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DOCTORAL THESIS

Strong Primeness in Fuzzy Environment

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ABSTRACT

The main aim of this investigation is to propose the notion of uniform and strong primeness in fuzzy environment. First, it is proposed and investigated the concept of fuzzy strongly prime and fuzzy uniformly strongly prime ideal. As an additional tool, the concept of t/m systems for fuzzy environment gives an alternative way to deal with primeness in fuzzy. Second, a fuzzy version of correspondence theorem and the radical of a fuzzy ideal are proposed. Finally, it is proposed a new concept of prime ideal for Quantales which enable us to deal with primeness in a noncommutative setting.

DEDICATION

This investigation is dedicated to you, the reader, in hopes that you will find what you are looking for.

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LIST OF SYMBOLS

An_r	right annihilator
An_l	left annihilator
F_x	insulator of x
$[a]$	the set $\{x \in R : x \equiv a\}$
$R \setminus I$	the complement of the I in R
R/I	the quotient ring by I
x^F	the set $x^F = \{xf : f \in F\}$
Fx	the set $Fx = \{fx : f \in F\}$
x^Fy	the set $x^Fy = \{xfy : f \in F\}$
$sp(S)$	the set of all strongly prime ideals of S
$sp_f(S)$	the set of all strongly prime ideals of S that contains the $Ker(f)$
$SP(S)$	the set of all strongly prime fuzzy ideals of S
$SP_f(S)$	the set of all strongly prime fuzzy ideals where $I_\alpha \subset sp_f(S)$
$Ker(f)$	the kernel of f
μ_I	the membership function of the I
$I(x)$	equivalent to $\mu_I(x)$
$f(I)_\alpha$	the α -cut of the fuzzy set $f(I)$
$\bigwedge F$	minimum of the set F
$\bigvee F$	maximum of the set F
$x \wedge y$	minimum of the set $\{x, y\}$
$x \vee y$	maximum of the set $\{x, y\}$
$r + I$	left coset
$\sqrt[3]{I}$	strongly radical or Levitzki radical
$\sqrt[4]{I}$	uniformly strongly radical of a fuzzy ideal
$US(I)$	uniformly strongly radical
I_*	the set $\{x \in R : I(x) = I(0)\}$
I_α	the α -cut or α -level, $\{x \in R : I(x) \geq \alpha\}$
$ASSP$	almost special strongly prime ideal
SSP	special strongly prime ideal
usp	uniformly strongly prime ideal
$uspf$	uniformly strongly prime fuzzy ideal

LIST OF SYMBOLS

$2Z$	the set $\{2q : q \in Z\}$
$FI_{FV}(R)$	the set of all finite-valued fuzzy ideals of R
$L_{FI}(R)$	the set of all fuzzy ideals of R
$Im(\mu)$	image of the membership μ
\perp	least element
\top	greatest
$\langle A \rangle$	ideal generated by A in a quantale

Chapter 1

Introduction

1.1 Historical Facts

In 1871, Dedekind generalized the concept of prime numbers to *prime ideals*, which were defined in a similar way, namely, as a proper subset of integers that contains a product of two elements if and only if it contains one of them. For example, the set of integers divisible by a fixed prime p form a prime ideal in the ring of integers. Also, the integer decomposition into the product of powers of primes has an analogue in rings. We can replace prime numbers with prime ideals, as long as powers of prime integers are not replaced by powers of prime ideals but by primary ideals. The uniqueness of the latter decomposition was proved in 1915 by Macaulay. Thus, we can think about of prime ideals as atoms, like prime numbers are atoms in the ring of integers.

From the properties of the integers we can develop a general structure called ring. Then, the concept of primeness may be extended to a commutative ring in a certain way, for example: a prime ring R is a ring where the (0) zero ideal is prime that is, given a and b nonzero elements in R , there exists $r \in R$, such that arb is nonzero in R . In commutative ring theory prime rings are integral domains, i.e. rings where $ab = 0$ implies $a = 0$ or $b = 0$. Suppose that, given $a \neq 0$ in R there exists a finite nonempty subset $F_a \subseteq R$, such that $afr = 0$ implies $r = 0$, for all $f \in F_a$. In that case we have a strong primeness condition on a ring. If the ring satisfies this last condition it is called strongly prime ring (shortened sp ring). If the same F can be chosen for any nonzero element in R , then the ring R is called uniformly strongly prime (usp) ring.

Strongly prime rings were introduced in 1974, as a prime ring with finite condition in the generalization of results on group rings proved by Lawrence in his PhD.'s thesis [1]. In 1975, Lawrence and Handelman [2] came up with properties of these rings and proved

important results, showing that all prime rings can be embedded in an sp ring; and that all sp rings are nonsingular.

After that, in 1987, Olson [3] published a relevant paper about usp rings and usp radical. He proved that the usp rings generate a radical class which properly contains both the right and left sp radicals and which is not contained in Jacobson and Brown-McCoy radicals.

In 1965, Zadeh [4] introduced fuzzy sets and in 1971, Rosenfeld [5] introduced fuzzy sets in the realm of group theory and formulated the concept of fuzzy subgroups of a group. Since then, many researchers have been engaged in extending the concepts/results of abstract algebra to the broader framework of the fuzzy setting. Thus, in 1982, Liu [6] defined and studied fuzzy subrings as well as fuzzy ideals. Subsequently, Liu himself (see [7]), Mukherjee and Sen [8], Swamy and Swamy [9], and Zhang Yue [10], among others, fuzzified certain standard concepts/results on rings and ideals. For example: Mukherjee was the first to study the notion of prime ideal in a fuzzy setting. Those studies were further carried out by Kumar in [11] and [12], where the notion of nil radical and semiprimeness were introduced.

After Mukherjee's definition of prime ideals in the fuzzy setting, many investigations extended crisp (classic) results to fuzzy setting. But Mukherjee's definition was not appropriate to deal with noncommutative rings. In 2012, Navarro, Cortadellas and Lobillo [13] drew attention to this specific problem. They proposed a new definition of primeness for fuzzy ideals for noncommutative rings holding the idea of "fuzzification" of primeness introduced by Kumbhojkar and Bapat [14, 15] to commutative rings, which is coherence with α -cuts. Thus, Navarro et. al. [13] reopened the possibility of developing fuzzy results for general rings of prime ideals.

1.2 The Main Problem

As it is known, after the Lawrence and Handelman's paper many researchers developed results about strong/uniform primeness (see [16–19]), but nothing was made in fuzzy and quantale setting. Hence, this is the question: Could we have the strong/uniform primeness in fuzzy and quantale setting? This thesis attempts to fill this gap by proposing/investigating this concept in both environments. Therefore, motivated by translating the concept of strong primeness for fuzzy setting, I decided to build a definition of strongly prime fuzzy ideal in which the first attempt was based on α -cuts (chapter 2 and published in [20]). In this approach every result for fuzzy environment has its counterpart in a classical crisp setting. Afterwards, I realized that all proofs were based on α -cuts

and the results were only translated to fuzzy setting. Then, a new definition of strongly prime fuzzy ideal was required, and should not be based on α -cuts, yet compatible in a certain way. Thus, the second attempt was to propose such definition introduced in [21] which can be found in the section 3.5. In this approach, we have the coherency with α -cuts in “only one side”; namely: if a fuzzy ideal is strongly prime, then all α -cuts are crisp strongly prime ideals. But the converse of this statement is still open up to now.

Instead of the concept of strongly prime ideal, the concept of uniformly strongly prime ideal is more suitable to translate to fuzzy setting. Thus, in [22] (chapter 3) I introduce the uniform fuzzy concept, compatible with α -cuts. In this approach I rediscovered some crisp results on uniformly strongly prime (usp) ideals for fuzzy setting and the compatibility with Navarro’s definition of fuzzy prime ideals. For example it is shown that a fuzzy usp ideal is a fuzzy prime ideal without using α -cuts to prove this statement. Also, some crisp results are no longer valid in fuzzy setting. For example, in crisp setting, an ideal I is usp iff the quotient ring R/I is a usp ring, but as you shall soon see (example 7) this is not true in fuzzy setting.

When I began to study primeness in quantales setting I realized that some authors (see Chapter 7) were working on noncommutative quantale with an elementwise definition for prime ideals. As it is known, in noncommutative ring theory, prime ideals are defined based on ideals instead of on elements. Thus, I firstly decided to provide a concept of prime ideal for a general (commutative and noncommutative) quantale in which the elementwise prime ideal definition was replaced by another based on ideals over the quantale. Hence, it was required to develop a crisp study for prime ideals before starting the investigation of sp/usp ideals for quantales.

This thesis is organized as follows: Chapter 2 provides an overview about the ring and fuzzy ring theory. It also contains the definition and results of sp/usp rings and ideals in a crisp setting; Chapter 3 contains the results discovered by the authors in [20] about fuzzy sp ideals; Chapter 4 introduces the usp fuzzy ideal and its radical, all results in this chapter are based on [22, 23], except the unpublished section 4.4 introduces a new tool for dealing with prime fuzzy ideals and usp fuzzy ideals called systems, where may extend the Navarro’s paper, since the complement of fuzzy prime ideal is a fuzzy system (see corollary 23); Chapter 5 shows some extra results on fuzzy ideals; Chapter 6 introduces a new concept of prime ideal in quantales; Chapter 7 contains some thoughts about the next studies; Finally, Appendix A contains published and unpublished studies.

Chapter 2

Rings and Ideals

This chapter introduces some definitions and results that will be required in this investigation. Here, we start by defining prime rings/ideals and uniformly strongly prime rings/ideals.

Definition 1. A *ring* is a nonempty set R of elements closed under two binary operations $+$ and \cdot with the following properties:

(i) $(R, +)$ (that is, the set R considered with the single operation of addition) is an abelian group (whose identity element is denoted 0_R , or just 0);

(ii) The operation \cdot is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for every $a, b, c \in R$. Thus, (R, \cdot) is a semigroup;

(iii). The operations $+$ and \cdot satisfy the two distributive laws: $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$, for every $a, b, c \in R$.

If R is a ring and there exists an element 1 such that $a \cdot 1 = a$ for every $a \in R$ we say that the ring has multiplicative identity. Also, if $a \cdot b = b \cdot a$ for $a, b \in R$ we call R a *commutative ring*.

Very often we omit writing the \cdot for multiplication, that is, we write ab to mean $a \cdot b$. Note that there can only be one additive identity in R (because $(R, +)$ is a group, and a group can only have one additive identity). Also, there can be only one multiplicative identity in R . If R is commutative and for any $a, b \in R$, $ab = 0$ implies $a = 0$ or $b = 0$ we call R an *integral domain*. Note that the ring of $n \times n$ matrices with integers entries is a *noncommutative ring* and not an integral domain.

Definition 2. Let R be a ring. A nonempty subset I of R is called a *right ideal* of R if:

(a) $a, b \in I$ implies $a + b \in I$;

(b) given $r \in R, a \in I$, then $ar \in I$ (that is, a right ideal absorbs right multiplication by the elements of the ring).

Similarly we can define *left ideal* replacing (b) by: (b') given $r \in R, a \in I$, then $ra \in I$. If I is both right and left ideal of R , we call I a *two-sided ideal* or simply an *ideal*.

For the next definition consider the following notation $I \cdot J$ and xRy in which:

$$I \cdot J = IJ = \{i_1j_1 + \cdots + i_nj_n : i_k \in I \text{ and } j_k \in J, k = 1, \dots, n; \text{ where } n \in \mathbb{Z}^+\}$$

$$xRy = \{xry : r \in R\}.$$

Definition 3. A *prime ideal* in an arbitrary ring R is any proper ($P \subseteq R$ and $P \neq R$) ideal P such that, whenever I, J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

Theorem 1. [[24], Proposition 10.2] An ideal P of a ring R is prime iff for $x, y \in R$, $xRy \subseteq P$ implies $x \in P$ or $y \in P$.

Definition 4. An ideal P of a ring R is called *completely prime* if given a and b two elements of R such that their product $ab \in P$, then $a \in P$ or $b \in P$.

Given a ring R and $a \in R$, the set $(a) = RaR = \{x_1ay_1 + \cdots + x_nay_n : n \in \mathbb{N}, x_i, y_i \in R\}$ is an ideal and is called the ideal generated by a .

For arbitrary rings, *completely prime* implies prime, but the converse is not true as we can see in the following example:

Example 1. [13] Let (0) as an ideal generated by 0, and let R be the ring of 2×2 matrices over the real numbers. Let us show that the (0) (zero ideal) is prime, but (0) is not completely prime by using the theorem 1. Thus, suppose that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are two matrices such that $XRY \subseteq (0)$. Hence $XTY = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for

any other matrix $T \in R$. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af \\ ce & cf \end{pmatrix} = 0 \Leftrightarrow a = c = 0 \text{ or } e = f = 0,$$

$$X \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ag & ah \\ cg & ch \end{pmatrix} = 0 \Leftrightarrow a = c = 0 \text{ or } g = h = 0,$$

$$X \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} be & bf \\ de & df \end{pmatrix} = 0 \Leftrightarrow b = d = 0 \text{ or } e = f = 0,$$

$$X \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} bg & bh \\ dg & dh \end{pmatrix} = 0 \Leftrightarrow b = d = 0 \text{ or } g = h = 0,$$

Hence, it should be the case that $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $X \in (0)$ or $Y \in (0)$ and then (0) is prime. Nevertheless, (0) is not completely prime, since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ although } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin (0).$$

2.1 Prime, Strongly Prime and Uniformly Strongly Prime Rings

A ring is called a simple ring if it is a nonzero ring that has no two-sided ideals besides the zero ideal and itself. In 1973, Formanek [25] proved that if D is an integral domain and G can be factored into a free product of a group, then the group ring DG is a simple ring. In the same year, Lawrence in his Master's thesis showed that a generalization of Formanek's result was possible, in which the integral domain is replaced by a prime ring with a finiteness condition called strong primeness. Although the condition of strong primeness was already used in a specific problem for group rings, the theory of strong primeness became more itself interesting. As a consequence, in 1975, Lawrence and Handelman [2] began to study the strongly prime rings for which some results were discovered, for example that every prime ring may be embedded in a strongly prime ring and that the Artinian strongly prime rings have a minimal right ideal.

Definition 5. A ring R is prime if for any two elements a and b of R , $arb = 0$ for all r in R implies that either $a = 0$ or $b = 0$.

We can think of prime rings as a simultaneous generalization of both integral domains and simple rings. In the commutative case R is prime iff R is an integral domain.

Definition 6. Let A be a subset of a ring R . The *right annihilator* of A is defined as $An_r(A) = \{x \in R : Ax = (0)\}$. Similarly, we can define the *left annihilator* An_l

Definition 7. [2] A ring R is called *right strongly prime* if for each nonzero $x \in R$ there exists a finite nonempty subset F_x of R such that the $An_r(xF_x) = (0)$.

When R is right strongly prime we can prove that F_x is unique and called a *right insulator* for x .

Parmenter, Stewart and Wiegandt [26] have shown that the definition of right strongly prime is equivalent to:

Proposition 1. A ring R is *right strongly prime* if each nonzero ideal I of R contains a finite subset F which has right annihilator zero.

It is clear that every right strongly prime ring is a prime ring. It is also possible to define left strongly prime in a manner analogous to that for right strong primeness. Handelman and Lawrence showed that these two concepts are distinct, by building a ring that is right strongly prime but not left strongly prime (see [2], Example 1).

Example 2. If I is an ideal in a simple ring R , then $I = (0)$ or $I = R$. Thus, if $I \neq (0)$, then $I = R$. Let $F = \{1\}$ then $An_r(F) = \{0\}$. Hence, according to definition 1 R is a right strongly prime ring.

Example 3. A *division ring* is a nonzero ring such that multiplicative identity in which every nonzero element a has a multiplicative inverse, i.e., an element x with $ax = xa = 1$. It is easy to see that a division ring is a simple ring. Therefore, it is a right strongly prime ring.

A *field* is a commutative division ring with multiplicative identity. Therefore strongly prime ring.

Example 4. Consider Z_n the commutative ring of integers mod n , for $n > 1$. If $a \in Z$, the class of a is $[a] = \{x \in Z : (x \text{ mod } n) = a\}$. Note that if n is not a prime number, then there exists $p, q \in Z$ such that $n = pq$, where $0 < p < n$ and $0 < q < n$. Hence, $[pq] = 0$ in Z_n , but $[p] \neq 0$ and $[q] \neq 0$. We conclude that Z_n is not an integral domain and as a consequence Z_n is not a prime ring. Thus, Z_n is not right strongly prime ring. On the other hand, if n is prime, Z_n is a field, hence right strongly prime ring.

Definition 8. A ring is a *bounded right strongly prime ring of bound n* , if each nonzero element has an insulator containing no more than n elements and at least one element has no insulator with fewer than n elements.

Definition 9. A ring is called *uniformly right strongly prime* if the same right insulator may be chosen for each nonzero element.

Since an insulator must be finite, it is clear that every uniformly strongly prime ring is a bounded right strongly prime ring of bound n . Again, analogous definitions of bounded left strongly prime and uniformly left strongly prime can be formulated. As in the case with the notation of strong primeness it is possible to find rings which are bounded left strongly prime but not bounded right strongly prime, and vice-versa (see [2], Example 1). However, Olson [3] showed that the concept of uniformly strongly prime ring is two-sided due to the following result:

Lemma 2. [3] A ring R is right/left uniformly strongly prime iff there exists a finite subset $F \subseteq R$ such that for any two nonzero elements x and y of R , there exists $f \in F$ such that $xfy \neq 0$.

Corollary 3. [3] R is uniformly right strongly prime ring if and only if R is uniformly left strongly prime ring.

Lemma 4. [3] The following are equivalent:

- i) R is a uniformly strongly prime ring;
- ii) There exists a finite subset $F \subseteq R$ such that $xFy = 0$ implies $x = 0$ or $y = 0$, where $x, y \in R$;
- iii) For every $a \neq 0, a \in R$, there exists a finite set $F \subset (a)$ such that $xFy = 0$ implies $x = 0$ or $y = 0$, where $x, y \in R$;
- iv) For every $a \neq 0, a \in R$, there exists a finite set $F \subset (a)$ such that $xFx = 0$ implies $x = 0$, where $x \in R$;
- v) For every ideal $I \neq 0$, there exists a finite set $F \subset I$ such that $xFy = 0$ implies $x = 0$ or $y = 0$, where $x, y \in R$;
- vi) For every ideal $I \neq 0$, there exists a finite set $F \subset I$ such that $xFx = 0$ implies $x = 0$, where $x \in R$;
- vii) For every $a \neq 0, a \in R$, there exists a finite set $F \subset R$ such that $xFaFx = 0$ implies $x = 0$, where $x \in R$;
- viii) For every $a \neq 0, a \in R$, there exists a finite set $F \subset R$ such that $xFaFy = 0$ implies $x = 0$ or $y = 0$, where $x, y \in R$.

