process, and the corresponding dynamic optimization problem is solved to yield an optimal harvesting rule. The form of the stochastic price process differs across studies. Some researchers have analyzed non-stationary random walk processes (e.g., geometric Brownian motion), while others have examined stationary autoregressive and serially uncorrelated processes. The basic insight provided by these studies is that, in most of the cases examined[2], optimal harvesting involves the use of a reservation price rule whereby timber managers harvest when price climbs above a reservation price and, otherwise, delay harvest and revisit the decision in the next period. It is shown that a reservation price rule weakly dominates a fixed-rotation rule consisting of Faustmann rotations evaluated at the mean of the price process.

The central question addressed in this study is: what is the appropriate model of timber prices? This is an important question given the prescriptive nature of the timber

---

[1] A representative, but but no means exhaustive, list of studies is Norstrom (1975), Brazee and Mendelsohn (1988), Morck *et al*. (1989), Clarke and Reed (1989), Haight and Holmes (1991), Haight and Smith (1991), Lohmander (1992), Thomson (1992), Reed (1993), Yin and Newman (1997), Plantinga (1998), Gong (1999), and Saphores *et al*. (2002). These studies complement the literatures on renewable and nonrenewable resource use under uncertainty (e.g., Brennan and Schwartz, 1985; Dasgupta and Heal, 1974; Pindyck, 1984) and irreversible land development under uncertainty (Arrow and Fisher, 1974; Fisher and Hanemann, 1986; Albers, 1996).

[2] The exception is when prices follow a random walk process and timber production involves no fixed costs (see, for example, Thomson, 1992).

harvesting literature[3] and evidence that the form of the price process strongly influences the performance of a reservation price rule relative to the Faustmann rotation (Haight and Holmes, 1991; Plantinga, 1998). Nonetheless, most authors appear to select the form of the price process for analytical convenience. When justification for the process is offered, one of two general arguments have been made. First, some authors have argued for random walk processes on the grounds that such prices are consistent with an informationally efficient timber market (Thomson, 1992; Reed, 1993). In the same vein, a number of authors have tested the hypothesis of efficient timber markets by applying unit root and other tests to time-series data on timber prices (Washburn and Binkley, 1990; Haight and Holmes, 1991; Hultkrantz, 1993; Yin and Newman, 1995; Abildtrup *et al*., 1997). The motivation for these studies is the claim that harvesting rules can work only if markets are inefficient since they rely on predictable price movements. Second, some authors take an empirical approach and fit time-series models to historical price series (Haight and Holmes, 1991; Gong, 1999; Saphores *et al*. 2002).

The objective of this paper is to examine the theoretical foundations for timber price dynamics. We focus on the use of relatively young forests, as distinct from the problem of extracting old-growth timber consider by Reed (1993) and Sapphores *et al*. (2002). With young forests, growth in the resource becomes a central feature of the problem and, as we show, has important implications for price dynamics. In the next section, we develop a simple model of a competitive timber market and examine the stochastic properties of efficient prices generated in this setting. In our model, 1) timber

[3] Particularly in studies published in forestry journals, authors advocate the use of reservation price rules by timber managers. For example, Brazee and Mendelsohn (1988) write, "When market demand and supply conditions are such that timber prices are relative high, individual land owners should respond by cutting more. By tailoring harvests to variations in prices, the present value of all future timber revenues can be greatly enhanced over the standard Faustmann model."

managers are price-takers with rational expectations who maximize the present discounted value of expected timber revenues over an infinite horizon, 2) timber demand is subject to exogenous i.i.d. shocks, and 3) the stock of timber evolves according to harvesting and a deterministic growth function. We show that under the stated model assumptions there exists near a perfect foresight steady-state a unique stationary rational expectations equilibrium that can be completely represented by a stationary ARMA(1,1) price process. This result indicates that stationary serially-correlated prices can arise in an informationally efficient timber market even when market shocks are i.i.d. Thus, our theoretical results indicate that market efficiency provides little justification for random walk prices and that the unit root tests applied in earlier studies do not distinguish between informationally efficient and inefficient markets for timber.

In the following section, simulations are employed to analyze a more general model of the timber market. In contrast to the theoretical model in which the timber inventory is represented by a single stock variable, we consider a more standard age-class inventory model. The model is used to simulate long sequences of prices, which, consistent with our theoretical results, display the features of a stationary and serially dependent price process. Moreover, simulations are used to show that the unconditional price distribution is asymmetric. A similar result is produced in models of optimal commodity storage (e.g., Williams and Wright, 1991), and is a consequence of stock-outs (depletion of the inventory). In contrast, price asymmetry in our model arises from the concavity of the growth function. This result indicates that researchers who rely on empirical evidence to justify a stochastic process for timber prices should exercise caution in fitting models to historical price series since the assumption of symmetric disturbances, adopted in all previous studies, may not hold. Incorrectly assuming symmetric disturbances can result in sub-optimal harvesting rules. These and other implications of our results are discussed in greater detail in the final section of the paper.

A Model of an Efficient Timber Market

In this section, we explore the dynamics of timber prices in a rational expectations equilibrium. Our purpose is to demonstrate that stationary and serially correlated prices can be generated even in a very simple market environment. By focusing on a simple model, we can derive concise analytical results and elucidate the factors determining the behavior of efficient prices. Simulations are used in the following section to examine price dynamics in a more complex setting. Our model has a similar structure to the models of competitive storage analyzed by Williams and Wright (1991) and Deaton and Laroque (1992, 1996), except that the resource in our model exhibits stock-dependent growth. As well, our model can be viewed as a stochastic version of Berck's (1981) model of a renewable resource market.

Agent Behavior

Agents in the market are timber growers who each have access to a stock that evolves according to a deterministic growth function. At each time *t*, agents choose some portion of the stock to sell in a competitive market, subject to the constraint that the quantity harvested (q_t) cannot exceed the available stock (s_t); that is, $q_t \geq s_t$. The stock remaining after harvest, defined as

$$x_t \equiv s_t - q_t, \tag{1}$$

grows according to the increasing and concave function $g(x_t)$. Thus, the deterministic stock dynamics in our model may be summarized as,

$$s_{t+1} = x_t + g(x_t)\,. \tag{2}$$

In the forest economics literature, timber volume is more commonly modeled as a function of the age of a timber stand. If a forest is composed of multiple stands of varying ages, then in general there is not a unique correspondence between total timber volume and timber growth. For simplicity, we ignore the age composition of the forest and treat the timber resource as homogeneous. In the simulations presented below, a more conventional age-class model of the timber resource is considered.

