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# Strategic Basins of Attraction, the Path Dominance Core, and Network Formation Games

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## Abstract

Given the preferences of players and the rules governing network formation, what networks are likely to emerge and persist? And how do individuals and coalitions evaluate possible consequences of their actions in forming networks? To address these questions we introduce a model of network formation whose primitives consist of a feasible set of networks, player preferences, the rules of network formation, and a dominance relation on feasible networks. The rules of network formation may range from noncooperative, where players may only act unilaterally, to cooperative, where coalitions of players may act in concert. The dominance relation over feasible networks incorporates not only player preferences and the rules of network formation but also assumptions concerning the degree of farsightedness of players. A specification of the primitives induces an abstract game consisting of (i) a feasible set of networks, and (ii) a path dominance relation defined on the feasible set of networks. Using this induced game we characterize sets of network outcomes that are likely to emerge and persist. Finally, we apply our approach and results to characterize the equilibrium of well known models and their rules of network formation, such as those of Jackson and Wolinsky (1996) and Jackson and van den Nouweland (2005).

**KEYWORDS:** basins of attraction, network formation games, stable sets, path dominance core, Nash networks

**JEL Classifications:** A14, C71, C72

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# 1 Introduction

## 1.1 Overview of the questions, the model and the main results

In many economic and social situations the totality of interactions between individuals and coalitions can be modeled as a network. We address the following question: given preferences of individuals and rules governing network formation, what networks are likely to emerge and persist? To address this question we introduce a model of network formation whose primitives consist of a feasible set of networks, player preferences, the rules of network formation, and a dominance relation. The rules of network formation may range from noncooperative, where players may only act unilaterally, to fully cooperative, where coalitions consisting of multiple players may act in concert. The dominance relation may be either direct or indirect. Under direct dominance players are concerned with immediate consequences of their network formation strategies whereas under indirect dominance players are farsighted and consider the eventual consequences of their strategies. As we will discuss, our framework can accommodate a wide variety of social and economic situations.

A specification of the primitives induces an abstract game consisting of (i) a feasible set of networks and (ii) a path dominance relation defined on the feasible set of networks. Under the path dominance relation, a network  $G$  path dominates another network  $G'$  if there is a finite sequence of networks, beginning with  $G$  and ending with  $G'$  where each network along the sequence dominates its predecessor.<sup>1</sup> Using this induced abstract game as our basic analytic tool we demonstrate that for any set of primitives the following results hold:

1. The feasible set of networks contains a unique, finite, disjoint collection of nonempty subsets each constituting a *strategic basin of attraction*. Given preferences and the rules of governing network formation, these basins of attraction are the absorbing sets of the process of network formation modeled via the game.
2. A stable set (in the sense of von Neumann Morgenstern) with respect to path dominance consists of one network from each basin of attraction.
3. The path dominance core, defined as a set of networks having the property that no network in the set is path dominated by any other feasible network, consists of one network from each basin of attraction containing a *single* network. Note that the path dominance core is contained in each stable set and is nonempty

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<sup>1</sup>Stated formally, given feasible set of networks  $\mathbb{G}$  and dominance relation  $>$ , network  $G' \in \mathbb{G}$  (weakly) path dominates network  $G \in \mathbb{G}$ , written  $G' \geq_p G$ , if  $G' = G$  or if there exists a *finite* sequence of networks  $\{G_k\}_{k=0}^n$  in  $\mathbb{G}$  with  $G = G_0$  and  $G' = G_n$  such that for  $k = 1, 2, \dots, n$

$$G_k > G_{k-1}.$$

The path dominance relation  $\geq_p$  induced by the dominance relation  $>$  is sometimes referred to as the transitive closure of  $>$ .

if and only if there is a basin of attraction containing a single network.<sup>2</sup> As a corollary, we conclude that any network contained in the path dominance core is constrained Pareto efficient. Thus, by considering the network formation game with respect to path dominance – and thus by considering the long run – we identify networks that, given the rules of network formation, are *both* stable and Pareto-efficient with respect to the original dominance relation.

4. From the results above it follows that if the dominance relation is transitive and irreflexive, then the path dominance core is nonempty.

We also demonstrate specializations of our model to treat network formation games over linking networks as well as hedonic games and we discuss how our results apply to these examples.

There are interesting connections between our notions of stability (basins of attraction, path dominance stable sets, and path dominance core) and some of the basic notions of stability and equilibrium found in the existing literature – such as, strong stability (Jackson and van den Nouweland 2005), pairwise stability (Jackson and Wolinsky 1996), consistency (Chwe 1994), and Nash equilibrium. We show that in general (for all primitives) the path dominance core is contained in the set of strongly stable networks. We conclude from our general results therefore that, for all primitives, the existence of at least one basin of attraction containing a single network is sufficient for the existence of a strongly stable network. We also demonstrate that, depending on how we specialize the primitives of the model, the path dominance core is equal to the set of strongly stable networks, the set of pairwise stable networks, or the set of Nash networks.

Of particular interest are the connections between the rules of network formation, the dominance relation inducing path dominance, and stability.<sup>3</sup> We provide a unified and systematic analysis of these connections. For example, we show that:

- (a) If path dominance is induced by a direct dominance relation (as opposed to an indirect dominance relation as in Chwe 1994, for example), then the path dominance core is equal to the set of strongly stable networks.
- (b) If, in addition, the rules of network formation are the Jackson-Wolinsky rules, then the path dominance core is equal to the set of pairwise stable networks.<sup>4</sup>
- (c) If path dominance is induced by a direct dominance relation and if the rules of network formation only allow network changes brought about by individuals, then the path dominance core is equal to the set of Nash networks.

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<sup>2</sup>Put differently, the path dominance core is empty if and only if *all* basins of attraction contain multiple networks.

<sup>3</sup>Although she treats a more specialized model, the questions addressed in Demange (2004) are related.

<sup>4</sup>Under the Jackson-Wolinsky rules arc addition is bilateral (i.e., the two players that would be involved in the arc must agree to adding the arc), arc subtraction is unilateral (i.e., at least one player involved in the arc must agree to subtract or delete the arc), and network changes take place one arc at a time (i.e., in any one play of the game, only one arc can be added or subtracted). See section 3.2.1 for a formal definition.

We then conclude from (3) above, the existence of at least one basin of attraction containing a single network is, depending on how we specialize primitives, both necessary and sufficient for either (i) the existence of a strongly stable network, or (ii) a pairwise stable network, or (iii) a Nash network.<sup>5</sup>

For path dominance induced by an indirect dominance relation as in Chwe (1994), we show that for all primitives - and in particular for all rules of network formation - each strategic basin of attraction has a nonempty intersection with the largest consistent set of networks (i.e., the Chwe set of networks, see Chwe 1994).<sup>6</sup> This fact, together with (2) above, implies that there always exists a path dominance stable set contained in the largest consistent set. Thus, the path dominance core is contained in the largest consistent set. In light of our results on the path dominance core and stability (both strong and pairwise), we conclude that if path dominance is induced by an indirect dominance relation, then any network contained in the path dominance core is not only consistent but also strongly stable, as well as pairwise stable.<sup>7</sup>

We remark that solution concepts defined using dominance relations have a distinguished history in the literature of game theory. First, consider the von-Neuman-Morgenstern stable set. The vN-M stable set is defined with respect to a dominance relation on a set of outcomes and consists of those outcomes that are externally and internally stable with respect to the given dominance relation.<sup>8</sup> Similarly, Gilles (1959) defines the core based on a given dominance relation. These solution concepts, with a few exceptions, have typically been applied to models of economies or cooperative games where the notion of dominance is based on what a coalition can achieve using only the resources owned by its members (cf., Aumann 1964) or a given set of utility vectors for each possible coalition (cf., Scarf 1967). Particularly notable exceptions are Schwartz (1974), Panzer, Kalai and Schmeidler (1976), Kalai and Schmeidler (1977) and Shenoy (1980). Their motivations are in part similar to ours in that they take as given a set of possible choices of a society and a dominance relation and, based on these, describe a set of possible or likely social outcomes called, by Kalai and Schmeidler, the admissible set. While their examples treat direct dominance, their general results have wider applications. We return to a discussion of the admissible set in our concluding section.

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<sup>5</sup>For Jackson-Wolinsky linking networks, Calvo-Armengol and Ilkilic (2004) provide necessary and sufficient conditions on the network link marginal payoffs such that the set of pairwise stable, pairwise Nash, and proper equilibrium networks coincide.

<sup>6</sup>Consistency with respect to indirect dominance and the notion of a largest consistent set were introduced by Chwe (1994) in an abstract game setting. We provide a detailed discussion of Chwe's notion in Section 5.3.

<sup>7</sup>Other papers on indirect dominance and consistency in games include Xue (1998), Diamantoudi and Xue (2003), and Mauleon and Vannetelbosch (2003).

<sup>8</sup>Richardson (1953) gives properties an irreflexive dominance relation must satisfy relative to a given set of outcomes in order to guarantee the existence of a vN-M stable set.

## 1.2 A further discussion of the model

In addition to introducing abstract games of network formation, our modeling approach contributes to the literature by extending the class of primitives used in the analysis of network formation in three respects. These extensions, listed below, significantly broaden the set of potential applications.

1. *Directed Networks with Heterogeneous Arcs and Multiple Uses of Arcs of the Same Type:* First, we focus on directed networks rather than on linking networks<sup>9</sup> and distinguish between nodes and decision making players (i.e., the set of players and the set of nodes are not necessarily the same). Connections are represented by arcs and each arc possesses an orientation or direction: arc  $a$  connecting nodes  $i$  and  $i'$  must either go from node  $i$  to node  $i'$  or must go from node  $i'$  to node  $i$ .<sup>10</sup> For example, an individual may have links on his web page to the web pages of all Nobel Laureates in economics but it may be that no Nobel Laureate has a link to that individual's web page. Connections between nodes (i.e., arcs), besides having an orientation, are allowed to be heterogeneous. To illustrate, if the nodes in a given network represent players, an arc  $a$  going from player  $i$  to player  $i'$  might represent a particular type and intensity of interaction (identified by the arc label  $a$ ) initiated by player  $i$  towards player  $i'$ . Player  $i$  might direct great affection toward player  $i'$  as represented by arc type  $a$ , but player  $i'$  may direct only lukewarm affection toward player  $i$  as represented by arc type  $a'$ .

Under our extended definition nodes are allowed to be connected by multiple, distinct arcs. Thus, we allow nodes to interact in multiple, distinct ways. For example, nodes  $i$  and  $i'$  might be connected by arcs  $a$  and  $a'$ , with arc  $a$  running from node  $i$  to  $i'$  and arc  $a'$  running in the opposite direction (i.e., from node  $i'$  to node  $i$ ).<sup>11</sup> If node  $i$  represents a seller and node  $i'$  a buyer, then arc  $a$  might represent a contract offer by the seller to the buyer, while arc  $a'$  might represent a counter offer or the acceptance or rejection of the contract offer. Finally, loops are allowed and arcs are allowed to be used multiple times in a given network.<sup>12</sup> For example, arc  $a$  might be used to connect nodes  $i$  and  $i'$  as well as nodes  $i'$  and  $i''$ . Thus, under our definition nodes  $i$  and  $i'$  as well as nodes  $i'$  and  $i''$  are allowed to engage in the same type of interaction as represented by arc type  $a$ .

Allowing each type of arc to be used multiple times makes it possible to distinguish coalitions by the type of interaction taking place between coalition members and to give a network representation of such coalitions. For example, if the nodes in a given network represent players, an ' $a$ -coalition' could consist of all players  $i$  having an  $a$ -connection with at least one other player  $i'$ . Such an  $a$ -coalition would then have a network representation as the directed subnetwork consisting of pairs of nodes,  $i$  and

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<sup>9</sup>In particular, we focus on the notion of directed networks introduced in Page, Wooders, and Kamat (2005).

<sup>10</sup>We denote arc  $a$  going from node  $i$  to node  $i'$  via the ordered pair  $(a, (i, i'))$ , where  $(i, i')$  is also an ordered pair. Alternatively, if arc  $a$  goes from node  $i'$  to node  $i$ , we write  $(a, (i', i))$ .

<sup>11</sup>Under our extended definition, arc  $a'$  might also run in the same direction as arc  $a$ . However, our definition does not allow arc  $a$  to go from node  $i$  to node  $i'$  multiple times.

<sup>12</sup>A loop is an arc going *from* a given node *to* that same node. For example, given arc  $a$  and node  $i$ , the ordered pair  $(a, (i, i))$  is a loop.

$i'$ , connected by arc type  $a$ .

Until now, most of the economic literature on networks has focused on linking networks (see Jackson 2005 for an excellent survey). In an undirected (or linking) network, an arc (or link) is identified with a nonempty subset of nodes consisting of exactly two distinct nodes, for example,  $\{i, i'\}$ ,  $i \neq i'$ . Thus, in an undirected network, a link has no orientation and simply indicates a connection between two players. Moreover, links are typically not distinguished by type (or by label) – that is, links are homogeneous. By allowing arcs to possess direction and the same type of arc to be used multiple times and by allowing loops and nodes to be connected by multiple arcs, our definition makes possible the application of networks to a rich set of economic environments. For example, a job opportunity market model may embody the features introduced above; individuals may have different relationships with their superiors in an organization and other individuals both within and outside of the organization. This may well affect social interactions and job opportunities.