2.2 Strongly Prime and Uniformly Strongly Prime Ideals

Let I be a two-sided ideal in R . We may define an equivalence relation \sim_I on R as follows: $a \sim b$ iff $b - a \in I$. In case $a \sim b$, we say that a and b are congruent modulo I . The equivalence class of the element a in R is given by:

$$[a] = a + I = \{a + r : r \in I\}.$$

The set of all such equivalence classes is denoted by R/I and it is a ring called the *quotient ring*, where the operations are:

$$[a] + [b] = (a + I) + (b + I) = (a + b) + I = [a + b];$$

$$[a] \cdot [b] = (a + I)(b + I) = (ab) + I = [ab].$$

The zero-element of R/I is $[0] = 0 + I = I$.

From this point forward, strongly prime means right strongly prime.

Definition 10. [2] An ideal I in a ring R is strongly prime if R/I is a strongly prime ring.

Proposition 2 ([27], Proposition 4.3, Chapter IX). An ideal I of a ring is prime iff R/I is a prime ring.

We reproved the following result.

Proposition 3. The ideal I is a strongly prime ideal in R iff for every $x \in R \setminus I$ there exists a finite subset F_x of R such that if $xF_x r \subseteq I$ implies $r \in I$.

Proof. Let I strongly prime ideal and R/I strongly prime ring. If $x \in R \setminus I$, then $(x + I) \neq I$ in R/I . The insulator of $x + I$ in R/I is a finite set $F_x^* = \{f_1 + I, \dots, f_k + I\}$ for some particular choice of the f_i . Let $F_x = \{f_1, \dots, f_n\} \subseteq R$ be such that $xF_x r \subseteq I$. Hence, $(xF_x^* r + I) = I$ in R/I . Thus, $(r + I) = I$ and $r \in I$. Conversely, given $(x + I) \neq I$ in R/I there exists a finite set $F_x \subset R$ such that $(x + I)(F_x + I)(y + I) = (xF_x y + I) = I$. This implies $xF_x y \in I$ and $y \in I$. Thus, $(y + I) = I$ and R/I is a strongly prime ring. \square

Corollary 5. P is strongly prime ideal in R iff for every $x, y \in R$, if $xPy \subseteq P$ and $xy \in P$, then either $x \in P$ or $y \in P$.

Proposition 4. [2] Let P be a proper ideal of a ring R . The following conditions are equivalent:

- (i) P is strongly prime.
- (ii) For every ideal $I \supset P$ there exists a finite set $F \subseteq I$ such that if $Fa \subseteq P$, then $a \in P$.

Proposition 5. [2] If R is a finite ring, then every prime ideal is a strongly prime ideal.

A ring homomorphism is a function between two rings which respects the structure. More explicitly, if R and S are rings, then a ring homomorphism is a function $f : R \rightarrow S$ such that: $f(a+b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$ for all a and b in R and $f(1_R) = 1_S$.

Let $f : R \rightarrow S$ be a homomorphism of rings. Let $sp(S)$ be the set of all strongly prime ideals of S and $sp_f(R) = \{I \in sp(R) : I \supseteq Ker(f)\}$ where $Ker(f) = \{r \in R : f(r) = 0\}$. According to the isomorphism theorem for rings, if f is an epimorphism, there is a bijection between $sp_f(R)$ and $sp(S)$. In the next chapter we will show its counterpart in a fuzzy setting (see proposition 12).

Theorem 6. [28] Let $f : R \rightarrow S$ be an epimorphism of rings. Then

- (i) $f(I) \in sp(S)$ for any $I \in sp_f(R)$;
- (ii) $f^{-1}(I) \in sp_f(R)$ for any $I \in sp(S)$;
- (iii) Define the mapping $\Psi : sp_f(R) \rightarrow sp(S)$, $\Psi(I) = f(I)$. Then Ψ is a bijection.

Proposition 6. Let $f : R \rightarrow S$ be an isomorphism of rings.

- a) $P \subseteq R$ is a prime ideal iff $f(P)$ is a prime ideal of S .
- b) $P \subseteq R$ is a strongly prime ideal iff $f(P)$ is a strongly prime ideal of S .

Proposition 7 ([29]). Let R be a ring and A, B be ideals of R with $A \subseteq B$: If B is strongly prime, then there exists a minimal element in $S = \{P \subseteq R : P \text{ is strongly prime ideal and } A \subseteq P \subseteq B\}$.

Definition 11. A proper ideal I of a ring R is a *uniformly strongly prime ideal* if R/I is a uniformly strongly prime ring.

We reproved the following two results, because they provide another characterization of uniformly strongly prime ideals.

Proposition 8. An ideal I of a ring R is uniformly strongly prime iff there exists a finite set $F \subseteq R$ such that $xFy \subseteq I$ implies $x \in I$ or $y \in I$, where $x, y \in R$.

Proof. Let I be a uniformly strongly prime ideal of the ring R . Then R/I is uniformly strongly prime ring. Let $F^* = \{f_1 + I, \dots, f_k + I\}$ be a insulator for R/I for some particular choice of the f_i and $F = \{f_1, \dots, f_k\}$. Choose $x, y \in R$ such that $xFy \subseteq I$. Hence $(x+I)F^*(y+I) = I$. By hypothesis $(x+I) = I$ or $(y+I) = I$. Thus, $x \in I$ or $y \in I$.

Conversely, let $(x + I), (y + I) \in R/I$. Suppose $(x + I)(F + I)(y + I) = xFy + I = I$. Hence, $xFy \subseteq I$, by hypothesis $x \in I$ or $y \in I$. Thus, $(x + I) = I$ or $(y + I) = I$. Therefore, R/I is a uniformly strongly prime ring. \square

Proposition 9. An ideal I of a ring R is uniformly strongly prime iff there exists a finite set $F \subseteq R$ such that for any two nonzero elements x and y of $R \setminus I$ (the complement of I in R), there exists $f \in F$ such that $xfy \notin I$.

Proof. Let I be a uniformly strongly prime ideal. Then, R/I is uniformly strongly prime. Let $\{f_1 + I, \dots, f_k + I\}$ be a insulator for R/I for some particular choice of the f_i . Choose $x, y \in R \setminus I$. Then $(x + I)$ and $(y + I)$ are nonzero elements in R/I and according to lemma 2 there exists $f_i + I \in F^*$ for some $i = 1, \dots, k$ such that $(x + I)(f_i + I)(y + I) = xf_iy + I \neq I$. Then $xf_iy \in R \setminus I$. Conversely, if $(x + I)$ and $(y + I)$ are nonzero elements of R/I then x and y are in $R \setminus I$. By hypothesis there exists $f \in F$ such that $xfy \in R \setminus I$. That is $(x + I)(f + I)(y + I) = xfy + I \neq I$. According to lemma 2 R/I is a uniformly strongly prime ring and $\{f + I : f \in F\}$ is a insulator for R/I . \square

Chapter 3

Strongly Prime Fuzzy Ideals

In this chapter the concept of strongly prime fuzzy ideal for rings is defined. Also, it is shown that the Zadeh's extension of homomorphism somewhat preserves strong primeness and that every strongly prime fuzzy ideal is a prime fuzzy ideal as well as every fuzzy maximal is a strongly prime fuzzy ideal. The concept of *strongly prime radical* of a fuzzy ideal and its properties are investigated. It is proved that Zadeh's extension preserves strongly prime radicals. A version of *theorem of correspondence* for strongly prime fuzzy ideals is also showed. Besides, we propose new algebraic fuzzy structures, namely: *strongly primary*, *strong radical*, *Special Strongly Prime (SSP)* and *Almost Special Strongly Prime (ASSP)*. At the end of this chapter it is shown the relation between strong primary and strong radicals as well as the connection between the classes SP, SSP and ASSP. All results in this chapter can be found in [20, 21, 30].

For fuzzy ideals and Prime fuzzy ideals we recommend first of all [13] and then [6–10].

3.1 Theory of fuzzy ideals

3.1.1 Fuzzy Subrings and Fuzzy Ideals

By a fuzzy set we mean the classical concept defined in [4], that is, a fuzzy set over a base set X is a set map $\mu : X \rightarrow [0, 1]$. The intersection and union of fuzzy sets is given by the point-by-point infimum and supremum. We shall use the symbols \wedge and \vee for denoting the infimum and supremum of a collection of real numbers.

Definition 12. A fuzzy subset $I : R \rightarrow [0, 1]$ of a ring R is called a *fuzzy subring* of R if, for all $x, y \in R$: the following requirements are met:

- 1) $I(x - y) \geq I(x) \wedge I(y)$;

2) $I(xy) \geq I(x) \wedge I(y)$;

If condition 2) is replaced by $I(xy) \geq I(x) \vee I(y)$, then I is called a *fuzzy ideal* of R .

Note that if I is a fuzzy ideal of a ring R , then $I(1) \leq I(x) \leq I(0)$ for all $x \in R$.

Definition 13. Let μ be any fuzzy subset of a set S and let $\alpha \in [0, 1]$. The set $\{x \in S : \mu(x) \geq \alpha\}$ is called a α -cut of μ which is symbolized by μ_α .

Clearly, if $t > s$, then $\mu_t \subseteq \mu_s$.

Proposition 10. [31] A fuzzy subset I of a ring R is a fuzzy subring/fuzzy ideal of R iff all α -cuts I_α are subrings/ideals of R .

Here is an example of a fuzzy subring of a ring R which is not a fuzzy ideal of R .

Example 5. Let \mathbb{R} denote the ring of real numbers under the usual operations of addition and multiplication. Define a fuzzy subset μ of \mathbb{R} by

$$\mu(x) = \begin{cases} t, & \text{if } x \text{ is rational,} \\ t', & \text{if } x \text{ is irrational,} \end{cases}$$

where $t, t' \in [0, 1]$ and $t > t'$. Note that $\mu_t = \mathbb{Q}$ and $\mu_{t'} = \mathbb{R}$. Thus μ_t is a subring according to the Proposition 10, but not a fuzzy ideal.

Definition 14 (Zadeh's Extension). [4] Let f be a function from set X into Y , and let μ be a fuzzy subset of X . The Zadeh extension of f is the fuzzy subset $f(\mu)$ of Y , where the membership function is: For all $y \in Y$,

$$f(\mu)(y) = \begin{cases} \vee \{\mu(x) : x \in X, f(x) = y\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

If λ is a fuzzy subset of Y , we define the fuzzy subset of X , denoted as $f^{-1}(\lambda)$, where $f^{-1}(\lambda)(x) = (\lambda \circ f)(x)$.

Proposition 11. [32] If $f : R \rightarrow S$ is a ring homomorphism and $I : R \rightarrow [0, 1]$ and $J : S \rightarrow [0, 1]$ are fuzzy ideals, then

i) $f^{-1}(J)$ (according to the last definition) is a fuzzy ideal which is constant on $\text{Ker}(f)$ (Kernel of f);

ii) $f^{-1}(J_\alpha) = f^{-1}(J)_\alpha$, where $\alpha = J(0)$;

iii) If f is an epimorphism, then $f(I)$ is a fuzzy ideal and $ff^{-1}(J) = J$ and $f(I_\alpha) = f(I)_\alpha$, where $\alpha = I(0)$;

iv) If I is constant on $\text{Ker}(f)$, then $f^{-1}f(I) = I$.

3.1.2 Fuzzy Prime Ideals

Definition 15. [13] Let R be a ring with unity. A non-constant fuzzy ideal $P : R \rightarrow [0, 1]$ is said to be prime or *fuzzy prime ideal* if for any $x, y \in R$, $\bigwedge P(xRy) = P(x) \vee P(y)$.

Proposition 12. [13] Let R be an arbitrary ring with unity and $P : R \rightarrow [0, 1]$ be a non-constant fuzzy ideal of R . The following conditions are equivalent:

- (i) P is prime;
- (ii) P_α is prime for all $P(1) < \alpha \leq P(0)$;
- (iii) R/P_α is a prime ring for all $P(1) < \alpha \leq P(0)$;
- (iv) For any fuzzy ideal J , if $J(xry) \leq P(xry)$ for all $r \in R$, then $J(x) \leq P(x)$ or $J(y) \leq P(y)$.

Note that if P is a fuzzy ideal, then $P(xry) \geq P(x) \vee P(r) \vee P(y) \geq P(x) \vee P(y)$ for all $r \in R$. Thus, $\bigwedge P(xRy) \geq P(x) \vee P(y)$.

Definition 16. Let I be a fuzzy ideal of a ring R . For all $r \in R$ define fuzzy left coset $r + I$, where $(r + I)(x) = I(x - r)$.

The definition above allow us to built the quotient ring R/I in the same way as we did in crisp setting.

3.1.3 Fuzzy Maximal Ideals

Definition 17. [33] Let M be a fuzzy ideal of a ring R . Then M is called fuzzy maximal of R if the following conditions are met:

- (i) M is non-constant;
- (ii) for any fuzzy ideal ν of R , if $M \subseteq \nu$ then either $M_* = \nu_*$ or $\nu = \mu_R$, where $M_* = \{x \in R : M(x) = M(0)\}$, $\nu_* = \{x \in R : \nu(x) = \nu(0)\}$ and $\mu_R(x) = 1$ if $x \in R$ and $\mu_R(x) = 0$ otherwise.

Proposition 13. [33] Let M be a fuzzy maximal ideal of a ring R . Then $M(0) = 1$.

Proposition 14. [33] Let M be a fuzzy maximal ideal of a ring R . Then $|Im(M)| = 2$.

3.2 Strongly Prime Fuzzy Ideals

In this section, the notion of strongly prime fuzzy ideal is introduced and the well-known crisp results in the fuzzy setting are proved.

Definition 18. (Strongly prime fuzzy ideal) Let R be an arbitrary ring with unity. A non-constant fuzzy ideal $P : R \rightarrow [0, 1]$ is said to be strongly prime if P_α is strongly prime for any $P(1) < \alpha \leq P(0)$.

Theorem 7. Every strongly prime fuzzy is prime fuzzy.

Proof. Let P be strongly prime fuzzy, then P_α is strongly prime for all $P(1) < \alpha \leq P(0)$. Hence P_α is prime. Based on Proposition 12 P is prime fuzzy. \square

Theorem 8. Let R be a finite ring with unity. P is a strongly prime fuzzy iff P is prime fuzzy.

Proof. Immediately from Proposition 5, definition 18 and Proposition 12. \square

The next two results show that Zadeh's extension preserves prime fuzzy and strongly prime fuzzy when f is an isomorphism.

Proposition 15. Let $f : R \rightarrow S$ be an isomorphism of rings. If P is a prime fuzzy ideal of R , then $f(P)$ is a prime fuzzy ideal of S .

Proof. Since f is bijective, given $y \in S$, there is a unique $x \in R$ such that $f(x) = y$. Hence, $f(P)(y) = P(x)$ and $f(x) = y$ for all $y \in S$. Then:

$$\begin{aligned} f(P)_\alpha &= \{y \in S : f(P)(y) \geq \alpha\} \\ &= \{f(x) \in S : P(x) \geq \alpha\} \\ &= f(P_\alpha). \end{aligned}$$

As P is prime fuzzy, by Proposition 12, P_α is prime $P(1) < \alpha \leq P(0)$ and by Proposition 6 $f(P_\alpha)$ is prime and then $f(P)_\alpha$ is prime for all $P(1) < \alpha \leq P(0)$. By Proposition 12 once more $f(P)$ is prime fuzzy. \square

Theorem 9. Let $f : R \rightarrow S$ be an isomorphism of rings. If P is a strongly prime fuzzy ideal of R , then $f(P)$ is a strongly prime fuzzy ideal of S .

Proof. Similar to demonstration of Proposition 15 \square

Proposition 16. Any strongly prime fuzzy ideal contains a minimal strongly prime fuzzy ideal.

Proof. Let P be a strongly prime fuzzy ideal over a ring R . Then, P_* is strongly prime and by Proposition 7 it has a minimal strongly prime $M \subseteq P_*$. Define

$$\nu(x) = \begin{cases} P(0) & \text{if } x \in M \\ P(1) & \text{otherwise.} \end{cases}$$

As $P(0) \neq P(1)$, ν_α is strongly prime for all $\alpha \in [0, 1]$. Thus, ν is equivalent to the characteristic map of M and $\nu \subseteq P$. \square

Proposition 17. Any strongly prime fuzzy ideal contains properly another strongly prime fuzzy ideal.

Proof. Let P be a strongly prime fuzzy. Consider the fuzzy set $\nu = \frac{1}{2} \cdot P \subset P$ defined by $\nu(x) = \frac{1}{2}P(x)$. Both fuzzy sets share the same level subsets. So ν is a strongly prime fuzzy ideal. \square

Theorem 10. Let R be a ring with unity. Any maximal fuzzy ideal is a fuzzy strongly prime ideal.

Proof. Let M be a maximal fuzzy ideal. By Proposition 13 and 14 $Im(M) = \{M(1), 1\}$, $M(0) = 1$ and M_* is a crisp maximal. Let $M(1) < \alpha \leq M(0)$ then $\alpha = M(0)$. Thus, $M_\alpha = M_*$ is a crisp maximal ideal. By crisp theory, every maximal ideal is strongly prime, and then M_α is strongly prime. Therefore, M is strongly prime fuzzy. \square

The converse of theorem 10 is not true as is shown by the following example.

Example 6. Let $R = Z$ be the ring of integers and $I(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$. Note that $I_\alpha = (0)$ for all $I(1) < \alpha \leq I(0)$. Thus, I is a strongly prime fuzzy ideal. Now let $\nu(x) = \begin{cases} 1 & \text{if } x \in 2Z \\ 0 & \text{otherwise} \end{cases}$. Where $2Z = \{n \in Z : n = 2q, q \in Z\}$.

Clearly $I \subseteq \nu$, but $\nu \neq Z$ and $\nu_* = 2Z \neq (0) = I_*$. Therefore, I is not maximal.

