The Instrumental Weighted Variables.
Part I. Consistency

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The Instrumental Weighted Variables. Part I. Consistency

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Abstract:
A robust version of the method of Instrumental Variables accommodating the idea of an implicit weighting the residuals is proposed and its properties studied. (The idea of implicit weighting down the “suspicious” residuals was firstly employed by the method of the Least Weighted Squares, see Víšek (2000c).) It means that at first, it is shown that all solutions of the corresponding normal equations are bounded in probability. Finally, the weak consistency of them is proved.

Keywords: Robustness, instrumental variables, implicit weighting, consistency of estimate by instrumental weighted variables

AMS classification: 62F35, 62J05

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Let \( N \) denote the set of all positive integers, \( R \) the real line and \( R^p \) the \( p \)-dimensional Euclidean space. We assume that all r.v.’s are defined on a basic probability space \((\Omega, \mathcal{A}, P)\). The linear regression model given as

\[
Y_i = X_i'\beta^0 + e_i = \sum_{j=1}^{p} X_{ij} \beta^0_j + e_i, \quad i = 1, 2, ..., n
\]  

(1)

will be considered. We shall assume that:

**C1** The sequence \( \{(X_i', e_i)\}'_{i=1}^{\infty} \) is sequence of independent and identically distributed \( p + 1 \)-dimensional random vectors (i.i.d. r.v.’s) with absolutely continuous distribution function \( F_{X,e}(x,v) \).

Moreover, \( \text{IE} \{(X_1', e) \cdot (X_1', e)\}' \) is positive definite matrix and the density \( f_{e|X}(v|X_1=x) \) is uniformly in \( x \) bounded in \( v \), say by \( U_e \).

**Remark 1** We may alternatively write that \( \{(X_i', e_i)\}'_{i=1}^{\infty} \subset R^{p+1} \times N \times \Omega \). As we have mentioned that all random variables will be defined on the (same) probability space \((\Omega, \mathcal{A}, P)\) and as the sequence (of any elements) is the mapping from \( N \) into (or onto) some space, we may abbreviate in what follows the expression \( \{(X_i', e_i)\}'_{i=1}^{\infty} \subset R^{p+1} \times N \times \Omega \) to \( \{(X_i', e_i)\}'_{i=1}^{\infty} \subset R^{p+1} \). We shall do it for the other (random) sequences as well.

In what follows \( F_X(x) \) and \( F_e(r) \) will denote the corresponding marginals of \( F_{X,e}(x,r) \). (All vectors throughout the paper will be considered to be the column ones. So if there will be a missprint, indicating a row vector, it is really misprint.)

**Remark 2** We shall consider the model with intercept, i.e. we shall assume that the first coordinate of explanatory variables \( X_i \) is degenerated and equal to \( 1 \). Notice please that we have not assumed independence between the explanatory variables \( X_i \)’s and the disturbances \( e_i \)’s.

**RECALLING REASONS FOR INSTRUMENTAL VARIABLES**

It is well known that in the case when the orthogonality condition \( \text{IE} \{X_i e_i\} = 0 \) is broken, the ordinary least squares are not consistent due to the fact that

\[
\tilde{\beta}^{(LS,n)} = \beta^0 + \left( \frac{1}{n} \sum_{k=1}^{n} X_k X_k' \right)^{-1} \frac{1}{n} \sum_{i=1}^{n} X_i e_i \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i e_i = \text{IE}X_1 e_1 \quad \text{in probability.}
\]  

(2)

where \( \tilde{\beta}^{(LS,n)} \) stays for the (Ordinary) Least Squares (due to **C1** \( \frac{1}{n} \sum_{k=1}^{n} X_k X_k' \) is, starting with some \( n_0 \), positive definite almost surely). The best known example of the situation when the orthogonality condition fails, is the model assuming that the explanatory variables are measured with random error. Assume that

\[
Y_i = V_i' \beta^0 + u_i, \quad i = 1, 2, ..., n
\]  