Timber is competitively supplied by n agents. Aggregate demand for timber is stochastic (see below) and, thus, risk-neutral agents will maximize expected discounted profits. By assuming that agents are identical, we can focus on the optimal supply decision for a single, representative timber grower. Specifically, the representative agent chooses the contingency plan $\{q_t\}$ to solve,

$$\begin{aligned} &\max_{\{q_t\}} E\sum_{t=0}^{\infty} \beta^t p_t q_t \\ &s.t. \quad x_t \geq 0 \\ &\qquad\;\; s_{t+1} = x_t + g(x_t) \\ &\qquad\;\; x_0 \text{ given} \end{aligned}$$

where β is the discount factor, p_t is the price of timber, and timber management costs are assumed to be zero. From the Kuhn-Tucker conditions, we obtain the Euler equations,

$$p_t = \beta(1 + g'(x_t))E_t p_{t+1} \qquad x_t > 0 \tag{3a}$$
$$p_t \geq \beta(1 + g'(x_t))E_t p_{t+1} \qquad x_t = 0 \tag{3b}$$

In what follows, we focus on the interior solution to the representative agent problem, and, thus, equation 3a is the relevant behavioral equation.[4] It says that agents will supply timber up to the point where the marginal benefit of selling in time *t* equals the discounted expected marginal benefit of selling in *t*+1, taking into account the growth in the stock between *t* and *t*+1. Conditions required for an interior solution to exist are discussed below. In the simulations section we consider corner solutions as well; that is, we solve the model with a version of equation 3b included.

Market Equilibrium

The supply of timber is determined by the agent's Euler equation (3a) and the stock dynamics in (2). Demand for timber is exogenous and subject to stochastic shocks transmitted through output markets for wood products. Inverse demand is denoted,

$$p_t = D(nq_t, \varepsilon_t) \tag{4}$$

where ε_t is an i.i.d. shock with known distribution. A rational expecations equilibrium (REE) of this model is a 4-tuple of stochastic processes $\{s_t, q_t, x_t, p_t\}$ that simultaneously satisfy equations (1), (2), (3a), and (4). Unfortunately, systems of non-linear, expectational difference equations cannot, in general, be solved. Instead, the standard approach in the macroeconomics literature on real business cycles (e.g., Kydland and Prescott, 1982; Farmer, 1999) is to log-linearize the system of equations about a perfect foresight steady-state. The idea is simple. If there is a perfect foresight steady-state, then the linearization about that steady-state should well approximate the local dynamic

[4] Here, we are considering the case in which stock-outs ($x_t = 0$) do not occur; we are not assuming that they *cannot* occur. If this were the case, timber would not be scarce and there would be no basis for positive prices.

behavior. If the steady-state is also robust to small perturbations, then adding stochastic shocks with small support should not cause the system to deviate from the neighborhood in which the approximation is valid. Once the linearization is complete, results from the literature on multivariate linear expectational difference equations may be applied to obtain the REE. Complete details on the procedure can be found in Evans and Honkaponja (2001).

We begin by identifying the perfect foresight steady-state. The demand shock is assumed to equal its mean value, denoted $\bar{\varepsilon}$. In the steady-state, prices are constant and, thus, the Euler equation (3a) implies $1+g'(\bar{x})=1/\beta$, where barred variables will represent their steady-state values. The invertibility of g' pins down $\bar{x}$ and the steady-state values of stock, harvest, and price are subsequently given by $\bar{q}=g(\bar{x})$, $\bar{s}=\bar{x}+\bar{q}$, and $\bar{p}=D(n\bar{q},\bar{\varepsilon})$, respectively. Existence of an economically reasonable steady-state is not guaranteed; that is, a steady-state with $\bar{p}>0$. It is clear that for most reasonable specifications of *g*, a positive $\bar{x}$ will obtain and yield reasonable (positive) values of $\bar{s}$ and $\bar{q}$. However, for a given demand specification, large values of $\bar{q}$ could result in negative steady-state prices. This issue will be considered in greater depth in an example provided below.

We are now in a position to obtain the linearized model by taking a Taylor series expansion about the steady-state. For equation (3a), the first-order expansion is,

$$\bar{p}+(p_t-\bar{p})=\beta(1+g'(\bar{x}))\bar{p}+\beta g''(\bar{x})(x_t-\bar{x})\bar{p}+\beta(1+g'(\bar{x}))E_t(p_{t+1}-\bar{p}).$$

Noting that $\bar{p}=\beta(1+g'(\bar{x}))\bar{p}$ and expressing variables as percent deviations from their steady-state values (indicated by hats), we obtain,

$$\hat{p}_t=E_t\hat{p}_{t+1}+\beta\bar{x}g''(\bar{x})\hat{x}_t \tag{5}$$

Applying the same procedure to the other equations yields,

$$\begin{aligned}
\hat{s}_{t+1} &= (\bar{x}/\bar{s})(1+g'(\bar{x}))\hat{x}_t \\
\hat{x}_t &= (\bar{s}/\bar{x})\hat{s}_t - (\bar{q}/\bar{x})\hat{q}_t \\
\hat{p}_t &= (\bar{q}/\bar{p})nD_1(n\bar{q},\bar{\varepsilon})\hat{q}_t + (\bar{\varepsilon}/\bar{p})D_2(n\bar{q},\bar{\varepsilon})\hat{\varepsilon}_t
\end{aligned} \tag{6}$$

where D_i is the partial derivative of the inverse demand function with respect to the *i*th argument and $\hat{\varepsilon}_t$ is the percent deviation from its mean.

We next compute a simplified reduced form for our model and analyze the REE. Combine equations (5) and (6) to eliminate the variables *q* and *s*. This leaves,

$$\begin{aligned}
\hat{p}_t &= E_t\hat{p}_{t+1} + \alpha\hat{x}_t \\
\hat{x}_t &= \gamma_1\hat{x}_{t-1} + \gamma_2\hat{p}_t + \hat{\mu}_t
\end{aligned}, \tag{7}$$

where $\alpha = \beta\bar{x}g''(\bar{x}) < 0$, $\gamma_1 = 1 + g'(\bar{x}) > 0$, $\gamma_2 = -\bar{p}/(\bar{x}nD_1(n\bar{q},\bar{\varepsilon})) > 0$, and $\hat{\mu}_t = \bar{\varepsilon}D_2(n\bar{q},\bar{\varepsilon})/(\bar{x}nD_1(n\bar{q},\bar{\varepsilon}))\hat{\varepsilon}_t$. A REE of the model may be obtained by finding a stationary solution to the linear expectational difference equation (7). A REE, however, may not exist and, even if it does, the equilibrium may not be unique. If a unique REE exists, then the model is said to be *determinate*. Whether or not the model is determinate depends on the magnitude of certain eigenvalues (details are found in Appendix A). We have the following proposition regarding the determinacy of the model:

Proposition 1. *If the steady-state is economically reasonable, then the model is determinate if and only if the reduced-form parameters satisfy,*

$$(1+\gamma_1-\alpha\gamma_2)^2 > 4\gamma_1; \tag{8}$$

otherwise, no equilibrium exists.