3.3 Strongly Prime Radical of a Fuzzy Ideal

The right strongly prime radical of a ring R is defined to be the intersection of all right strongly prime ideals of R and the left strongly primeness determines the left strongly prime radical. An example given by Parmenter, Passman and Stewart [34] showed that these two radicals are distinct. In this section we define the concept of right strongly radical (shortly strongly radical) of a fuzzy ideal. Also, it is shown a version of Correspondence Theorem and a right strongly prime radical (shortly sp radical) of a fuzzy ideal is defined and investigated. Throughout this section, unless stated otherwise, R has identity.

Proposition 18. Let $f : R \longrightarrow S$ be an epimorphism of rings such that $f^{-1}(Y)$ is a finite set for all $Y \subseteq S$. If I is a fuzzy set of R and J a fuzzy set of S , then $f(I_\alpha) = f(I)_\alpha$ and $f^{-1}(J_\alpha) = f^{-1}(J)_\alpha$.

Proof. Consider $f(I_\alpha) = \{y \in S : y = f(x), x \in I_\alpha\}$ and $f(I)_\alpha = \{y \in S : f(I)(y) \geq \alpha\}$. Let $y \in f(I_\alpha)$, $y = f(x_0)$ where $I(x_0) \geq \alpha$. Thus, $f(I)(y) = \sup\{I(x) : f(x) = y\} \geq I(x_0) \geq \alpha$ and then $y \in f(I)_\alpha$. On the other side, let $y \in f(I)_\alpha$, i.e. $f(I)(y) = \sup\{I(x) : f(x) = y\} \geq \alpha$. As f is surjective, there exists $x_0 \in R$, where $\alpha \leq I(x_0) \leq \sup\{I(x) : f(x) = y\} = f(I)(y)$. Thus, $x_0 \in I_\alpha$ and then $f(x_0) = y \in f(I_\alpha)$.

To prove $f^{-1}(J_\alpha) = f^{-1}(J)_\alpha$, let $x \in f^{-1}(J_\alpha)$, then $f(x) \in J_\alpha$. Thus, $f^{-1}(J)(x) = J(f(x)) \geq \alpha$ and, therefore, $x \in f^{-1}(J)_\alpha$. Now let $x \in f^{-1}(J)_\alpha$ then $J(f(x)) = f^{-1}(J)(x) \geq \alpha$ and therefore $f(x) \in J_\alpha$. In this case, it is not necessarily used $f^{-1}(Y)$ as a finite set. \square

Theorem 11. Let $f : R \longrightarrow S$ be an epimorphism of rings such that $f^{-1}(Y)$ is a finite set for all $Y \subseteq S$. If I is a sp fuzzy ideal of R such that $\text{Ker}(f) \subseteq I_\alpha$ for $I(1) < \alpha \leq I(0)$, then $f(I)$ is sp fuzzy ideal of R .

Proof. Let I be a sp fuzzy ideal of R , where $I_\alpha \in \text{sp}_f(R)$ for $I(1) < \alpha \leq I(0)$. Applying Theorem 6, (i) $f(I_\alpha) \in \text{sp}(S)$. By the Proposition 18 $f(I)_\alpha$ is sp fuzzy ideal of S . Thus, $f(I) \in \text{sp}(S)$. \square

Proposition 19. Let $f : R \longrightarrow S$ be an epimorphism of rings. If J is a sp fuzzy ideal of S , then $f^{-1}(J)$ is a sp fuzzy ideal of R , where $f^{-1}(J)_\alpha \supseteq \text{Ker}(f)$ for $J(1) < \alpha \leq J(0)$.

Proof. It is a consequence from proposition 18 and theorem 6 (ii). \square

For the next result, consider $\text{SP}_f(R) = \{I \text{ is sp fuzzy ideal of } R : I_\alpha \in \text{sp}_f, I(1) < \alpha \leq I(0)\}$ and $\text{SP}(S)$ is the set of all sp fuzzy ideals of S .

Theorem 12. (Correspondence Theorem) Let $f : R \longrightarrow S$ be an epimorphism of rings such that $f^{-1}(Y)$ is a finite set for all $Y \subseteq S$. Then, there exists a bijection between $\text{SP}_f(R)$ and $\text{SP}(S)$.

Proof. Define $\Psi : \text{SP}_f(R) \longrightarrow \text{SP}(S)$, $\Psi(I) = f(I)$. Let $I, M \in \text{SP}_f(R)$, where $I \neq M$. Thus, there exists $x \in R$, where $I(x) \neq M(x)$, if $\alpha = I(x)$, then $I_\alpha \neq M_\alpha$. According to proposition 18 and Theorem 6, $f(I)_\alpha = f(I_\alpha) \neq f(M_\alpha) = f(M)_\alpha$. Therefore, Ψ is injective. On the other hand, let $J \in \text{SP}(S)$. As J_α is SP by Theorem 6, we have

$f^{-1}(J_\alpha) \in sp_f(R)$, by Proposition 18, $f^{-1}(J_\alpha) = f^{-1}(J)_\alpha$. Thus, $f^{-1}(J)_\alpha$ is SP and $f^{-1}(J) \in SP_f(R)$. Moreover, $\Psi(f^{-1}(J)) = f(f^{-1}(J)) = J$. Therefore, Ψ is surjective.

□

Definition 19. Given a crisp ideal I of a ring R , the strongly radical(or Levitzki radical) of I is $\sqrt[I]{I} = \bigcap \{P : P \supseteq I, P \text{ is strongly prime}\}$.

Definition 20. Let I be a fuzzy ideal of R , the strongly radical of I is $\sqrt[I]{I} = \bigcap_{P \in \mathcal{S}_I} P$, where \mathcal{S}_I is the family of all sp fuzzy ideals P of R such that $I \subseteq P$.

Clearly $\sqrt[I]{I}$ is an ideal, and if I is a sp fuzzy ideal, then $\sqrt[I]{I} = I$

Proposition 20. Let I be a nonconstant fuzzy ideal of ring R . Then:

- i) $\sqrt[I]{I}_* \subseteq (\sqrt[I]{I})_*$, where $I_* = \{x \in R ; I(x) = I(0)\}$;
- ii) $\sqrt[I]{I}(x) = 1$ for all $x \in (\sqrt[I]{I})_*$;
- iii) $\sqrt[I]{I}(0) = I(0)$, $\sqrt[I]{I}(1) = I(1)$;
- iv) $I_* \subseteq (\sqrt[I]{I})_*$;
- v) $I \subseteq \sqrt[I]{I}$.

Proof. Straightforward.

□

Proposition 21. If I, J are a fuzzy ideal of a ring R , then:

- (i) if $I \subseteq J$, then $\sqrt[I]{I} \subseteq \sqrt[J]{J}$;
- (ii) $\sqrt[\sqrt[I]{I}]{\sqrt[I]{I}} = \sqrt[I]{I}$;
- (iii) $I_\alpha \subseteq (\sqrt[I]{I})_\alpha$;
- (iv) If I is SP fuzzy, then $\sqrt[I]{I}_\alpha = (\sqrt[I]{I})_\alpha$;
- (v) $\sqrt[I \cap J]{I \cap J} \subseteq \sqrt[I]{I} \cap \sqrt[J]{J}$.

Proof. (i) $\sqrt[J]{J} = \bigcap_{P \in \mathcal{S}_J} P \supseteq \bigcap_{P \in \mathcal{S}_I} P = \sqrt[I]{I}$. (ii) It is easy to see that $\sqrt[I]{I} \subseteq \sqrt[\sqrt[I]{I}]{\sqrt[I]{I}}$. On the other side, let's show $\mathcal{S}_I \subseteq \mathcal{S}_{\sqrt[I]{I}}$. In fact, let $P \in \mathcal{S}_I$, then $P \supseteq I$ using (i) $P = \sqrt[P]{P} \supseteq \sqrt[I]{I}$. (iii),(iv) and (v) is straightforward. □

Proposition 22. Let $f : R \rightarrow S$ be a homomorphism of rings and I a fuzzy ideal of R . Then:

$$1) f(I) \subseteq f(\sqrt[s]{I}) \subseteq \sqrt[s]{f(\sqrt[s]{I})};$$

$$2) I \subseteq f^{-1}(\sqrt[s]{f(I)}).$$

Proof. 1) Straightforward.

2) As $f(I) \subseteq \sqrt[s]{f(I)}$, then $f^{-1}(f(I)) \subseteq f^{-1}(\sqrt[s]{f(I)})$. Thus, $I \subseteq f^{-1}(f(I)) \subseteq f^{-1}(\sqrt[s]{f(I)})$.

□

Proposition 23. Let $f : R \rightarrow S$ be a homomorphism of rings and I a SP fuzzy ideal of R . Then, $f(\sqrt[s]{I}) \subseteq \sqrt[s]{f(I)}$.

Proof. As I is SP fuzzy $\sqrt[s]{I} = I$, then $\sqrt[s]{f(I)} = \sqrt[s]{f(\sqrt[s]{I})}$. Thus, $f(\sqrt[s]{I}) \subseteq \sqrt[s]{f(\sqrt[s]{I})} = \sqrt[s]{f(I)}$.

□

Proposition 24. Let $f : R \rightarrow S$ be an epimorphism of rings and I a sp fuzzy ideal of R , such that $\text{Ker}(f) \subseteq I_\alpha$ for $I(1) < \alpha \leq I(0)$. Then, $f(\sqrt[s]{I}) = \sqrt[s]{f(I)}$.

Proof. As I is SP fuzzy ideal, $I = \sqrt[s]{I}$, $f(I) = f(\sqrt[s]{I})$. Using the theorem 11 $f(I)$ is SP fuzzy ideal and then $f(I) = \sqrt[s]{f(I)}$. Thus, $f(\sqrt[s]{I}) = \sqrt[s]{f(I)} = \sqrt[s]{\sqrt[s]{f(I)}}$.

□

Proposition 25. Let $f : R \rightarrow S$ be an epimorphism of rings and I a fuzzy ideal of R such that $\text{Ker}(f) \subseteq I_\alpha$ for $I(1) < \alpha \leq I(0)$ and $\sqrt[s]{I}$ is SP fuzzy ideal. Then, $f(\sqrt[s]{I})$ is SP fuzzy ideal of S .

Proof. Straightforward.

□

3.4 Semi-Strongly Prime and Strongly Primary Ideals

The aim of this section is to prove a strong prime fuzzy version (proposition 27) of the following *theorem*:

“In a commutative ring, I is a prime ideal iff I is semi-prime and primary ideal.”

Definition 21. A crisp or fuzzy ideal I of ring R is semi-strongly prime (or semi-sp), iff $\sqrt[s]{I} = I$.

Proposition 26. Let $f : R \rightarrow S$ be a homomorphism of rings and I semi-sp fuzzy ideal of R , then $f(I)$ is semi-sp fuzzy ideal of S .

$$\textit{Proof. } f(I) = f(\sqrt[s]{I}) = f\left(\bigcap_{J \in \mathcal{P}_I} J\right) = \bigcap_{J \in \mathcal{P}_I} f(J) \supseteq \bigcap_{P \in \mathcal{P}_{f(I)}} P = \sqrt[s]{f(I)}.$$

Thus, $f(I) = \sqrt[s]{f(I)}$.

□

Corollary 13. Let $f : R \rightarrow S$ be a homomorphism of rings.

- 1) If I is semi-sp fuzzy ideal of R , then $f(\sqrt[s]{I}) = \sqrt[s]{f(I)}$.
- 2) If I is sp fuzzy ideal of R , then $f(I)$ is sp fuzzy ideal of S .

Proof. Straightforward.

□

According to the classical definition, a proper ideal I in a ring R is said to be primary whenever xy is an element of I we have $x \in I$ or $y^n \in I$, for some $n > 0$. Moreover, the last condition can be replaced by $x \in I$ or $y \in \sqrt{I}$, where \sqrt{I} is the radical of I defined by $\sqrt{I} = \{r \in R : r^n \in I \text{ for some positive integer } n\}$.

Definition 22. (Crisp) A proper ideal I of a ring R is said to be *strongly primary*, whenever $xy \in I$, we have $x \in I$ or $y \in \sqrt{I}$.

According [35], Malik and Moderson, if a fuzzy ideal is primary, its α -cuts may not be necessarily primary. Thus, we decided to define primary fuzzy from α -cuts as follows:

Definition 23. A non-constant fuzzy ideal I of a ring R is said to be *primary fuzzy* iff its α -cuts are primary ideals of R .

Definition 24. A non-constant fuzzy ideal I of a ring R is said to be *strongly primary fuzzy* iff its α -cuts are strongly primary ideals of R .

We observe that whenever I is a primary ideal, then I is strongly primary.

Proposition 27. (Crisp) I is SP iff I is semi-strongly prime and primary ideal of R .

Proof. (\Rightarrow) Straightforward. (\Leftarrow) Suppose I is not strongly prime. Then, there exists an element $x \in R - I$, such that for every finite subset $F \subseteq R$, there exists $r \in R$, such that $xFr \subseteq I$ and $r \notin I$. Let $F = \{1\}$, then there exists $r \in R$, such that $xr \in I$ and $r \notin I$. That contradicts the fact of $I = \sqrt[s]{I}$ and strongly primary. □

Corollary 14. If I is SP fuzzy ideal, then I is semi-strongly prime and primary fuzzy ideal of R .

Proof. Straightforward. □

Sometimes we define a fuzzy structure appealing to α -cuts. However, may all results will be depending of them. Note that for the converse of Corollary 14 it is necessary to have $\sqrt[s]{I_\alpha} = (\sqrt[s]{I})_\alpha$. This fact occurs because our definition on primary was based on α -cuts.

Corollary 15. If I is semi-strongly prime and primary fuzzy ideal of R and $\sqrt[s]{I_\alpha} = (\sqrt[s]{I})_\alpha$ for all $I(1) < \alpha I(0)$, then I is SP fuzzy.

Proof. Straightforward. □

Theorem 16. I is SP fuzzy ideal iff I is semi-strongly prime and primary fuzzy ideal and $\sqrt[s]{I_\alpha} = (\sqrt[s]{I})_\alpha$ for all $I(1) < \alpha \leq I(0)$.

Proof. Straightforward. □

3.5 Special Strongly Prime Fuzzy Ideals

In this section we provide two fuzzy structures which do not have correspondence in crisp Algebra and are not built from α -cuts.

Definition 25. (ASSP) Let R be an arbitrary ring with unity. A non-constant fuzzy ideal $I : R \rightarrow [0, 1]$ is said to be *almost special strongly prime*, if for every $x \in R$ there exists a subset F_x of R , such that $I(r) \geq \bigwedge I(xF_x r)$ for all $r \in R$.

Definition 26. (SSP) Let R be an arbitrary ring with unity. A non-constant fuzzy ideal $I : R \rightarrow [0, 1]$ is said to be *special strongly prime*, if for every $x \in R$ there exists a *finite* subset F_x of R , such that $I(r) \geq \bigwedge I(xF_x r)$ for all $r \in R$.

Proposition 28. If I is SSP, then I is SP fuzzy ideal.

Proof. Let's show that I_α is SP for $I(1) < \alpha \leq I(0)$. Let $x \in I_\alpha \setminus R$. As I is SSP, there exists a finite set F_x . Suppose $xF_x r \subseteq I_\alpha$, hence $I(r) \geq \bigwedge I(xF_x r) \geq \alpha$ and then $r \in I_\alpha$.

□

Proposition 29. If I is SSP, then I is ASSP.

Proof. Straightforward. □

Proposition 30. If I is SP fuzzy ideal, then I is ASSP.

Proof. Let $x \in R$, then $x \notin I_\beta$ for all $I(x) < \beta \leq I(0)$. As I_β is strongly prime, there exists a finite set $F_x^\beta \subseteq R$, such that $xF_x^\beta r \subseteq I_\beta$ implies $r \in I_\beta$ for all $r \in R$, i.e. $\bigwedge I(xF_x^\beta r) \geq \beta$ implies $I(r) \geq \beta$. Let $J = \{\beta \in [0, 1] : I(x) < \beta \leq I(0)\}$ and $F_x = (\bigcup_{\beta \in J} F_x^\beta) \cup \{y\}$, where $I(y) > I(x)$. Consider $r \in R$ and $t = \bigwedge I(xF_x r) > I(x)$.

Observe that $x \notin I_t$ and I_t are SP, i.e. $\bigwedge I(xF_x^t r) \geq t$ implies $I(r) \geq t$.

As $F_x^t \subseteq F_x$, then $\bigwedge I(xF_x^t r) \geq \bigwedge I(xF_x r) = t$ and $I(r) \geq t$. Therefore, I is ASSP.

□

Corollary 17. Let I be a fuzzy ideal and $Im(I)$ is a finite set. I is SSP iff I is SP.

Proof. Straightforward.

□

Question 1. In which conditions do the classes of ASSP and SSP coincide? Furthermore, does Zadeh's extension preserve ASSP and SSP?

Chapter 4

Uniformly Strongly Prime Fuzzy Ideals

We proposed in section 3.2 a notion of sp ideals for the fuzzy environment. This definition of sp fuzzy ideal, or shortened spf ideal, was based on α -cuts. In this approach we can realize that all results for fuzzy environment have similar counterpart in classical algebra. Although we could not find (like Navarro in Definition 15) a pure fuzzy definition of spf ideals, these ideas led them to propose the concept of uspf (uniformly strongly prime fuzzy) ideal. Thus, as we shall see, it is possible to propose a notion of uspf ideals which is not based on α -cuts. This approach is proposed in order to investigate a fuzzy algebraic structure which is somehow independent of the crisp setting. For example, in classical ring theory an ideal is a usp ideal if and only if its quotient is a usp ring. However, as we shall prove in the example 7, this statement is not true for uspf ideals.

Section 4.1 provides the definition of uspf ideals and results about them. We prove that the inverse image of Zadeh's extension of uspf ideal is an uspf ideal which are constant on $Ker(f)$ (Proposition 32). On the other hand, the direct image of a uspf ideal of Zadeh's extension is not a uspf ideal (Example 8). Also, it is shown that all uspf ideals are prime fuzzy ideals in accordance with the new definition of prime fuzzy ideal given in Definition 15. It is shown how we can build a uspf ideal based on usp crisp ideal and section 4.2 has new results on uspf ideals and contains questions and conjectures about it. Finally, section 4.3 introduces the uniform strong radical in fuzzy setting.