(3)

with \( \text{IE}u_i = 0 \) and \( \text{IE}u_i^2 = \sigma^2 \in (0, \infty) \) and that we observe

\[
\tilde{V}_i = V_i + \eta_i
\]  

(4)
assuming usually that \( \mathbb{E}\eta_i = 0, \mathbb{E}\eta_i \cdot \eta_i' = \Sigma_\eta \) with \( \Sigma_\eta \) nonsingular and \( \mathbb{E}\eta_i \cdot u_i = 0 \). Then, substituting (4) into (3), we obtain

\[
Y_i = \left( \tilde{V}_i - \eta_i \right)' \beta^0 + u_i = \tilde{V}_i' \beta^0 - \eta_i' \beta^0 + u_i = \tilde{V}_i' \beta^0 + w_i. \tag{5}
\]

where \( w_i = -\eta_i' \beta^0 + u_i \). But then

\[
\mathbb{E} \left( \tilde{V}_i \cdot w_i \right) = \mathbb{E} \left[ \left( V_i + \eta_i \right) \cdot \left( -\eta_i' \beta^0 + u_i \right) \right] = -\Sigma_\eta \beta^0. \]

Then \( \beta^0 \neq 0 \) implies that \( \Sigma_\eta \beta^0 \neq 0 \) and then (2) indicates that the LS-estimate of regression coefficients of model (3) is inconsistent.

Another example considers the lagged response variable as explanatory one, see Judge et al. (1985) or Višek (1998a).

The problem is usually treated by means of the Method of Instrumental Variables. (The word "usually" means "usually in econometrics". Another possibility how to solve the problem is to find so called the "Total Least Squares", see e.g. Van Huffel (2004). The approach via the "Total Least Squares" implicitly assumes that disturbances \( e_i \)'s in (1) represents just only the measurement errors of the response variable. In other words, the disturbances don’t contain any "random part" of the response variable which we cannot explain by available explanatory variables or which we don’t want to explain, e. g. due to the fact that it would require a lot of additional, although available (but may be e.g. costly to measure), explanatory variables. Of course, there may be some other reasons why the idea that the disturbances represent only measurement errors of the response variable is not acceptable. It means, the approach via the "Total Least Squares" is accommodated mainly for technical applications (after all, the title of Van Huffel’s paper indicates it).)

**Definition 1** For any sequence of \( p \)-dimensional random vectors \( \{Z_i\}_{i=1}^{\infty} \) the solution(s) of the (vector) equation

\[
\sum_{i=1}^{n} Z_i \left( Y_i - X_i' \beta \right) = 0 \tag{6}
\]

will be called the estimator obtained by means of the method of Instrumental Variables (or Instrumental Variables, for short) and denoted by \( \hat{\beta}^{(IV,n)} \).

For the heuristics which show the reasons for defining \( \hat{\beta}^{(IV,n)} \) in this way see Bowden, Turkington (1984) or Judge et al. (1985). In nineties the method became a standard tool in many case studies of dynamic regression model since the correlation of explanatory variables and disturbances frequently appeared. Moreover, many papers considering possibilities how to select the instruments for explanatory variables brought applicable results (including also easy available implementations), see e.g. Arellano, Bond (1991), Arellano, Bover (1995) or Sargan (1988) (and for examples of implementation see for SAS - Der and Everitt (2002), for R and S-PLUS - Fox, J. (2002)).

As (6) is an analogy of the normal equations for the Ordinary Least Squares, \( \hat{\beta}^{(IV,n)} \) is not robust with respect to the outliers and/or leverage points. Hence we are going to define its robustified version. We shall use the idea of implicit weighting the squared residuals which was firstly employed in the method of the Least Weighted Squares, see Višek (2000c).
WHY THE LEAST WEIGHTED SQUARES

Prior to continuing, we need to enlarge a bit the notations. For any \( \beta \in \mathbb{R}^p \), put the random variable \( r_i(\beta) = Y_i - X_i'\beta \), i.e. \( r_i(\beta) \) denotes the \( i \)-th residual and \( r_{(h)}^2(\beta) \) the \( h \)-th order statistic among the squared residuals, i.e. we have

\[
r_{(1)}^2(\beta) \leq r_{(2)}^2(\beta) \leq \ldots \leq r_{(m)}^2(\beta).
\]

Without loss of generality we may assume that \( \beta^0 = 0 \) (otherwise we should write in what follows \( \beta - \beta^0 \) instead of \( \beta \)).