Proof of Proposition 1. *The proof is omitted, but available from the authors upon request. The restriction in (8) is equivalent to the assumption that the eigenvalues* λ_i *(see Appendix A) are real.*

If the steady-state parameters are economically reasonable and the reduced-form parameters satisfy (8), then the unique REE may be obtained using standard techniques, as described in the Appendix. From equation (A4), we see that the equilibrium must satisfy,

$$\begin{aligned} \hat{p}_t &= \phi_1 \hat{x}_{t-1} + \phi_2 \hat{\mu}_t \\ \hat{x}_t &= \gamma_1 \hat{x}_{t-1} + \gamma_2 \hat{p}_t + \hat{\mu}_t \end{aligned} . \tag{9}$$

Equation (9) could be placed in standard VAR form, but the one-dimensional nature of the dynamics permits a simple ARMA(1,1) representation, as follows:

$$\hat{p}_t = (\gamma_1 + \phi_1 \gamma_2) \hat{p}_{t-1} + (\phi_1 - \phi_2 \gamma_1) \hat{\mu}_{t-1} + \phi_2 \hat{\mu}_t . \tag{10}$$

We conclude that, provided an interior steady-state exists, the associated unique efficient stationary price process will have an ARMA(1,1) structure.

Before providing an example, two points about the solution are noted. First, the above analysis was conducted under the assumption that an interior solution to the agent's optimization problem exists. Now notice that if a reasonable steady-state exists, then there is an interior solution to the corresponding non-stochastic model. Further, provided the shocks are small enough to maintain the linear approximation, we have found an interior solution to the stochastic problem as well. Second, a consequence of linearizing the model is that the disturbance terms in (10) are proportional to the demand shock $\hat{\varepsilon}_t$. This implies that the unconditional price distribution will be symmetric if the demand shock distribution is symmetric. This need not be the case for a non-linearized model, as our simulations will demonstrate.

Example

We provide an example to show that an economically reasonable determinate case exists, and to give additional intuition for the preceding results. For simplicity, we adopt the following linear specification of the inverse demand function,

$$p_t = D(nq_t, \varepsilon_t) = \varepsilon_t - c_1 nq_t \tag{11}$$

where $c_1 > 0$ and ε_t is i.i.d. with mean $c_0 > 0$ and variance σ^2. For the growth function, we specify,

$$g(x) = g_0 + x^\theta,$$

where $g_0 \geq 0$ is the timber growth with zero stock and θ captures the curvature of the growth function.

With the functional forms specified above, there are seven structural parameters in our model. We assign the following values to these parmeters: $\beta = 0.95$, $g_0 = 10$, $\theta = 0.1$, $c_0 = 10$, $c_1 = 0.8$, $\sigma^2 = 1$, and $n = 1$. The steady-state values are then $\bar{x} = 2.04$, $\bar{q} = 11.07$, $\bar{p} = 1.14$, and $\bar{s} = 13.11$.[5] Thus, we see that there exists an economically reasonable steady-state with positive prices, quantities, and stock levels. We can then compute the reduced-form parameters of the model in (7): $\alpha = -0.045$, $\gamma_1 = 1.05$, and $\gamma_2 = 0.7$. Following the procedure described in the Appendix, we obtain the stationary ARMA(1,1) process defined by,

$$\hat{p}_t = 0.86\hat{p}_{t-1} - 0.31\hat{\mu}_{t-1} + 0.04\hat{\mu}_t. \tag{12}$$

[5] Lowering the mean intercept to c_0=5 shifts the demand curve far enough inward so that the equilibrium price level is negative and, thus, the associated steady-state is not reasonable.

Given the assumed parameter values, the price expression in (12) completely characterizes the REE of our model.

Simulation of an Efficient Timber Market with Multiple Timber Age Classes

In this section, we simulate the infinite-horizon REE using an age-class model of the timber inventory commonly applied in the forestry literature. In the model, the fixed amount of land dedicated to timber production is normalized to one, and allocated across six timber age classes. The volume of timber per unit land (v_j), beginning with age class 1, is 1.0, 2.2, 3.35, 4.45, 5.5, and 6.5.[6] Thus, growth is concave, as timber grows by 1.2 units from age 1 to age 2, by 1.15 units from age 2 to age 3, and so on. At age 6, timber obtains old-growth status, in the sense that it no longer grows. With the total land in forests normalized to one, the amount of land in age class one, l_1, is simply,

$$l_1 = 1 - \sum_{j=2}^{6} l_j ,$$

and so the state of the forest is fully described by the amount of land in each of the age classes 2 through 6.

Over time timber grows from one age class to the next, unless it is cut, in which case it reverts to age class 1, or unless it attains old growth status, in which case it remains in age class 6 until harvested. Formally, then, the state of the forest evolves according to the following system, where h_{jt} is the amount of land in age class *j* harvested at time *t*:

[6] The results are robust to other representations of the timber growth function.

$$
\begin{aligned}
l_{1t} &= 1-\sum_{j=2}^{6} l_{jt} \\
l_{2,t+1} &= l_{1t}-h_{1t} = 1-\sum_{j=2}^{6} l_{jt}-h_{1t} \quad , \\
l_{j,t+1} &= l_{j-1,t}-h_{j-1,t} \qquad j=3,4,5 \\
l_{6,t+1} &= l_{5t}+l_{6t}-h_{5t}-h_{6t}
\end{aligned} \tag{13}
$$

with initial conditions on land,

$$
\begin{aligned}
& l_{j0} \geq 0 \qquad j=2,\ldots,6 \\
& \sum_{j=2}^{6} l_{j0} \leq 1 \qquad ,
\end{aligned} \tag{14}
$$

and with the harvested land in each age class constrained to be positive and less than the total land in the age class:

$$
0 \leq h_{jt} \leq l_{jt} \qquad j=1,\ldots,6 \quad \forall t \ . \tag{15}
$$

This last condition permits the harvest of all the timber in an age class, as well as all timber in all age classes. As such, we broaden the focus of the theoretical model to consider corner, in addition to interior, solutions of the model.

Whereas in the theoretical analysis the state of the forest is completely described by the stock of timber, here the stock of timber is not sufficient to identify the state of the forest; there is an infinite number of ways to arrange the age structure of the forest to obtain a given stock of standing timber. With the above notation, the total stock of timber is defined as:

$$
s_t = \sum_{j=1}^{6} l_{jt} v_j \ .
$$

The demand for timber takes the form in (11) with $c_1 = n = 1$, where q_t is the aggregate harvest at time *t*, $q_t = \sum_{j=1}^{6} h_{jt} v_j$, and ε_t is drawn from a discrete uniform distribution with values ranging from 9.6 to 10.4 in increments of 0.1.