4.1 Introduction

Definition 27. [23] Let R be an associative ring with unity. A non-constant fuzzy ideal

$I : R \rightarrow [0, 1]$ is said to be *uspf* ideal if there exists a finite subset F such that $\bigwedge I(xFy) = I(x) \vee I(y)$, for any $x, y \in R$. The set F is called *insulator* of I .

Proposition 31. I is *uspf* ideal of R iff I_α is *usp* ideal of R for all $I(1) < \alpha \leq I(0)$.

Proof. Suppose I a *USPf* ideal and let $F \subseteq R$ be a finite set given by definition 27. Let $x, y \in R$ and $I(1) < \alpha \leq I(0)$ such that $xFy \subseteq I_\alpha$. Hence, $I(x) \vee I(y) = \bigwedge I(xFy) \geq \alpha$, and thus $I(x) \geq \alpha$ or $I(y) \geq \alpha$. Therefore, $x \in I_\alpha$ or $y \in I_\alpha$. On the other hand, suppose I_α is a *usp* ideal of R for all $I(1) < \alpha \leq I(0)$. According to Proposition 8 each I_α has a finite set F_α such that if $xF_\alpha y \subseteq I_\alpha$ implies $x \in I_\alpha$ or $y \in I_\alpha$. Let a finite set $F = \bigcap_{I(1) < \alpha \leq I(0)} F_\alpha$. Suppose $\bigwedge I(xFy) > I(x) \vee I(y)$ and $t = \bigwedge I(xFy)$ for any $x, y \in R$.

Note that $t > I(x) \vee I(y)$ and $t \leq I(xfy)$ for all $f \in F$. Hence, $x, y \notin I_t$, but $xFy \subseteq I_t$ and thus (by hypothesis) $x \in I_t$ or $y \in I_t$, where we have a contradiction. Therefore, $\bigwedge I(xFy) = I(x) \vee I(y)$. \square

Corollary 18. If I is a *uspf* ideal of a ring R , then R/I_α is a *usp* ring for all $I(1) < \alpha \leq I(0)$.

Proof. It stems from the definition of *usp* ideal and the last proposition. \square

Corollary 19. If I is *uspf* ideal, then I is prime fuzzy ideal.

Proof. Since I is *uspf* ideal, there exists a finite set F , where $\bigwedge I(xFy) = I(x) \vee I(y)$, for any $x, y \in R$. Note that $xFy \subseteq xRy$. Hence, $\bigwedge I(xFy) \geq \bigwedge I(xRy)$. Therefore, $\bigwedge I(xFy) = I(x) \vee I(y) \geq \bigwedge I(xRy)$. \square

Proposition 32. If $f : R \rightarrow S$ is a epimorphism of rings and J is a *uspf* ideal of S , then $f^{-1}(J)$ is a *uspf* ideal of R which is constant on $\text{Ker}(f)$.

Proof. As J is *uspf* ideal, then there exists a finite set F_J (according to definition of *uspf* ideal) and $f^{-1}(J)$ is a fuzzy ideal which is constant on $\text{Ker}(f)$ by Proposition 11. Let $F = f^{-1}(F_J)$, hence

$$\begin{aligned}
\bigwedge f^{-1}(J)(xFy) &= \bigwedge J(f(xFy)) \\
&= \bigwedge J(f(x)f(F)f(y)) \\
&= \bigwedge J(f(x)f(f^{-1}(F_J))f(y)) \\
&= \bigwedge J(f(x)F_Jf(y)) \\
&= J(f(x)) \vee J(f(y)) \\
&= f^{-1}(J)(x) \vee f^{-1}(J)(y).
\end{aligned}$$

Thus, $f^{-1}(F_J)$ is the insulator of a fuzzy ideal $f^{-1}(J)$. \square

Proposition 33. If I is a uspf ideal of a ring R , then R/I is a uspf ring.

Proof. As I is uspf, there exists a finite set F such that $\bigwedge I(xFy) = I(x) \vee I(y)$, for any $x, y \in R$. Let $F' = \Psi(F)$, where Ψ is the natural homomorphism from R to R/I . Given $x + I, y + I \neq \bar{0}$ (i.e. $I(x), I(y) \neq I(0)$) in R/I . As I is uspf, then we have $\bigwedge I(xFy) = I(x) \vee I(y)$. Hence, there exists $f \in F$ such that $I(xfy) = I(x) \vee I(y) \neq I(0)$. Therefore, $xfy + I \neq \bar{0}$, where $f + I \in F'$ and according to Lemma 2 R/I is uspf ring. \square

For the next result, consider $I_* = I_{I(0)} = \{x \in R : I(x) = I(0)\}$.

Proposition 34. If I is a uspf ideal of a ring R , then $R/I_* \cong R/I$.

Proof. Consider $f : R \rightarrow R/I$, where $f(x) = x + I$. Note that $r + I = 0$ iff $I(r) = I(0)$. Thus, $\text{Ker}(f) = I_*$ and by the isomorphism theorem [36] we have $R/I_* \cong R/I$. \square

Corollary 20. If $f : R \rightarrow S$ is an epimorphism and I uspf ideal of R which is constant on $\text{Ker}(f)$, then $R/I \cong S/f(I)$.

Proof. Define $h : R \rightarrow S/f(I)_*$ and $h(x) = f(x) + f(I)_*$. Thus, h is onto and $\text{Ker}(h) = I_*$. Applying the isomorphism theorem, $R/I_* \cong S/f(I)_*$. Thus, $R/I \cong R/I_* \cong S/f(I)_* \cong S/f(I)$; \square

The next proposition shows us how it is possible to build a uspf ideal based on usp crisp ideal.

Proposition 35. Let J be an ideal (crisp) of R . Define $I : R \rightarrow [0, 1]$ as

$$I(x) = \begin{cases} 1, & \text{if } x = 0; \\ \alpha, & \text{if } x \in J \setminus \{0\}; \\ 0, & \text{if } x \notin J, \end{cases}$$

where $0 < \alpha < 1$. Then:

- i) I is a fuzzy ideal;
- ii) I is uspf ideal iff J is usp ideal.

Proof. i) Note that all α -cuts of I are $I_* = (0)$, $I_\alpha = J$ and $I_0 = R$, according to Proposition 10 I is a fuzzy ideal of R . ii) Suppose I is uspf ideal. We will prove that J is an usp ideal according to Proposition 9. Thus, let $x, y \notin J$, as I is uspf, there is a finite set F , where $\bigwedge I(xFy) = I(x) \vee I(y) = 0$. Since F is finite, there exists $f \in F$ where $I(xfy) = 0$, then $xfy \notin J$. On the other hand, suppose J is usp ideal of R , hence there exists a finite set F for J according to definition of usp crisp ideal. Thus, given $x, y \in R$ we have the following cases: 1) If $x, y = 0$, then we have triviality $\bigwedge I(xFy) = I(0) = I(x) \vee I(y) = I(0)$; 2) If $x \in J$ or $y \in J$, then $xFy \subseteq J$. Thus, $\bigwedge I(xFy) = \alpha = I(x) \vee I(y)$; 3) If $x \notin J$ and $y \notin J$, then there exists $f \in J$ such that $xfy \notin J$. Thus, $\bigwedge I(xFy) = 0 = I(x) \vee I(y)$.

□

Corollary 21. Let I be a non-constant fuzzy ideal of R and define:

$$M(x) = \begin{cases} I(0), & \text{if } x = 0; \\ \alpha, & \text{if } x \in I_* \setminus \{0\}; \\ I(1), & \text{if } x \notin I_*. \end{cases}$$

Then, M is uspf ideal of R iff I_* is usp ideal of R .

Proof. Straightforward

□

Corollary 22. Let $I_1 \subset I_2 \subset \dots \subset I_n = R$ be any chain of usp ideals of a ring R . Let t_1, t_2, \dots, t_n be some numbers in $[0, 1]$ such that $t_1 > t_2 > \dots > t_n$. Then the fuzzy subset I defined by

$$I(x) = \begin{cases} t_1, & \text{if } x \in I_1 \\ t_i, & \text{if } x \in I_i \setminus I_{i-1}, i = 2, \dots, n, \end{cases}$$

is a uspf ideal of R .

Example 7. Consider Z the ring of integers and $4Z = \{x \in Z : x = 4q, q \in Z\}$. Define

$$\text{a fuzzy set as: } I(x) = \begin{cases} 1, & \text{if } x = 0; \\ 1/2, & \text{if } x \in 4Z \setminus \{0\}; \\ 0, & \text{if } x \notin 4Z. \end{cases}$$

I is a fuzzy ideal, since its all α -cuts ($I_1 = (0), I_{1/2} = 4Z, I_0 = Z$) are ideals. Moreover, I is not uspf ideal, since $4Z$ is not prime ideal, according to Proposition 35. Note that $I_* = (0)$ is uspf ideal. Hence, R/I_* is a uspf ring. Applying the Proposition 34 $R/I \cong R/I_*$. Therefore, R/I is a uspf ring, but I is not uspf ideal.

Example 8. Let $f : Z \rightarrow Z_4$ be defined by $f(x) = [x]_4 = x \text{ mod } 4$. The function f is an epimorphism with kernel $4Z$. Consider

$$I(x) = \begin{cases} 1, & \text{if } x = 0; \\ 1/2, & \text{if } x \in 3Z \setminus \{0\}; \\ 0, & \text{if } x \notin 3Z. \end{cases}$$

and then

$$f(I)(y) = \begin{cases} 1, & \text{if } x = 0; \\ 1/2, & \text{if } x \neq 0. \end{cases}$$

Clearly I is uspf ideal of Z , but $f(I)$ is not uspf ideal of Z_4 , since $I_{1/2} = 3Z$ is not uspf ideal.

4.2 Extra Results on Uspf Ideals

This section amplifies results about uspf ideals. The first one (Proposition 36) is geared to commutative rings. However, it may be valid for noncommutatives (Conjecture 2). The Proposition 8 brings the difference between crisp and fuzzy setting by showing the behavior of Zadeh's extension on uspf ideals. The results in this section was published in [37].

Proposition 36. If I is a non-constant fuzzy ideal of a commutative ring R , then there exists a uspf ideal K such that $I \subseteq K$.

Proof. Consider the crisp ideal $I^* = \{x \in R : I(x) > I(1)\}$. By Zorn's Lemma, there exists a maximal ideal M of R containing I^* . Now we can define the following fuzzy set:

$$K(x) = \begin{cases} I(0) & \text{if } x \in M, \\ I(1) & \text{otherwise.} \end{cases}$$

Clearly, K is a fuzzy ideal and $I \subseteq K$. Now, consider the finite set $F = \{1\}$. Thus, $\bigwedge K(xFy) = K(xy)$ for any $x, y \in R$.

If $x \in M$ or $y \in M$, then $xy \in M$ and then $K(xy) = I(0) = K(x) \vee K(y)$. On the other hand, as R is commutative, M is completely prime, hence if $x \notin M$ and $y \notin M$, then $xy \notin M$. Therefore, $K(xy) = I(1) = I(1) \vee I(1) = K(x) \vee K(y)$.

□

Conjecture 1. According to the definition of fuzzy maximal ideal given in Definition 17, the set K in the demonstration of Proposition 36 is a fuzzy maximal ideal.

Conjecture 2. The Proposition 36 can be extended to noncommutative rings.

Proposition 37. Let $f : R \rightarrow S$ is a epimorphism of commutative and non usp rings. If I is a uspf ideal of R which is constant on $\text{Ker}(f)$, then $f(I)$ is not a uspf ideal of S .

Proof. As I is constant on $\text{Ker}(f)$, then by Proposition 34 and Corollary 20 we have: $R/I_* \cong R/I \cong R/f(I) \cong R/f(I)_*$. As I_* is uspf ideal, then R/I_* is uspf ring. Hence, $R/f(I)_*$ is uspf ring. Thus, $f(I)_*$ is uspf ideal. As we know $f(I)_* \subseteq f(I)_\alpha$ for all $\alpha \in [0, 1]$. But S is commutative and $f(I)_*$ is Prime, hence $f(I)_*$ is maximal, this last statement implies $f(I)_\alpha = S$ for all $\alpha \neq I(0)$ and by hypotheses S is not uspf ring. Therefore, $f(I)$ is not uspf. □

Question 2. The Proposition 37 shows us that uspf ideals cannot be preserved by Zadeh's extension. Thus, we ask: Under which conditions can Zadeh's extension preserve the uspf ideals? This question still open.

Proposition 38. If I and P are fuzzy ideals of a ring R with P uspf ideal, then $I \cap P$ is uspf ideal of R .

Proof. Note that: $\bigwedge (I \cap P)(xFy) = (\bigwedge I(xFy)) \wedge (\bigwedge P(xFy)) = (\bigwedge I(xFy)) \wedge (P(x)) \vee P(y) \leq P(x) \vee P(y)$.

□

Proposition 39. Any uspf ideal contains properly another uspf ideal

Proof. Suppose I uspf ideal of a ring R . Let $P = \frac{1}{2}I \subset I$ defined by $P(x) = \frac{1}{2}I(x)$ for all $x \in R$. Hence, $\bigwedge P(xFy) = \bigwedge \frac{I(xFy)}{2} = \frac{I(x)}{2} \vee \frac{I(y)}{2} = P(x) \vee P(y)$. □

The next proposition tell us about the following question: If a fuzzy ideal has at least one usp α -cut, what can we say about this ideal. Is it a uspf ideal?

Proposition 40. Let I a nonconstant fuzzy Ideal of a Integral Domain R and R is not a usp ring and I_t is usp ideal for some $I(1) < t \leq I(0)$. If $k \neq t$ and $I_k \neq I_t$, then I_k is not usp ideal. Hence, I is not a uspf ideal.

Proof. When $I_k = R$ is trivial. Now suppose $I_k \neq R$ and note that in a Integral Domain if I is usp ideal, then I is a Maximal ideal. Thus, $I_t \subset I_k$ is impossible, since I_t is Maximal. If $I_k \subset I_t$ implies I_k not maximal, then I_k not a usp ideal. \square

4.3 US Fuzzy Radical

Since its inception, general theory of radicals has proved to be fundamental for the structure of ring theory. A better understanding of radical of a fuzzy ideal can give to us some information about its nature. The crisp Uniformly Strongly Prime radical (US radical) of a ring R was defined to be the intersection of all usp ideals of R . Olson [3] located this radical in the lattice of radical classes and proved that US radical is independent of Jacobson and Brown-McCooy radical. In this section we defined the US radical of a fuzzy ideal in the standard way by comparing with two new notions.

Definition 28. [3] The US radical of a crisp ideal I is $US(I) = \cap\{P \subseteq R : P \supseteq I \text{ and } P \text{ is usp ideal of } R\}$.

Definition 29. [23] Let I be a fuzzy ideal of R . The uniformly strongly fuzzy radical of I is $\sqrt[usp]{I} = \cap\{P \subseteq R : P \supseteq I \text{ and } P \text{ is uspf}\}$.

The radical $usp\mathcal{R}$ of a ring R is defined as $\sqrt[usp]{0}$. Clearly $\sqrt[usp]{R} = \cap\{P \subseteq R : P \text{ is uspf ideal of } R\}$.

Remark 1. According to [23] the quotient $R/\sqrt[usp]{R}$ is usp ring and $R/(\sqrt[usp]{R})_* \cong R/\sqrt[usp]{R}$.

Clearly, $\sqrt[usp]{I}$ is an ideal, and if I is a unif. strongly prime fuzzy ideal, then $\sqrt[usp]{I} = I$.

Proposition 41. If I is a fuzzy ideal of a ring R , then $\sqrt[usp]{I}$ is a uspf ideal of R .

Proof. Consider $S_I = \{P \subseteq R : P \supseteq I \text{ and } P \text{ is uspf}\}$ and $F = \bigcap_{P \in S_I} F_P$, where F_P is a finite set (insulator) of P . Clearly F is a finite set. Given $x, y \in R$, hence,

$$\begin{aligned} \bigwedge \sqrt[4]{I}(xFy) &= \bigwedge \left(\bigcap_{P \in \mathcal{S}_I} P(xFy) \right) = \bigwedge \left(\bigwedge_{P \in \mathcal{S}_I} P(xFy) \right) = \bigwedge_{P \in \mathcal{S}_I} \left(\bigwedge P(xFy) \right) = \\ & \bigwedge_{P \in \mathcal{S}_I} (P(x) \vee P(y)) = \bigwedge_{P \in \mathcal{S}_I} P(x) \vee \bigwedge_{P \in \mathcal{S}_I} P(y) = \sqrt[4]{I}(x) \vee \sqrt[4]{I}(y). \quad \square \end{aligned}$$

Proposition 42. If I, J are a fuzzy ideals of a ring R , then:

- (i) if $I \subseteq J$, then $\sqrt[4]{I} \subseteq \sqrt[4]{J}$;
- (ii) $\sqrt[4]{\sqrt[4]{I}} = \sqrt[4]{I}$;
- (iii) $I_\alpha \subseteq (\sqrt[4]{I})_\alpha$;
- (iv) If I is unif. strongly prime fuzzy ideal, then $\sqrt[4]{I}_\alpha = (\sqrt[4]{I})_\alpha$;
- (v) $\sqrt[4]{I \cap J} \subseteq \sqrt[4]{I} \cap \sqrt[4]{J}$.

Proof. (i) $\sqrt[4]{J} = \bigcap_{P \in \mathcal{S}_J} P \supseteq \bigcap_{P \in \mathcal{S}_I} P = \sqrt[4]{I}$. (ii) It is easy to see that $\sqrt[4]{I} \subseteq \sqrt[4]{\sqrt[4]{I}}$. On the other side, let's show $\mathcal{S}_I \subseteq \mathcal{S}_{\sqrt[4]{I}}$. In fact, let $P \in \mathcal{S}_I$, then $P \supseteq I$ using (i) $P = \sqrt[4]{P} \supseteq \sqrt[4]{I}$. (iii),(iv) and (v) are straightforward. \square

Proposition 43. Let $f : R \rightarrow S$ be a homomorphism of rings and I a fuzzy ideal of R . Then:

- 1) $f(I) \subseteq f(\sqrt[4]{I}) \subseteq \sqrt[4]{f(\sqrt[4]{I})}$;
- 2) $I \subseteq f^{-1}(\sqrt[4]{f(I)})$.

Proof. 1) Straightforward.

