Working paper
UK FSV - IES

No. 48

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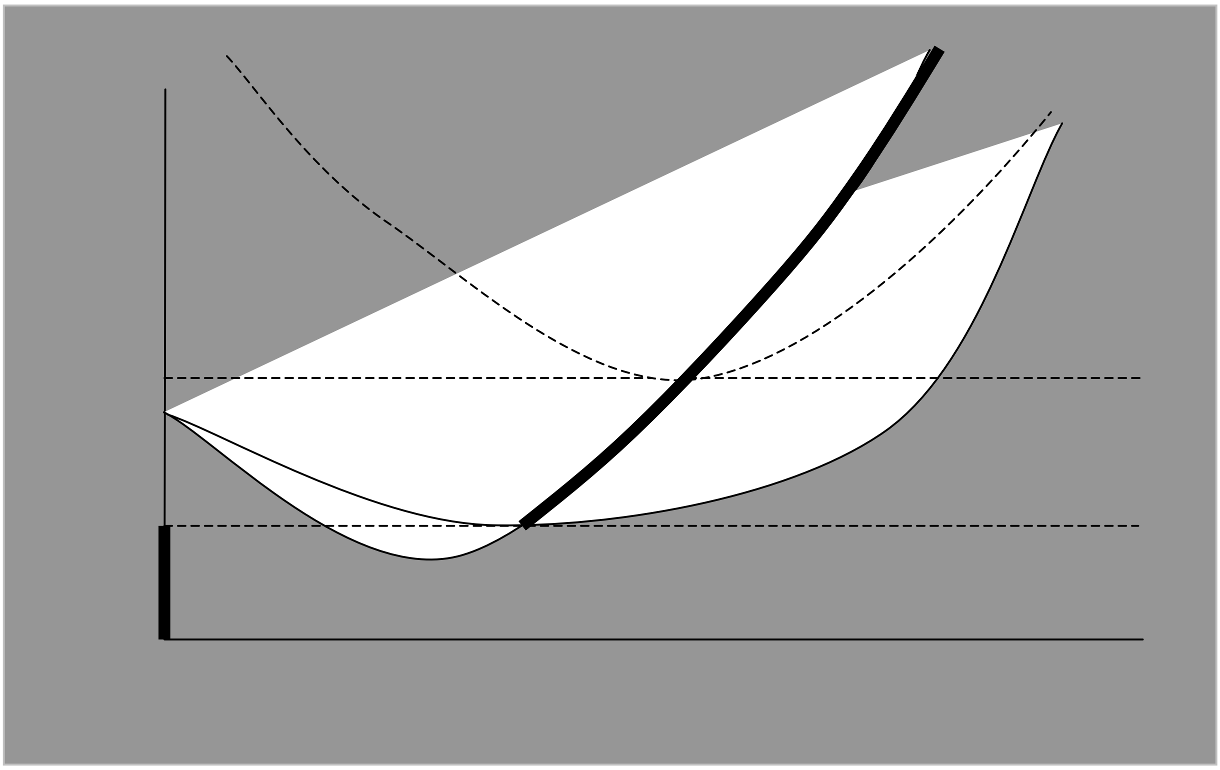

2004

POWER INDICES: SHAPLEY-SHUBIK OR PENROSE-BANZHAF?

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Abstract:

EN: Shapley-Shubik and Penrose-Banzhaf (absolute and relative) power measures and their interpretations are analysed. Both of them could be successfully derived as cooperative game values, and at the same time both of them can be interpreted as probabilities of some decisive position (pivot, swing) without using cooperative game theory at all. In the paper we show that one has to be very careful in interpretation of results based on relative PB-power index and not to use it without absolute PB-power index, what is frequently the case in many published studies.

CZ: Felsenthal, Machover a Zwicker (1998) zavedli pojmy tzv. I-hlasovací síly (vliv na výsledek hlasování) a P-hlasovací síly (podíl na vysledcích vítězství). V článku upozorňujeme na nezbytnou opatrnost při používání této klasifikace v souvislosti s aplikacemi apriorních indexů hlasovací síly Shapleyho-Shubika a Penrose-Banzhafa a na nesprávnost interpretací, založených na relativním indexu Penrose-Banzhafa bez použití absolutního indexu, což je velmi frekventovaný případ v mnoha publikovaných studiích.

Keywords: Absolute power, cooperative games, I-power, pivot, power indices, relative power, P-power, swing.