In a competitive timber market, REE prices are those that maximize the expected discounted value of timber. Formally, we define $\mathbf{l}_t = \left(l_{jt}\right)_{j=2}^{6}$ as the vector of land state variables; $\mathbf{h}_t = \left(h_{jt}\right)_{j=1}^{6}$ as the vector of land areas harvested at time *t*; $w\left(\mathbf{l}_t, \varepsilon_t\right)$ as the state-dependent value of the forest at time *t*, given optimal decisions are made; and β as the discount factor (in the simulation, $\beta = 1/1.05$). Then REE prices are found by solving the infinite-horizon problem,

$$w\left(\mathbf{l}_t, \varepsilon_t\right) = \max_{\mathbf{h}_t} \left[\varepsilon_t q_t - \frac{1}{2} q_t^2 + \beta E_t w\left(\mathbf{l}_{t+1}, \varepsilon_{t+1}\right) \right], \tag{16}$$

subject to (13)-(15).

With the value function $w\left(\mathbf{l}_t, \varepsilon_t\right)$ unknown, solving for the infinite-horizon REE price function for a model such as this one involves the solution of a stochastic dynamic programming (SDP) problem with *J* state variables: *J*-1 variables describing the state of the forest, and the disturbance term ε. An obvious obstacle to numerically solving for REE prices is the so-called curse of dimensionality—the size of the programming problem increases exponentially as the number of variables in the state space increases linearly. Our choice of six age classes strikes a balance between complexity—in particular, having enough age classes so that the concavity of timber growth comes into play—and computational ease.

It is important to emphasize that, in the algorithm discussed below, we approximate the derivatives of $w(\mathbf{l}_t, \varepsilon_t)$, rather than in $w(\mathbf{l}_t, \varepsilon_t)$ itself, because good approximations of a function obtained from discrete methods often yield poor approximations of the function's derivatives, and, as revealed below, REE prices are tightly bound to the derivatives of $w(\mathbf{l}_t, \varepsilon_t)$. With this in mind, the solution algorithm involves the first-order necessary conditions and the adjoint equations. The former include,

$$\begin{aligned} & p_t v_j - \beta E\left\{\frac{\partial w_{t+1}}{\partial l_{j+1,t+1}}\right\} - \rho_{jt}^1 + \rho_{jt}^2 = 0, \quad j = 1,\ldots,5 \\ & p_t v_6 - \beta E\left\{\frac{\partial w_{t+1}}{\partial l_{6,t+1}}\right\} - \rho_{6t}^1 + \rho_{6t}^2 = 0, \end{aligned} \tag{17}$$

where ρ_{jt}^1 and ρ_{jt}^2 are Lagrange multipliers associated with constraint (15), and the time subscript on *w* indexes the state at which the derivative is evaluated. The adjoint equations are:

$$\begin{aligned} & \frac{\partial w_t}{\partial l_{jt}} = \rho_{jt}^1 + \beta E\left\{\frac{\partial w_{t+1}}{\partial l_{j+1,t+1}} - \frac{\partial w_{t+1}}{\partial l_{2,t+1}}\right\}, \quad j = 2,3,4,5 \\ & \frac{\partial w_t}{\partial l_{6t}} = \rho_{6t}^1 + \beta E\left\{\frac{\partial w_{t+1}}{\partial l_{6,t+1}} - \frac{\partial w_{t+1}}{\partial l_{2,t+1}}\right\}. \end{aligned} \tag{18}$$

Details on the solution algorithm are provided in Appendix B.

Figure 1 presents a typical 100-year sequence of REE prices from the model. Consistent with the results of the theoretical model, this price sequence exhibits the classical symptoms of a stationary, serially-correlated price series: prices cross the long-term mean price (8.89, denoted by the red-dashed line) frequently and some persistence in the price level is evident. We fit the AR(1) model,

$$p_t = d_0 + d_1 p_{t-1} + \eta_t ,$$

to a sequence of five thousand prices. This yields parameter values $d_0 = 5.95$ and $d_1 = 0.331$ with standard errors 0.119 and 0.0134, respectively, and an estimated standard deviation of η equal to 0.405. The coefficient on lagged price suggests a stationary process with a modest degree of serial dependence. We present these results purely for descriptive purposes. Below, we discuss a number of challenging econometric issues that arise with the estimation of ARMA models of timber prices.

As noted above, our timber model is similar to models of optimal commodity storage (e.g., Williams and Wright, 1991), except that these models typically assume the stock depreciates at a constant rate. An important result from this literature is that price changes are asymmetric: positive changes above the mean (spikes) tend to be larger in absolute value than negative changes below the mean (troughs). This asymmetry results from periodic stock-outs. When there is a positive demand shock (or negative supply shock) and a stock-out occurs, price arbitrage cannot moderate increases in the current price due to the impossibility of drawing on future stocks. In contrast, during periods of low demand when stock-outs do not occur, price declines are cushioned by carrying inventory forward to the future.

In our simulations, stock-outs never occur in a simulation of 10,000 periods. This accurately reflects the situation on the ground—the depletion of the timber inventory would never arise in the U.S. market, nor in most foreign markets. Yet, despite the absence of stock-outs, our simulated timber prices display the same asymmetric pattern of high peaks and low troughs. Figure 1 suggests that prices are right-skewed: prices are usually below average. This is confirmed in Figure 2, which presents the estimated unconditional probability density function for prices in the simulation model. The function was obtained via kernel density estimation using a normal kernel and the approximately optimal bandwidth described by Silverman (1986). The estimation involved 1000 prices, each of which was generated as the terminal price of a 500-period simulation, with the initial state of the forest chosen randomly. The null hypothesis of no skew in the distribution is rejected at the 1% level.

For timber, then, the explanation of asymmetric prices in an efficient market subject to symmetric shocks is not stock-outs *per se*, yet the fundamentals of the explanation are the same as in the storage literature. Essentially, there are “stock-outs” of slower-growing timber in the oldest age classes. When the older timber stock is depleted during periods of high demand, owners of younger, faster-growing timber may withhold timber from the market, preferring to forego high prices in the current period for the high growth in their timber. In contrast, during periods with low demand and a relative abundance of old timber, owners of older, slower-growing timber are more willing to supply in the current period because the opportunity cost of doing so is lower.

We demonstrate this dependence of prices on the stock level by plotting the difference between p_{t+1} and p_t against the stock level s_t for a typical 1,000-year sequence of prices (Figure 3). Note, first, that prices are more volatile at low stocks than at high stocks, even though stock-outs never arise.[7] Second, when stocks are low, prices at time t+1 are more likely to be lower than prices at time t (the price difference is less than zero). This is because when stocks are low, the only timber remaining is the fast-

[7] A stock-out is implied by a stock level of 1.0, which indicates that the entire forest is in stand age 1 (recall that the forest area is normalized to unity, and the volume of timber in stand age 1 is 1.0 per unit area). This is possible only if all stock in the previous period was harvested. The minimum stock in the 1000-year sequence is 1.364.

growing timber in the younger age classes, and the owners of this timber are willing to forego a high current price for a lower price in the future, in order to reap the benefits of relatively high timber growth. Note, too, that when stocks are high, the reverse relationship holds: prices are more likely to be higher in period *t*+1 than in period *t* (the price difference is greater than zero). When stocks are high, a large amount of timber is in older, slower-growing timber. The owners of this timber are willing to sell at a low price—foregoing the opportunity to fetch a higher price by postponing harvest—in order to move their land into younger, faster-growing trees.