2) As $f(I) \subseteq \sqrt[4]{f(I)}$, then $f^{-1}(f(I)) \subseteq f^{-1}(\sqrt[4]{f(I)})$. Thus, $I \subseteq f^{-1}(f(I)) \subseteq f^{-1}(\sqrt[4]{f(I)})$. \square

Proposition 44. Let $f : R \rightarrow S$ be a homomorphism of rings and I a SP fuzzy ideal of R . Then, $f(\sqrt[4]{I}) \subseteq \sqrt[4]{f(I)}$.

Proof. As I is SP fuzzy $\sqrt[4]{I} = I$, then $\sqrt[4]{f(I)} = \sqrt[4]{f(\sqrt[4]{I})}$. Thus, $f(\sqrt[4]{I}) \subseteq \sqrt[4]{f(\sqrt[4]{I})} = \sqrt[4]{f(I)}$. \square

4.3.1 Extra Results on US Radical

Proposition 45. If I is a fuzzy ideal of R and $P \supseteq I$ a uspf ideal, then $\cap P_t = US(I_t)$ for any $t \in (I(1), I(0)]$, where P_t, I_t are t -cuts of P and I respectively.

Proof. Straightforward. □

Definition 30. The radical *uspfR-inf* of a fuzzy ideal I is $\alpha_I : R \rightarrow [0, 1]$, where $\alpha_I(x) = \wedge \{t : x \in US(I_t)\}$. The radical *uspfR-sup* is $\beta_I : R \rightarrow [0, 1]$, where $\beta_I(x) = \vee \{t : x \in US(I_t)\}$.

Proposition 46. If I is a fuzzy ideal of R , then $\sqrt[uspf]{I} \geq \beta_I \geq \alpha_I$.

Proof. Clearly, $\beta_I \geq \alpha_I$. For $\sqrt[uspf]{I} \geq \beta_I$ consider $x \in US(I_t)$. According to Proposition 45, $x \in \cap P_t$. Thus, $\cap P(x) \geq t$, where $I \subseteq P$ and P is uspf ideal. Therefore, $\sqrt[uspf]{I} \geq \beta_I$. □

Question 3. Under which conditions $\sqrt[uspf]{I} = \beta_I$?

Proposition 47. If I is a non-constant fuzzy Ideal of a commutative ring R , then $\sqrt[uspf]{I}(0) = I(0)$ and $\sqrt[uspf]{I}(1) = I(1)$.

Proof. According to Proposition 36 there exists a uspf ideal $K \supseteq I$. Thus, $\sqrt[uspf]{I}(0) \leq K(0) = I(0)$ and $\sqrt[uspf]{I}(1) \leq K(1) = I(1)$. By the definition of uspfR radical $\sqrt[uspf]{I} \supseteq I$, then $\sqrt[uspf]{I}(0) \geq I(0)$ and $\sqrt[uspf]{I}(1) \geq I(1)$. □

Conjecture 3. The Proposition 47 is valid in associative rings with unit.

4.4 The Fuzzy m - and t -systems

An m -system is a generalization of multiplicative systems. In the ring theory a set M is a m -system if for any two elements x, y in M there exists r in R such that the product xry belongs M . It is not hard to see that an ideal is prime iff its complement is a m -system (see McCoy [38]). On the other hand we have the t -systems which are sets where if any two elements x, y in M there exists a finite set F such that xfy belongs M for some f in F . Clearly a t -system is a m -system. Olson [3] proved that I is a uniformly strongly prime ideal iff its complement is a t -system. Therefore, we have a tool to deal with primeness and uniform strong primeness.

In this chapter we will introduce the m -system in a fuzzy setting based on the definition of prime fuzzy ideals without α -cut dependence, given by Navarro [13] in 2012. The

t -system is also introduced, and we have another characterization of uspf ideal. At the end of the chapter a method to count the number of the uspf ideals in a finite ring is introduced.

Definition 31. [38] A subset K of a ring R is called a m -system if for any two elements $x, y \in K$ there exists $r \in R$ such that $xry \in K$.

Definition 32. [3] A subset M of a ring R is called a t -system if there exists a finite set $F \subseteq R$ such that for any two elements $x, y \in M$ there exists $f \in F$ such that $xfy \in M$.

In the last Definition F will be called the insulator of M . The empty set will be a t -system by definition.

Proposition 48. [38] If M is a t -system, then M is a m -system.

Proposition 49. [38] I is a prime ideal of a ring R iff $R \setminus I$ (the complement of I in R) is a m -system.

Proposition 50. [3] An ideal I is uspf of a ring R iff $R \setminus I$ (the complement of I in R) is a t -system.

Proposition 51. [3] If I and P are ideals of a ring R with P uspf ideal, then $I \cap P$ is uspf ideal.

4.4.1 The Fuzzy m - and t -Systems

For the next definition consider $xRy = \{xry : r \in R\}$.

Definition 33. Let R be an associative ring with unity. A non-constant fuzzy set $K : R \rightarrow [0, 1]$ is said to be *fuzzy m -system* if $\bigvee K(xRy) = K(x) \wedge K(y)$, for any $x, y \in R$.

Proposition 52. If K is fuzzy subset of a ring R such that K_α is a m -system for all α -cuts, then $\bigvee K(xRy) \geq K(x) \wedge K(y)$,

Proof. Let $x, y \in R$ and $t = K(x) \wedge K(y)$. As K_t is m -system and $x, y \in K_t$ then there exists $r \in R$ such that $xry \in K_t$ i.e $K(xry) \geq t$. Hence, $\bigvee K(xRy) \geq t$. \square

Question 4. Under which conditions can we have the following result: is K a fuzzy m -system of R iff K_α is an m -system for all α -cuts?

For the next results consider P the fuzzy ideal and $P_c = 1 - P$ the complement of P in R .

Corollary 23. P is a prime fuzzy ideal of R iff P_c (the complement of P in R) is a fuzzy m -system.

Proof. Suppose P prime fuzzy, then $\bigwedge P(xRy) = P(x) \vee P(y)$ for any $x, y \in R$. Hence, $\bigvee P_c(xRy) = \bigvee (1 - P(xRy)) = 1 - \bigwedge P(xRy) = 1 - (P(x) \vee P(y)) = (1 - P(x)) \wedge (1 - P(y)) = I_c(x) \wedge I_c(y)$. On the other side, suppose P_c a fuzzy m -system, then $\bigvee P_c(xRy) = P_c(x) \wedge P_c(y)$ for any $x, y \in R$. Thus, $1 - \bigwedge P(xRy) = 1 - (P(x) \vee P(y))$. Therefore $\bigwedge P(xRy) = P(x) \vee P(y)$. \square

For the next definition consider the subset $xFy = \{xfy : f \in F\}$ of ring R .

Definition 34. Let R be an associative ring with unity. A non-constant fuzzy set $M : R \rightarrow [0, 1]$ is said to be *fuzzy t -system* if there exists a finite subset F such that $\bigvee M(xFy) = M(x) \wedge M(y)$, for any $x, y \in R$.

Proposition 53. I is a uspf ideal of R iff I_c (the complement of I in R) is a fuzzy t -system.

Proof. Suppose I uspf, then there exists a finite set F where $\bigwedge I(xFy) = I(x) \vee I(y)$ for any $x, y \in R$. Hence, $\bigvee I_c(xFy) = \bigvee (1 - I(xFy)) = 1 - \bigwedge I(xFy) = 1 - (I(x) \vee I(y)) = (1 - I(x)) \wedge (1 - I(y)) = I_c(x) \wedge I_c(y)$. On the other side, suppose I_c a fuzzy t -system, then there exists a finite set F where $\bigvee I_c(xFy) = I_c(x) \wedge I_c(y)$ for any $x, y \in R$. Thus, $1 - \bigwedge I(xFy) = 1 - (I(x) \vee I(y))$. Therefore $\bigwedge I(xFy) = I(x) \vee I(y)$. \square

Proposition 54. If M is a fuzzy t -system of R , then M_α is a t -system for all α -cuts.

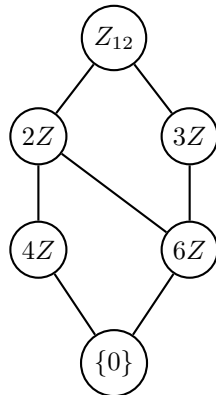
Proof. As M is a fuzzy t -system there exists a finite set F , where $\bigwedge I(xFy) = I(x) \vee I(y)$ for any $x, y \in R$. Let $x, y \in M_\alpha$, then $\bigvee M(xFy) = M(x) \wedge M(y) \geq \alpha$. Since F is a finite set, there exists $f \in F$ such that $M(xfy) \geq \alpha$. Thus, $xfy \in M_\alpha$. Therefore, M_α is a t -system. \square

Question 5. Under which conditions can we have the following result: if K is a fuzzy t -system of R , then is K an m -system?

The next example help us to count (in certain way) the number of uspf ideals of a finite ring R . We recommend the reading of Chapter 5 before.

Example 9. Let I, J fuzzy subsets of a ring R . Define the following relation: $I \sim J$ iff I, J induce the same α -cuts. Clearly, \sim is an equivalence relation.

Consider $R = Z_{12}$ the integers mod 12. According to the diagram below we can count the possible chains of crisp ideals ending with Z_{12} . Hence, we have 15 following chains:

FIGURE 4.1: Ideals of Z_{12}

1. $\{0\} \subset Z_{12}$
2. $\{0\} \subset 6Z \subset Z_{12}$
3. $\{0\} \subset 6Z \subset 3Z \subset Z_{12}$
4. $\{0\} \subset 3Z \subset Z_{12}$
5. $\{0\} \subset 6Z \subset 2Z \subset Z_{12}$
6. $\{0\} \subset 2Z \subset Z_{12}$
7. $\{0\} \subset 4Z \subset Z_{12}$
8. $\{0\} \subset 4Z \subset 2Z \subset Z_{12}$
9. $2Z \subset Z_{12}$
10. $3Z \subset Z_{12}$
11. $3Z \subset 6Z \subset Z_{12}$
12. $4Z \subset 2Z \subset Z_{12}$
13. $4Z \subset Z_{12}$
14. $6Z \subset Z_{12}$
15. $6Z \subset 2Z \subset Z_{12}$

According to Proposition 75 if I is a uspf ideal of Z_{12} , then the α -cuts are uspf ideals for all $I(1) < \alpha \leq I(0)$. Hence, we can count the numbers of uspf ideals of Z_{12} . Since Z_{12} has only two uspf ideals $(2Z, 3Z)$, then it has only 5 uspf ideals in Z_{12} under the equivalence relation \sim .

Chapter 5

Some Properties of Fuzzy Ideals

In this chapter it is investigated the homomorphic image of fuzzy subring, fuzzy ideal, fuzzy prime, and fuzzy irreducible ideals of a ring by Zadeh's extension and how it may influence in the homomorphism of the fuzzy ideal lattices. Finite-valued fuzzy ideals and their relations with Artinian (Noetherian) rings are also described.

Section 5.1 provides an overview on the theory of fuzzy rings. Section 5.2 contains the demonstration of the following result: if G and H are isomorphic rings, then the respective family of ideals are isomorphic lattices. Section 5.3 contains the investigation of some results about fuzzy ring theory similarly to [39]. It is proved that $FI_{FV}(R)$, which is the set of finite-valued fuzzy ideals of R , is a sublattice of $L_{FI}(R)$ (the lattice of all fuzzy ideals of R). Moreover, a condition is shown in order to a fuzzy set to belong to $FI_{FV}(R)$ (based in a chain of ideals of R) *i.e.* the fuzzy set is a finite-valued fuzzy ideal iff there exists a certain kind of chain of ideals. This result entails that, if R is Artinian ring with unity, then a fuzzy ideal can be written in terms of the chain of ideals. At the end of the section, it is proved that $L_I(R)$, *i.e.* the lattice of ideals of R , is isomorphic to a sublattice of $FI_{FV}(R)$.

5.1 Preliminaries

This section explains some definitions and results that will be required in the next sections.

A fuzzy set μ is finite-valued, whenever $Im(\mu)$ is a finite set.

The set of all fuzzy subrings and ideals of the ring R are denoted by $L_F(R)$ and $L_{FI}(R)$, respectively.

Definition 35. Let μ, ν be any two fuzzy subsets of ring R the product $\mu\nu$ is defined as follows:

$$(\mu\nu)(x) = \begin{cases} \bigvee_{x=y \cdot z} (\mu(y) \wedge \nu(z)), & \text{where } y, z \in R \\ 0, & \text{if } x \text{ is not expressible as } x = y \cdot z \\ & \text{for all } y, z \in R. \end{cases}$$

Definition 36. A non-constant fuzzy ideal μ of a ring R is called fuzzy prime if for any fuzzy ideals μ_1 and μ_2 of R the condition $\mu_1\mu_2 \subseteq \mu$ implies that either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

According to Navarro [13] if I is a fuzzy prime ideal, then I is a fuzzy prime. Then, all results proved for fuzzy prime can be used for fuzzy prime ideal.

Definition 37. A fuzzy ideal μ of a ring R is called fuzzy irreducible if it is not a finite intersection of two fuzzy ideals of R properly containing μ : otherwise μ is termed fuzzy reducible.

Some properties of fuzzy rings/ideals can be verified in the works [31, 40–42]. Note that for any fuzzy subring/fuzzy ideal μ of a ring R , if for some $x, y \in R$, $\mu(x) < \mu(y)$, then $\mu(x - y) = \mu(x) = \mu(y - x)$.

Theorem 24. [31] If μ is any fuzzy subring/fuzzy ideal of a ring R , then each level subset $\mu_t = \{x \in R : \mu(x) \geq t\}$ where $0 \leq t \leq \mu(0)$ is a subring/an ideal crisp of R . In particular, if R has unity, $Im(\mu) \subseteq [\mu(1), \mu(0)]$.

Theorem 25. [31] A fuzzy subset μ of a ring R is a fuzzy subring/fuzzy ideal of R iff the level subsets μ_t , ($t \in Im \mu$) are subrings/ideals of R .

In general, the product of two fuzzy ideals may not be a fuzzy ideal.

Proposition 55. The family of fuzzy subrings/fuzzy ideals of a ring R is closed under intersection.

Proposition 56. Let μ be any fuzzy subring and ν any fuzzy ideal of a ring R . Then $\mu \cap \nu$ is a fuzzy ideal of the crisp subring $\{x \in R : \mu(x) = \mu(0)\}$.

Proposition 57. Let $I_1 \subset I_2 \subset \dots \subset I_n = R$ be any chain of ideals of a ring R . Let t_1, t_2, \dots, t_n be some numbers in the interval $[0, 1]$ such that $t_1 > t_2 > \dots > t_n$. Then the fuzzy subset μ of R defined by

$$\mu(x) = \begin{cases} t_1, & \text{if } x \in I_1 \\ t_i, & \text{if } x \in I_i \setminus I_{i-1}, \quad i = 2, \dots, n, \end{cases}$$

is a fuzzy ideal of R .

Theorem 26 ([40]). Let R be a ring with unity. R is Artinian iff every fuzzy ideal of R is finite-valued.

Theorem 27 ([41]). If a Γ -ring M is Artinian, then every fuzzy ideal of M is finite-valued.

5.2 The Isomorphism

In this section it is proved that for a given pair of rings R, S the lattices of fuzzy ideals of R and S are isomorphic whenever R and S are isomorphic rings.

Proposition 58. Zadeh's extension preserves fuzzy subrings. Let $f : R \rightarrow S$ be a homomorphism of rings and μ a fuzzy subring of the ring R . Then $f(\mu)$ is a fuzzy subring of the ring S .

Proof. Let $x, y \in S$. Suppose either $x \notin f(R)$ or $y \notin f(R)$. Then according definition 14 $f(\mu)(x) = 0$ or $f(\mu)(y) = 0$. Thus $f(\mu)(x) \wedge f(\mu)(y) = 0 \leq f(\mu)(x - y)$ and $f(\mu)(x) \wedge f(\mu)(y) = 0 \leq f(\mu)(xy)$.

Now suppose $x, y \in f(R)$. Observe that f is a homomorphism, $f(R)$ is a subring and $x - y, xy \in f(R)$. Moreover, for demonstration below we will use the following equality $\vee\{A \cup B\} = (\vee\{A\}) \vee (\vee\{B\})$.

$$\begin{aligned} f(\mu)(x - y) &= \vee\{\mu(z) : f(z) = x - y\} \geq \vee\{\mu(m - n) : f(m) = x \text{ and } f(n) = y\} \geq \\ &\vee\{\mu(m) \wedge \mu(n) : f(m) = x \text{ and } f(n) = y\} = \vee\{\cup\{\mu(m) \wedge \mu(n) : f(m) = x\} : f(n) = y\} \\ &= \vee\{\vee\{\mu(m) \wedge \mu(n) : f(m) = x\} : f(n) = y\} = \vee\{\vee\{\mu(m) : f(m) = x\} \wedge \\ &\mu(n) : f(n) = y\} = (\vee\{\mu(m) : f(m) = x\}) \wedge (\vee\{\mu(n) : f(n) = y\}) = f(\mu)(x) \wedge f(\mu)(y). \end{aligned}$$

$$\begin{aligned} f(\mu)(xy) &= \vee\{\mu(z) : f(z) = xy\} \geq \vee\{\mu(mn) : f(m) = x \text{ and } f(n) = y\} \geq \vee\{\mu(m) \wedge \\ &\mu(n) : f(m) = x \text{ and } f(n) = y\} = \vee\{\cup\{\mu(m) \wedge \mu(n) : f(m) = x\} : f(n) = y\} = \\ &\vee\{\vee\{\mu(m) \wedge \mu(n) : f(m) = x\} : f(n) = y\} = \vee\{\vee\{\mu(m) : f(m) = x\} \wedge \mu(n) : f(n) = \\ &y\} = (\vee\{\mu(m) : f(m) = x\}) \wedge (\vee\{\mu(n) : f(n) = y\}) = f(\mu)(x) \wedge f(\mu)(y). \end{aligned}$$

□

Proposition 59 ([43]). Epimorphism preserves fuzzy ideals. Let $f : R \rightarrow S$ be an epimorphism of rings and μ a fuzzy ideal of the ring R . Then $f(\mu)$ is a fuzzy ideal of the ring S .

Proof. Let $x, y \in S = f(R)$, then $x - y, xy \in f(R)$.