JEL Classification: D710, D740

1. Introduction

Let N = {1, ..., n} be the set of members (players, parties) and ω_i (i = 1, ..., n) be the (real, non-negative) weight of the i-th member such that

$$\sum_{i \in N} \omega_i = 1, \omega_i \geq 0$$

(e.g. the share of votes of party i, or the ownership of i as a proportion of the total number of shares, etc.). Let γ be a real number such that $0 < \gamma < 1$. The (n+1)-tuple

$$[\gamma, \boldsymbol{\omega}] = [\gamma, \omega_1, \omega_2, \ldots, \omega_n]$$

such that

$$\sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0, 0 \leq \gamma \leq 1$$

we shall call a committee (or a weighted voting body) of the size n = card N with quota γ and allocation of weights

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$$

(by *card* S we denote the cardinality of the finite set S, for empty set *card* $\emptyset = 0$)

Any non-empty subset $S \subseteq N$ we shall call a voting configuration. Given an allocation $\boldsymbol{\omega}$ and a quota γ, we shall say that $S \subseteq N$ is a winning voting configuration, if

$$\sum_{i \in S} \omega_i \geq \gamma$$

and a losing voting configuration, if

$$\sum_{i \in S} \omega_i < \gamma$$

(i.e. the configuration S is winning, if it has a required majority, otherwise it is losing).

Let

$$G = \left[(\gamma, \boldsymbol{\omega}) \in R^{n+1} : \sum_{i=1}^{n} \omega_i = 1, \omega_i \geq 0, 0 \leq \gamma \leq 1 \right]$$

be the space of all committees of the size n and

$$E = \left[\ \mathbf{e} \in \mathrm{R}^n : \sum_{i \in N} e_i = 1, e_i \geq 0 \ (i = 1, ..., n) \ \right]$$

be the unit simplex.

A *power index* is a vector valued function

$$\pi : \mathrm{G} \rightarrow \mathrm{E}$$

that maps the space G of all committees into the unit simplex E. A power index represents a reasonable expectation of the share of decisional power among the various members of a committee, given by their ability to contribute to formation of winning voting configurations. We shall denote by $\pi_i(\gamma, \boldsymbol{\omega})$ the share of power that the index $\boldsymbol{\pi}$ grants to the i-th member of a committee with weight allocation $\boldsymbol{\omega}$ and quota γ. Such a share is called a *power index of the i-th member*.

2. Penrose-Banzhaf and Shapley-Shubik power indices

Two most widely used power indices were proposed by Penrose and Banzhaf (1946, 1965) and Shapley and Shubik (1954). We shall refer to them as PB-power index and SS-power index.

The PB-power measure is based on the concept of swing. Let S be a winning configuration in a committee $[\gamma, \boldsymbol{\omega}]$ and $i \in S$. We say that a member i has a swing in configuration S if

$$\sum_{k \varepsilon S} \omega_k \geq \gamma \quad and \quad \sum_{k \varepsilon S \setminus \{ i \}} \omega_k < \gamma$$

Let s_i denote total number of swings of the member i in the committee $[\gamma, \boldsymbol{\omega}]$. Then PB-power index is defined as

$$\pi_i^{PB}(\gamma, \boldsymbol{\omega}) = \frac{s_i}{\sum_{k \in N} s_k}$$

In the literature this form is usually called a relative PB-index. Original Penrose definition of power of the member i was

$$\pi_i^{P}(\gamma, \omega) = \frac{s_i}{2^{n-1}}$$

which (assuming that all coalitions are equally likely) is nothing else but the probability that the given member will be decisive (probability to have a swing). In literature this form is usually called an absolute PB-index. The relative PB-index is obtained by normalization of the absolute PB-index.

Let the numbers 1, 2, ..., n be fixed names of committee members. Let

$$(i_1, i_2, \ldots, i_n)$$

be a permutation of those numbers, members of the committee, and let member k be in position r in this permutation, i.e. $k = i_r$. We shall say that member k of the committee is in a pivotal situation with respect to a permutation $(i_1, i_2, ..., i_n)$, if