Discussion and Conclusions

The key result from our theoretical model is that stationary serially-correlated prices can arise in an efficient timber market even when market shocks are i.i.d. Before we can explore the implications of this result for the optimal harvesting literature, we must be clear about the meaning of "efficient market." The notion of an efficient market is frequently associated with Eugene Fama. In an influential paper, Fama (1970) attempts to formalize the idea that efficient markets fully reflect available information by requiring the forecast errors to form a martingale difference sequence. More specifically, he says the price process p_t is efficient with respect to the information sets I_t if

$$x_{t+1} = p_{t+1} - E_t p_{t+1} \tag{19}$$

satisfies $E_t x_{t+1} = 0$, where $E_t(\cdot) = E\left(\cdot \middle| I_t\right)$. Various forms of efficiency differ according to the content of the information set; for example, weak-form efficiency refers to the case in which I_t contains exactly the history of past prices. Unfortunately, this formal definition is of little use. As LeRoy (1989) notes, Fama's definition is a tautology and hence has no meaningful implications. To see this, simply apply the expectations operator to both sides of (19) and note that $E_t(E_t(p_{t+1})) = E_t(p_{t+1})$ from the law of iterated expectations.

In a subsequent paper that attempts to address these difficulties, Fama (1976) offers the following alternative: a market is efficient if agents are rational and use all relevant available information to form expectations. A slightly more formal statement is that a market is efficient with respect to the information sets I_t provided agents form their expectations mathematically conditional on these information sets. Note, then, that any rational expectations equilibrium is efficient with respect to the information set imposed by the model. In particular, the ARMA(1,1) price process derived from our theoretical model is efficient with respect to the information set containing all past values of prices and shocks. It is important to keep in mind that these prices are an equilibrium result of rational, optimizing behavior. Thus, even if prices are serially correlated, as in (12), there is no scope for agents to earn even higher returns by exploiting stochastic price variations. While this statement may seem at odds with the central conclusion of the optimal harvesting literature, it should be remembered that the point of comparison in these studies is a myopic Faustmann rule. In an efficient market, agents cannot generate higher returns because all relevant information about the structure of the market is already incorporated into their decision calculus.

As noted above, authors of previous studies have motivated the selection of non-stationary random walk processes for timber prices on the grounds that such prices are consistent with an efficient timber market. One finds a source for this claim in Fama (1970): "… it is best to regard the random walk model as an extension of the general …

efficient markets model in the sense of making a more detailed statement of the economic environment." However, our results show that market efficiency provides little justification, at least in the case of timber markets, for random walk prices. The ARMA (1,1) process derived above represents efficient prices and may be stationary depending on the values of model parameters. In our example, we show that a stationary process can be obtained with economically reasonable choices of parameters. Of course, there is no reason to think that an efficient market cannot also produce non-stationary prices. Thus, market efficiency alone provides little guidance for the specification of a price process.

A similar critique applies to studies that use time-series data on timber prices to test for the efficiency of timber markets (Washburn and Binkley, 1990; Haight and Holmes, 1991; Hultkrantz, 1993; Yin and Newman, 1995; Abildtrup *et al*., 1997). These studies focus on weak-form efficiency; that is, efficiency is defined with respect to the information set containing past prices. The general approach is to apply Dickey-Fuller unit root tests to historical price series, where failure to reject the unit root (indicating non-stationarity) is taken as evidence of market efficiency.[8] In its simplest form, the Dickey-Fuller test requires estimation of,

$$\Delta p_t = b_0 + b_1 p_{t-1} + \varepsilon_t \tag{20}$$

where p_t is the logged price in time t, $\Delta p_t = p_t - p_{t-1}$, b_0 and b_1 are model parameters, and ε_t is a normally distributed disturbance term with zero mean.[9] The null hypothesis is

[8] The test in Washburn and Binkley (1990) has a slightly different form; see their reply (Washburn and Binkley, 1993) to the comment by Hultkrantz (1993).

[9] Haight and Holmes, Yin and Newman, and Abildtrup *et al*. also include lagged, first-differenced price terms, which accommodates a more complicated moving average error structure (this is an augmented Dickey-Fuller test). The null hypothesis for this version of the model is also $b_1 = 0$.

$b_1 = 0$, implying that prices follow a random walk process with drift rate b_0. In most of the timber price studies, the null hypothesis of a unit root is rejected. Our results suggest, however, that a finding of stationarity does not distinguish between efficient and inefficient markets. For the ARMA(1,1) process in (12), $b_1 = 0.14$. For the simulated prices series discussed in the previous section, $b_1 = 0.67$. Thus, we see that rejection of the unit root should not be accepted as evidence of market inefficiency. Nor should it be viewed as evidence of efficiency since there is no reason to think that inefficient markets cannot generate stationary prices.

A key insight provided by the simulations is that the concavity of the timber growth function may give rise to an asymmetric unconditional price distribution. If prices are modeled as an ARMA(p,q), then the asymmetry of the unconditional price distribution implies that the disturbance term in the econometric model is asymmetric as well.[10] In earlier studies, researchers who fit empirical models to historical price series have assumed symmetric errors. While an incorrect assumption of symmetric disturbances will not bias least squares coefficient estimates, it can produce sub-optimal harvesting rules. In particular, under an assumption of symmetric prices, landowners will tend to regard high prices as a rarer event than they actually are. Thus, landowners will be induced to harvest in cases where postponing the harvest decision is optimal.

The discussion thus far would appear to argue the case for empirically-based models of timber prices, provided appropriate attention is paid to the structure of the error

[10] For an ARMA(p,q) process with symmetric disturbances, it can be shown that the unconditional price distribution is symmetric. The proof involves rewriting the right-hand side of the ARMA model as a linear combination of disturbances and then showing that a linear combination of symmetric disturbances is itself symmetric. It follows, therefore, that if the unconditional price distribution is asymmetric, then the disturbance term in the ARMA price model is asymmetric.

terms. However, it must be recognized that not all REEs can be represented as ARMA processes. That it could in the model analyzed above is a consequence of the simple stock dynamics in (2). If multiple stock variables are required to represent the timber inventory, as with the age-class model considered in the simulations, then the solution cannot be represented by a single equation. In general, a system of equations (a vector autoregressive moving average model, or VARMA) that includes stock variables in addition to prices and shocks will need to estimated. Given the heterogeneity of timber resources, such models could be difficult to estimate and apply to actual harvesting decisions. Thus, future research might compare the performance of simple models of timber prices to models providing a more complex representation of market equilibria.

References

Abildtrup, J., Riis, J., and B.J. Thorsen. 1997. The Reservation Price Approach and Informationally Efficient Markets. *J. For. Econom*. 3(3):229-45.