*2. The Rules of Network Formation:* We explicitly model the rules of network formation and thus provide a systematic treatment of the relationship between rules and stability. The rules of network formation specify which players must be involved in adding, subtracting, or replacing an arc as well as how many and what types of arcs can be added, subtracted, or replaced in any one play of the game.

In much of the literature, it is assumed (sometimes implicitly) that network formation is governed by the Jackson-Wolinsky rules.<sup>13</sup> Other rules are possible. For example, the addition of an arc might require that a simple majority of the players agree to the addition while the deletion of an arc might require that a two-thirds majority agree to the deletion. Under our approach, such rules are allowed. We achieve this flexibility by representing the rules of network formation via a collection of coalitional effectiveness relations,  $\{\rightarrow_S\}_S$ , defined on the feasible set of networks. Given feasible networks  $G$  and  $G'$ , if the relation  $G \rightarrow_S G'$  holds, the players in coalition  $S$  can change network  $G$  to network  $G'$ . In constructing our abstract game of network formation, we will equip the feasible set of networks with a dominance relation which incorporates - or represents - *both* the preferences of individuals and coalitions *and* the rules of network formation as represented via the coalitional effectiveness relations  $\{\rightarrow_S\}_S$ . Thus, the stability results we obtain using the path dominance relation will reflect both preferences and rules.

*3. The Dominance Relation Defined on Feasible Networks:* While all of our main results (Section 4) hold for path dominance induced by any binary relation, we will focus primarily on path dominance induced by either direct dominance or indirect dominance (Sections 3.3.1 and 3.3.2).

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<sup>13</sup>Jackson-van den Nouweland (2005) focus on linking networks and assume that link addition is bilateral while link subtraction is unilateral. But in their model, network changes are not required to take place one link at a time - multiple link changes can take place in any one play of the game. We shall refer to these rules as the Jackson-van den Nouweland rules. Calvo-Armengol and Ilklic (2004) also consider linking networks under bilateral-unilateral rules and allow multiple link changes.

### 1.3 Examples

To demonstrate the flexibility of our approach and the wide applicability of our results, we consider three examples. Our first example treats noncooperative network formation games and shows that any such network formation game possessing a potential function has basins of attraction each consisting of a single network - and thus shows that any noncooperative network formation game possessing a potential function has a nonempty path dominance core. Our second example demonstrates how our approach can be applied to Jackson-Wolinsky linking networks and provides necessary and sufficient conditions for nonemptiness of the set of pairwise stable linking networks. Finally, our third example, proposed to us by Salvador Barbera and Michael Maschler in private correspondence, shows how our framework also encompasses hedonic games – games where players’ preferences are defined over the set of coalitions in which they may be members. The example illustrates how, through indirect dominance, outcomes in a game might move from one hedonic core point to another. From our prior results, this demonstrates that, even though the hedonic core, that is the core with respect to direct dominance, is nonempty, the hedonic farsighted core, that is the core with respect to indirect dominance, is empty. (In related work Diamantoudi and Xue 2003 also investigate hedonic games with indirect dominance, but with a different set of effectiveness relations than we consider here).

## 2 Directed Networks

### 2.1 The Definition

Let  $N$  be a finite set of nodes, with typical element denoted by  $i$ , and let  $A$  be a finite set of arcs types (or simply arcs), with typical element denoted by  $a$ . Arcs represent potential types of connections between nodes, and depending on the application, nodes can represent economic players or economic objects such as markets or firms. The following definition is from Page, Wooders, and Kamat (2001).

**Definition 1** (*Directed Networks*)

*Given node set  $N$  and arc set  $A$ , a directed network,  $G$ , is a nonempty subset of  $A \times (N \times N)$ . The collection of all directed networks is denoted by  $P(A \times (N \times N))$ .*

A directed network  $G \in P(A \times (N \times N))$  specifies how the nodes in  $N$  are connected via the arcs in  $A$ . Note that in a directed network order matters. In particular, if  $(a, (i, i')) \in G$ , this means that arc  $a$  goes from node  $i$  to node  $i'$ . Also, note that loops are allowed - that is, we allow an arc to go from a given node back to that given node. For example, in a network model of journal citations loops could represent self-cites.<sup>14</sup> Finally, an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, under our definition an arc  $a$  is not allowed to go from a node  $i$  to a node  $i'$  multiple times.

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<sup>14</sup>Other examples could be developed. For example, in a network model of information sharing, the fact that each player knows his own information would be represented by a loop.



The following notation is useful in describing changes in networks and the properties of networks. Given directed network  $G \in P(A \times (N \times N))$ , let  $G \cup (a, (i, i'))$  denote the network obtained by adding arc  $a$  from node  $i$  to node  $i'$  to network  $G$ , and let  $G \setminus (a, (i, i'))$  denote the network obtained by subtracting (or deleting) arc  $a$  from node  $i$  to node  $i'$  from network  $G$ . Also, let

$$\left. \begin{aligned} G(a) &:= \left\{ (i, i') \in N \times N : (a, (i, i')) \in G \right\}, \\ &\text{and} \\ G(i) &:= \left\{ a \in A : \text{for some } i' \in N \text{ either } (a, (i, i')) \in G \text{ or } (a, (i', i)) \in G \right\}. \end{aligned} \right\} \quad (1)$$

Thus,  $G(a)$  is the *set of node pairs* connected by arc  $a$  in network  $G$ , and  $G(i)$  is the *set of arcs* going from node  $i$  or coming to node  $i$  in network  $G$ .

Note that if for some arc  $a \in A$ ,  $G(a)$  is empty, then arc  $a$  is not used in network  $G$ . Moreover, if for some node  $i \in N$ ,  $G(i)$  is empty then node  $i$  is not used in network  $G$ , and node  $i$  is said to be isolated relative to network  $G$ .

Suppose that the node set  $N$  is given by  $N = \{i_1, i_2, \dots, i_5\}$ , while the arc set  $A$  is given by  $A = \{a_1, a_2, \dots, a_5, a_6, a_7\}$ . Consider network  $G$  in Figure 1.

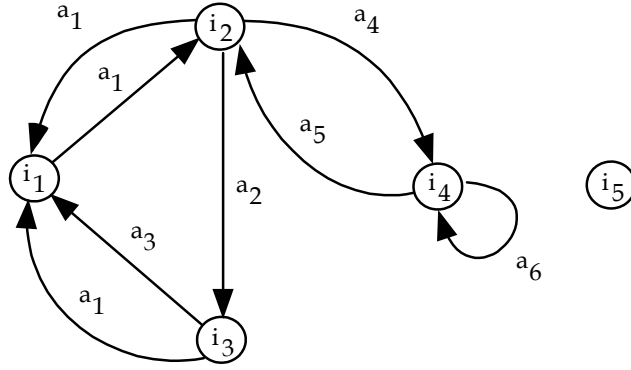


Figure 1: Network  $G$

Note that in network  $G$  nodes  $i_1$  and  $i_2$  are connected by two  $a_1$  arcs running in opposite directions and that nodes  $i_1$  and  $i_3$  are connected by two arcs,  $a_1$  and  $a_3$ , running in the same directions from node  $i_3$  to node  $i_1$ . Thus,  $G(a_1) = \{(i_1, i_2), (i_2, i_1), (i_3, i_1)\}$  and  $G(a_3) = \{(i_3, i_1)\}$ . Observe that  $(a_6, (i_4, i_4)) \in G$  is a loop. Thus,  $G(a_6) = \{(i_4, i_4)\}$ . Also, observe that arc  $a_7$  is not used in network  $G$ . Thus,  $G(a_7) = \emptyset$ .<sup>15</sup>

<sup>15</sup>The fact that arc  $a_7$  is not used in network  $G$  can also be denoted by writing

$$a_7 \notin \text{proj}_A G,$$

where  $\text{proj}_A G$  denotes the projection onto  $A$  of the subset

$$G \subseteq A \times (N \times N)$$

representing the network.

Finally, observe that  $G(i_4) = \{a_4, a_5, a_6\}$ , while  $G(i_5) = \emptyset$ . Thus, node  $i_5$  is *isolated* relative to  $G$ .<sup>16</sup>

## 2.2 Linking Networks, Directed Graphs, and Directed Networks

As before, let  $N$  denote a finite set of nodes. A linking network, say  $g$ , consists of a finite collection of subsets of the form  $\{i, i'\}$ ,  $i \neq i'$ . Thus,  $\{i, i'\} \in g$  means that nodes  $i$  and  $i'$  are linked in network  $g$ . For example,  $g$  might be given by  $g = \{\{i, i'\}, \{i', i''\}\}$  for  $i, i'$ , and  $i''$  in  $N$ . Note that all connections or links are the same (i.e., connection types are homogeneous), direction does not matter, and loops are ruled out. Letting  $g^N$  denote the collection of all subsets of  $N$  of size 2, the collection of all linking networks given  $N$  is given by  $P(g^N)$  where  $P(g^N)$  denotes the collection of all nonempty subsets of  $g^N$  (e.g., see the definition in Jackson and Wolinsky 1996).<sup>17</sup>

A directed graph, say  $E$ , consists of a finite collection of ordered pairs  $(i, i') \in N \times N$ . For example,  $E$  might be given by  $E = \{(i, i'), (i', i')\}$  for  $(i, i')$  and  $(i', i')$  in  $N \times N$ . Stated more compactly, a directed graph  $E$  is simply a subset of  $N \times N$ . Thus, in any directed graph connection types are again homogeneous but direction does matter and loops are allowed.

Under our definition, a directed network  $G$  is a subset of  $A \times (N \times N)$ , where as before  $A$  is a finite set of arcs. Thus, in a directed network, say  $G \in P(A \times (N \times N))$ , connection types are allowed to be heterogeneous (distinguished by arc labels), direction matters, and loops are allowed.

Formally, linking networks are not a special cases of directed networks. However, any linking network can be given an alternative representation as a directed network. To see this, consider linking network  $g \in P(g^N)$  and suppose nodes  $i$  and  $i'$  are linked in network  $g$  (i.e.,  $\{i, i'\} \in g$ ). Next consider a directed network  $G \in P(A \times (N \times N))$  where the set of arc types  $A$  contains one arc,  $A = \{1\}$ , and say that nodes  $i$  and  $i'$  are directly linked in  $G$  if and only if there is an arc from  $i$  to  $i'$  and another arc from  $i'$  to  $i$ .<sup>18</sup> We say that directed network  $G$  is an alternative representation of linking network  $g$  provided

$$\{i, i'\} \in g \text{ if and only if } i \text{ and } i' \text{ are directly linked in } G.$$

With multiple arc types, directed networks allow us to differentiate links by types or intensity levels, and thus allow us to consider a richer collection of links between nodes. For example, suppose that  $A$  contains multiple arc types each specifying a type of connection or an intensity level of a connection. We say that  $i$  and  $i'$  are  $a$ -linked in network  $G \in P(A \times (N \times N))$  provided both  $(a, (i, i'))$  and  $(a, (i', i))$  are in  $G$ . Thus, various sorts of links between players can be modelled and analyzed.

<sup>16</sup>If the loop  $(a_7, (i_5, i_5))$  were part of network  $G$  in Figure 1, then node  $i_5$  would no longer be considered isolated under our definition. Moreover, we would have  $G(i_5) = \{a_7\}$ .

<sup>17</sup>In section 6.3, we show how our approach to network formation games, as well as some of our main results, can be applied to linking networks.

<sup>18</sup>Thus, nodes  $i$  and  $i'$  are directly linked in  $G$  if and only if  $(1, (i, i'))$  and  $(1, (i', i))$  are in  $G$ . Whereas, nodes  $i$  and  $i'$  are connected if and only if  $(1, (i, i'))$  or  $(1, (i', i))$  is in  $G$  (i.e., mutual arcs raise a connection to the level of a link).

As we shall show in Section 6.2, in addition to the fact that linking networks can be given alternative representations as directed networks, the game theoretic approach to network formation we shall develop here can be applied directly to linking networks.

### 3 Preferences, Rules, and Dominance Relations

#### 3.1 Preferences

Let  $D$  denote the set of players (or economic decision making units) with typical element denoted by  $d$ , and let  $P(D)$  denote the collection of all coalitions (i.e., non-empty subsets of  $D$ ) with typical element denoted by  $S$ . Note that, the set of players  $D$  and the set of nodes  $N$  are not necessarily the same set.

Given a feasible set of directed networks  $\mathbb{G} \subseteq P(A \times (N \times N))$ , we shall assume that each player's preferences over networks in  $\mathbb{G}$  are specified via an *irreflexive* binary relation  $\succ_d$ . Thus, player  $d \in D$  prefers network  $G' \in \mathbb{G}$  to network  $G \in \mathbb{G}$  if  $G' \succ_d G$  and for all networks  $G \in \mathbb{G}$ ,  $G \not\succeq_d G$  (irreflexivity). Coalition  $S' \in P(D)$  prefers network  $G'$  to network  $G$ , written  $G' \succ_{S'} G$ , if  $G' \succ_d G$  for all players  $d \in S'$ .