In 1992, Hettmansperger and Sheather, utilizing Engine Knock Data (Mason et al. (1989)) demonstrated that a small change of data may cause a large change of the Least Median of Squares estimator (Rousseeuw (1984)). Later their result (at the first glance surprising) appeared to be due to the unreliable algorithm, they used. In Víšek (1994), the application of algorithm based on the simplex method (Boček and Lachout (1995)) corrected the result. (An implementation of this algorithm, evaluating a tight approximation to the solution of extremal problem which defines the Least Median of Squares (see either Rousseeuw (1984) or Hampel et al. (1986)), was available - due to Pavel Boček - at 1993 (although the referred paper - Boček and Lachout (1995) - discussing it, appeared later). The implementation of algorithm was tested by Víšek (1996b, 2000a) and it proved to be quick and reliable and it is available on request from Pavel Boček or from the author of present paper.) Nevertheless, employing Engine Knock Data and evaluating the Least Trimmed Squares (Hampel et al. (1986)) by the algorithm which gave precise value of the estimator (due to the small number of observations - 16 cases - it was possible to apply the Least Squares on the all subsamples of size 11; 11 is the optimal number for to reach the maximal breakdown point, see Rousseeuw and Leroy (1987)), we have confirmed that a small change of data can really cause a large change of the estimate (Víšek (1994)). In Víšek (1996b, 2000a) instructive academic examples demonstrated why an arbitrarily small shift of one observation may change the estimator as much as you want. The example is as follows (the observation denoted by small circle was shifted)

It means: Robust, especially high breakdown point estimators can be very sensitive to a very small change of data.

On the other hand, Víšek (1992, 1996a, 2002c) revealed that for the M-estimator with discontinuous \( \psi \)-function, the deletion of even one observation may cause very large change of the estimate. Víšek (2000b) conjectured and Víšek (2006d) established the same result for the Least Trimmed Squares. The example for the Least Trimmed Squares is as follows (the observation at the left upper corner of the first picture was deleted)
It means: **The sensitivity of robust estimators with respect to the deletion of even one point from data can be very high** (see also Chatterjee, Hadi (1988) or Zvára (1989) where the classical formula for the difference of the Least Squares estimators before and after deleting one observation is given).

**Remark 3** In both examples the changes of estimator are however bounded, hence the examples do not indicate that the classical criteria of robust statistics (as bounded influence of one observation, or finite breakdown point) are wrong. On the other hand they show that from the practical point of view they should be enlarged by the requirements of the small subsample sensitivity and of the small sensitivity to the (small) shift of observation, see Višek (2000b).

Both these unpleasant consequences of (high) robustness have one denominator, namely that the estimators rely to much on a group of observations, they have selected (considering these observations to be “clean” or “proper”, as you want), while the others are assumed to be contamination, i.e. they are deleted from the data. A remedy can be to weight down the observations which seem to be suspicious, i.e. to depress their influence on the value of the estimator smoothly. It led to a proposal of the **Least Weighted Squares** in the form (Višek (2000c), see also (2002a,b))

$$\hat{\beta}(LWS,n,w) = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} w\left(\frac{i-1}{n}\right) r^2_i(\beta)$$  \hspace{1cm} (8)

where $w : [0, 1] \rightarrow [0, 1]$ is a weight function.