Albers, H.J. 1996. Modeling Ecological Constraints on Tropical Forest Management: Spatial Interdependence, Irreversibility, and Uncertainty. *J. Environ. Econ. Manage.* 30:73-94.

Arrow, K.J., and A.C. Fisher. 1974. Environmental Preservation, Uncertainty, and Irreversibility. *Quart. J. Econ*. 88:312-19.

Berck, P. 1981. Optimal Management of Renewable Resources with Growing Demand for Stock Externalities. *J. Environ. Econ. Manage.* 8(2):105-17.

Brazee, R., and R. Mendelsohn. 1988. Timber Harvesting with Flucuating Prices. *For. Sci.* 34:359-372.

Brennan, M.J., and E.S. Schwartz. 1985. Evaluating Natural Resource Investments. *J. Business* 58(2):135-57.

Clarke, H.R., and W.J. Reed. 1989. The Tree-Cutting Problem in a Stochastic Environment: The Case of Age-Dependent Growth. *J. Econ. Dynamics Control* 13:569-595.

Dasgupta, P.S., and G.M. Heal. 1974. The Optimal Depletion of Exhaustible Resources. *Rev. Econ. Studies* 41:3-28.

Deaton, A., and G. Laroque. 1992. On the Behaviour of Commodity Prices. *Rev. Econ. Studies* 59:1-23.

Deaton, A., and G. Laroque. 1996. Competitive Storage and Commodity Price Dynamics. *J. Polit. Economy* 104:896-923.

Evans, G., and E. Honkaponja. 2001. *Learning and Expectations in Macroeconomics*. Princeton University Press.

Evans, G., and B. McGough. 2002. Stable Sunspot Solutions in Models with Predetermined Variables. Unpublished paper, Department of Economics, Oregon State University.

Fama, E.F. 1970. Efficient Capital Markets: A Review of Theory and Empirical Work. *Journal of Finance* 25:383-417.

Fama, E.F. 1976. Efficient Capital Markets: Reply. *Journal of Finance* 31(1):143-45.

Farmer, R. 1999. *The Macroeconomics of Self-Fulfilling Prophecies*. MIT Press, Cambridge, 2nd edition.

Fisher, A.C., and W.M. Hanemann. 1986. Option Value and the Extinction of Species. *Advances Appl. Micro-Econ*.(4):169-90.

Gong, P. 1999. Optimal Harvest Policy with First-Order Autoregressive Price Process. *Journal of Forest Economics* 5(3):413-39.

Haight, R.W., and T.P. Holmes. 1991. Stochastic Price Models and Optimal Tree Cutting: Results for Loblolly Pine. *Natural Resour. Modeling* 5:423-443.

Haight, R.W., and W.D. Smith. 1991. Harvesting Loblolly Pine Plantations with Hardwood Competition and Stochastic Prices. *Forest Science* 37(5):1266-82.

Hultkrantz, L. 1993. Informational Efficiency of Markets for Stumpage: Comment. *American Journal of Agricultural Economics* 75:234-238.

Kydland, F., and C. Prescott. 1982. Time to Build and Aggregate Flucuations. *Econometrica* 50:1345-70.

LeRoy, S.F. 1989. Efficient Capital Markets and Martingales. *J. Econom. Lit.* 27(4):1583-1621.

Lohmander, P. 1992. The Multi-Species Forest Stand, Stochastic Prices and Adaptive Selective Thinning. *Syst. Anal. Model. Simul*. 9:229-50.

Miranda, Mario J., and Paul L. Fackler. 2002. *Applied Computational Economics and Finance*. MIT Press, Cambridge.

Morck, R., E. Schwartz, and D. Stangeland. 1989. The Valuation of Forestry Resources Under Stochastic Prices and Inventories. *J. Financ. Quant. Econom*. 24:473-487.

Norstrom, C.J. 1975. A Stochastic Model for the Growth Period Decision in Forestry. *Swed. J. Econ*. 77:329-37.

Pindyck, R.S. 1984. Uncertainty in the Theory of Renewable Resource Markets. *Rev. Econ. Studies* 51:289-303.

Plantinga, A.J. 1998. The Optimal Rotation Decision: An Option Value Approach. *For. Sci*. 44(2):192-202.

Reed, W.J. 1993. The Decision to Conserve or Harvest Old-Growth Forest. *Ecol. Econom.* 8:45-69.

Saphores, J.-D., Khalaf, L., and D. Pelletier. 2002. On Jumps and ARCH Effects in Natural Resource Prices: An Application to Pacific Northwest Stumpage Prices. *American Journal of Agricultural Economics* 84(2):387-400.

Silverman, B.W. 1986. *Density Estimation*. Chapman and Hall.

Thomson, T.A. 1992. Optimal Forest Rotation When Stumpage Prices Follow a Diffusion Process. *Land Econom*. 68:329-342.

Washburn, C., and C. Binkley. 1990. Informational Efficiency of Markets for Stumpage. *Amer. J. Agric. Econom.* 72:394-405.

Washburn, C.L., and C.S. Binkley. 1993. Informational Efficiency of Markets for Stumpage. *Amer. J. Agric. Econom*. 72:239-242.

Williams, J.C., and B.D. Wright. 1991. Storage and Commodity Markets. Cambridge University Press.

Yin, R., and D.H. Newman. 1995. Are Markets for Stumpage Informationally Efficient? *Can. J. For. Res*. 26:1032-1039.

Yin, R., and D.H. Newman. 1997. When to Cut a Stand of Trees? *Natural Resource Modeling* 10(3):251-61.