In many applications, a player's preferences are specified via a real-valued network payoff function,  $v_d(\cdot)$ . For each player  $d \in D$  and each directed network  $G \in \mathbb{G}$ ,  $v_d(G)$  is the payoff to player  $d$  in network  $G$ . Player  $d$  then prefers network  $G'$  to network  $G$  if  $v_d(G') > v_d(G)$ . Moreover, coalition  $S' \in P(D)$  prefers network  $G'$  to network  $G$  if  $v_d(G') > v_d(G)$  for all  $d \in S'$ . Note that the payoff  $v_d(G)$  to player  $d$  depends on the entire network. Thus, the player may be affected by directed links between other players even when he himself has no direct or indirect connection with those players. Intuitively, 'widespread' network externalities are allowed.

**Remark 1** *All of our results on basins of attraction, path dominance stable sets, and the path dominance core (Theorems 1-4 below) remain valid even if coalitional preferences  $\{\succ_S\}_{S \in P(D)}$  over networks are based on weak preference relations  $\{\succeq_d\}_{d \in D}$ . If  $G' \succeq_d G$  then player  $d$  either strictly prefers  $G'$  to  $G$  (denoted  $G' \succ_d G$ ) or is indifferent between  $G'$  and  $G$  (denoted  $G' \sim_d G$ ). Given weak preferences  $\{\succeq_d\}_{d \in D}$ , coalition  $S' \in P(D)$  prefers network  $G'$  to network  $G$ , written  $G' \succ_{S'} G$ , if for all players  $d \in S'$ ,  $G' \succeq_d G$  and if for at least one player  $d' \in S'$ ,  $G' \succ_{d'} G$ . Note that if coalitional preferences  $\{\succ_S\}_{S \in P(D)}$  are defined in this way (i.e., using weak preferences  $\{\succeq_d\}_{d \in D}$ ), then they are irreflexive (i.e.,  $G \not\succeq_S G$  for all  $G \in \mathbb{G}$  and  $S \in P(D)$ ).*

#### 3.2 Rules

The rules of network formation are specified via a collection of coalitional effectiveness relations  $\{\rightarrow_S\}_{S \in P(D)}$  defined on the feasible set of networks  $\mathbb{G}$ . Each effectiveness relation  $\rightarrow_S$  represents what a coalition  $S$  can do. Thus, if  $G \rightarrow_S G'$  this means that under the rules of network formation coalition  $S \in P(D)$  can change network  $G \in \mathbb{G}$  to network  $G' \in \mathbb{G}$  by adding, subtracting, or replacing arcs in  $G$ .

### 3.2.1 Examples of Network Formation Rules

*Jackson-Wolinsky Rules:* To illustrate, consider Figure 2 depicting two networks  $G_1$  and  $G_2$  in which the nodes represent players. Thus,  $D = N = \{i_1, i_2, i_3\}$ .

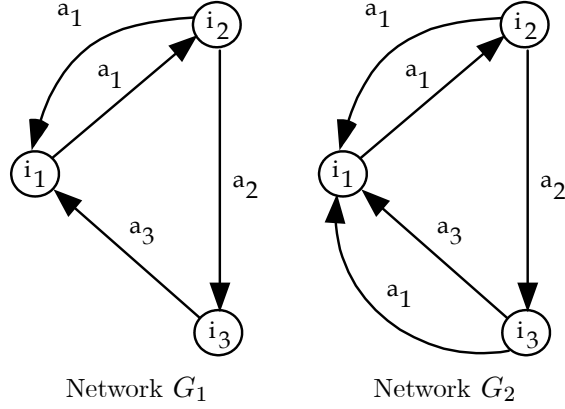


Figure 2

Observe that

$$G_2 = G_1 \cup (a_1, (i_3, i_1)) \quad \text{and} \quad G_1 = G_2 \setminus (a_1, (i_3, i_1)).$$

Assume that

- (i) adding an arc  $a$  from player  $i$  to player  $i'$  requires that both players  $i$  and  $i'$  agree to add arc  $a$  (i.e., arc addition is bilateral);
- (ii) subtracting an arc  $a$  from player  $i$  to player  $i'$  requires that player  $i$  or player  $i'$  agree to subtract arc  $a$  (i.e., arc subtraction is unilateral);
- (iii) for any pair of networks  $G$  and  $G'$  in  $\mathbb{G}$ , if  $G \rightarrow_S G'$ , then  $G \neq G'$  and

$$\begin{aligned} &\text{either } G' = G \cup (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N) \\ &\quad \text{or} \\ &G' = G \setminus (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N). \end{aligned}$$

For the case  $D = N$  (i.e., players = nodes), we shall refer to rules (i)-(iii) above as Jackson-Wolinsky rules. Note that rules (i) and (ii) imply that if  $G \rightarrow_S G'$ , then  $1 \leq |S| \leq 2$ . Referring to Figure 2, the effectiveness relations over networks  $G_1$  and  $G_2$  under Jackson-Wolinsky rules are given by

$$G_1 \xrightarrow[\{i_1, i_3\}]{} G_2 \quad G_2 \xrightarrow[\{i_1, i_3\}]{} G_1 \quad G_2 \xrightarrow[\{i_1\}]{} G_1 \quad G_2 \xrightarrow[\{i_3\}]{} G_1.$$

*Jackson-van den Nouweland rules:* Consider networks  $G_0$  and  $G_3$  depicted in Figure 3 and again suppose that nodes represent players.

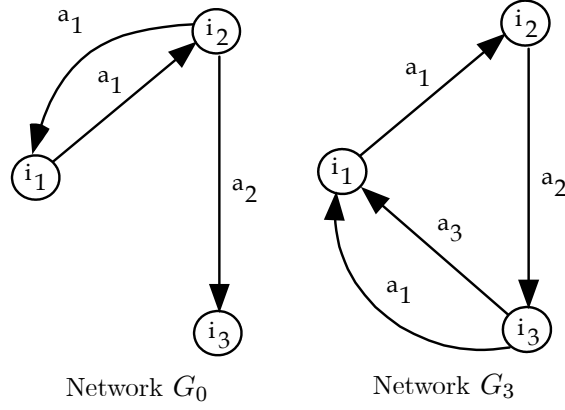


Figure 3

Observe that

$$G_3 = (G_0 \setminus (a_1, (i_2, i_1))) \cup (a_1, (i_3, i_1)) \cup (a_3, (i_3, i_1))$$

and

$$G_0 = (G_3 \setminus ((a_1, (i_3, i_1)) \cup (a_3, (i_3, i_1)))) \cup (a_1, (i_2, i_1)).$$

Assume that

- (i) adding an arc  $a$  from player  $i$  to player  $i'$  requires that both players  $i$  and  $i'$  agree to add arc  $a$  (i.e., arc addition is bilateral);
- (ii) subtracting an arc  $a$  from player  $i$  to player  $i'$  requires that player  $i$  or player  $i'$  agree to subtract arc  $a$  (i.e., arc subtraction is unilateral);

For the case  $D = N$  (i.e., players = nodes), we shall refer to rules (i)-(ii) above as Jackson-van den Nouweland rules. Thus, the Jackson-van den Nouweland rules are the Jackson-Wolinsky rules without the one-arc-at-a-time restriction. Note that if arc addition is bilateral and arc subtraction is unilateral (i.e., if rules (i) and (ii) hold), then  $G \rightarrow_S G'$  implies that  $G'$  is obtainable from  $G$  via coalition  $S$ , that is,

$$(i) \quad (a, (i, i')) \in G' \text{ and } (a, (i, i')) \notin G \\ \Rightarrow \{i, i'\} \subseteq S;$$

$$(ii) \quad (a, (i, i')) \notin G' \text{ and } (a, (i, i')) \in G \\ \Rightarrow \{i, i'\} \cap S \neq \emptyset.$$

Referring to Figure 3, the effectiveness relations over networks  $G_0$  and  $G_3$  under Jackson-van den Nouweland rules are given by

$$G_0 \xrightarrow[\{i_1, i_2, i_3\}]{} G_3 \quad G_0 \xrightarrow[\{i_1, i_3\}]{} G_3 \quad G_3 \xrightarrow[\{i_1, i_2\}]{} G_0 \quad G_3 \xrightarrow[\{i_1, i_2, i_3\}]{} G_0.$$

*Noncooperative Rules:* Again suppose that nodes represent players and assume that

- (i) adding an arc  $a$  from player  $i$  to player  $i'$  requires only that player  $i$  agree to add the arc (i.e., arc addition is unilateral and can be carried out only by the initiator, player  $i$ );
- (ii) subtracting an arc  $a$  from player  $i$  to player  $i'$  requires only that player  $i$  agree to subtract the arc (i.e., arc subtraction is unilateral and can be carried out only by the initiator, player  $i$ );
- (iii)  $G \rightarrow_S G'$  implies that  $|S| = 1$  (i.e., only network changes brought about by individual players are allowed).

We shall refer to rules (i)-(iii) as noncooperative. Note that a player  $i$  can add or subtract an arc to player  $i'$  without regard to the preferences of player  $i'$ . Thus in general under noncooperative rules, effectiveness relations display a type of symmetry, and in particular, if  $G \xrightarrow[\{i\}]{} G'$ , then  $G' \xrightarrow[\{i\}]{} G$ .

Under noncooperative rules, the effectiveness relations over networks  $G_1$  and  $G_2$  in Figure 2 are given by

$$G_1 \xrightarrow[\{i_3\}]{} G_2 \quad G_2 \xrightarrow[\{i_3\}]{} G_1.$$

Note that under noncooperative rules, networks  $G_0$  and  $G_3$  in Figure 3 are *not* related under the effectiveness relations  $\{\rightarrow_{\{i\}}\}_{i \in N}$ . However, referring to the networks in Figures 2 and 3, under the noncooperative rules we have, for example, the following effectiveness relations

$$\begin{aligned} G_3 \rightarrow_{\{i_2\}} G_2 \quad G_2 \rightarrow_{\{i_3\}} G_0 \\ \text{and} \\ G_0 \rightarrow_{\{i_3\}} G_2 \quad G_2 \rightarrow_{\{i_2\}} G_3. \end{aligned}$$

$(\frac{1}{2}, \frac{2}{3})$ -*Voting Rules*: All of the rules above require that arc addition and arc subtraction involve at least one player who is a party to the arc. Consider now arc addition and arc subtraction based on voting. If nodes represent players, then under certain voting rules, arcs can be imposed on players. To see this, consider the following rules for arc addition and arc subtraction.

- (i) adding an arc  $a$  from player  $i$  to player  $i'$  requires a simple majority agree to add arc  $a$ ;
- (ii) subtracting an arc  $a$  from player  $i$  to player  $i'$  requires a  $\frac{2}{3}$  majority agree to subtract arc  $a$ ;
- (iii) for any pair of networks  $G$  and  $G'$  in  $\mathbb{G}$ , if  $G \rightarrow_S G'$ , then  $G \neq G'$  and

$$\begin{aligned} \text{either } G' = G \cup (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N) \\ \text{or} \\ G' = G \setminus (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N) \end{aligned}$$

(i.e., networks changes take place one arc at a time).

We shall refer to rules (i)-(iii) above as  $(\frac{1}{2}, \frac{2}{3})$ -voting rules. Thus, under rules (i)-(iii), if  $G \rightarrow_S G'$ , then  $G \neq G'$  and either

$$G' = G \cup (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N) \text{ and } \frac{|S|}{|D|} > \frac{1}{2}$$

or

$$G' = G \setminus (a, (i, i')) \text{ for some } (a, (i, i')) \in A \times (N \times N) \text{ and } \frac{|S|}{|D|} > \frac{2}{3}.$$

Referring to Figure 2, the effectiveness relations over networks  $G_1$  and  $G_2$  under  $(\frac{1}{2}, \frac{2}{3})$ -voting rules are given by

$$G_1 \xrightarrow[\{i_1, i_2\}]{} G_2 \quad G_1 \xrightarrow[\{i_1, i_3\}]{} G_2 \quad G_1 \xrightarrow[\{i_2, i_3\}]{} G_2$$

and

$$G_2 \xrightarrow[\{i_1, i_2, i_3\}]{} G_1.$$

Note that under  $(\frac{1}{2}, \frac{2}{3})$ -voting rules the move from network  $G_1$  to network  $G_2$  may involve the imposition of arc  $a_1$  from player  $i_3$  to player  $i_1$  upon player  $i_1$  by players  $i_2$  and players  $i_3$ . Also, note that under  $(\frac{1}{2}, \frac{2}{3})$ -voting rules in order to move from network  $G_2$  back to network  $G_1$  (i.e., in order to remove arc  $a_1$  from player  $i_3$  to player  $i_1$ ) requires the agreement of all three players.

*Nonuniform Rules and the Network Representation of Network Formation Rules:* In all of the examples above, the rules for arc addition and arc subtraction are uniform across pairs of networks. In some applications, such uniformity is not present. One very concise way to write down such nonuniform network formation rules is to use a network representation. In particular, suppose we write

$$(S, (G, G')) \text{ if and only if } G \rightarrow_S G'.$$

Thus,  $(S, (G, G'))$  if and only if under the rules coalition  $S \in P(D)$  can change network  $G$  to network  $G'$ . Letting the set of arcs be given by the collection of all coalitions  $P(D)$  and letting the set of nodes be given by the feasible set of networks  $\mathbb{G}$ , the rules of network formation can be represented by a network  $\mathbf{G} \subseteq P(D) \times (\mathbb{G} \times \mathbb{G})$ . Then the set of all possible network formation rules is given by the set of all such networks.