As the **Least Trimmed Squares** and the **Least Median of Squares** are special cases of the **Least Weighted Squares**, it may adapt to various situations. It hints that by “tailoring” the weight function to the character of data, we can create the estimator which is “appropriately robust” but avoiding the problems we have discussed a few lines earlier. On the other hand, when we put some lower bound on values on the weight function, we facilitate the use of the estimator also for the panel data where we cannot afford to delete any observation completely - since otherwise we disturb the correlation structure of data. Moreover, avoiding the discontinuous weight function we get rid of the high subsample sensitivity while keeping all plausible (robust) properties for finite sizes of data sets. Finally, it seems that when proving asymptotic properties for the estimators with discontinuous object function, we get into inevitable complicated technicalities (see e.g. Višek (2006b)). That is why in what follows we shall assume that the weight function has following properties:

**C2** Weight function $w : [0, 1] \rightarrow [0, 1]$ is absolutely continuous and nonincreasing, with the derivative $w'(\alpha)$ bounded from below by $-L$, $w(0) = 1$.

(Please, see also Čížek (2002) where the estimator is called the **Smoothed Least Trimmed Squares**. Although this name indicates that for a special case of weight function, we obtain the Least
Trimmed Squares (LTS) as a special case of the Least Weighted Squares (LWS), it may however obscure the fact that LWS are able to control subsample sensitivity (see Víšek (1996a, 2000c, 2002c)). The same is true about the behaviour of LTS versus LWS with respect to a small shift of an observation (see Víšek (1996b, 2000a)). The last but not least, LWS can be used for panel data processing (especially when we put some lower bound on the values of the weight function \(w\)), while LTS can’t because the deletion of (even only) one observation from panel data may destruct the correlation structure of disturbances and/or of explanatory variables.)

For any \(i \in \{1, 2, ..., n\}\) let us define the random rank of the \(i\)-th residual as

\[
\pi(\beta, i) = j \in \{1, 2, ..., n\} \quad \Leftrightarrow \quad r_i^2(\beta) = r_j^2(\beta)
\]

(the definition is an analogy of rank which is used nonparametric statistics, see e.g. Hájek, Šidák (1967)). Then we have

\[
\hat{\beta}_{LWS,n,w} = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} w \left( \frac{\pi(\beta, i) - 1}{n} \right) r_i^2(\beta).
\]

Finally, let us denote for any \(n \in \mathbb{N}\) by \(P_n\) be the set of all permutations of the indices \(\{1, 2, ..., n\}\) and denote \(\pi_i\) the \(i\)-th coordinate of the vector \(\pi \in P_n\).

Now, we are going to show that (10) (and hence also (8)) has always a solution. Let us consider following steps:

- For any \(\beta \in \mathbb{R}^p\) and arbitrary \(\pi \in P_n\) put \(S(\beta, \pi) = \sum_{i=1}^{n} w \left( \frac{\pi_i - 1}{n} \right) r_i^2(\beta)\).

- Recalling that we have defined \(\pi(\beta, i)\) in (9) \((i = 1, 2, ..., p)\), for any \(\beta \in \mathbb{R}^p\) put \(\pi(\beta) = (\pi(\beta, 1), \pi(\beta, 2), ..., \pi(\beta, n))^{t}\). As \(\pi(\beta) \in P\) we have

\[
\arg\min_{\beta \in \mathbb{R}^p} \arg\min_{\pi \in P_n} \sum_{i=1}^{n} w \left( \frac{\pi_i - 1}{n} \right) r_i^2(\beta) \leq \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} w \left( \frac{\pi(\beta, i) - 1}{n} \right) r_i^2(\beta),
\]

i.e.

\[
\arg\min_{\beta \in \mathbb{R}^p} \arg\min_{\pi \in P_n} S(\beta, \pi) \leq \arg\min_{\beta \in \mathbb{R}^p} S(\beta, \pi(\beta)).
\]

(11)

- For any \(\beta \in \mathbb{R}^p\) and \(\pi \in P_n\)

\[
S(\beta, \pi(\beta)) \leq S(\beta, \pi).
\]

(12)

(Please realize that

\[
S(\beta, \pi(\beta)) = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} w \left( \frac{\pi(\beta, i) - 1}{n} \right) r_i^2(\beta) = \arg\min_{\beta \in \mathbb{R}^p} \sum_{i=1}^{n} w \left( \frac{i - 1}{n} \right) r_i^2(\beta),
\]

i.e. the smallest residual obtains the largest weight, the second smallest residuals obtain the second largest weight, etc.. Finally, any sum, in which the weights are prescribed to residuals in any other way, can’t be smaller.)