$f(\mu)(x - y) \geq f(\mu)(x) \wedge f(\mu)(y)$ is similar to Proposition 58.

$$\begin{aligned} f(\mu)(xy) &= \vee\{\mu(z) : f(z) = xy\} \geq \vee\{\mu(mn) : f(m) = x \text{ and } f(n) = y\} \geq \vee\{\mu(m) \vee \\ &\mu(n) : f(m) = x \text{ and } f(n) = y\} = (\vee\{\mu(m) : f(m) = x\}) \vee (\vee\{\mu(n) : f(n) = y\}) = \\ &f(\mu)(x) \vee f(\mu)(y). \end{aligned}$$

□

Since for any homomorphism of rings $f : A \rightarrow B$, $f(A)$ is also a ring, and $f : A \rightarrow f(A)$ is an epimorphism, then it is reasonable to say that homomorphism induces the preservation of fuzzy ideals.

Definition 38. For any two fuzzy ideals μ and ν of R , define $\mu \wedge \nu = \mu \cap \nu$ and $\mu \vee \nu = \cap\{\eta : \eta \text{ is a fuzzy ideal of } R \text{ such that } \eta \geq \mu, \nu\}$.

Proposition 60. Let R be a ring. Then $L_{FI}(R)$ is a complete lattice under \wedge and \vee .

Proof. See [44].

□

As it is known, homomorphism preserves algebraic structures. As we will prove, the theorem below shows that Zadeh's extension preserves certain algebraic properties.

Theorem 28. Let R, S be rings. If $R \cong S$, then the lattices $L_{FI}(R)$ and $L_{FI}(S)$ are isomorphic.

Proof. As f is an isomorphism, then $f(\mu)(y) = \mu(x)$ for $y = f(x)$ (1). Let $\mu, \nu \in L_{FI}(R)$ and $y \in R$. Then:

$$f(\mu \wedge \nu)(y) = \mu \wedge \nu(x) \quad (1) = \mu(x) \wedge \nu(x) = f(\mu)(y) \wedge f(\nu)(y)$$

$$\begin{aligned} f(\mu) \vee f(\nu)(y) &= \wedge\{\eta : \eta \geq f(\mu), f(\nu)\}(y) = \cap\{\eta(y) : \eta(y) \geq f(\mu)(y), f(\nu)(y)\} = \\ &\cap\{\eta(f(x)) : \eta(f(x)) \geq \mu(x), \nu(x)\} = \cap\{\eta \circ f(x) : \eta \circ f(x) \geq \mu(x), \nu(x)\} = \\ &\cap\{\eta'(x) : \eta'(x) \geq \mu(x), \nu(x)\} \quad \eta' = \eta \circ f = \mu \vee \nu(x) = f(\mu \vee \nu)(y) \quad (1) \end{aligned}$$

To prove that Zadeh's extension is bijective, let $f(\mu) = f(\nu)$ then $f(\mu)(y) = f(\nu)(y)$ for all $y \in S$. By definition $\vee\{\mu(x) : f(x) = y\} = \vee\{\nu(x) : f(x) = y\}$. By (1) $\mu(x) = \nu(x)$. Therefore, $\mu(x) = \nu(x)$ for all $x \in R$ and then $\mu = \nu$. On the other hand, let $\mu \in L_{FI}(S)$ and define ν such that $\nu(x) = \mu(y)$ where $y = f(x)$. Thus $f(\nu)(y) = \vee\{\nu(x) : f(x) = y\} = \nu(x) = \mu(y)$ for all $y \in S$. Therefore $f(\nu) = \mu$. □

The converse of this theorem is not true as shown by the following example:

Example 10. Consider the rings Q and R of rational and real numbers, respectively. Both contain only two ideals: (0) and their own ring. Then $|L_{FI}(Q)| = |L_{FI}(R)|$ and in this case it is not difficult to show that the lattices $L_{FI}(Q), L_{FI}(R)$ are isomorphic.

The next propositions are corollaries of the Theorem 28.

Proposition 61. Let $f : R \rightarrow S$ be an isomorphism of rings. If μ is a prime fuzzy of R , then $f(\mu)$ is a fuzzy prime of S .

Proof. Let ν_1, ν_2 be fuzzy ideals of S such that $\nu_1\nu_2 \subseteq f(\mu)$, then by using the Theorem 28 there exist μ_1, μ_2 fuzzy ideals such that $\nu_1 = f(\mu_1)$ and $\nu_2 = f(\mu_2)$. Since $\nu_1\nu_2 \subseteq f(\mu)$, then $\mu_1\mu_2 \subseteq \mu$. In accordance with the hypothesis, $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. Therefore $\nu_1 \subseteq f(\mu)$ or $\nu_2 \subseteq f(\mu)$. \square

Proposition 62. Let $f : R \rightarrow S$ be an isomorphism of rings. If μ is a fuzzy irreducible of R , then $f(\mu)$ is a fuzzy irreducible of S .

Proof. We will show the contrapositive. Suppose $f(\mu)$ is a fuzzy reducible of S , then there exist ν_1, ν_2 such that $f(\mu) = \nu_1 \cap \nu_2$. By the Theorem 28, $\nu_1 \cap \nu_2 = f(\mu_1) \cap f(\mu_2) = f(\mu_1 \cap \mu_2)$. Thus $\mu = \mu_1 \cap \mu_2$. Therefore μ is reducible. \square

5.3 An Equivalence of Fuzzy Ideals

In this section we look at finite-valued fuzzy subsets. We prove that $FI_{FV}(R)$ is a sublattice of $L_{FI}(R)$, and we give a condition for the ideal to belong to $FI_{FV}(R)$ based on chain of ideals of R . A consequence of this fact is that if R is an Artinian ring with unity, then a fuzzy ideal can be written in terms of the chain of ideals of R . At the end of the section, we prove that $L_I(R)$, *i.e.* the lattice of ideals of R , is isomorphic to a sublattice of $FI_{FV}(R)$.

Theorem 29. $FI_{FV}(R)$ is a sublattice of $L_{FI}(R)$.

Proof. If μ, ν is a finite-valued fuzzy ideal, clearly $\mu \wedge \nu$ and $\mu \vee \nu$ are finite-valued fuzzy ideal. \square

Theorem 30. Let μ be a fuzzy subset of R such that $Im(\mu) = \{t_1, t_2, \dots, t_n\}$ where $t_1 > t_2 > \dots > t_n$. Hence μ can be written as

$$\mu(x) = \begin{cases} t_1, & \text{if } x \in B_1 \\ t_2, & \text{if } x \in B_2 \\ \vdots & \\ t_n, & \text{if } x \in B_n, \end{cases}$$

with $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^n B_i = R$.

Proof. Let $B_1 = \{x \in R : \mu(x) = t_1\}$, $B_k = \{x \in R \setminus \bigcup_{j=1}^{k-1} B_j : \mu(x) = t_k\}$, $k \in \{2, \dots, n\}$. \square

Observation 1. Given the sets B_i introduced in the proof of theorem 30, let $I_m = \bigcup_{i=1}^m B_i$, then $I_m = \{x \in R : \mu(x) \geq t_m\}$, $B_{i+1} = I_{i+1} \setminus I_i$ and $I_n = \bigcup_{i=1}^n B_i = R$. Thus it can be written

$$\mu(x) = \begin{cases} t_1, & \text{if } x \in I_1 \\ t_2, & \text{if } x \in I_2 \setminus I_1 \\ \vdots & \\ t_n, & \text{if } x \in I_n \setminus I_{n-1}. \end{cases}$$

Where I_i , $i \in \{1, \dots, n\}$ are ideals of ring R by Theorem 24.

The next Corollary 31 gives a condition to μ be a finite-valued fuzzy ideal.

Corollary 31. μ is a finite-valued fuzzy ideal, if and only if, there exists a chain of ideals $I_1 \subset I_2 \subset \dots \subset I_n = R$ such that $I_i = \{x \in R : \mu(x) \geq t_i\}$, $i \in \{1, \dots, n\}$ and

$$\mu(x) = \begin{cases} t_1, & \text{if } x \in I_1 \\ t_2, & \text{if } x \in I_2 \setminus I_1 \\ \vdots & \\ t_n, & \text{if } x \in I_n \setminus I_{n-1}. \end{cases}$$

Proof. (\Rightarrow) If μ is a finite-valued fuzzy ideal, then it is possible to build the sequence like observation 1.

(\Leftarrow) Theorem 57. \square

In particular, if R is a finite ring, then all fuzzy ideals are completely determined by the chains of ideals in R .

Corollary 32. Let R be an Artinian ring with unity. μ is a fuzzy ideal of R , if and only if, there exists a finite chain $I_1 \subset I_2 \cdots \subset I_n$ such that μ such that $I_i = \{x \in R : \mu(x) \geq t_i\}$, $i \in \{1, \dots, n\}$ and

$$\mu(x) = \begin{cases} t_1, & \text{if } x \in I_1, \\ t_2, & \text{if } x \in I_2 \setminus I_1 \\ \vdots & \\ t_n, & \text{if } x \in I_n \setminus I_{n-1}. \end{cases}$$

Proof. Immediately from Theorem 26 and Corollary 31. □

Corollary 33. Let M be an Artinian Γ -ring M . μ is a fuzzy ideal of R , if and only if, there exists a finite chain $I_1 \subset I_2 \cdots \subset I_n$ such that $I_i = \{x \in R : \mu(x) \geq t_i\}$, $i \in \{1, \dots, n\}$ and

$$\mu(x) = \begin{cases} t_1, & \text{if } x \in I_1, \\ t_2, & \text{if } x \in I_2 \setminus I_1 \\ \vdots & \\ t_n, & \text{if } x \in I_n \setminus I_{n-1}. \end{cases}$$

Proof. Immediately from Theorem 27 and Corollary 31. □

In $FI_FV(R)$ it is defined the following relation: let $\mu, \nu \in FI_FV(R)$ then:

- 1) $\mu \sim \nu$ iff μ, ν induce the same ideal chains.
- 2) $\mu \equiv \nu$ iff $I_1^\mu = I_1^\nu$.

Clearly \sim and \equiv are equivalence relations. Moreover if $\mu \sim \nu$ then $\mu \equiv \nu$.

Example 11. Consider the ring of integers Z and define:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in 2Z \\ 1/2, & \text{if } x \in Z \setminus 2Z \end{cases}$$

$$\eta(x) = \begin{cases} 1/3, & \text{if } x \in 2Z \\ 1/4, & \text{if } x \in Z \setminus 2Z \end{cases}$$

$$\nu(x) = \begin{cases} 1/3, & \text{if } x \in 4Z \\ 1/4, & \text{if } x \in Z \setminus 4Z \end{cases}$$

$$\beta(x) = \begin{cases} 1, & \text{if } x \in 4Z \\ 1/2, & \text{if } x \in 2Z \setminus 4Z \\ 1/4, & \text{if } x \in Z \setminus 2Z \end{cases}$$

$$\mu \sim \eta, \mu \approx \nu, \mu \equiv \eta, \nu \equiv \beta.$$

As it has been shown before, any finite-valued fuzzy ideal of R determines a chain of ideals in R of type $I_1 \subset I_2 \subset \dots \subset I_n = R$ such that $I_i = \{x \in R : \mu(x) \geq t_i\}$, $i \in \{1, \dots, n\}$. Now it is possible to count the number of fuzzy ideals of a finite ring based on \sim or \equiv since in a finite ring it is possible to count the number of ideal chains.

Example 12. Consider Z_5 the integers mod 5. Herein there are only 2 chains, i.e. $\{0\} \subset Z_5$ and Z_5 . Hence under \sim there are only 2 fuzzy ideals.

Although this work provides two equivalence relations in $FI_{FV}(R)$ namely \sim and \equiv , in what follows, only \equiv is investigated. Consider $L_I(R)$ which is the lattice of all crisp ideals of R . The equivalence classes modulo \equiv are:

$$L = \{\mu_I : I \in L_I(R)\}, \text{ where}$$

$$\mu_I : R \longrightarrow [0, 1], \mu_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if } x \in R \setminus I \end{cases}, \text{ for all } I \in L_I(R).$$

Proposition 63. The set $L = \{\mu_I : I \in L_I(R)\}$ is a sublattice of $FI_{FV}(R)$.

Proof. It is easy to see that for any two ideals I and J of R , $\mu_I \wedge \mu_J = \mu_{I \cap J}$ and $\mu_I \vee \mu_J = \mu_{I \cup J}$. \square

Proposition 64. The map $f : L_I(R) \longrightarrow L_{FI}(R)$ defined by $f(I) = \mu_I$, for all $I \in L_I(R)$, is a lattice embedding and $L_I(R), L$ are isomorphic lattices.

Proof. It is enough to see that: if $I, J \in L_I(R)$, then

$$f(I \cap J) = \mu_{I \cap J} = \mu_I \wedge \mu_J = f(I) \wedge f(J).$$

$$f(I \cup J) = \mu_{I \cup J} = \mu_I \vee \mu_J = f(I) \vee f(J)$$

\square

5.4 Final Remarks

The theorem 28 seems very simple, but it brings relevant information to the study of fuzzy algebra, because it tells us that we can look at the lattice of fuzzy ideals in the same manner that we look at the lattice of crisp ideals. Finally, it worth to think the theorem 28 for semisimple rings based on pure definition of prime fuzzy ideal.

Chapter 6

Prime Ideals and Fuzzy Prime Ideals Over Noncommutative Quantales

In this chapter we propose a new concept of prime ideals in noncommutative quantales. The usual definition of prime ideal is preserved as a completely prime ideal. In this investigation it is proved that these two concepts coincide in commutative quantales, but are no longer valid in the noncommutative setting. Also, the notions of strong and uniform strong primeness as well as the fuzzy version of prime ideal and uniformly strongly prime ideal are introduced in quantales. All these studies in this chapter were submitted to the fuzzy sets and systems journal.

6.1 Introduction

In 2013, Lingyun Yang and Luoshan Xu [45] defined a prime ideal in quantales based on elements of quantale. After that they built the rough prime ideal in quantales over this concept. In 2014, Qingjun Luo and Guojun Wang [46] used the same definition of prime ideals of quantales to write an investigation called roughness and fuzziness where the first ideas on semi-prime, primary and strong primeness are presented. As it is known, ideals are the main object in the investigation of ring theory and provide important information about the rings because they are structural pieces. The same may occur in quantales. The definition of prime ideals proposed in [45, 46] is based on elements of a quantale and we ponder it is geared to commutative environment. When we move from commutative to the noncommutative setting, elementwise should be replaced by an approach based on ideals. Nevertheless, some authors defined the concept of primeness for commutative and noncommutative cases without realizing that this concept may not be suitable for noncommutative setting as it was well shown by Navarro et. al. in [13]. We state that the

concept of prime ideal of general quantales could be defined as it is done in ring theory, i.e. based on ideals. The concept of prime ideal provided for quantales by Lingyun Yang and Luoshan Xu is more suitable for commutative quantales. Therefore, this chapter provides a new concept of prime ideal for a general (commutative and noncommutative) quantale which the elementwise prime ideal definition proposed by Lingyun Yang and Luoshan Xu is called completely prime ideal.

The first aim is to study the notion of primeness in the following perspective: I renamed prime ideal defined in [45] to *completely prime ideal* and define a new concept of prime ideal for quantales. Then It is translated an important result in ring theory for quantales environment (theorem 35) to prove that these two concepts coincide in the commutative setting, but are no longer valid in the noncommutative setting (see Proposition 67). Besides, based on the studies of Lawrence and Handelman [2], started in 1975 I developed the notion of strong primeness for general quantales. The second aim is to propose the concept of fuzzy primeness and fuzzy strong primeness as well as fuzzy uniform strong primeness for quantales following the ideas developed in previous chapters.

At the end of this chapter, I introduce the initial ideas of t-systems and m-systems for quantales. As a consequence an ideal is prime iff its complement is an m-system.

6.2 Primeness in Quantales

This section proposes a new concept of prime ideals suitable for commutative and noncommutative quantales. The definition of prime ideal used in [45, 46] will be called herein completely prime ideals. We drew attention to the theorem 35 where prime ideals can be characterized in a certain way via elements. The Proposition 68 shows that in the commutative case, prime and completely prime concepts coincide, which are no longer valid in the noncommutative setting according to Proposition 67. Finally, the concept of quantale prime is proposed.

Definition 39. [47] A quantale is a complete lattice Q with an associative binary operation \circ satisfying:

$$a \circ \left(\bigvee_{k \in K} b_k \right) = \bigvee_{k \in K} (a \circ b_k), \quad \left(\bigvee_{k \in K} a_k \right) \circ b = \bigvee_{k \in K} (a_k \circ b)$$

for all $a, b, a_k, b_k \in Q$ and $k \in K$.

A quantale Q is called commutative whenever $a \circ b = b \circ a$ for $a, b \in Q$. In this work we denote the least and greatest elements of a quantale by \perp and \top respectively. If there

exists an element e in Q such that $x \circ e = e \circ x = x$ for all x in Q the quantale is called a quantale with identity. In this work we consider quantales with identity.

Definition 40. [46] Let Q be a quantale. A non-empty subset $I \subseteq Q$ is called a right ideal of Q if it satisfies the following conditions:

- i) $a, b \in I$ implies $a \vee b \in I$;
- ii) for all $a, b \in Q$, $a \in I$ and $b \leq a$ imply $b \in I$,
- iii) for all $x \in Q$ and $a \in I$, we have $a \circ x \in I$.

Similarly we may define left ideal replacing (iii) by: (iii') for all $x \in Q$ and $a \in I$, we have $x \circ a \in I$. If I is both right and left ideal of Q , we call I a two-sided ideal or simply an ideal of Q .

Clearly by (ii) $\perp \in I$. Also, the set of all ideals of Q is closed under arbitrary intersections.

In Q we denote the subset $I \circ J = \{i \circ j \in Q : i \in I \text{ and } j \in J\}$ and $A \vee B = \{a \vee b : a \in A \text{ and } b \in B\}$. Since the operation \circ is associative, we have $(A \circ B) \circ C = A \circ (B \circ C)$. Also, if A is an two-sided ideal, then $A \circ Q, Q \circ A, Q \circ A \circ Q \subseteq A$.

As usual, \vee induces an order relation \leq on Q by putting $x \leq y \Leftrightarrow x \vee y = y$. Moreover, \leq is a congruence i.e. for every $x, y, u, v \in Q$ if $x \leq y$ and $u \leq v$, then $x \circ u \leq y \circ v$. To prove this, we first observe that if $w \leq z$ then, for any $s \in Q$, $s \circ w \leq s \circ z$ and $w \circ s \leq z \circ s$ because $z = w \vee z$ implies $s \circ z = s \circ (w \vee z) = (s \circ w) \vee (s \circ z)$ and $z \circ s = (w \vee z) \circ s = (w \circ s) \vee (z \circ s)$; now suppose $x \leq y$ and $u \leq v$, then $x \circ u \leq y \circ u$ and $y \circ u \leq y \circ v$. Hence, $x \circ u \leq y \circ v$ by transitivity.