### 3.3 Dominance Relations

We will focus primarily on two types of dominance relations on the feasible set of networks  $\mathbb{G} \subseteq P(A \times (N \times N))$ , direct and indirect dominance.

#### 3.3.1 Direct Dominance

Network  $G' \in \mathbb{G}$  *directly dominates* network  $G \in \mathbb{G}$ , sometimes written  $G' \triangleright G$ , if for some coalition  $S \in P(D)$ ,

$$G \prec_S G'$$

and

$$G \xrightarrow[S]{} G'.$$

Thus, network  $G'$  directly dominates network  $G$  if some coalition  $S$  prefers  $G'$  to  $G$  and if under the rules of network formation coalition  $S$  has the power to change  $G$  to  $G'$ .

### 3.3.2 Indirect Dominance

Network  $G' \in \mathbb{G}$  *indirectly dominates* network  $G \in \mathbb{G}$ , written  $G' \triangleright \triangleright G$ , if there is a *finite* sequence of networks,

$$G_0, G_1, \dots, G_h,$$

with  $G = G_0$ ,  $G' = G_h$ , and  $G_k \in \mathbb{G}$  for  $k = 0, 1, \dots, h$ , and a corresponding sequence of coalitions,

$$S_1, S_2, \dots, S_h,$$

such that for  $k = 1, 2, \dots, h$

$$\begin{aligned} G_{k-1} &\xrightarrow{S_k} G_k, \\ &\text{and} \\ G_{k-1} &\prec_{S_k} G_h. \end{aligned}$$

Note that if network  $G'$  indirectly dominates network  $G$  (i.e., if  $G' \triangleright \triangleright G$ ), then what matters to the initially deviating coalition  $S_1$ , as well as all the coalitions along the way, is that the ultimate network outcome  $G' = G_h$  be preferred. Thus, for example, the initially deviating coalition  $S_1$  will not be deterred from changing network  $G_0$  to network  $G_1$  even if network  $G_1$  is not preferred to network  $G = G_0$ , as long as the ultimate network outcome  $G' = G_h$  is preferred to  $G_0$ , that is, as long as  $G_0 \prec_{S_1} G_h$ .<sup>19</sup>

### 3.3.3 Path Dominance

Each dominance relation  $>$  induces a path dominance relation on the set of networks. In particular, corresponding to dominance relation  $>$  on networks  $\mathbb{G}$  there is a corresponding path dominance relation  $\geq_p$  on  $\mathbb{G}$  specified as follows: network  $G' \in \mathbb{G}$  (weakly) path dominates network  $G \in \mathbb{G}$  with respect to  $>$  (i.e., with respect to the underlying dominance relation  $>$ ), written  $G' \geq_p G$ , if  $G' = G$  or if there exists a *finite* sequence of networks  $\{G_k\}_{k=0}^h$  in  $\mathbb{G}$  with  $G_h = G'$  and  $G_0 = G$  such that for  $k = 1, 2, \dots, h$

$$G_k > G_{k-1}.$$

We refer to such a finite sequence of networks as a *finite domination path* and we say network  $G'$  is *>-reachable* from network  $G$  if there exists a finite domination path from  $G$  to  $G'$ . Thus,

$$G' \geq_p G \text{ if and only if } \begin{cases} G' \text{ is } > \text{-reachable from } G, \text{ or} \\ G' = G. \end{cases} \quad (2)$$

---

<sup>19</sup>In order to capture the idea of farsightedness in strategic behavior, Chwe (1994) analyzes abstract games equipped with indirect dominance relations in great detail, introducing the equilibrium notions of consistency and largest consistent set. The basic idea of indirect dominance goes back to the work of Guilbaud (1949) and Harsanyi (1974).



If network  $G$  is reachable from network  $G$ , that is, if there is a finite domination path from  $G$  back to  $G$  then we call this path a *circuit*. Finally, if network  $G$  is *not* reachable from any network in  $\mathbb{G}$  and if no network in  $\mathbb{G}$  is reachable from  $G$ , then network  $G$  is *isolated* (i.e., network  $G \in \mathbb{G}$  is isolated if there does not exist a network  $G' \in \mathbb{G}$  with  $G' \geq_p G$  or  $G \geq_p G'$ ).

### 3.3.4 The Directed Graph of a Dominance Relation

It is often useful to represent the dominance relation over networks using a directed graph. For example, Figure 3 depicts the graph of dominance relation  $>$  on the feasible set of networks  $\mathbb{G} = \{G_0, G_1, \dots, G_7\}$ .

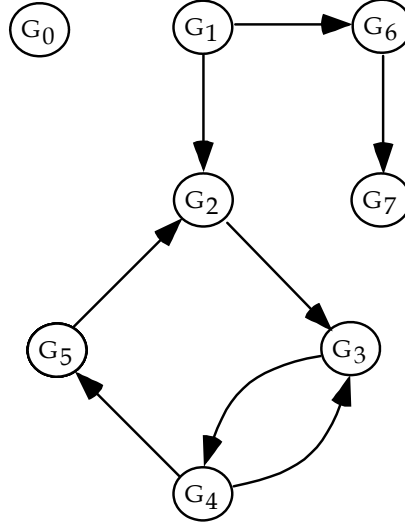


Figure 4: Directed Graph of Dominance Relation  $>$

The arrow (or  $>$ -arc) *from* network  $G_3$  *to* network  $G_4$  in Figure 4 indicates that  $G_4$  dominates  $G_3$ . Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  and given that  $>$  is a *direct* dominance relation, the  $>$ -arc from network  $G_3$  to network  $G_4$  means that for some coalition  $S$ ,  $G_4$  is preferred to  $G_3$  and more importantly, that coalition  $S$  has the power to change network  $G_3$  to network  $G_4$ . Thus,  $G_3 \prec_S G_4$  and  $G_3 \rightarrow_S G_4$ . But notice also that there is a  $>$ -arc in the opposite direction, from network  $G_4$  to network  $G_3$ . Thus,  $G_3$  also dominates  $G_4$ , and thus for some other coalition  $S'$  distinct from coalition  $S$ , that is, some coalition  $S'$  with  $S' \cap S = \emptyset$ ,  $G_4 \prec_{S'} G_3$  and  $G_4 \rightarrow_{S'} G_3$ .<sup>20</sup>

Note that network  $G_3$  is  $>$ -reachable from network  $G_3$  via two paths. Thus, the graph of dominance relation  $>$  depicted in Figure 4 contains two circuits. Defining the *length* of a domination path to be the number of  $>$ -arcs in the path, these two circuits are of length 4 and length 2.

<sup>20</sup>Note that if preferences over networks are weak as in Remark 1, then the statement, ‘for some other coalition  $S'$  distinct from coalition  $S$ ’ can be weakened to ‘for some other coalition  $S'$  not equal to coalition  $S$ ’. With this weakening, the requirement that the intersection of  $S$  and  $S'$  be empty is no longer required.

Because networks  $G_2$  and  $G_5$  in Figure 4 are on the same circuit,  $G_5$  is  $>$ -reachable from  $G_2$  and  $G_2$  is  $>$ -reachable from  $G_5$ . Thus,  $G_5$  path dominates  $G_2$  (i.e.,  $G_5 \geq_p G_2$ ) and  $G_2$  path dominates  $G_5$  (i.e.,  $G_2 \geq_p G_5$ ). The same cannot be said of networks  $G_1$  and  $G_5$  in Figure 4. In particular, while  $G_5 \geq_p G_1$ , it is not true that  $G_1 \geq_p G_5$  because  $G_1$  is not  $>$ -reachable from  $G_5$ . Finally, note that network  $G_0$  is isolated.

## 4 Network Formation Games and Stability

We can now present our main results. Using the abstract network formation game with respect to path dominance given by the pair

$$(\mathbb{G}, \geq_p) \tag{3}$$

and induced by primitives

$$(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}, \tag{4}$$

we introduce and characterize the notions of (i) strategic basins of attraction, (ii) path dominance stable sets, and (iii) the path dominance core. All of the results presented in this section hold for any path dominance relation induced by an irreflexive dominance relation constructed from coalitional preferences,  $\{\succ_S\}_{S \in P(D)}$  and coalitional effectiveness relations,  $\{\rightarrow_S\}_{S \in P(D)}$ .<sup>21</sup>

### 4.1 Networks Without Descendants

If  $G_1 \geq_p G_0$  and  $G_0 \geq_p G_1$ , networks  $G_1$  and  $G_0$  are *equivalent*, written  $G_1 \equiv_p G_0$ . If networks  $G_1$  and  $G_0$  are equivalent then either networks  $G_1$  and  $G_0$  coincide or  $G_1$  and  $G_0$  are on the same circuit (see Figure 4 above for a picture of a circuit). If  $G_1 \geq_p G_0$  but  $G_1$  and  $G_0$  are not equivalent (i.e., not  $G_1 \equiv_p G_0$ ), then network  $G_1$  is a *descendant* of network  $G_0$  and we write

$$G_1 >_p G_0. \tag{5}$$

Referring to Figure 4, observe that network  $G_5$  is a descendant of network  $G_1$ , that is,  $G_5 >_p G_1$ .

Network  $G' \in \mathbb{G}$  has *no descendants in  $\mathbb{G}$*  if for any network  $G \in \mathbb{G}$

$$G \geq_p G' \text{ implies that } G \equiv_p G'.$$

Thus, if  $G'$  has no descendants then  $G \geq_p G'$  implies that  $G$  and  $G'$  coincide or lie on the same circuit.<sup>22</sup>

In attempting to identify those networks which are likely to emerge and persist, networks *without descendants* are of particular interest. Here is our main result concerning networks without descendants.

<sup>21</sup>In fact, all the results in this section hold for any abstract game  $(\mathbb{G}, \geq_p)$  where  $\mathbb{G}$  is a finite set of outcomes and  $\geq_p$  is a path dominance relation induced by any binary relation on  $\mathbb{G}$ .

<sup>22</sup>Note that any isolated network is by definition a network without descendants (e.g., network  $G_0$  in Figure 3).

**Theorem 1** (All path dominance network formation games have networks without descendants)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game. For every network  $G \in \mathbb{G}$  there exists a network  $G' \in \mathbb{G}$  such that  $G' \geq_p G$  and  $G'$  has no descendants.

**Proof.** Let  $G_0$  be any network in  $\mathbb{G}$ . If  $G_0$  has no descendants then we are done. If not choose  $G_1$  such that  $G_1 >_p G_0$ . If  $G_1$  has no descendants then we are done. If not, continue by choosing  $G_2 >_p G_1$ . Proceeding iteratively, we can generate a sequence,  $G_0, G_1, G_2, \dots$ . Now observe that in a finite number of iterations we must come to a network  $G_{k'}$  without descendants. Otherwise, we could generate an infinite sequence,  $\{G_k\}_k$  such that for all  $k$ ,

$$G_k >_p G_{k-1}.$$

However, because  $\mathbb{G}$  is finite this sequence would contain at least one network, say  $G_{k'}$ , which is repeated an infinite number of times. Thus, all the networks in the sequence lying between any two consecutive repetitions of  $G_{k'}$  would be on the same circuit, contradicting the fact that for all  $k$ ,  $G_k$  is a descendant of  $G_{k-1}$  (i.e.,  $G_k >_p G_{k-1}$ ). ■

By Theorem 1, in any network formation game  $(\mathbb{G}, \geq_p)$ , corresponding to any network  $G \in \mathbb{G}$  there is a network  $G' \in \mathbb{G}$  without descendants which is  $>$ -reachable from  $G$ . Thus, in any network formation game the set of networks without descendants is nonempty. Referring to Figure 4, the set of networks without descendants is given by

$$\{G_0, G_2, G_3, G_4, G_5, G_7\}.$$

We shall denote by  $\mathbb{Z}$  the set of networks without descendants.

## 4.2 Basins of Attraction

Stated loosely, a basin of attraction is a set of *equivalent* networks to which the strategic network formation process represented by the game might tend and from which there is no escape. Formally, we have the following definition.

**Definition 2** (*Basin of Attraction*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game. A set of networks  $\mathbb{A} \subseteq \mathbb{G}$  is said to be a basin of attraction for  $(\mathbb{G}, \geq_p)$  if

1. the networks contained in  $\mathbb{A}$  are equivalent (i.e., for all  $G'$  and  $G$  in  $\mathbb{A}$ ,  $G' \equiv_p G$ ) and for no set  $\mathbb{A}'$  having  $\mathbb{A}$  as a strict subset is this true that all the networks in  $\mathbb{A}'$  are equivalent,<sup>23</sup> and
2. no network in  $\mathbb{A}$  has descendants (i.e., there does not exist a network  $G' \in \mathbb{G}$  such that  $G' >_p G$  for some  $G \in \mathbb{A}$ ).

---

<sup>23</sup> $\mathbb{A}$  is a strict subset of  $\mathbb{A}'$  if

$$\mathbb{A} \subset \mathbb{A}' \text{ and } \mathbb{A}' \setminus \mathbb{A} \neq \emptyset.$$

As the following characterization result shows, there is a very close connection between networks without descendants and basins of attraction.