- (11) and (12) yield

\[
\arg\min_{\beta \in \mathbb{R}^p} \arg\min_{\pi \in P_n} S(\beta, \pi) = \arg\min_{\beta \in \mathbb{R}^p} S(\beta, \pi(\beta)).
\]

(13)
• Fix \( \omega_0 \in \Omega \), \( \pi \in \mathcal{P}_n \), and evaluate the (classical) \textit{Weighted Least Squares} with the weight matrix \( W(\pi) = \text{diag}\{ w\left( \frac{\pi_1-1}{n}\right), w\left( \frac{\pi_2-1}{n}\right), \ldots, w\left( \frac{\pi_n-1}{n}\right) \} \) as follows (notice please the notation \( \hat{\beta}(WLS, n, \pi) \)):

\[
\hat{\beta}(WLS, n, \pi) = \arg \min_{\beta \in \mathbb{R}^p} \sum_{i=1}^n w\left( \frac{\pi_i-1}{n}\right) (Y_i - X_i'\beta)^2 = (X'W(\pi)X)^{-1}X'W(\pi)Y
\]

where \( Y = (Y_1, Y_2, \ldots, Y_n)' \) and \( X = (X_1, X_2, \ldots, X_n)' \). Then we have for any \( \beta \in \mathbb{R}^p \)

\[
S(\hat{\beta}(WLS, n, \pi), \pi) \leq S(\beta, \pi). \tag{14}
\]

• Repeat it for all \( p \in \mathcal{P}_n \).

• For our \( \omega_0 \in \Omega \) (we have fixed in last but one item) define \( \pi(\omega_0) \) by

\[
\pi(\omega_0) = \arg \min_{\pi \in \mathcal{P}_n} S(\hat{\beta}(WLS, n, \pi), \pi).
\]

• Then for any \( \pi \in \mathcal{P}_n \)

\[
S(\hat{\beta}(WLS, n, \pi(\omega_0)), \pi(\omega_0)) \leq S(\hat{\beta}(WLS, n, \pi), \pi). \tag{15}
\]

• Due to (14), for any \( p \in \mathcal{P}_n \) and any \( \beta \in \mathbb{R}^p \)

\[
S(\hat{\beta}(WLS, n, p(\omega)), p(\omega)) \leq S(\hat{\beta}(WLS, n, p), p) \leq S(\beta, p),
\]

i. e.

\[
S(\hat{\beta}(WLS, n, \pi(\omega)), \pi(\omega)) = \arg \min_{\beta \in \mathbb{R}^p} \arg \min_{\pi \in \mathcal{P}_n} S(\beta, \pi) \tag{15}
\]

and due to (13)

\[
S(\hat{\beta}(WLS, n, \pi(\omega)), \pi(\omega)) = \arg \min_{\beta \in \mathbb{R}^p} S(\beta, \pi(\beta)).
\]

• Repeating just described steps for all \( \omega \)'s, we conclude demonstration of existence of solution of (10).

As a byproduct of the previous demonstration we have found that the \textit{Least Weighted Squares} estimator is, at fixed \( \omega \in \Omega \), equal to the (classical) \textit{Weighted Least Squares} estimator (with the weights \( w(\pi(\omega)) = \left( w\left( \frac{\pi_1(\omega)-1}{n}\right), w\left( \frac{\pi_2(\omega)-1}{n}\right), \ldots, w\left( \frac{\pi_n(\omega)-1}{n}\right) \right)' \)). Since \( \hat{\beta}(WLS,n,w(\pi(\omega))) \) is the solution of corresponding normal equations, considering successively all \( \omega \in \Omega \), we verify that \( \hat{\beta}(LWS,n,w) \) is one of solutions of \textit{normal equations}

\[
\text{NE}_{Y,X,n}(\beta) = \sum_{i=1}^n w\left( \frac{\pi(\beta,i)-1}{n}\right) X_i \left( Y_i - X_i'\beta \right) = 0. \tag{16}
\]

(An alternative way is to show that \( \frac{\partial \pi(\beta,i)}{\partial \beta} = 0 \), see Višek (2006b).)
INSTRUMENTAL WEIGHTED VARIABLES

As we have already recalled the estimator obtained by means of the method of Instrumental Variable is not robust. On the other hand, the inconsistency of the Least Squares when the orthogonality condition is broken, as it was explained in INTRODUCTION), takes place generally also for the Least Weighted Squares. That is why we define an estimator which will be an analogy of the estimator obtained by the method of Instrumental Variables but which will weight down the residuals of those observations which seem to be atypical.