Figure 1. Typical Sequence of REE Prices

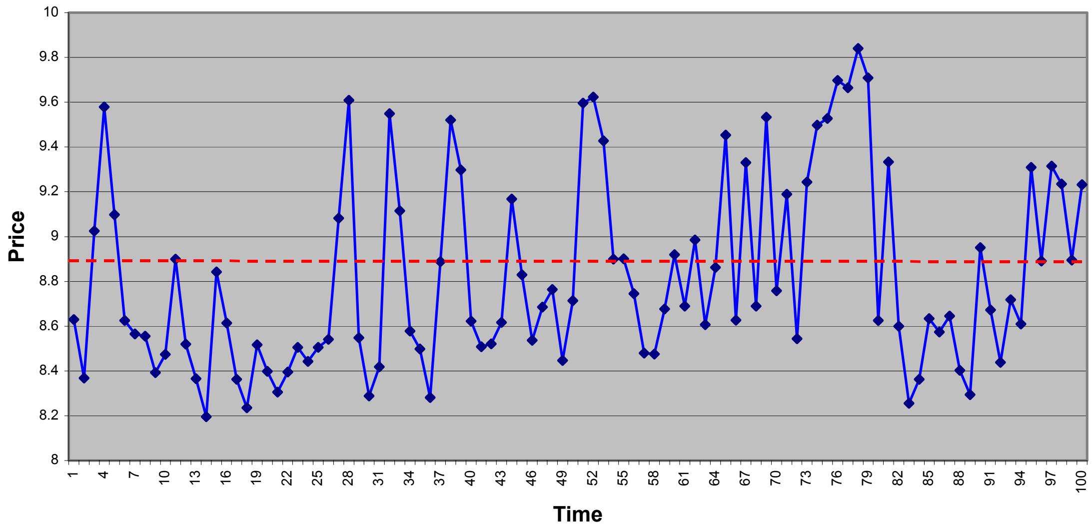


Figure 2. Estimated Unconditional Probability Density Function of REE Prices

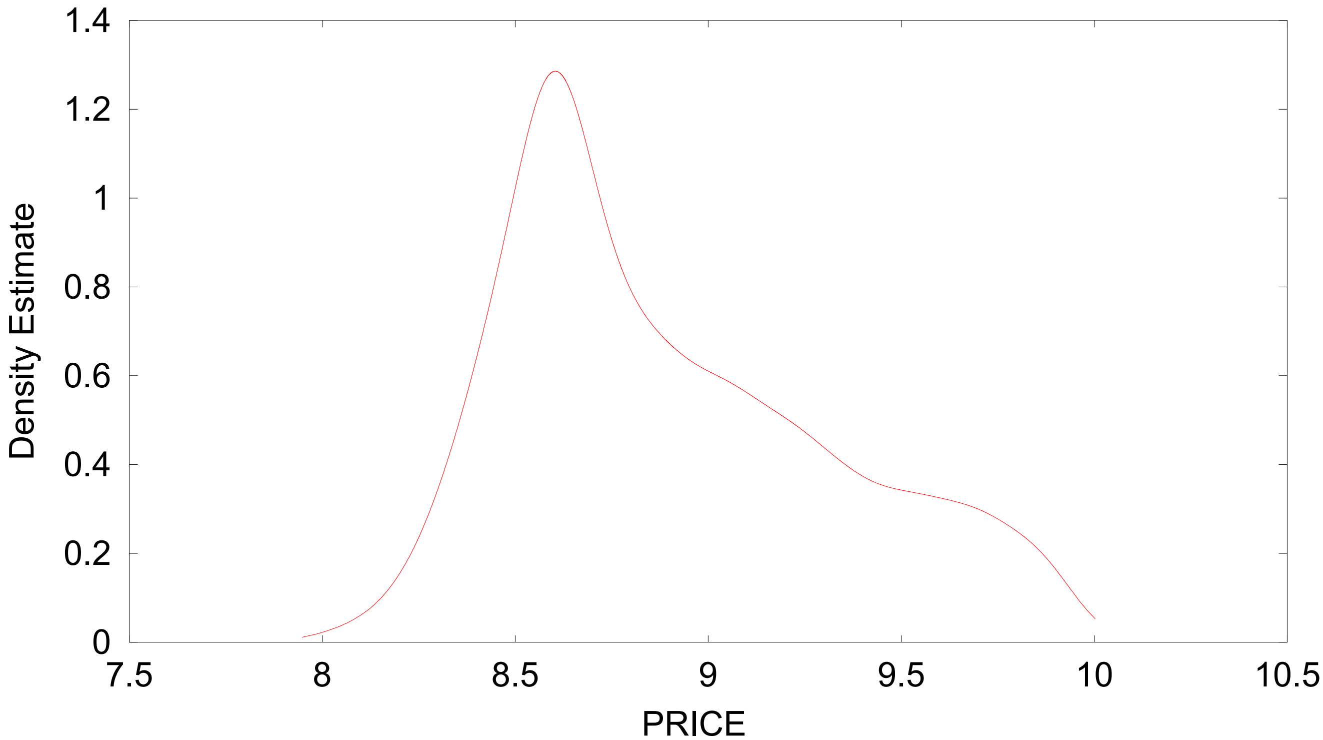
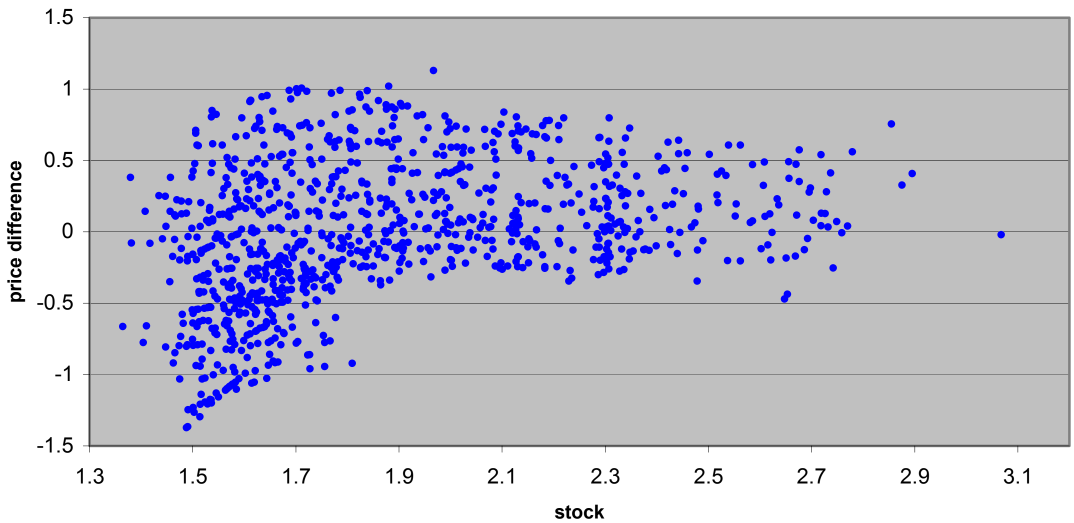


Figure 3. Effect of the Timber Stock on the Price Difference, $p_{t+1} - p_t$



Appendix A

In this appendix, we obtain the REE of the reduced-form model in (7). For notational ease, we make the following variable change: $\tilde{x}_{t+1} = \hat{x}_t$. Then, (7) can be written,

$$\begin{bmatrix} 1 & 0 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} \hat{p}_t \\ \tilde{x}_t \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_t \hat{p}_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{\mu}_t \end{bmatrix},$$

or, more compactly, $Hy_t = FE_t y_{t+1} + \eta_t$. Recall that a martingale difference sequence is any stochastic process y_t that satisfies $E_t y_{t+1} = 0$. Now define the forecast error as the 2-vector $\xi_t = y_t - E_{t-1} y_t$. Because agents in the model are assumed to be rational, the expectations operator is the conditional expectations operator (in particular, the information set includes the structural equations (5) and (6)). It follows from the law of iterated expectations that ξ_t is a martingale difference sequence. Furthermore, because $\tilde{x}_{t+1}$ is known at time *t*, $E_t(\tilde{x}_{t+1}) = \hat{x}_t$. Thus, the forecast error for the second element of ξ_t is zero; that is, $\xi_{2,t+1} = \tilde{x}_{t+1} - E_t \tilde{x}_{t+1} = 0$.