**Theorem 2** (*A characterization of basins of attraction*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game and let  $\mathbb{A}$  be a subset of networks in  $\mathbb{G}$ . The following statements are equivalent:

1.  $\mathbb{A}$  is a basin of attraction for  $(\mathbb{G}, \geq_p)$ .
2. There exists a network without descendants,  $G \in \mathbb{Z}$ , such that

$$\mathbb{A} = \{G' \in \mathbb{Z} : G' \equiv_p G\}.$$

**Proof.** (1) implies (2): Because the sets  $\mathbb{A}$  and  $\{G' \in \mathbb{Z} : G' \equiv_p G\}$ ,  $G \in \mathbb{Z}$ , are equivalence classes,  $\mathbb{A} \neq \{G' \in \mathbb{Z} : G' \equiv_p G\}$  implies that

$$\mathbb{A} \cap \{G' \in \mathbb{Z} : G' \equiv_p G\} = \emptyset \text{ for all } G \in \mathbb{Z}.$$

Thus, if (2) fails, this implies that  $\mathbb{A}$  contains a network with descendants. Thus,  $\mathbb{A}$  cannot be a basin of attraction for  $(\mathbb{G}, \geq_p)$ , and thus, (1) implies (2).<sup>24</sup>

(2) implies (1): Suppose now that

$$\mathbb{A} = \{G' \in \mathbb{Z} : G' \equiv_p G\}$$

for some network  $G \in \mathbb{Z}$ . If  $\mathbb{A}$  is not a basin of attraction, then for some network  $G'' \in \mathbb{G}$ ,  $G'' >_p G'$  for some  $G' \in \mathbb{A}$ . But now  $G'' >_p G'$  and  $G' \equiv_p G$  imply that  $G'' >_p G$ , contradicting the fact that  $G \in \mathbb{Z}$ . Thus, (2) implies (1). ■

In light of Theorem 2, we conclude that in any network formation game  $(\mathbb{G}, \geq_p)$ ,  $\mathbb{G}$  contains a *unique*, finite, disjoint collection of basins of attraction, say  $\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_m\}$ , where for each  $k = 1, 2, \dots, m$  ( $m \geq 1$ )

$$\mathbb{A}_k = \mathbb{A}_G := \{G' \in \mathbb{Z} : G' \equiv_p G\}$$

for some network  $G \in \mathbb{Z}$ . Note that for networks  $G'$  and  $G$  in  $\mathbb{Z}$  such that  $G' \equiv_p G$ ,  $\mathbb{A}_{G'} = \mathbb{A}_G$  (i.e. the basins of attraction  $\mathbb{A}_{G'}$  and  $\mathbb{A}_G$  coincide). Also, note that if network  $G \in \mathbb{G}$  is isolated, then  $G \in \mathbb{Z}$  and

$$\mathbb{A}_G := \{G' \in \mathbb{Z} : G' \equiv_p G\} = \{G\}$$

is, by definition, a basin of attraction - but a very uninteresting one.

**Example 1** (*Basins of attraction*)

In Figure 4 above the set of networks without descendants is given by

$$\mathbb{Z} = \{G_0, G_2, G_3, G_4, G_5, G_7\}.$$

Even though there are six networks without descendants, because networks  $G_2, G_3, G_4$ , and  $G_5$  are equivalent, there are only three basins of attraction:

$$\mathbb{A}_1 = \{G_0\}, \mathbb{A}_2 = \{G_2, G_3, G_4, G_5\}, \text{ and } \mathbb{A}_3 = \{G_7\}.$$

Moreover, because  $G_2, G_3, G_4$ , and  $G_5$  are equivalent,

$$\mathbb{A}_{G_2} = \mathbb{A}_{G_3} = \mathbb{A}_{G_4} = \mathbb{A}_{G_5} = \{G_2, G_3, G_4, G_5\}.$$

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<sup>24</sup>Note that if  $G \in \mathbb{Z}$  and  $G' \equiv_p G$ , then  $G' \in \mathbb{Z}$ .

### 4.3 Stable Sets with Respect to Path Dominance

The formal definition of a  $\geq_p$ -stable set is as follows.<sup>25</sup>

**Definition 3** (*Stable Sets with Respect to Path Dominance*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game. A subset  $\mathbb{V}$  of networks in  $\mathbb{G}$  is said to be a stable set for  $(\mathbb{G}, \geq_p)$  if

(a) (internal  $\geq_p$ -stability) whenever  $G_0$  and  $G_1$  are in  $\mathbb{V}$ , with  $G_0 \neq G_1$ , then neither  $G_1 \geq_p G_0$  nor  $G_0 \geq_p G_1$  hold, and

(b) (external  $\geq_p$ -stability) for any  $G_0 \notin \mathbb{V}$  there exists  $G_1 \in \mathbb{V}$  such that  $G_1 \geq_p G_0$ .

In other words, a nonempty subset of networks  $\mathbb{V}$  is a stable set for  $(\mathbb{G}, \geq_p)$  if  $G_0$  and  $G_1$  are in  $\mathbb{V}$ , with  $G_0 \neq G_1$ , then  $G_1$  is not reachable from  $G_0$ , nor is  $G_0$  reachable from  $G_1$ , and if  $G_0 \notin \mathbb{V}$ , then there exists  $G_1 \in \mathbb{V}$  reachable from  $G_0$ .

We now have our main results on the existence, construction, and cardinality of stable sets.<sup>26</sup>

**Theorem 3** (*Stable sets: existence, construction, and cardinality*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game, and without loss of generality assume that  $(\mathbb{G}, \geq_p)$  has basins of attraction given by

$$\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_m\},$$

where basin of attraction  $\mathbb{A}_k$  contains  $|\mathbb{A}_k|$  many networks (i.e.,  $|\mathbb{A}_k|$  is the cardinality of  $\mathbb{A}_k$ ). Then the following statements are true:

1.  $\mathbb{V} \subseteq \mathbb{G}$  is a stable set for  $(\mathbb{G}, \geq_p)$  if and only if  $\mathbb{V}$  is constructed by choosing one network from each basin of attraction, that is, if and only if  $\mathbb{V}$  is of the form

$$\mathbb{V} = \{G_1, G_2, \dots, G_m\},$$

where  $G_k \in \mathbb{A}_k$  for  $k = 1, 2, \dots, m$ .

2.  $(\mathbb{G}, \geq_p)$  possesses

$$|\mathbb{A}_1| \cdot |\mathbb{A}_2| \cdot \dots \cdot |\mathbb{A}_m| := M$$

many stable sets and each stable set,  $\mathbb{V}_q$ ,  $q = 1, 2, \dots, M$ , has cardinality

$$|\mathbb{V}_q| = |\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_m\}| = m.$$

---

<sup>25</sup>By equipping the abstract network formation game with the path dominance relation rather than the original dominance relation, we entirely avoid the famous Lucas (1968) example of a game with no stable set.

<sup>26</sup>These results can be viewed as applications of some classical results from graph theory to the theory of network formation games (e.g., see Berge 2001, Chapter 2).

**Proof.** It suffices to prove (1). Given (1), the proof of (2) is straightforward. To begin, let

$$\mathbb{V} = \{G_1, G_2, \dots, G_m\},$$

where  $G_k \in \mathbb{A}_k$  for  $k = 1, 2, \dots, m$ , and suppose that for  $G_k$  and  $G_{k'}$  in  $\mathbb{V}$ ,  $G_{k'} \geq_p G_k$ . Since  $G_k \in \mathbb{A}_k$  has no descendants, this would imply that  $G_{k'} \equiv_p G_k$ . But this is a contradiction because  $G_k \in \mathbb{A}_k$  and  $G_{k'} \in \mathbb{A}_{k'}$  and the basins of attraction  $\mathbb{A}_k$  and  $\mathbb{A}_{k'}$  are disjoint. Thus,  $\mathbb{V}$  is internally  $\geq_p$ -stable. Now suppose that network  $G$  is not contained in  $\mathbb{V}$ . By Theorem 1, there exists a network  $G' \in \mathbb{G}$  without descendants such that  $G' \geq_p G$ . By Theorem 2,  $G'$  is contained in some basin of attraction  $\mathbb{A}_k$  and therefore  $G' \equiv_p G_k$  where  $G_k$  is the  $k^{\text{th}}$  component of  $\{G_1, G_2, \dots, G_m\}$ . Thus, we have  $G_k \geq_p G' \geq_p G$  implying that  $G_k \geq_p G$ , and thus  $\mathbb{V}$  is externally  $\geq_p$ -stable.

Suppose now that  $\mathbb{V} \subseteq \mathbb{G}$  is a stable set for  $(\mathbb{G}, \geq_p)$ . First note that each network  $G$  in  $\mathbb{V}$  is a network without descendants. Otherwise there exists  $G' \in \mathbb{G} \setminus \mathbb{V}$  such that  $G' >_p G$ . But then because  $\mathbb{V}$  is externally  $\geq_p$ -stable, there exists  $G'' \in \mathbb{V}$ ,  $G'' \neq G$ , such that  $G'' \geq_p G'$  implying that  $G'' \geq_p G$  and contradicting the internal  $\geq_p$ -stability of  $\mathbb{V}$ . Because each  $G \in \mathbb{V}$  is without descendants, it follows from Theorem 2 that each  $G \in \mathbb{V}$  is contained in some basin of attraction  $\mathbb{A}_k$ . Moreover, because  $\mathbb{V}$  is internally  $\geq_p$ -stable and because all networks contained in any one basin of attraction are equivalent, no two distinct networks contained in  $\mathbb{V}$  can be contained in the same basin of attraction. It only remains to show that for each basin of attraction,  $\mathbb{A}_k$ ,  $k = 1, 2, \dots, m$ ,

$$\mathbb{V} \cap \mathbb{A}_k \neq \emptyset.$$

Suppose not. Then for some  $k'$ ,  $\mathbb{V} \cap \mathbb{A}_{k'} = \emptyset$ . Because all networks in  $\mathbb{A}_{k'}$  are without descendants, for no network  $G \in \mathbb{A}_{k'}$  is it true that there exists a network  $G' \in \mathbb{V}$  such that  $G' \geq_p G$ . Thus, we have a contradiction of the external  $\geq_p$ -stability of  $\mathbb{V}$ . ■

**Example 2** (*Basins of attraction and stable sets*)

Referring to Figure 4, it follows from Theorem 3 that because

$$|\mathbb{A}_1| \cdot |\mathbb{A}_2| \cdot |\mathbb{A}_3| = 1 \cdot 4 \cdot 1 = 4,$$

the network formation game  $(\mathbb{G}, \geq_p)$  has 4 stable sets, each with cardinality 3. By examining Figure 4 in light of Theorem 3, we see that the stable sets for  $(\mathbb{G}, \geq_p)$  are given by

$$\begin{aligned} \mathbb{V}_1 &= \{G_0, G_2, G_7\}, \\ \mathbb{V}_2 &= \{G_0, G_3, G_7\}, \\ \mathbb{V}_3 &= \{G_0, G_4, G_7\}, \\ \mathbb{V}_4 &= \{G_0, G_5, G_7\}. \end{aligned}$$

#### 4.4 The Path Dominance Core

**Definition 4** (*The Path Dominance Core*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game. A network  $G \in \mathbb{G}$  is contained in the path dominance core  $\mathbb{C} \subset \mathbb{G}$  if and only if there does not exist a network  $G' \in \mathbb{G}$ ,  $G' \neq G$ , such that  $G' \geq_p G$ .

Our next results give necessary and sufficient conditions for the path dominance core of a network formation game to be nonempty, as well as a recipe for constructing the path dominance core.

**Theorem 4** (*Path dominance core: nonemptiness and construction*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game, and without loss of generality assume that  $(\mathbb{G}, \geq_p)$  has basins of attraction given by

$$\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_m\},$$

where basin of attraction  $\mathbb{A}_k$  contains  $|\mathbb{A}_k|$  many networks. Then the following statements are true:

1.  $(\mathbb{G}, \geq_p)$  has a nonempty path dominance core if and only if there exists a basin of attraction containing a single network, that is, if and only if for some basin of attraction  $\mathbb{A}_k$ ,  $|\mathbb{A}_k| = 1$ .

2. Let

$$\{\mathbb{A}_{k_1}, \mathbb{A}_{k_2}, \dots, \mathbb{A}_{k_n}\} \subseteq \{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_m\},$$

be the subset of basins of attraction containing all basins having cardinality 1. Then the path dominance core  $\mathbb{C}$  of  $(\mathbb{G}, \geq_p)$  is given by

$$\mathbb{C} = \{G_{k_1}, G_{k_2}, \dots, G_{k_n}\},$$

where  $G_{k_i} \in \mathbb{A}_{k_i}$ , for  $i = 1, 2, \dots, n$ .