Definition 2 For any sequence of random vectors \( \{Z_i\}_{i=1}^{\infty} \subset \mathbb{R}^p \) the solution(s) of the (vector) equation

\[
\mathbb{N}E_{Y,Z,n}(\beta) = \sum_{i=1}^{n} w \left( \frac{\pi(\beta, i) - 1}{n} \right) Z_i \left( Y_i - X_i' \beta \right) = 0
\] (17)

will be called the Instrumental Weighted Variables estimator and denoted by \( \hat{\beta} \text{IWV},n,w \).

Remark 4 The elements of the sequence \( \{Z_i\}_{i=1}^{\infty} \) are usually called instruments. Without loss of generality we may assume that \( Z_{i1} = 1 \) and \( \mathbb{E}Z_{ij} = 0, j = 2, 3, ..., p \) and \( i = 1, 2, ... \). We do not lose generality firstly, due to the fact that \( Z_{i1} = 1 \) represents constants and hence they cannot be correlated with disturbances (in fact we have then \( Z_{i1} = X_{i1} \)). Secondly, what concerns the assumption that \( \mathbb{E}Z_{ij} = 0, j = 2, 3, ..., p \), if it would not be fulfilled, we can “move” \( \mathbb{E}Z_{ij} \) into the intercept of the original model (1).

For any \( \beta \in \mathbb{R}^p \) the distribution of the absolute value of residual will be denoted \( F_\beta(r) \), i. e.

\[
F_\beta(r) = P(|Y_i - X_i' \beta| < r) = P(|e_i - X_i' \beta| < r)
\] (18)

(remember, we have assumed \( \beta^0 = 0 \)). Similarly, for any \( \beta \in \mathbb{R}^p \) the empirical distribution of the absolute value of residual will be denoted \( F_\beta^{(n)}(r) \). It means that, denoting the indicator of a set \( A \) by \( I \{ A \} \), we have

\[
F_\beta^{(n)}(r) = \frac{1}{n} \sum_{j=1}^{n} I \{|r_j(\beta)| < r\} = \frac{1}{n} \sum_{j=1}^{n} I \{|e_j - X_j' \beta| < r\}.
\] (19)

Realize now that denoting \( |r_j(\beta)| = a_i(\beta) \), the order statistics \( a_i(\beta) \)'s and the order statistics of the squared residuals \( r_j^2(\beta) \)'s assign to given fix observation the same rank, i. e. the residual of given fix observation (say for \( i = i_0 \), for some \( i_0 \in \{1, 2, ..., n\} \)) is in the sequence

\[
r_{(1)}^2(\beta) \leq r_{(2)}^2(\beta) \leq ... r_{(n)}^2(\beta)
\] (20)

and in the sequence

\[
a_{(1)}(\beta) \leq a_{(2)}(\beta) \leq ... a_{(n)}(\beta)
\] (21)

on the same position. In other words, if the squared residual of the \( j \)-th observation is the \( \ell \)-th smallest among the squared residuals, also the absolute value of the \( j \)-th residual is the \( \ell \)-th smallest among the absolute values of residuals. Then looking for the empirical distribution function of the absolute values of residuals, we observe that the first “jump” (having the magnitude \( \frac{1}{n} \)) is at the smallest absolute value of residuals, i. e. at \( a_{(1)}(\beta) \). But due to the sharp inequality in the definition (19) of the empirical distribution function (see (19)), it holds \( F_\beta^{(n)}(a_{(1)}(\beta)) = 0. \)
Hence, at the $\ell$-th “jump” at $a(\ell) (\beta)$, we have $F_\beta^{(n)}(a(\ell) (\beta)) = \frac{\ell - 1}{n}$. Now, let us realize that $a(x(\beta,i)) (\beta) = |r_i(\beta)|$. It means that at the $\pi(\beta,i)$-th “jump”, we have

$$F_\beta^{(n)}(a(x(\beta,i)) (\beta)) = F_\beta^{(n)}(|r_i(\beta)|) = \frac{\pi(\beta,i) - 1}{n}$$