Using the definition of the forecast error, the reduced-form model may be written,

$$y_t = F^{-1} H y_{t-1} - F^{-1} \eta_{t-1} + \xi_t . \tag{A1}$$

We may conclude that if y_t is a REE, then there exists a martingale difference sequence ξ_t with $\xi_{2,t} = 0$ so that y_t satisfies (A1). On the other hand, it may not be the case that any such martingale difference sequence yields an REE. To more precisely determine the number and nature of the equilibria, we now analyze which martingale difference sequences yield reasonable solutions to (A1).

The linearization technique used to obtain (5) and (6) requires us to focus on (asymptotically) stationary equilibria, and for obvious reasons. If the equilibrium is non-stationary, the process will tend to drift away from the steady-state and, thus, from the neighborhood in which the linearization applies. Even near a steady-state there may exist many stationary REE. The steady-state is determinate if there is associated a unique stationary REE; if many exist the steady-state is indeterminate and if none exist it is explosive. Whether or not our steady-state is determinate will depend on the modulus of the eigenvalues λ_i of the matrix $F^{-1}H$. This dependence is most easily seen by stacking our potential REE (A1) in VAR form. We write $w_t = [\hat{p}_t, \tilde{x}_t, \hat{\mu}_t]'$ and,

$$G = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$F^{-1}H$}} & -\alpha \\ & & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$w_t = Gw_{t-1} + [\xi_t^{'}, \hat{\mu}_t]'. \tag{A2}$$

It is straightforward to show that G is diagonalizable with eigenvalues λ_i and zero, and so we write $G = S\Lambda S^{-1}$, where S is the matrix of eigenvectors and $\Lambda = Diag[\lambda_1, \lambda_2, 0]$. Changing coordinates to $z_t = S^{-1}w_t$ allows us to write (A2) as,

$$z_t = \Lambda z_{t-1} + S^{-1}[\xi_t^{'}, \hat{\mu}_t]'. \tag{A3}$$

We see now that if both eigenvalues lie in the unit circle then for any martingale difference sequence, $\xi_{1,t}$, the resulting VAR is stationary (in this case, $z_{1,t}$ and $z_{2,t}$ are first-order autoregressive processes with coefficients on the lagged variables less than one). This is the indeterminate case and the associated martingale difference sequences are sometimes referred to as sunspots. We also see that if both eigenvalues lie outside the unit circle, then for any $\xi_{1,t}$ the result VAR is explosive. In this case, the process would tend to move away from the steady-state and the non-linear dynamics of the model would come into play. Finally, if one eigenvalue is in the unit circle and one is outside, the determinate case obtains; there is a unique martingale difference sequence for which (A3) is stationary and the associated dynamics of w_t take place entirely in the contracting eigenspace. Proposition 1 characterizes the regions of the parameter space corresponding to these types of steady-states; see text equation (8).[11]

We will assume that the reduced-form parameters satisfy (8) and, without loss of generality, that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Stationarity of (A3) then requires that $z_{1,t} = 0$ and $S^{11}\xi_{1,t} = -S^{13}\hat{\mu}_t$, where S^{ij} is the *ij*-component of S^{-1}. The associated REE may then be written,

$$\begin{aligned} \hat{p}_t &= -\frac{S^{12}}{S^{11}}\tilde{x}_t - \frac{S^{13}}{S^{11}}\hat{\mu}_t \\ \tilde{x}_t &= \gamma_1\tilde{x}_{t-1} + \gamma_2\hat{p}_{t-1} + \hat{\mu}_{t-1} \end{aligned} \quad . \tag{A4}$$

Equation (9) is derived from (A4) by defining $\phi_1 = -S^{12}/S^{11}$, $\phi_2 = -S^{13}/S^{11}$, replacing $\tilde{x}_t$ with $\hat{x}_{t-1}$, and moving the second equation forward by one period.

[11] This method of obtaining the REE is discussed extensively in Evans and McGough (2002).

Appendix B

In this appendix, we provide details on the algorithm for solving the simulation model. The algorithm involves three steps.

1. With the five expected marginal value functions $\beta E\left\{\frac{\partial w(\mathbf{l},\varepsilon)}{\partial l_j}\right\}$ approximated from the previous iteration (initially these functions are identically equal to zero), for each grid point in the state space a search across harvest volumes is made to find the harvest decision $\mathbf{h}$ satisfying the six necessary conditions. This search exploits the structure of the problem; namely, that because the rate of growth of timber declines with stand age, harvest must proceed monotonically from age class six to age class one. So long as some, but not all, timber is harvested, the solution is characterized by the result that

$$p\left(q^{REE}(\mathbf{h},\varepsilon),\varepsilon\right)v_j - \beta E\left\{\frac{\partial w_{t+1}}{\partial l_{j+1,t+1}}\right\} = 0 \tag{B1}$$

for the age class *j* for which the acreage harvested is an interior solution ($0 < h_j < l_j$) or a degenerate corner solution (i.e., both B1 holds and $\rho^1 = 0$). Otherwise, the equilibrium price is found from the inverse demand function with all timber stock consumed. The solution is thus quickly bracketed within an age class, and quasi-Newton methods are then used to find the solution to (B1) for the candidate age class.

2. Given the solution of the problem in step 1 at all of the state grid points, the solution values of ρ_j^1 are used in the adjoint equations to find new values of $\frac{\partial w(\mathbf{l},\varepsilon)}{\partial l_j}$. Taking the expectation of these values over the disturbance term yields new approximations of the functions $E\left\{\frac{\partial w(\mathbf{l},\varepsilon)}{\partial l_j}\right\}$ at the grid points. To these grid points a 5-dimensional Chebyshev polynomial of order five is fit (note, then, that the grid points are the Chebyshev nodes).[12] In summary, the outcome at each iteration of the algorithm is a set of approximations of the five expected marginal value functions, each approximation being a five-dimensional Chebyshev polynomial.

The advantage of using Chebyshev polynomials to approximate functions is well-documented in Miranda and Fackler (2002). Not only do Chebyshev polynomials satisfy certain minimax theorems of approximation (theorems concerned with whether a polynomial minimizes the maximum approximation error), but coefficients of the polynomials are obtained by exceptionally rapid algorithms.

[12] We also fit a polynomial of order seven and found little effect on the results.

3. The algorithm returns to step 1 with the new approximations of $E\left\{\frac{\partial w(\mathbf{l},\varepsilon)}{\partial l_j}\right\}$, and terminates when the new approximations of equilibrium prices, $p\left(q^{REE}(\mathbf{h},\varepsilon),\varepsilon\right)$, defined in (B1), are "close enough" to the old approximations.[13] Sequences of REE prices are then generated by substituting sequences of random draws from the distribution of demand shocks into the final version of (17) and solving for the REE prices, using the state equations (13) to update the state of the forest.

[13] The convergence criterion used in the algorithm is that at each grid point, prices are within 1.0^{-5} percent of their values in the previous iteration.