**Proof.** It suffices to show that a network  $G$  is contained in the path dominance core  $\mathbb{C}$  if and only if  $G \in \mathbb{A}_k$  for some basin of attraction  $\mathbb{A}_k$ ,  $k = 1, 2, \dots, m$ , with  $|\mathbb{A}_k| = 1$ . First note that if  $G$  is in the path dominance core, then  $G$  is a network without descendants. Thus,  $G \in \mathbb{A}_k$  for some basin of attraction  $\mathbb{A}_k$ . If  $|\mathbb{A}_k| > 1$ , then there exists another network  $G' \in \mathbb{A}_k$  such that  $G' \equiv_p G$ . Thus,  $G' \geq_p G$  contradicting the fact that  $G$  is in the path dominance core. Conversely, if  $G \in \mathbb{A}_k$  for some basin of attraction  $\mathbb{A}_k$  with  $|\mathbb{A}_k| = 1$ , then there does not exist a network  $G' \neq G$  such that  $G' \geq_p G$ . ■

**Remark 2** If coalitional preferences  $\{\succ_S\}_{S \in P(D)}$  over networks are based on weak preference relations  $\{\succsim_d\}_{d \in D}$  rather than on strong preference relations  $\{\succ_d\}_{d \in D}$  (see Remark 1 above), then the corresponding path dominance core - the weak path dominance core - is contained in the path dominance core based on strong preference relations.

**Example 3** (*Basins of attraction and the path dominance core*)

It follows from Theorem 4 that the path dominance core of the network formation game  $(\mathbb{G}, \geq_p)$  with feasible set

$$\mathbb{G} = \{G_0, G_1, \dots, G_7\}$$

and path dominance relation  $\geq_p$  induced by the dominance relation depicted in Figure 4 is

$$\mathbb{C} = \{G_0, G_7\}.$$

Figure 5 contains the graph of a different dominance relation on  $\mathbb{G} = \{G_0, G_1, \dots, G_7\}$ .

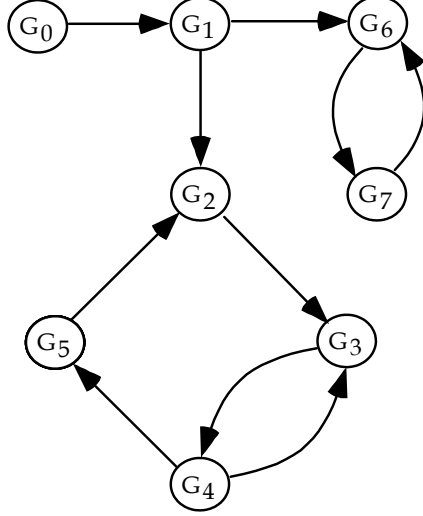


Figure 5: Graph of a different dominance relation  $\succ$

Denoting the new dominance relation by  $\succ$ , the network formation game  $(\mathbb{G}, \geq_p)$  with respect to the path dominance relation  $\geq_p$  induced by the dominance relation  $\succ$  has 3 circuits and 2 basins of attraction,

$$\mathbb{A}_1 = \{G_2, G_3, G_4, G_5\} \text{ and } \mathbb{A}_2 = \{G_6, G_7\}.$$

Because  $|\mathbb{A}_1| = 4$  and  $|\mathbb{A}_2| = 2$ , by Theorem 4 the path dominance core of  $(\mathbb{G}, \geq_p)$  is empty. By Theorem 3,  $(\mathbb{G}, \geq_p)$  has 8 stable sets each containing 2 networks (i.e., each with cardinality 2). These stable sets are given by

$$\begin{aligned} \mathbb{V}_1 &= \{G_2, G_6\}, \\ \mathbb{V}_2 &= \{G_3, G_6\}, \\ \mathbb{V}_3 &= \{G_4, G_6\}, \\ \mathbb{V}_4 &= \{G_5, G_6\}, \\ \mathbb{V}_5 &= \{G_2, G_7\}, \\ \mathbb{V}_6 &= \{G_3, G_7\}, \\ \mathbb{V}_7 &= \{G_4, G_7\}, \\ \mathbb{V}_8 &= \{G_5, G_7\}. \end{aligned}$$

#### 4.4.1 The Path Dominance Core and Constrained Pareto Efficiency

Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, \succ)_{S \in P(D)}$ , we say that a network  $G \in \mathbb{G}$  is constrained Pareto efficient if and only if there does not exist another network  $G' \in \mathbb{G}$



such that (i) some coalition  $S$  can change network  $G$  to network  $G'$  (that is,  $G \rightarrow_S G'$  for some coalition  $S \in P(D)$ ) and (ii)  $G'$  is preferred by all players (that is,  $G \prec_d G'$  for *all* players  $d \in D$ ). Letting  $\mathbb{E}$  denote the set of all constrained Pareto efficient networks, it is easy to see that the path dominance core  $\mathbb{C}$  of network formation game  $(\mathbb{G}, \geq_p)$  is a subset of  $\mathbb{E}$ , that is,  $\mathbb{C} \subseteq \mathbb{E}$ .

Under the classical notion of Pareto efficiency, a network  $G$  is said to be Pareto efficient if and only if there does not exist another network  $G'$  such that  $G \prec_d G'$  for *all* players  $d \in D$ , regardless of whether or not some coalition  $S$  can change network  $G$  to network  $G'$ . Letting  $\mathbb{PE}$  denote the set of all classically Pareto efficient networks, it is easy to see that  $\mathbb{PE} \subseteq \mathbb{E}$ . Note, however, that if under primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ , any network  $G$  can be changed to any other network  $G'$  via the actions of some coalition  $S$ , then the notions of constrained Pareto efficiency and classical Pareto efficiency are equivalent. Thus, if the collection of coalitional effectiveness relations  $\{\rightarrow_S\}_{S \in P(D)}$  on  $\mathbb{G}$  is complete, that is, if for any pair of networks  $G$  and  $G'$  in  $\mathbb{G}$ ,  $G \rightarrow_S G'$  for some coalition  $S \in P(D)$ , then  $\mathbb{PE} = \mathbb{E}$ , and we have  $\mathbb{C} \subseteq \mathbb{PE} = \mathbb{E}$ .

## 5 Other Stability Notions for Network Formation Games

### 5.1 Strongly Stable Networks

We begin with a formal definition of strong stability for abstract network formation games.

**Definition 5** (*Strong Stability*)

Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  and network formation game  $(\mathbb{G}, \geq_p)$ , network  $G \in \mathbb{G}$  is said to be strongly stable in  $(\mathbb{G}, \geq_p)$  if for all  $G' \in \mathbb{G}$  and  $S \in P(D)$ ,  $G \rightarrow_S G'$  implies that  $G \not\prec_S G'$ .

Thus, a network is strongly stable if whenever a coalition has the power to change the network to another network, the coalition will be deterred from doing so because not *all* members of the coalition are made better off by such a change.<sup>27</sup> If nodes represent players and arc addition is bilateral while arc subtraction is unilateral, then our definition of strong stability is essentially that of Jackson-van den Nouweland but for directed networks rather than linking networks. Note that under our definition of strong stability a network  $G \in \mathbb{G}$  that cannot be changed to another network by any coalition is strongly stable.

We now have our main result on the path dominance core and strong stability. Denote the set of strongly stable networks by  $\mathbb{SS}$ .

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<sup>27</sup>Our definition of a strongly stable network differs slightly from the definition given in Jackson-van den Nouweland (2005). In particular, under their definition, a network is strongly stable if whenever a coalition has the power to change the network to another network, the coalition will be deterred from doing so because at least one member of the coalition is made *worse off* by the change. If coalitional preferences,  $\{\succ_S\}_{S \in P(D)}$  are based upon weak players preferences,  $\{\prec_d\}_{d \in D}$ , then our definition of strong stability is equivalent to that of Jackson-van den Nouweland (see Remark 1).

**Theorem 5** (*The path dominance core and strong stability*)

Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  and network formation game  $(\mathbb{G}, \geq_p)$ , where path dominance  $\geq_p$  is induced by either a direct relation or an indirect dominance relation, the following statements are true.

1. If the path dominance core  $\mathbb{C}$  of  $(\mathbb{G}, \geq_p)$  is nonempty, then  $\mathbb{SS}$  is nonempty and  $\mathbb{C} \subseteq \mathbb{SS}$ .
2. If the dominance relation  $>$  underlying  $\geq_p$  is a direct dominance relation, then  $\mathbb{C} = \mathbb{SS}$  and  $\mathbb{SS}$  is nonempty if and only if there exists a basin of attraction containing a single network.

**Proof.** 1. Let  $\mathbb{C} \subseteq \mathbb{G}$ ,  $\mathbb{C} \neq \emptyset$ , be the path dominance core of  $(\mathbb{G}, \geq_p)$  and let network  $G$  be contained in  $\mathbb{C}$ . Then there does not exist a network  $G' \in \mathbb{G}$ ,  $G' \neq G$ , such that  $G' \geq_p G$ . If for some coalition  $S$  and some network  $G' \in \mathbb{G}$ ,  $G \rightarrow_S G'$  and  $G \prec_S G'$ , then  $G' \geq_p G$  trivially, a contradiction. Thus, for  $G$  contained in  $\mathbb{C}$ ,  $G \rightarrow_S G'$  for coalition  $S$  implies that  $G \not\prec_S G'$ , and thus  $G \in \mathbb{C}$  implies  $G \in \mathbb{SS}$ .

2. To see that  $\mathbb{SS} \subseteq \mathbb{C}$  if the dominance relation  $>$  underlying  $\geq_p$  is a direct dominance relation, consider the following. If  $G \notin \mathbb{C}$ , then there exists a network  $G' \neq G$  which path dominates  $G$ , that is,  $G' \geq_p G$ . This implies that there exists a network  $G''$  such that  $G' \geq_p G'' > G$ . Because  $>$  is a direct dominance relation, for some coalition  $S$  we have  $G \rightarrow_S G''$  and  $G \prec_S G''$ . Thus,  $G \notin \mathbb{SS}$ . By part 1 of Theorem 4,  $\mathbb{C} = \mathbb{SS}$  is nonempty if and only if there exists a basin of attraction containing a single network. ■

Note that the set of strongly stable networks is contained in the set of constrained Pareto efficient networks. Thus,  $\mathbb{C} \subseteq \mathbb{SS} \subseteq \mathbb{E}$ .

## 5.2 Pairwise Stable Networks

The following definition is a formalization of Jackson-Wolinsky (1996) pairwise stability for abstract network formation games.

**Definition 6** (*Pairwise Stability*)

Given networks  $P(A \times (N \times N))$  where nodes represent players (i.e.,  $N = D$ ) and given feasible networks  $\mathbb{G} \subseteq P(A \times (N \times N))$  and primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$ , network  $G \in \mathbb{G}$  is said to be pairwise stable in network formation game  $(\mathbb{G}, \geq_p)$  if for all  $(a, (i, i')) \in A \times (N \times N)$ ,

1.  $G \rightarrow_{\{i, i'\}} G \cup (a, (i, i'))$  implies that  $G \not\prec_{\{i, i'\}} G \cup (a, (i, i'))$ ;
2. (a)  $G \rightarrow_{\{i\}} G \setminus (a, (i, i'))$  implies that  $G \not\prec_{\{i\}} G \setminus (a, (i, i'))$ , and  
 (b)  $G \rightarrow_{\{i'\}} G \setminus (a, (i, i'))$  implies that  $G \not\prec_{\{i'\}} G \setminus (a, (i, i'))$ .

Thus, a network is pairwise stable if there is no incentive for any pair of players to add an arc to the existing network and there is no incentive for any player who is party to an arc in the existing network to dissolve or remove the arc. Note that

under our definition of pairwise stability a network  $G \in \mathbb{G}$  that cannot be changed to another network by any coalition, or can only be changed by coalitions of size greater than 2, is pairwise stable.

Let  $\mathbb{P}\mathbb{S}$  denote the set of pairwise stable networks. It follows from the definitions of strong stability and pairwise stability that  $\mathbb{S}\mathbb{S} \subseteq \mathbb{P}\mathbb{S}$ . Moreover, if the full set of Jackson-Wolinsky rules are in force, then  $\mathbb{S}\mathbb{S} = \mathbb{P}\mathbb{S}$ . Jackson-van den Nouweland (2005) provide two examples of the potential for strong stability to refine pairwise stability (i.e., two examples where  $\mathbb{S}\mathbb{S}$  is a strict subset of  $\mathbb{P}\mathbb{S}$ ). However, *under Jackson-Wolinsky rules* because network changes can occur only one arc at a time and because deviations by coalitions of more than two players are not possible such refinements are not possible driving  $\mathbb{S}\mathbb{S}$  and  $\mathbb{P}\mathbb{S}$  to equality.<sup>28</sup>

We now have our main result on the path dominance core and pairwise stability.

**Theorem 6** (*The path dominance core and pairwise stability*)

Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  where nodes represent players (i.e.,  $N = D$ ) and given network formation game  $(\mathbb{G}, \geq_p)$ , where path dominance  $\geq_p$  is induced by either a direct relation or an indirect dominance relation, the following statements are true.

1. If the path dominance core  $\mathbb{C}$  of  $(\mathbb{G}, \geq_p)$  is nonempty, then  $\mathbb{P}\mathbb{S}$  is nonempty and  $\mathbb{C} \subseteq \mathbb{P}\mathbb{S}$ .
2. If the dominance relation  $>$  underlying  $\geq_p$  is a direct dominance relation and if the Jackson-Wolinsky rules hold, then  $\mathbb{C} = \mathbb{P}\mathbb{S}$  and  $\mathbb{P}\mathbb{S}$  is nonempty if and only if there exists a basin of attraction containing a single network.