In what follows we propose a more general definition of prime ideals which encompasses commutative and non-commutative quantales.

Definition 41. A prime ideal in a quantale Q is any proper ideal P such that, whenever I, J are ideals of Q with $I \circ J \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

Definition 42. A subset P of a quantale Q is called completely prime ideal if x and y are two elements of Q such that their product $x \circ y \in P$, then $x \in P$ or $y \in P$.

As we will see the concept of prime and completely prime ideals are different and coincide whenever Q is commutative.

Proposition 65. If P is completely prime, then P is prime.

Proof. Suppose that P is completely prime and $I \circ J \subseteq P$, but $J \not\subseteq P$, where I, J are ideals of Q . Thus, there exists $j \in J$ such that $j \notin P$. For all $i \in I$ we have $i \circ j \in I \circ J \subseteq P$, as P is completely prime and $j \notin P$, then $i \in P$. Therefore $I \subseteq P$.

□

The Proposition 67 will show that the converse of this Proposition is not true.

Definition 43. [46] Let Q be a quantale and $A \subseteq Q$. The least ideal containing A is called the ideal generated by A , and denoted as $\langle A \rangle$.

Clearly, $\langle \emptyset \rangle = \{\perp\}$. If $\emptyset \neq A \subseteq Q$, then we have the following result.

Proposition 66. [46] Let A be a non-empty subset of a quantale Q . Then $\langle A \rangle = \{x \in Q : x \leq \bigvee_{i=1}^n a_i, \text{ for some positive integer } n \text{ and } a_1, \dots, a_n \in A \cup (A \circ Q) \cup (Q \circ A) \cup (Q \circ A \circ Q)\}$.

We may denote $\langle a \rangle = \langle \{a\} \rangle$ and $a \circ Q = \{a\} \circ Q$.

Lemma 34. $\langle a \rangle \circ Q \subseteq \langle a \rangle$ for all $a \in Q$. If there exists an unit 1 in Q , then $\langle a \rangle \circ Q = \langle a \rangle$.

Proof. Let $x \circ q \in \langle a \rangle \circ Q$, where $x \in \langle a \rangle$ and $q \in Q$. Hence, $x \circ q \leq \bigvee_{i=1}^n (a_i \circ q)$, where $a_i \circ q \in a \circ Q \cup Q \circ a \cup Q \circ a \circ Q$. Thus, $x \circ q \in \langle a \rangle$. On the other hand if there exists unit 1 in Q , we write $z \in \langle a \rangle$ as $z = z \circ 1$. Thus, $z \in \langle a \rangle \circ Q$ and we have $\langle a \rangle \circ Q = \langle a \rangle$.

□

Theorem 35. For an ideal P in Q the following statements are equivalent:

- (1) P is prime ideal;
- (2) $\langle a \rangle \circ \langle b \rangle \subseteq P$ implies $a \in P$ or $b \in P$;
- (3) $a \circ Q \circ b \subseteq P$ implies $a \in P$ or $b \in P$.

Proof. For (1) \Rightarrow (2) note that $\langle a \rangle$ and $\langle b \rangle$ are ideals of Q . As P is prime and $\langle a \rangle \circ \langle b \rangle \subseteq P$, then $\langle a \rangle \subseteq P$ or $\langle b \rangle \subseteq P$. Hence, $a \in P$ or $b \in P$. For (2) \Rightarrow (1), assume that $I \circ J \subseteq P$, but $J \not\subseteq P$, where I, J are ideals of Q . Thus, there exists $j \in J$ such that $j \notin P$. Given $i \in I$ we have $\langle i \rangle \subseteq I$. Hence, $\langle i \rangle \circ \langle j \rangle \subseteq I \circ J \subseteq P$. By hypothesis $i \in P$ or $j \in P$. As $j \notin P$ then we have $i \in P$. Therefore, $I \subseteq P$.

For (3) \Rightarrow (1), assume that $I \circ J \subseteq P$, but $J \not\subseteq P$, where I, J are ideals of Q . Thus, there exists $j \in J$ such that $j \notin P$. Given $i \in I$ we have $i \circ Q \circ j \subseteq I \circ J \subseteq P$. Hence, $i \in P$ or $j \in P$, as $j \notin P$ then we have $i \in P$. Thus, $I \subseteq P$.

For (1) \Rightarrow (3), suppose $a \circ Q \circ b \subseteq P$, we first shall show that $\langle a \rangle \circ Q \circ \langle b \rangle \subseteq P$. For this, let $x \circ q \circ y \in \langle a \rangle \circ Q \circ \langle b \rangle$, where $x \in \langle a \rangle$, $y \in \langle b \rangle$ and $q \in Q$. Hence, by Proposition 66, $x \leq \bigvee_{i=1}^n a_i$ and $y \leq \bigvee_{j=1}^m b_j$, where $a_i \in (a \circ Q) \cup (Q \circ a) \cup (Q \circ a \circ Q)$

and $b_i \in (b \circ Q) \cup (Q \circ b) \cup (Q \circ b \circ Q)$. Hence $x \circ q \circ y \leq (\bigvee_{i=1}^n a_i) \circ q \circ (\bigvee_{j=1}^m b_j) = (\bigvee_{i=1}^n (a_i \circ q)) \circ (\bigvee_{j=1}^m b_j) = \bigvee_{i=1}^n (a_i \circ q \circ \bigvee_{j=1}^m b_j) = \bigvee_{i=1}^n (\bigvee_{j=1}^m (a_i \circ q \circ b_j))$.

Observe that $a_i \in a \circ Q \cup Q \circ a \cup Q \circ a \circ Q$ and $b_j \in b \circ Q \cup Q \circ b \cup Q \circ b \circ Q$ it is no hard to see that $a_i \circ q \circ b_j \in a \circ Q \circ b \subseteq P$ for all i, j . As P is an ideal we have $x \circ q \circ y \in P$. Thus, $\langle a \rangle \circ Q \circ \langle b \rangle \subseteq P$. By the Lemma 34 $\langle a \rangle \circ Q = \langle a \rangle$. Then, $\langle a \rangle \circ Q \circ \langle b \rangle = \langle a \rangle \circ \langle b \rangle \subseteq P$. By the first proof ((1) \Leftrightarrow (2)) we have $a \in P$ or $b \in P$.

□

Proposition 67. There exists a noncommutative quantale where a prime ideal is not a completely prime ideal.

Proof. Consider G the invertible 2×2 matrices under multiplication over the real interval $[0, 1]$ and the partial order $A \leq B \Leftrightarrow a_{ij} \leq b_{ij}$. According to Rosenthal ([47], page 19, example 16) any complete partially ordered group (written multiplicatively) is a quantale with $a \circ b = a \cdot b$. Thus, G is a noncommutative quantale.

Let $\langle 0 \rangle$ as an ideal generated by 0, clearly $\langle 0 \rangle = \{0\}$. We will show that the $\langle 0 \rangle$ (zero ideal) is prime, but not completely prime by using the Theorem 35 (3). Thus, suppose that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are two matrices such that $X \circ G \circ Y \subseteq \langle 0 \rangle$.

Hence $X \circ T \circ Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all matrix $T \in G$. Then, in particular,

$$X \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af \\ ce & cf \end{pmatrix} = 0 \Leftrightarrow a = c = 0 \text{ or } e = f = 0,$$

$$X \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ag & ah \\ cg & ch \end{pmatrix} = 0 \Leftrightarrow a = c = 0 \text{ or } g = h = 0,$$

$$X \circ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} be & bf \\ de & df \end{pmatrix} = 0 \Leftrightarrow b = d = 0 \text{ or } e = f = 0,$$

$$X \circ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} bg & bh \\ dg & dh \end{pmatrix} = 0 \Leftrightarrow b = d = 0 \text{ or } g = h = 0,$$

Hence, a solution must verify that $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ or $Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $X \in \langle 0 \rangle$ or $Y \in \langle 0 \rangle$ and then $\langle 0 \rangle$ is prime. Nevertheless, $\langle 0 \rangle$ is not completely prime, since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ although } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin \langle 0 \rangle.$$

□

Proposition 68. In a commutative quantale an ideal is completely prime iff it is prime.

Proof. If P is a completely prime ideal of a quantale Q , then by the Proposition 65 P is prime. On the other hand, suppose P is a prime ideal and $a \circ b \in P$ for any $a, b \in Q$. Let $x \circ y \in \langle a \rangle \circ \langle b \rangle$, where $x \in \langle a \rangle$ and $y \in \langle b \rangle$. Thus, $x \circ y \leq \bigvee_{i=1}^n a_i \circ \bigvee_{j=1}^m b_j = \bigvee_{i=1}^n \left(\bigvee_{j=1}^m (a_i \circ b_j) \right)$, where $a_i \in a \circ Q \cup Q \circ a \cup Q \circ a \circ Q$ and $b_j \in b \circ Q \cup Q \circ b \cup Q \circ b \circ Q$. As Q is commutative $a \circ Q = Q \circ a = Q \circ a \circ Q$ and $b \circ Q = Q \circ b = Q \circ b \circ Q$. Thus, $a_i \circ b_j \in a \circ Q \circ b \circ Q = a \circ b \circ Q$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Hence, $a_i \circ b_j = a \circ b \circ q \in P$ and then $x \circ y \leq \bigvee_{i=1}^n \left(\bigvee_{j=1}^m (a_i \circ b_j) \right) \in P$. Therefore, $\langle a \rangle \circ \langle b \rangle \subseteq P$ and by the Theorem 35 we have $a \in P$ or $b \in P$.

□

In what follows, we will introduce the notion of quantale prime. As we know, in ring theory, a quantale is prime iff the ideal generated by 0 is a prime ideal. Then, the next Proposition translates this result into quantale environment and opens the investigation on quantales prime.

Definition 44. A quantale Q is called prime if given $a, b \in Q$ with $a \neq \perp$ and $b \neq \perp$, there exists $f \in Q$ such that $a \circ f \circ b \neq \perp$.

Proposition 69. A quantale Q is prime iff $\langle \perp \rangle$ is a prime ideal.

Proof. Suppose Q prime and assume that $I \circ J \subseteq \langle \perp \rangle$, but $I, J \not\subseteq \langle \perp \rangle$, where I, J are ideals of Q . Thus, there exists $i \in I, j \in J$ such that $i, j \neq \perp$. As Q is prime, there exists $q \in Q$ such that $i \circ q \circ j \neq \perp$, then we have a contradiction because $i \circ q \circ j \in I \circ J \subseteq \langle \perp \rangle$. Hence, $I \subseteq \langle \perp \rangle$ or $J \subseteq \langle \perp \rangle$. On the other hand, suppose $\langle \perp \rangle$ is a prime ideal of Q . Given $a, b \neq \perp$ in Q , suppose $a \circ q \circ b = \perp$ for all $q \in Q$. Hence, $a \circ Q \circ b \subseteq \langle \perp \rangle$. As $\langle \perp \rangle$ is a prime ideal, then $a \in \langle \perp \rangle$ or $b \in \langle \perp \rangle$, but $a, b \neq \perp$. □

6.3 Strong Primeness in Quantales

Strongly prime rings were introduced in 1973, as a prime ring with finite condition in the generalization of results on group rings proved by Lawrence in his master's thesis. In 1975, Lawrence and Handelman [2] came up with properties of those rings and proved important results, for instance all prime rings may be embedded in a strongly prime ring; and all strongly prime rings are nonsingular. After such relevant paper, Olson [3] published a paper about uniform strong primeness and its radical. On the contrary of the concept of strong primeness, Olson proved that the concept of uniform strong primeness is two-sided. In this section we bring this concept to quantales making specific adaptations for this environment. Finally, it is proposed t - and m -systems for quantales, since it gives another characterization of prime and uniformly strongly prime ideal.

Definition 45. Let A be a subset of a quantale Q . The right annihilator of A is defined as $An_r(A) = \{x \in Q : Ax = \langle \perp \rangle\}$. Similarly, we can define the *left annihilator* An_l .

Definition 46. [2] A quantale Q is called right strongly prime if for each $x \in Q - \{\perp\}$ there exists a finite nonempty subset F_x of Q such that $An_r(x \circ F_x) = \langle \perp \rangle$.

Clearly if Q is right strongly prime, then Q is prime. The set F_x is called an insulator of x in Q .

Proposition 70. If Q is right strongly prime, then every nonzero ideal I of Q contains a finite subset F which has right annihilator zero.

Proof. Suppose Q right strongly prime. Let $x \in I$ and $x \neq \perp$ and $F = x \circ F_x \subseteq I$. Thus, $An_r(F) = \langle \perp \rangle$.

□

It is clear that every right strongly prime quantale is a prime quantale. It is also possible to define left strongly prime in a similar manner for right strong primeness.

Definition 47. A quantale Q is called uniformly strongly prime (**usp**) if the same right insulator may be chosen for each nonbottom element.

Proposition 71. A quantale Q is a right uniformly strongly prime iff there exists a finite subset $F \subseteq Q$ such that for any two nonbottom elements x and y of Q , there exists $f \in F$ such that $x \circ f \circ y \neq \perp$.

Proof. Let Q be uniformly right strongly prime quantale. Hence Q has a uniform right insulator F which is a finite set such that for any element $x \in Q$, $x \circ F$ has no nonbottom right annihilators. Thus, if x and y are any two nonbottom elements in Q , y cannot be

in the annihilator of $x \circ F$. Hence there is an $f \in F$ such that $x \circ f \circ y \neq \perp$. For the reverse implication it is easy to see that if the condition is satisfied then for any $x \neq \perp$ in Q , no nonbottom element annihilates $x \circ F$ on the right. Hence Q is uniformly right strongly prime

□

It is clear that the condition in Proposition 71 is not one-sided; consequently, this condition is also equivalent to uniformly left strongly prime, and we have:

Corollary 36. Q is uniformly right strongly prime iff Q is uniformly left strongly prime.

Corollary 37. A quantale Q is uniformly strongly prime iff there exists a finite subset $F \subseteq Q$ such that $a \circ F \circ b = \perp$ implies $a = \perp$ or $b = \perp$ for all $a, b \in Q$.

Proof. Straightforward. □

Definition 48. An ideal $P \neq \langle \perp \rangle$ of a quantale Q is called uniformly strongly prime (**usp**) ideal if there exists a finite subset $F \subseteq Q$ such that $a \circ F \circ b \subseteq P$ implies $a \in P$ or $b \in P$.

Proposition 72. An ideal I of a quantale Q is a usp ideal iff there exists a finite subset $F \subseteq Q$ such that for any two elements $a, b \in Q \setminus I$ (complement of I in Q), there exists $f \in F$ such that $x \circ f \circ y \notin I$.

Proof. Suppose I a usp ideal of Q . If $a \notin I$ and $b \notin I$ by Definition 48 $a \circ F \circ b$ is not a subset of I . Hence, there exists $f \in F$ such that $a \circ f \circ b \notin I$. For the converse, note that by hypothesis it is impossible to have $a \circ F \circ b \subseteq I$ and $a \notin I$ and $b \notin I$. □

Subsequently we introduce the t -/ m -systems. They will give us another characterization of prime and usp ideals.

Definition 49. A subset M of a quantale Q is called an m -system if for any two elements $x, y \in M$ there exists $q \in Q$ such that $x \circ q \circ y \in M$.

Definition 50. A subset T of a quantale Q is called a t -system if there exists a finite subset $F \subseteq Q$ such that for any two elements $x, y \in T$ there exists $f \in F$ such that $x \circ f \circ y \in T$.

Proposition 73. I is a prime ideal of a quantale Q iff $Q \setminus I$ (the complement of I in Q) is an m -system.

Proof. Suppose I a prime ideal. If $a, b \in Q \setminus I$, then $a, b \notin I$. By Proposition 35 the subset $a \circ Q \circ b$ is not a subset of I . Thus, there exists $q \in Q$ such that $a \circ q \circ b \notin I$. For the converse, let $a, b \in Q$ such that $a \circ Q \circ b \subseteq I$, if a and b not in I , then $a, b \in Q \setminus I$. By hypothesis there exists $q \in Q$ such that $a \circ q \circ b \in Q \setminus I$, but $a \circ Q \circ b \subseteq I$. □

Proposition 74. An ideal I is a usp ideal of a quantale Q iff $Q \setminus I$ (the complement of I in R) is a t -system.

Proof. This Proposition is similar to Proposition 72. □

At the end of section 6.2 we introduced a quantale prime where it was proposed a right/left strongly quantale prime. Lawrence and Handelman proved that all prime rings may be embedded in a strongly prime ring. Then, a question arises: based on their studies, may we have the similar result in quantales?

6.4 Fuzzy Prime and Fuzzy usp Ideals in Quantales

This section introduces the first version of fuzzy prime ideals and fuzzy uniformly strongly prime ideals in quantales compatible with the ideas developed in [13] i.e. a fuzzy concept on membership function is defined and after that it is proved a coherency with α -cuts. It is also proved that every fuzzy completely prime ideal is a fuzzy prime ideal but the converse is not true in noncommutative quantale according to Proposition 67.

The intersection and union of fuzzy sets are given by the point-by-point infimum and supremum. I will use the symbols \wedge and \vee for denoting the infimum and supremum of a collection of real numbers. Hence, $\bigvee A$ is the supremum of a set A and $\bigwedge A$ is the infimum of a set A . Again, $x \circ A$ denotes the set $\{x \circ a : a \in A\}$ and $x \circ A \circ y = \{x \circ a \circ y : a \in A\}$.

Definition 51. [46] Let Q be a quantale. A fuzzy subset I of Q is called a fuzzy ideal of Q if it satisfies the following conditions for $x, y \in Q$:

- (1) if $x \leq y$, then $I(x) \leq I(y)$;
- (2) $I(x \vee y) \geq I(x) \wedge I(y)$;
- (3) $I(x \circ y) \geq I(x) \vee I(y)$.

From (1) and (2) it follows that $I(x \vee y) = I(x) \wedge I(y)$. Thus, a fuzzy subset I is a fuzzy ideal of Q iff $I(x \vee y) = I(x) \wedge I(y)$ and $I(x \circ y) \geq I(x) \vee I(y)$.