(for $\pi(\beta)$ see (9)) and so (17) can be written as

$$\sum_{i=1}^{n} w \left( F_\beta^{(n)}(|r_i(\beta)|) \right) Z_i \left( Y_i - X_i' \beta \right) = 0.$$  

(23)

In what follows we shall denote the joint d. f. of explanatory variables, of instrumental variables and of disturbances by $F_{X,Z,e}(x,z,r)$ and of course the marginal d. f.’s by $F_{X,Z}(x,z)$, $F_{X,e}(x,r)$, $F_X(x)$, $F_Z(z)$ etc. We will need also the following notation. For any $\beta \in R^p$ the distribution of the product $\beta' Z' X' \beta$ will be denoted $F_{\beta' Z' X' \beta}(u)$, i. e.

$$F_{\beta' Z' X' \beta}(u) = P(\beta' Z_1 X_1' \beta < u)$$

(24)

and similarly as in (18) and (19), the corresponding empirical distribution will be denoted $F_{\beta' Z' X' \beta}^{(n)}(u)$, so that

$$F_{\beta' Z' X' \beta}^{(n)}(u) = \frac{1}{n} \sum_{j=1}^{n} I \left\{ \beta' Z_j X_j' \beta < u \right\} = \frac{1}{n} \sum_{j=1}^{n} I \left\{ \omega \in \Omega : \beta' Z_j(\omega) X_j(\omega) \beta < u \right\}.$$  

(25)

For any $\lambda \in R^+$ and any $\alpha \in R$ put

$$\gamma_{\lambda,\alpha} = \sup_{\|\beta\| = \lambda} F_{\beta' Z' X' \beta}(\alpha).$$  

(26)

Notice please that due to the fact that the surface of ball $\{ \beta \in R^p, \|\beta\| = \lambda \}$ is compact, there is $\beta_\lambda \in \{ \beta \in R^p, \|\beta\| = \lambda \}$ so that

$$\gamma_{\lambda,\alpha} = F_{\beta_\lambda' Z' X' \beta_\lambda}(\alpha).$$  

(27)

For any $\lambda \in R^+$ let us denote

$$\tau_\lambda = \inf_{\|\beta\| \leq \lambda} \beta \mathbb{E} \left[ Z_1 X'_1 \cdot I \{ \beta' Z_1 X_1' \beta < 0 \} \right] \beta.$$  

(28)

Notice please that $\tau_\lambda \geq 0$ and that again due to the fact that the ball $\{ \beta \in R^p, \|\beta\| \leq \lambda \}$ is compact, the infimum is finite, since there is a $\bar{\beta} \in \{ \beta \in R^p, \|\beta\| \leq \lambda \}$ so that

$$\tau_\lambda = \bar{\beta} \mathbb{E} \left[ Z_1 X'_1 \cdot I \{ \bar{\beta}' Z_1 X_1' \bar{\beta} < 0 \} \right] \bar{\beta}.$$  

(29)