$$\sum_{j=1}^{r-1} w_{i_j} < q \quad and \quad \sum_{j=1}^{r} w_{i_j} \geq q$$

The SS-power measure is based on the concept of pivot. Let us assume that an ordering of members in a given permutation expresses an intensity of their support (preferences) for a particular issue in the sense that, if a member i_s precedes in this permutation a member i_t, then i_s support for the particular proposal to be decided is stronger than support by the member i_t. One can assume that the group supporting the proposal will be formed in the order of the positions of members in the given permutation. If it is so, then the member k will be in situation when the group composed from preceding members in the given permutation still does not have enough of votes to pass the proposal, and a group of members place behind him in the permutation has not enough of votes to block the proposal. The group that will manage his support will win. Member in a pivotal situation has a decisive influence on the final outcome. Assuming many voting acts and all possible preference orderings equally likely, under the full veil of ignorance about other aspects of individual members preferences, it makes sense to evaluate an a priori voting power of each committee member as a probability of being in pivotal situation. This probability is measured by the SS-power index:

$$\pi_i^{SS}(\gamma, \boldsymbol{\omega}) = \frac{p_i}{n!}$$

where p_i is the number of pivotal positions of the committee member i and n! is the number of permutations of the committee members (number of different orderings).

3. The I-power, P-power and cooperative games

Felsenthal, Machover and Zwiker (1998) introduced concept of so called I-power and P-power.

By I-power they mean “voting power conceived of as a voter’s potential influence over the outcome of divisions of the decision making body: whether proposed bills are adopted or blocked. Penrose’s approach was clearly based on this notion, and his measure of voting power is a proposed formalization of a priori I-power:” By P-power they mean “voting power conceived as a voter’s expected relative share in a fixed prize available to the winning coalition under a decision rule, seen in the guise of a simple TU (transferable utility) cooperative game. The Shapley-Shubik approach was evidently based on this notion, and their index is a proposed quantification of a priori P-power” (in the both cases we are quoting Felsenthal and Machover, 2003, p. 8). Hence, the fundamental distinction between I-power and P-power is in the fact that the I-power notion takes the outcome to be the immediate one, passage or defeat of the proposed bill, while the P-power view is that passage of the bill is merely the ostensible and proximate outcome of a division; the real and ultimate outcome is the distribution of fixed purse - the prize of power – among the victors (Felsenthal and Machover, 2003, p. 9-10). As a conclusion it follows that SS-power index does not measure a priori voting power, but says how to divide the “pie” (benefits of victory).

As the major argument of this classification the authors provide a historical observation: Penrose paper from 1946 was ignored and unnoticed by mainstream – predominantly American – social choice theorists, and Shapley and Shubik’s 1954 paper was seen as inaugurating the scientific study of voting power. Because the Shapley-Shubik paper was wholly based on cooperative game theory, it induced among social scientists an almost universal unquestioning belief that the study of power was necessarily and entirely a branch of that theory (Felsenthal and Machover, 2003, p. 8). Conclusion follows, that since the cooperative game theory with transferable utility is about how to divide a pie, and SS-power index was derived as a special case of Shapley value of cooperative game, the SS-power index is about P-power and does not measure voting power as such.

We demonstrated above, that one does not need cooperative game theory to define and justify SS-power index. SS-power index is a probability to be in a pivotal situation in an intuitively plausible process of forming a winning configuration, no division of benefits is involved. Incidentally SS-power index originally appeared as an interesting special case of Shapley value for cooperative games with the transferable utility, but in exactly the same way one can handle the PB-index. Let us make a short excursion into the cooperative game theory.

Let N be the set of players in a cooperative game (cooperation among the players is permitted and the players can form coalitions and transfer utility gained together among themselves) and 2^N its power set, i.e. the set of all subsets $S \subseteq N$, called coalitions, including empty coalition. Characteristic function of the game is a mapping

$$v : 2^N \rightarrow R$$

with

$$v(\emptyset) = 0$$

The interpretation of v is that for any subset S of N the number v(S) is the value (worth) of the coalition S, in terms how much "utility" the members of S can divide among themselves in any way that sums to no more than v(S) if they all agree. The characteristic function is said to be super-additive if for any two disjoint subsets S, T

$$v(S \cup T) \geq v(S) + v(T)$$

$\subseteq$ N

i.e. the worth of the coalition S $\cup$ T is equal to at least the worth of its parts acting separately. Let us denote cooperative characteristic function form by [N, v]. The game [N, v] is said to be super-additive if its characteristic function is super-additive. By a value of the game [N, v] we mean a non-negative vector $\varphi(v)$ such that

$$\sum_{i \in N} \varphi_i(v) = v(N)$$

By

$$c(i,T) = v(T) - v\{T - \{i\})$$

we shall denote marginal contribution of the player $i \in N$ to the coalition $T \subseteq N$. Then in an abstract setting the value $\varphi_i(v)$ of the i-th player in the game [N, v] can be defined as a weighted sum of his marginal contributions to all possible coalitions he can be member of:

$$\varphi_i(v) = \sum_{T \subseteq N, i \in T} \alpha(T) c(i,T)$$

Different weights α(T) leads to different definitions of values.