**Proof.** The proof of part 1 follows from part 1 of Theorem 5 and the fact that  $\mathbb{S}\mathbb{S} \subseteq \mathbb{P}\mathbb{S}$ . For the proof of part 2, note that under the Jackson-Wolinsky rules  $\mathbb{S}\mathbb{S} = \mathbb{P}\mathbb{S}$ . Thus, we have  $\mathbb{C} \subseteq \mathbb{S}\mathbb{S} = \mathbb{P}\mathbb{S}$ . If in addition the path dominance relation is induced by a direct dominance relation, then we have  $\mathbb{P}\mathbb{S} = \mathbb{S}\mathbb{S} \subseteq \mathbb{C}$ . Thus, if the path dominance is induced by a direct dominance and if the Jackson-Wolinsky rules hold, then we have  $\mathbb{C} = \mathbb{S}\mathbb{S} = \mathbb{P}\mathbb{S}$ . By part 1 of Theorem 4,  $\mathbb{C} = \mathbb{S}\mathbb{S} = \mathbb{P}\mathbb{S}$  is nonempty if and only if there exists a basin of attraction containing a single network. ■

Theorem 6 can be viewed as an extension of a result due Jackson and Watts (2002) on the existence of pairwise stable linking networks for network formation games

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<sup>28</sup>In particular, under Jackson-Wolinsky rules, if

$$G \rightarrow_S G',$$

then there are only three possibilities:

- (i)  $G' = G \cup (a, (i, i'))$  for some  $a \in A$  and  $S = \{i, i'\}$ ;
- (ii)  $G \setminus (a, (i, i'))$  for some  $a \in A$  and  $S = \{i\}$ ; or
- (iii)  $G \setminus (a, (i, i'))$  for some  $a \in A$  and  $S = \{i'\}$ .

Thus, under Jackson-Wolinsky rules, if a network is not strongly stable, automatically it is not pairwise stable - and thus under Jackson-Wolinsky rules

$$\mathbb{P}\mathbb{S} \subseteq \mathbb{S}\mathbb{S}.$$

induced by Jackson-Wolinsky rules. In particular, Jackson and Watts (2002) show that for this particular class of Jackson-Wolinsky network formation games, if there does not exist a closed cycle of networks, then there exists a pairwise stable network. Our notion of a strategic basin of attraction containing *multiple* networks corresponds to their notion of a closed cycle of networks. Thus, stated in our terminology, Jackson and Watts show that for this class of network formation games, if there does not exist a basin of attraction containing multiple networks, then there exists a pairwise stable network. Following our approach, if we specialize to this class of Jackson-Wolinsky network formation games, then by part 2 of Theorem 6 the existence of *at least one* strategic basin containing a single network is both necessary and sufficient for the existence of a pairwise stable network.

### 5.3 Consistent Networks

We begin with a formal definition of farsighted consistency (Chwe 1994).

**Definition 7** (*Consistent Sets*)

Let  $(\mathbb{G}, \geq_p)$  be a network formation game where path dominance  $\geq_p$  is induced by an indirect dominance relation  $\triangleright\triangleright$ . A subset  $\mathbb{F}$  of directed networks in  $\mathbb{G}$  is said to be consistent in  $(\mathbb{G}, \geq_p)$  if

$$\begin{aligned} & \text{for all } G_0 \in \mathbb{F}, \\ G_0 \rightarrow_{S_1} G_1 & \text{ for some } G_1 \in \mathbb{G} \text{ and some coalition } S_1 \text{ implies that} \\ & \text{there exists } G_2 \in \mathbb{F} \\ & \text{with } G_2 = G_1 \text{ or } G_2 \triangleright\triangleright G_1 \text{ such that,} \\ & G_0 \not\prec_{S_1} G_2. \end{aligned}$$

In words, a subset of directed networks  $\mathbb{F}$  is said to be consistent in  $(\mathbb{G}, \geq_p)$  if given any network  $G_0 \in \mathbb{F}$  and any deviation to network  $G_1 \in \mathbb{G}$  by coalition  $S_1$  (via adding, subtracting, or replacing arcs in accordance with effectiveness relations  $\rightarrow_S$ ), there exists further deviations leading to some network  $G_2 \in \mathbb{F}$  where the initially deviating coalition  $S_1$  is not better off - and possibly worse off. A network  $G \in \mathbb{G}$  is said to be consistent if  $G \in \mathbb{F}$  where  $\mathbb{F}$  is a consistent set in  $(\mathbb{G}, \geq_p)$ .

There can be many consistent sets in  $(\mathbb{G}, \geq_p)$ . We shall denote by  $\mathbb{F}^*$  the *largest consistent set*. Thus, if  $\mathbb{F}$  is a consistent set, then  $\mathbb{F} \subseteq \mathbb{F}^*$ . By Proposition 1 in Chwe (1994) there exists uniquely a largest consistent set in  $(\mathbb{G}, \geq_p)$ . Moreover, by the Corollary to Proposition 2 in Chwe (1994) this largest consistent set is nonempty and externally stable with respect to indirect dominance  $\triangleright\triangleright$ .<sup>29</sup>

We now have our main result on the relationship between basins of attraction, stable sets, the path dominance core, and the largest consistent set.

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<sup>29</sup>Page and Kamat (2005) provide an alternative proof of the nonemptiness and external stability of the largest consistent set (with respect to indirect dominance). In particular, Page and Kamat modify the indirect dominance relation so as to make it transitive as well as irreflexive. They then show that the unique stable set with respect to path dominance induced by this new transitive indirect dominance relation is contained in the largest consistent set - and in this way show that the largest consistent set is nonempty and externally stable.

**Theorem 7** (*Basins of attraction, the path dominance core, and the largest consistent set*)

Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  and given network formation game  $(\mathbb{G}, \geq_p)$ , where path dominance is induced by an indirect dominance relation  $\triangleright \triangleright$ , assume without loss of generality that  $(\mathbb{G}, \geq_p)$  has nonempty largest consistent set given by  $\mathbb{F}^*$  and basins of attraction given by

$$\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_m\}.$$

Then the following statements are true:

1. Each basin of attraction  $\mathbb{A}_k$ ,  $k = 1, 2, \dots, m$ , has a nonempty intersection with the largest consistent set  $\mathbb{F}^*$ , that is

$$\mathbb{F}^* \cap \mathbb{A}_k \neq \emptyset, \text{ for } k = 1, 2, \dots, m.$$

2. If  $(\mathbb{G}, \geq_p)$  has a nonempty path dominance core  $\mathbb{C}$ , then

$$\mathbb{C} \subseteq \mathbb{F}^*.$$

**Proof.** In light of Theorem 4, (2) easily follows from (1). Thus, it suffices to prove (1). Suppose that for some basin of attraction  $\mathbb{A}_{k'}$

$$\mathbb{F}^* \cap \mathbb{A}_{k'} = \emptyset.$$

Let  $G'$  be a network in  $\mathbb{A}_{k'}$ . Because  $\mathbb{F}^*$  is externally stable with respect to the indirect dominance relation  $\triangleright \triangleright$ ,  $G' \notin \mathbb{F}^*$  implies that there exists some network  $G^* \in \mathbb{F}^*$  such that  $G^* \triangleright \triangleright G'$ . Thus,  $G^* \geq_p G'$ . Because the networks in  $\mathbb{A}_{k'}$  are without descendants, it must be true that  $G' \geq_p G^*$ . But this implies that  $G^* \equiv_p G'$ , and therefore that  $G^* \in \mathbb{A}_{k'}$ , a contradiction. ■

**Remark 3** *Recently, Herings, Mauleon, and Vannetelbosch (2005) introduced a notion of pairwise farsighted stability. If in our model coalitional preferences  $\{\succ_S\}_{S \in P(D)}$  over networks are based on weak preference relations  $\{\succ_d\}_{d \in D}$  (see Remark 1 above), if nodes represent players (i.e.,  $N = D$ ), and if the dominance relation underlying the path dominance relation is indirect, then under Jackson-Wolinsky rules the corresponding weak path dominance core is contained in the set of pairwise farsightedly stable networks.*

## 5.4 Nash Networks

**Definition 8** (*Nash Networks*)

Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  and network formation game  $(\mathbb{G}, \geq_p)$ , network  $G \in \mathbb{G}$  is said to be a Nash network in  $(\mathbb{G}, \geq_p)$  if for all  $G' \in \mathbb{G}$  and  $S \in P(D)$  such that  $|S| = 1$ ,  $G \rightarrow_S G'$  implies that  $G \not\prec_S G'$ .

Thus, a network is Nash if whenever an individual player has the power to change the network to another network, the player will have no incentive to do so. We shall denote by  $\mathbf{NE}$  the set of Nash networks. Note that our definition of a Nash network does not require that the network formation rules, as represented via the effectiveness relations  $\{\rightarrow_S\}_{S \in P(D)}$ , be noncooperative (see subsection 3.2.1). Also, note that under our definition any network that cannot be changed to another network by a coalition of size 1 is a Nash network. Finally, note that the set of strongly stable networks  $\mathbf{SS}$  is contained in the set of Nash networks  $\mathbf{NE}$ .

We now have our main result on the path dominance core and strong stability.

**Theorem 8** (*The path dominance core and Nash networks*)

*Given primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  and network formation game  $(\mathbb{G}, \geq_p)$ , where path dominance  $\geq_p$  is induced by either a direct relation or an indirect dominance relation, the following statements are true.*

1. *If the path dominance core  $\mathbb{C}$  of  $(\mathbb{G}, \geq_p)$  is nonempty, then  $\mathbf{NE}$  is nonempty and  $\mathbb{C} \subseteq \mathbf{NE}$ .*
2. *If the dominance relation  $>$  underlying  $\geq_p$  is a direct dominance relation and if the rules of network formation are such that  $G \rightarrow_S G'$  implies that  $|S| = 1$ , then  $\mathbb{C} = \mathbf{NE}$  and  $\mathbf{NE}$  is nonempty if and only if there exists a basin of attraction containing a single network.*

**Proof.** The proof of part 1 follows from part 1 of Theorem 5 and the fact that  $\mathbf{SS} \subseteq \mathbf{NE}$ . For the proof of part 2, note that if the rules of network formation are such that  $G \rightarrow_S G'$  implies that  $|S| = 1$ , then  $\mathbf{SS} = \mathbf{NE}$ . Thus, we have  $\mathbb{C} \subseteq \mathbf{SS} = \mathbf{NE}$ . If in addition the path dominance relation is induced by a direct dominance relation, then we have  $\mathbf{NE} = \mathbf{SS} \subseteq \mathbb{C}$ , and we conclude that  $\mathbb{C} = \mathbf{SS} = \mathbf{NE}$ . Thus, if the path dominance is induced by a direct dominance and if the rules are such that  $G \rightarrow_S G'$  implies that  $|S| = 1$ , then we have  $\mathbb{C} = \mathbf{SS} = \mathbf{NE}$ . By part 1 of Theorem 4,  $\mathbb{C} = \mathbf{SS} = \mathbf{NE}$  is nonempty if and only if there exists a basin of attraction containing a single network. ■

We close this section by noting that if the dominance relation  $>$  underlying  $\geq_p$  is a direct dominance relation and if the rules of network formation are such that  $G \rightarrow_S G'$  implies that  $|S| = 1$ , then the set of Nash networks  $\mathbf{NE}$  is contained in the set of constrained Pareto efficient networks  $\mathbf{E}$ . Thus, for this case we have  $\mathbb{C} = \mathbf{SS} = \mathbf{NE} \subseteq \mathbf{E}$ .

## 6 Examples

In the abstract games,  $(\mathbb{G}, \geq_p)$ , that we have considered, the set of outcomes  $\mathbb{G}$  is a set of directed networks and we have focused on path dominance induced by either direct dominance or indirect dominance. However, our main results, Theorems 1-4, hold for any abstract game with a finite set of outcomes equipped with path dominance induced by any dominance relation. With this in mind, in this section we will

demonstrate the flexibility of our approach and the wide applicability of our results by first considering network games with a potential function and then considering games where the set of outcomes is, in one case, a set of linking networks and, in another case, a set of coalition structures, and where the path dominance relation is induced by a dominance relation other than a direct or an indirect dominance relation (as defined in sections 3.3.1 and 3.3.2). In particular, in our first example, we consider noncooperative network formation games and show that any noncooperative network formation game possessing a potential function has basins of attraction each consisting of a single network and thus has a nonempty path dominance core. In our second example, we show how our approach can be applied to Jackson-Wolinsky linking networks and we provide necessary and sufficient conditions for nonemptiness of the set of pairwise stable linking networks. Finally, we show via an example proposed to us by Salvador Barbera and Michael Maschler (2006) how our approach can be used to analyze hedonic games, and in particular, we show how farsightedness can lead to instability (i.e., emptiness of the path dominance core) in hedonic games.