Let μ be a fuzzy subset of X and let $\alpha \in [0, 1]$. Then the set $\{x \in X : \mu(x) \geq \alpha\}$ is called the α -cut. Clearly, if $t > s$, then $\mu_t \subseteq \mu_s$. Again, it is proved in [46] that I is a fuzzy ideal of Q iff I_α is an ideal of Q for all $\alpha \in (I(\top), 1]$.

Definition 52. A non-constant fuzzy ideal $P : Q \rightarrow [0, 1]$ is a fuzzy prime ideal of Q if for any $x, y \in Q$, $\bigwedge P(x \circ Q \circ y) = P(x) \vee P(y)$.

Definition 53. A non-constant fuzzy ideal $P : Q \rightarrow [0, 1]$ is said to be fuzzy completely prime (fcp) ideal of Q if for any $x, y \in Q$, $P(x \circ y) = P(x)$ or $P(x \circ y) = P(y)$.

Definition 54. A non-constant fuzzy ideal $I : Q \rightarrow [0, 1]$ is said to be **fuzzy uniformly strongly prime (fusp)** ideal if there exists a finite subset $F \subseteq Q$ such that $\bigwedge I(x \circ F \circ y) = I(x) \vee I(y)$, for any $x, y \in Q$. The subset F is called insulator of I in Q .

The following Proposition says that the definition of fuzzy uniformly strongly prime is coherent with the α -cuts.

Proposition 75. I is a fuzzy prime ideal of Q iff I_α is a prime ideal of Q for all $\alpha \in (I(\top), 1]$.

Proof. Suppose I a fuzzy ideal of Q . Let $x, y \in Q$ such that $x \circ Q \circ y \subseteq I_\alpha$. Thus, $x \circ q \circ y \in I_\alpha$ for all $q \in Q$. As I is a fuzzy prime, then we have $I(x) \vee I(y) = \bigwedge I(x \circ Q \circ y) \geq \alpha$ hence $I(x) \geq \alpha$ or $I(y) \geq \alpha$ i.e. $x \in I_\alpha$ or $y \in I_\alpha$. Thus, by Theorem 35 I_α is a prime ideal. On the other hand, suppose I_α prime ideal of Q for all $\alpha \in (I(\top), 1]$ and $\bigwedge I(x \circ Q \circ y) > I(x) \vee I(y)$. Let $t = \bigwedge I(x \circ Q \circ y)$, and thus $t > I(x) \vee I(y)$ and $x, y \notin I_t$, but this is a contradiction because $I(x \circ q \circ y) \geq t$ for all $q \in Q$ i.e. $x \circ Q \circ y \subseteq I_t$, as I_t is a prime ideal then $x \in I_t$ or $y \in I_t$. Therefore, I is a fuzzy prime ideal. □

Proposition 76. [46] For a fuzzy ideal P in Q the following statements are equivalent:

- (1) P is fuzzy completely prime ideal;
- (2) $I(x \circ y) = I(x) \vee I(y)$ for all $x, y \in Q$;
- (3) I_α is completely prime ideal of Q for all $\alpha \in (I(\top), 1]$.

Proposition 77. If P is a completely fuzzy prime ideal of Q , then P is a fuzzy prime ideal of Q .

Proof. Use Proposition 75 and Proposition 76. □

Proposition 78. In a commutative quantale an ideal is completely fuzzy prime iff is fuzzy prime.

Proof. Use Propositions 68,75 and 76. □

Theorem 38. For a fuzzy ideal P in Q the following statements are equivalent:

- (1) P is a fuzzy prime ideal;
- (2) Given $x, y \in Q$ and J a fuzzy ideal of Q we have: $J(x \circ r \circ y) \leq P(x \circ r \circ y)$ for all $r \in Q$ implies $J(x) \leq P(x)$ or $J(y) \leq P(y)$.

Proof. (1) \Rightarrow (2), suppose P fuzzy prime ideal, if $J(x \circ q \circ y) \leq P(x \circ q \circ y)$ for all $q \in Q$, then $\bigwedge J(x \circ Q \circ y) \leq \bigwedge P(x \circ Q \circ y)$. As J is a fuzzy ideal, by Definition 51 (3) we have $J(x \circ r \circ y) \geq J(x) \vee J(r) \vee J(y) \geq J(x) \vee J(y)$, hence $\bigwedge J(x \circ Q \circ y) \geq J(x) \vee J(y)$. Thus, $J(x) \vee J(y) \leq \bigwedge J(x \circ Q \circ y) \leq \bigwedge P(x \circ Q \circ y) = P(x) \vee P(y)$. Hence, $J(x) \vee J(y) \leq P(x) \vee P(y)$. Therefore, $J(x) \leq P(x)$ or $J(y) \leq P(y)$. For (2) \Rightarrow (1), suppose that $\bigwedge P(x \circ Q \circ y) > P(x) \vee P(y)$ for some $x, y \in Q$. Then there exists $t \in (0, 1)$ such that $\bigwedge P(x \circ Q \circ y) > t > P(x) \vee P(y)$. Now, define the ideal $I : Q \rightarrow [0, 1]$ given by:

$$I(z) = \begin{cases} P(z), & \text{if } P(z) \geq t \\ t, & \text{otherwise} \end{cases}$$

This is a fuzzy ideal with $t < I(x \circ q \circ y) = P(x \circ q \circ y)$ for all $q \in Q$, but $t = I(x) = I(y) > P(x) \vee P(y)$. □

Proposition 79. I is a fusp ideal of Q iff I_α is a usp ideal of Q for all $\alpha \in (I(\top), 1]$.

Proof. Suppose I a fusp ideal and $F \subseteq Q$ a finite subset given by Definition 54. Let $x, y \in Q$ and $\alpha \in (I(\top), 1]$ such that $x \circ F \circ y \subseteq I_\alpha$. Hence, $I(x) \vee I(y) = \bigwedge I(x \circ F \circ y) \geq \alpha$, and thus $I(x) \geq \alpha$ or $I(y) \geq \alpha$. Therefore, $x \in I_\alpha$ or $y \in I_\alpha$. On the other hand, suppose I_α a usp ideal of Q for all $\alpha \in (I(\top), 1]$. According to Definition 48 each I_α has a finite set F_α such that if $x \circ F_\alpha \circ y \subseteq I_\alpha$ implies $x \in I_\alpha$ or $y \in I_\alpha$. Consider the finite set $F = \bigcap_{\alpha \in (I(\top), 1]} F_\alpha$. Suppose $\bigwedge I(x \circ F \circ y) > I(x) \vee I(y)$ and $t = \bigwedge I(x \circ F \circ y)$ for some $x, y \in Q$. Note that $t > I(x) \vee I(y)$ and $t \leq I(x \circ f \circ y)$ for all $f \in F$. Hence, $x, y \notin I_t$, but $x \circ F \circ y \subseteq I_t$ and thus (by hypothesis) $x \in I_t$ or $y \in I_t$, where we have a contradiction. Therefore, $\bigwedge I(x \circ F \circ y) = I(x) \vee I(y)$. □

Corollary 39. If P is a fusp ideal of Q , then P is a fuzzy prime ideal of Q .

Proof. Use Proposition 75 and 79 □

The next proposition enables us to build a fuzzy ideal from a crisp ideal. Also, it is another way to verify if a subset S is an ideal of Q or not.

Proposition 80. Let J be an ideal of Q and $\alpha \in (I(\top), 1)$. Define $I : Q \longrightarrow [0, 1]$ as

$$I(x) = \begin{cases} 1, & \text{if } x = \perp; \\ \alpha, & \text{if } x \in J \setminus \{\perp\}; \\ 0, & \text{if } x \notin J. \end{cases}$$

Then:

- i) I is a fuzzy ideal of Q ;
- ii) I is a fusp ideal iff J is a usp ideal .

Proof. i) Note that for all α -cut ($\alpha \in (I(\top), 1)$) we have $I_\alpha = J$ and $I_1 = \{\perp\}$. Hence, I is a fuzzy ideal since I_α and I_1 are ideals. ii) Suppose I a usfp ideal and let $x, y \notin J$. As I is usfp, there exists a finite set F , where $\bigwedge I(x \circ F \circ y) = I(x) \vee I(y) = 0$. Since F is finite, there exists $f \in F$ where $I(x \circ f \circ y) = 0$, then $x \circ f \circ y \notin J$. On the other hand, suppose J is usp ideal of Q . According to Proposition 37 there exists a finite subset $F \subseteq Q$ such that $a \circ F \circ b = \perp$ implies $a = \perp$ or $b = \perp$ for all $a, b \in Q$. Thus, given $x, y \in Q$ we have the following cases: 1) If $x, y = \perp$, then we have triviality $\bigwedge I(x \circ F \circ y) = I(\perp) = I(x) \vee I(y)$; 2) If $x \in J$ or $y \in J$, then $x \circ F \circ y \subseteq J$. Thus, $\bigwedge I(x \circ F \circ y) = \alpha = I(x) \vee I(y)$; 3) If $x \notin J$ and $y \notin J$, then by Proposition 72 there exists $f \in J$ such that $x \circ f \circ y \notin J$. Therefore, $\bigwedge I(x \circ F \circ y) = 0 = I(x) \vee I(y)$.

□

6.5 Final Remarks

Prime ideals have developed an important role in ring theory and have attracted the attention of some researchers in the investigation of quantales. As prime ideals are structural pieces of a ring it is relevant to study its concept in order to establish a well-founded quantale theory. With this in mind, it is necessary to investigate primeness over arbitrary quantales, i.e. commutative and noncommutative setting.

Chapter 7

Next Steps

In this work i introduced the m -systems and proved that the fuzzy ideal is prime iff its complement is an m -system. The next steps must develop this theory for noncommutative rings and must extend the paper of Navarro [13].

As I have said in the introduction of this work, I can think of uniformly strongly prime fuzzy ideals in two manners: One, to solve crisp problems and two, to develop a pure fuzzy ideals theory. With respect to solve crisp problems, a future work may increase the connections between crisp and fuzzy ideal theory, starting by Proposition 35. It is necessary to improve this connection if we want to solve crisp problems. Concerning the development of fuzzy ideal theory we have some conjectures and answers, for instance:

1 - The Conjecture 1 in the Chapter 3: If the ideal K is Maximal, then we have the maximality principle for uniformly strongly prime fuzzy ideals. On the other hand, if this conjecture is false, then we have another example which shows the difference between fuzzy and crisp algebra theory. The answer of Conjecture 2 can extend the Proposition 36 to noncommutative rings.

2 - Zadeh's Extension is important for us because it can be associated with problems involving isomorphism theorem. For instance, the Corollary 20 shows how we can solve problems for a ring S using the ring R . Thus, the Question 2 discusses about the classes of rings, where the uniformly strongly prime fuzzy ideals are preserved by Zadeh's Extension.

3 - As we know the radical theory was very important to understand rings and their structures. This work introduced the uniformly strongly radical of a fuzzy ideal and proved some initial results on it. But it is necessary to increase this work by making connections between the radical and fuzzy ideals. It is also important to decide which definition of a fuzzy radical is more appropriate, since this work suggests three concepts.

Another issue is the definition of strongly prime fuzzy ideals given in this work, which was based on α -cuts. This definition is not appropriate because it does not produce new results in a pure fuzzy environment or does not show the differences between crisp and fuzzy setting.

The concept of strong primeness introduced in this work brings some questions about strong primeness for quantale environment. For instance: Is the concept of right and left strong primeness in quantales distinct? May all prime quantales be embedded in a strongly prime quantale?

Appendix A

Publications

In this I highlight the published papers with their abstracts. I draw attention to the fourth work in the next sequence, where the difference between crisp and fuzzy ring theory on usp ideals is presented. **This paper (see [22]) won the third place as the best work at NAFIPS 2014, Boston - USA.** The papers 8 and 9 do not deal with Strongly Primeness in the fuzzy environment, they are extra publications.

A.1 Published Studies

1 - Strongly Prime Fuzzy Ideals Over Noncommutative Rings [20]

Abstract: In this paper it is defined the concept of strongly prime fuzzy ideal for non-commutative rings. Also, it is proved that the Zadeh's extension preserves strongly fuzzy primeness and that every strongly prime fuzzy ideal is a prime fuzzy ideal as well as every fuzzy maximal is a strongly prime fuzzy ideal.

2 - On Properties of fuzzy ideals [48]

Abstract: The main goal of this paper is to investigate the properties of fuzzy ideals of a ring R . It provides a proof that there exists an isomorphism of lattices of fuzzy ideals when ever the rings are isomorphic. Finite-valued fuzzy ideals are also described and a method is created to count the number of fuzzy ideals in finite and Artinian rings.

3 - Uniformly Strongly Prime Fuzzy Ideals [23]

Abstract: In this paper we define the concept of *uniformly strongly prime fuzzy ideal* for associative rings with unity. This concept is proposed without dependence of level cuts.

We show a pure fuzzy demonstration that all uniformly strongly prime fuzzy ideals are a prime fuzzy ideal according to the newest definition given by Navarro, Cortadellas and Lobillo [13] in 2012. Also, some properties about fuzzy strongly prime radical and their relations with Zadeh's extension are shown.

4 - A Fuzzy Version of Strongly Prime Ideals [22]

Abstract: This paper is a step forward in the field of fuzzy algebra. Its main target is the investigation of some properties about uniformly strongly prime fuzzy ideals (uspf) based on a definition without α -cuts dependence. This approach is relevant because it is possible to find pure fuzzy results and to see clearly how the fuzzy algebra is different from classical algebra. For example: in classical ring theory an ideal is uniformly strongly prime (usp) if and only if its quotient is a usp ring, but as we shall demonstrate here, this statement does not happen in the fuzzy algebra. Also, we investigate the Zadeh's extension on uspf ideals.

5 - The Strongly Prime Radical of a Fuzzy Ideal [30]

Abstract: In 2013, Bergamaschi and Santiago [20] proposed *Strongly Prime Fuzzy (SP)* ideals for commutative and noncommutative rings with unity, and investigated their properties. This paper goes a step further since it provides the concept of *Strongly Prime Radical* of a fuzzy ideal and its properties are investigated. It is shown that Zadeh's extension preserves strongly prime radicals. Also, a version of *Theorem of Correspondence* for strongly prime fuzzy ideals is proved.

6 - New Types of Strongly Prime Fuzzy Ideal [21]

Abstract: Inspired on the ideas of Malik, Moderson and Navarro about fuzzy primeness, the current paper goes a step further since it provides a characterization of *Strongly Prime fuzzy ideals*. To achieve that, new kind of fuzzy ideals are introduced: *Semi-prime*, *Primary*, *Special Strongly Prime (SSP)* and *Almost Special Strongly Prime (ASSP)*. The last two types of ideals have no crisp correspondents in Algebra. All the ideals together play a fundamental role to prove that crisp results are also valid in the fuzzy environment. The paper also shows how Zadeh's extension behaves in such new fuzzy ideals.

7 - On Properties of Uniformly Strongly Prime Fuzzy Ideals [37]

Abstract: The main purpose of this paper is to continue the study of uniform strong primeness in fuzzy setting started in 2014. A pure fuzzy notion of this structure allows us to develop specific fuzzy results on Uniformly Strongly Prime (USP) ideals over commutative and noncommutative rings. Besides, the differences between crisp and fuzzy setting are investigated. For instance, in crisp setting an ideal I of a ring R is a USP

ideal iff the quotient R/I is a USP ring. Nevertheless, when working over fuzzy setting this is no longer valid. This paper shows new results on USP fuzzy ideals and proves that the concept of uniform strong primeness is compatible with α -cuts. Also, the t - and m -systems are introduced in a fuzzy setting and their relations with fuzzy prime and uniformly strongly prime ideals are investigated.

8 - Fuzzy Quaternion Numbers [49] - Extra Paper

Abstract: In this paper we build the concept of fuzzy quaternion numbers as a natural extension of fuzzy real numbers. We discuss some important concepts such as their arithmetic properties, distance, supremum, infimum and limit of sequences.

9 - Rotation of Triangular Fuzzy Numbers via Quaternion [50] - Extra Paper

Abstract: In this paper we introduced the concept of three-dimensional triangular fuzzy number and their properties are investigated. It is shown that this set has important metrical properties, e.g. *convexity*. The paper also provides a rotation method for such numbers based on quaternion and aggregation operator.

A.2 Unpublished Studies

10 - On Properties of Uniformly Strongly Prime Fuzzy Ideals - Full Version

Paper accepted in 10/15/2015: Journal of Communication and Computer, USA.

Abstract: The main purpose of this paper is to continue the study of uniform strong primeness in fuzzy setting started in 2014. A pure fuzzy notion of this structure allows us to develop specific fuzzy results on Uniformly Strongly Prime (USP) ideals over commutative and noncommutative rings. Besides, the differences between crisp and fuzzy setting are investigated. For instance, in crisp setting an ideal I of a ring R is a USP ideal iff the quotient R/I is a USP ring. Nevertheless, when working over fuzzy setting this is no longer valid. This paper shows new results on USP fuzzy ideals and proves that the concept of uniform strong primeness is compatible with α -cuts. Also, the t - and m -systems are introduced in a fuzzy setting and their relations with fuzzy prime and uniformly strongly prime ideals are investigated.

11 - Prime Ideals and Fuzzy Prime Ideals Over Noncommutative Quantaes

This paper will be submitted to Fuzzy Sets and Systems Journal.

Abstract: In this paper we propose a new concept of prime ideals in noncommutative quantales. The usual definition of prime ideal is preserved as a completely prime ideal. In this investigation it is proved that these two concepts coincide in commutative quantales, but are no longer valid in the noncommutative setting. Also, the notions of strong and uniform strong primeness as well as the fuzzy version of prime ideal and uniformly strongly prime ideal are introduced in quantales.

12 Uniform Primeness in Fuzzy - Survey.

Paper submitted to the Journal of Fuzzy Mathematics in 10/20/2015.

Abstract: The main aim of this paper is to introduce some results discovered by Bergamaschi and Santiago about the strong and uniform strong primeness in the fuzzy environment. The study of strong primeness in fuzzy setting was initially motivated by crisp problems on ring and group-ring theory, but after a short time it became itself more interesting for instance strongly prime ideals may be defined without α -cut dependence but compatible in a certain way; some true statements about uniform strong primeness in crisp case are not true in the fuzzy setting; the Zadeh's principle over ring's homomorphism does not preserve uniform strong primeness; the t - and m -systems may be developed to fuzzy setting.

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