Shapley (1953) defined his value by the weights

$$\alpha(T) = \frac{(t-1)!(n-t)!}{n!}$$

where t = card (T). He proved that it is the only value that satisfies three axioms: dummy axiom (null player, i.e. the player that contributes nothing to any coalition, has zero value), permutation axiom (for any game [N, u] that is generated from the game [N, v] by a permutation of players, the value $\varphi(u)$ is a corresponding permutation of the value $\varphi(v)$) and additivity axiom (for the sum [N, v+u] of two games [N, v] and [N, u] the value $\varphi(v+u) = \varphi(v) + \varphi(u)$).

As Owen (1995) noticed, the relative PB-index is meaningful for general cooperative games with transferable utilities. One can define Banzhaf value by setting the weights

$$\alpha(T) = \frac{v(N)}{\sum_{k \in N, S \subseteq N} c(k,S)}$$

Owen (1995) shows a certain relation between the Shapley value and Banzhaf value of cooperative game with transferable utilities: both give averages of player's marginal contributions, the difference lies in the weighting coefficients (in the Shapley value coefficients depends on size of coalitions, in the Banzhaf value they are independent of coalition size).

The relation between values and power indices is straightforward: power index of a decision maker is identical with his value in corresponding game. A cooperative characteristic function game represented by a characteristic function v such that v takes only the values 0 and 1 is called a simple game. With any committee with quota q and allocation **w** we can associate a super-additive simple game such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

(i.e. a coalition has value 1 if it is winning and value 0 if it is losing). Super-additive simple games can be used as natural models of voting in committees. Shapley and Shubik (1954) applied the concept of the Shapley value for general cooperative characteristic function games to the super-additive simple games as a measure of voting power in committees. Here we generalized the Penrose-Banzhaf relative power index as a value for general cooperative characteristic function games. But we do not need cooperative games to model voting in committees.

4. Concluding remarks

Using the Felsenthal and Machover classification, there is no reason why not consider Banzhaf value to be a plausible rule for dividing the cake: If Shapley-Shubik expresses P-power, then Penrose-Banzhaf expresses the same. On the other hand, we have demonstrated I-power interpretation of the both indices: they provide the probability of being in a "decisive position", defined either as pivot, or as swing. Both of the indices can be defined and interpreted in terms of cooperative game theory, both
of them can be introduced and analysed without any reference to cooperative games.

Dispute about I-power and P-power is not useless, it raises question about what the voting power is about. In this we see the contribution of Felsenthal and Machover. But dismissing the well defined concept of the SS-power index on the basis of its origin leads to nowhere. We need both PB-power measure and SS-power measure. PB-power measure opens an intriguing question about absolute and relative power: is there something like that? Does power of a committee as an entity depend on quota and distribution of votes among its members or not? The absolute PB-power index

$$\pi_i^P(\gamma,\omega) = \frac{s_i}{2^{n-1}}$$

provides the probability that member i of the committee will be decisive. Then

$$1 - \pi_i^P(\gamma,\omega)$$

is the probability that i will not be decisive, but not the probability that somebody else will be decisive. This can lead to so called donation paradox for the relative PB-power index: one member of the committee can increase his relative power by transferring part of his votes to another member, while his absolute power is decreasing. SS-power index does not exhibit such a paradox (see Turnovec, 1998). Another paradox (let us call it a paradox of abstention) says, that by not using systematically part of his votes, when the quota remains to be fixed (systematic abstention) a member i can increase his relative power, while its absolute power is decreasing. Again, nothing like that happens for SS-power index. The message is that we have to be very careful in interpretation of results based on relative PB-power index and not to use it without absolute PB-power index, what is frequently the case in many published studies.

Acknowledgements

This research was supported by the Polish Research Foundation, project No. 5 H02B 001 21.

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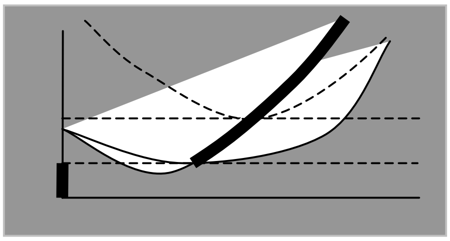

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