## 6.1 Noncooperative Network Formation Games Possessing a Potential Function.

Suppose the primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, \triangleright)_{S \in P(D)}$  underlying the network formation game  $(\mathbb{G}, \geq_p)$  are such that:

1. the set of nodes  $N$  and the set of players  $D$  are one and the same (i.e.,  $N = D$  and  $\mathbb{G} \subseteq P(A \times (D \times D))$ );
2. preferences  $\{\succ_S\}_{S \in P(D)}$  over networks  $\mathbb{G}$  are specified via player payoff functions  $v_d(\cdot)$ , that is, coalition  $S' \in P(D)$  prefers network  $G'$  to network  $G$  if  $v_d(G') > v_d(G)$  for all  $d \in S'$ ; <sup>30</sup>
3. effectiveness relations  $\{\rightarrow_S\}_{S \in P(D)}$  over networks  $\mathbb{G}$  are such that,
  - (i) adding an arc  $a$  from player  $i$  to player  $i'$  requires only that player  $i$  agree to add the arc (i.e., arc addition is unilateral and can be carried out only by the initiator, player  $i$ ),
  - (ii) subtracting an arc  $a$  from player  $i$  to player  $i'$  requires only that player  $i$  agree to subtract the arc (i.e., arc subtraction is unilateral and can be carried out only by the initiator, player  $i$ ), and
  - (iii)  $G \rightarrow_S G'$  implies that  $|S| = 1$  (i.e., only network changes brought about by individual players are allowed);
4. the dominance relation  $>$  over  $\mathbb{G}$  is given by a direct dominance relation  $\triangleright$ , that is,  $G' \triangleright G$  if and only if for some player  $d' \in D$ ,  $v_{d'}(G') > v_{d'}(G)$  and  $G \rightarrow_{d'} G'$ .

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<sup>30</sup>This is a frequently used way of defining payoffs to coalitions; see for example, Jackson (2005) and van den Nouweland (2005).

We say that the noncooperative network formation game  $(\mathbb{G}, \geq_p)$  is a potential game if there exists a function

$$P(\cdot) : \mathbb{G} \rightarrow \mathbb{R}$$

such that for all  $G$  and  $G'$  with  $G \rightarrow_{d'} G'$  for some player  $d'$ ,

$$v_{d'}(G') > v_{d'}(G) \text{ if and only if } P(G') > P(G).$$

It is easy to see that any noncooperative network formation game  $(\mathbb{G}, \geq_p)$  possessing a potential function has no circuits, and thus possesses strategic basins of attraction each consisting of a single network without descendants.<sup>31</sup> Thus, we can conclude from our Theorem 4 that any noncooperative network formation game possessing a potential function has a nonempty path dominance core. In addition, we know from our Theorem 8 that in this example the path dominance core  $\mathbb{C}$  is equal to the set of Nash networks  $\mathbb{NE}$ .<sup>32</sup>

## 6.2 Jackson-Wolinsky Linking Networks

Consider primitives  $(\mathbb{G}, \{\succ_S\}, \{\rightarrow_S\}, >)_{S \in P(D)}$  with corresponding network formation game  $(\mathbb{G}, \geq_p)$  where  $\mathbb{G}$  is given by a feasible set of linking networks, coalitional preferences  $\{\succ_S\}_{S \in P(D)}$  are based on weak preferences (see Remarks 1 and 2 above), effectiveness relations  $\{\rightarrow_S\}_{S \in P(D)}$  are specified via Jackson-Wolinsky rules, and the dominance relation  $>$  is direct. In particular, assume that the set of nodes  $N$  and the set of players  $D$  are equal, let  $g^N$  denote the collection of all subsets of  $N$  of size 2, and let  $\mathbb{G}$  be a nonempty subset of  $P(g^N)$ , where  $P(g^N)$  denotes the collection of all nonempty subsets of  $g^N$  (i.e., the set of all linking networks - see the definition in Jackson and Wolinsky 1996). To simplify comparisons, we use the standard notation for linking networks and let  $g$  denote a typical linking network.

Under Jackson-Wolinsky rules, if  $g \rightarrow_{S'} g'$  then  $g \neq g'$  and either (i)  $g' = g \cup \{i, i'\}$  (a link is added between players  $i$  and  $i'$ ) and  $S' = \{i, i'\}$  or (ii)  $g' = g \setminus \{i, i'\}$  (the link between players  $i$  and  $i'$  is removed) and  $S' = \{i\}$  or  $S' = \{i'\}$  or  $S' = \{i, i'\}$ . Moreover, if coalitional preferences  $\{\succ_S\}_{S \in P(D)}$  are based on weak preference relations  $\{\succ_d\}_{d \in D}$ , then coalition  $S' \in P(D)$  *prefers* network  $G'$  to network  $G$ , written  $G' \succ_{S'} G$ , if for all players  $d \in S'$ ,  $G' \succ_d G$  and if for at least one player  $d' \in S'$ ,  $G' \succ_{d'} G$ .<sup>33</sup> Finally, if the dominance relation  $>$  is direct with underlying weak preferences, then  $g' \triangleright g$  if and only if either (i)  $g \rightarrow_{\{i, i'\}} g'$  and  $g' \succ_{\{i, i'\}} g$  where  $g' = g \cup \{i, i'\}$  or (ii) (a)  $g \rightarrow_{\{i\}} g'$  and  $g' \succ_{\{i\}} g$  where  $g' = g \setminus \{i, i'\}$  or (b)  $g \rightarrow_{\{i'\}} g'$  and  $g' \succ_{\{i'\}} g$  where  $g' = g \setminus \{i, i'\}$ .

<sup>31</sup>As has been shown by Monderer and Shapley (1996), potential games are closely related to congestion games introduced by Rosenthal (1973).

<sup>32</sup>Page and Wooders (2007) introduce a club network formation game which is a variant of the noncooperative network formation game described above and show that this game possesses a potential function. Prior papers studying potential games in the context of linking networks include Slikker, Dutta, and van den Nouweland (2000) and Slikker and van den Nouweland (2002). These papers have focused on providing the strategic underpinnings of the Myerson value (Myerson 1977).

<sup>33</sup>Recall from Remark 1 that if  $G' \succ_d G$  then player  $d$  either strictly prefers  $G'$  to  $G$  (denoted  $G' \succ_d G$ ) or is indifferent between  $G'$  and  $G$  (denoted  $G' \sim_d G$ ).



It follows from our Theorem 4 that any network formation game  $(\mathbb{G}, \geq_p)$  induced by Jackson-Wolinsky primitives (i.e., primitives as specified above) has a nonempty path dominance core if and only if there is at least one strategic basin of attraction containing a single network. Moreover, it follows from our Theorem 6 that for any such network formation game the path dominance core is equal to the set of pairwise stable networks (as defined for linking networks in Jackson and Wolinsky 1996).

### 6.3 Hedonic games

Consider a hedonic game where a move from one coalition structure to another can be initiated by any group of players defecting from the original structure, but in order for the change to prevail all players in coalitions augmented or created by the defecting players must prefer their new coalitions to their old coalitions - or must prefer their eventual coalitions to their old coalitions if players are farsighted. Call the path dominance core with respect to direct dominance, the hedonic direct core and the path dominance core with respect to indirect dominance the hedonic farsighted core. Note that the hedonic direct core is equivalent to the usual hedonic core. As the following example will show, the hedonic farsighted core may be empty even when the hedonic core is not.

Consider the following hedonic game proposed to us by Barbera and Maschler (2006). Let the player set be given by  $D = \{1, 2, 3, 4, 5, 6, 7, 8\}$ . Player preferences over coalitions are as follows:

player 1	(1, 2, 3, 4)	(1, 2, 3)	(1, 2)	(1)	...
player 2	(1, 2, 3, 4)	(1, 2, 3)	(1, 2)	(2)	...
player 3	(1, 2, 3, 4)	(3, 4, 5, 6)	(1, 2, 3)	(3)	(3, 6)
player 4	(1, 2, 3, 4)	(3, 4, 5, 6)	(4, 5)	(4)	...
player 5	(3, 4, 5, 6)	(5, 6, 7, 8)	(4, 5)	(5)	...
player 6	(3, 4, 5, 6)	(5, 6, 7, 8)	(6, 7, 8)	(6)	(3, 6)
player 7	(5, 6, 7, 8)	(6, 7, 8)	(7, 8)	(7)	...
player 8	(5, 6, 7, 8)	(6, 7, 8)	(7, 8)	(8)	...

Players' Preferences Over Coalitions

Consider the row for player 1 in the table above. The interpretation is that 1 prefers the coalition (1, 2, 3, 4) to the coalition (1, 2, 3), to the coalition (1, 2), and so on. Player 1's preferences over the remaining coalitions are irrelevant to the following example so they are not specified. The same interpretation applies to the rows corresponding to other players.

A partition of the player set is in the *hedonic core* if there does not exist a coalition that is preferred by all its members to their coalitions of membership in the original partition (i.e., a partition is in the hedonic core if it is not directly dominated by another partition). Consider the partition  $((1, 2, 3, 4), (5, 6, 7, 8)) \in \mathbb{G}$ . This is a core point for the hedonic game because the only coalition that is preferred by players 5 and 6 is (3, 4, 5, 6) but two members of this coalition, 3 and 4, do not prefer it (i.e.,  $((1, 2), (3, 4, 5, 6), (7, 8))$  does not directly dominate  $((1, 2, 3, 4), (5, 6, 7, 8))$ ). If players

4 and 5 are farsighted, however, and domination is indirect, 4 and 5 can decide to form a coalition (4, 5) - thus bringing about the partition  $((1, 2, 3), (4, 5), (6, 7, 8))$ . Now players 3, 4, 5, and 6 could all benefit from forming a coalition. This brings us to the partition  $((1, 2), (3, 4, 5, 6), (7, 8))$  a hedonic core point in which 4 and 5 are better off than in the original hedonic core point. Thus,  $((1, 2), (3, 4, 5, 6), (7, 8))$  indirectly dominates  $((1, 2, 3, 4), (5, 6, 7, 8))$ . But the story is not finished. Starting from  $((1, 2), (3, 4, 5, 6), (7, 8))$ , players 3 and 6 can separate and form their own coalition. Using an argument similar to the one above, this move by 3 and 6 can then lead back to the original partition. Thus,  $((1, 2, 3, 4), (5, 6, 7, 8))$  indirectly dominates  $((1, 2), (3, 4, 5, 6), (7, 8))$ .

We see here that, even though the hedonic core is nonempty, the hedonic farsighted core is empty. Another point illustrated is that for path dominance, it is only necessary that a coalition perceive *some* path that would lead to a preferred situation; it is not required that a coalition perceive some preferred *final* (and presumably stable) outcome. The example also suggests for those special cases of hedonic games where the hedonic direct core (i.e., the hedonic core) is non-empty and not a singleton, then the path dominance core with respect to indirect dominance (i.e., the hedonic farsighted core) is empty. (See Diamantoudi and Xue 2003 for related work applying indirect dominance to hedonic games).<sup>34</sup>

## 7 Conclusions

From the viewpoint of the path dominance core with direct or indirect dominance, there are a number of potential questions to be addressed. For example, what is the relationship, if any, between basins of attraction and the path dominance core and partnered (or separating) collections of coalitions, as in for example Page and Wooders (1996), Reny and Wooders (1997) or Maschler and Peleg (1967) and Maschler, Peleg and Shapley (1971)? Or what is relationship between basins of attraction and the path dominance core and the inner core, as in Qin (1993,1994)?

To conclude, we return to the prior research introducing concepts similar to the abstract game defined in this paper and the union of basins of attractions; see Schwartz (1974), Panzer, Kalai and Schmeidler (1976), Kalai and Schmeidler (1977) and Shenoy (1980).<sup>35</sup> For specificity, we focus on Kalai and Schmeidler (1977). These authors take as given a set of feasible alternatives, denoted by  $S$ , a dominance relation, denoted by  $M$ , and the transitive closure of  $M$ , denoted by  $\widehat{M}$ . Their admissible set is the set

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<sup>34</sup>In brief, the effectiveness relations in Diamantoudi and Xue differ from the effectiveness relations in our rendition of the Barbera-Maschler example. In particular, in Diamantoudi and Xue all defecting players must form a coalition in the new partition, whereas in the Barbera-Maschler example, defecting players can join already existing coalitions in forming the new partition. Moreover, in Diamantoudi and Xue only defecting players must prefer their new coalition in order for the change to take place, whereas in the Barbera-Maschler example, not only must defecting players prefer their new coalitions, but also all players in coalitions joined by the defecting players must prefer their new coalitions in order for the change to take place.

<sup>35</sup>We thank Sylvia Thoron for brining this to our attention.

$A(S, M) := \{x \in S : y \in S \text{ and } y\widehat{M}x \text{ imply } x\widehat{M}y\}$ .<sup>36</sup> Besides non-emptiness of the admissible set, they also shown that the admissible set is equal to the union of certain subsets – in our terminology, basins of attraction. While Kalai and Schmeidler apply their concept to cooperative games and games in normal (strategic) form, they do not consider networks, the focus of our research. Once our model of network formation is developed, then our abstract game is a particular case of the abstract game of these earlier authors. Our contribution differs in that we develop the network framework and characterize several equilibrium concepts from network theory in terms of their relationships to each other and to basins of attraction and the path dominance core. In addition, we characterize the set of von-Neumann-Morgenstern solutions and the path-dominance core (a case of the abstract core notion introduced in Gilles 1959) in terms of their relationships to basins of attraction. It may well be that the insightful examples developed by these authors will lead to new sorts of examples for networks, a question we are currently addressing. Also, Kalai and Schmeidler (1977) allow an infinite set of possibilities, which, in a network framework, introduces a host of new questions. We plan to address some of these in future research.

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<sup>36</sup>Kalai and Schmeidler (1977) also cite Schwartz (1974) for the origins of this concept.

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