The classical regression analysis accepted the assumption that $\mathbb{E} Z_1 X'_1$ is regular and $\mathbb{E} \{ e_i | Z_1 \} = 0$ (see e. g. Bowden, Turkington (1984) or Judge et al. (1985)) to be able to prove consistency of the estimator obtained by the method of Instrumental Variables. We need to assume similar ones. The following more or less academic considerations give us an inspiration.
Transforming the variables so that we put $\tilde{X}_{11} = X_{11}$ and for any $j = 2, 3, \ldots, p$

$$\tilde{X}_{1j} = X_{1j} - \sum_{k=1}^{j-1} \lambda_{jk} \tilde{X}_{1k}$$

where $\lambda_{jk}$ are selected so that $\text{cov}(\tilde{X}_{1j}, \tilde{X}_{1k}) = 0$ for $j \neq k$, we have the matrix $\mathbb{E} \tilde{X}_1 \tilde{X}_1'$ diagonal and the model for transformed data, namely $Y_i = X'_i \beta + u_i$ has the same "explanatory" abilities as (1). New explanatory variables $\{\tilde{X}_i\}_{i=1}^\infty$ would not allow presumably so direct (physical, biological, economic etc.) interpretation, nevertheless they have also at least one advantage, namely that overfitting the model does not imply automatically a decrease of efficiency of the estimates of regression coefficients, see Chatterjee and Hadi (1988).

Assuming that we shall look for a sequence of instrumental variables $\{\tilde{Z}_i\}_{i=1}^\infty$ for the sequence of transformed explanatory variables $\{\tilde{X}_i\}_{i=1}^\infty$. We would like to find it so that also $\mathbb{E} \tilde{Z}_1 \tilde{X}_i'$ is regular and diagonal. In other words, we would like to find the instrumental variables so that $\tilde{Z}_{1j}$ is correlated only with $\tilde{X}_{1j}$ (of course for all $j = 2, 3, \ldots, p$). Assume that it is possible. Then we may assume that $\mathbb{E} \tilde{Z}_{1j} \tilde{X}_{1j} > 0$ (otherwise we take instead of $\tilde{Z}_{1j}$ the instrumental variable $-\tilde{Z}_{1j}$). Then however $\mathbb{E} \tilde{Z}_1 \tilde{X}_i'$ is positive definite.

These (let us repeat academic) considerations can inspire us to made following assumptions about the instrumental variables:

**C3** The instrumental variables $\{Z_i\}_{i=1}^\infty \subset \mathbb{R}^p$ are independent and identically distributed with distribution function $F_Z(z)$. Moreover, they are independent from the sequence $\{e_i\}_{i=1}^\infty$. Further, the joint distribution function $F_{X,Z}(x,z)$ is absolutely continuous, $\mathbb{E} \{w(F_{X,Z}(e_1)))Z_1 X'_1\}$ as well as $\mathbb{E} Z_1 Z'_1$ are positive definite (one can compare C3 with Víšek (1998) where we considered instrumental $M$-estimators and the discussion of assumptions for $M$-instrumental variables was given) and there is $q > 1$ so that $\mathbb{E} \{\|Z_i\| \cdot \|X_i\|\}^q < \infty$. Finally, there is $a > 0$, $b \in (0,1)$ and $\lambda > 0$ so that

$$a \cdot (b - \gamma_{\lambda,a}) \cdot w(b) > \tau_{\lambda}$$

for $\gamma_{\lambda,a}$ and $\tau_{\lambda}$ given by (26) and (28).

**Remark 5** Let us briefly discuss assumptions we have made. Let us recall that the Least Squares ($\hat{\beta}^{(LS,n)}$) are optimal only under normality of disturbances. Here the optimality means that they reach the lower Rao-Cramer bound (of course, in multivariate Rao-Cramer lemma we consider the ordering of the covariance matrices in the sense of ordering the positive definite matrices). On the other hand, a small departure from normality may cause (and usually does) a large decrease of efficiency (see e.g. Fisher (1920), (1922)). So, without the assumption of normality of disturbances $\hat{\beta}^{(LS,n)}$ is much worse, in fact they are the best unbiased estimator only in the class of linear unbiased estimators, for a discussion showing that restriction on linear estimators can be drastic see Hampel et al. (1986). Sometimes, however we may meet with the statement that we do not need necessarily the normality of disturbances, just because $\hat{\beta}^{(LS,n)}$ is still (without normality) the best unbiased estimator in the class of linear unbiased estimators. And the restriction on the class of linear unbiased estimators is justified by a claim that we have to restrict ourselves on the class of linear estimators, as in the the class of linear unbiased estimators, the estimators are scale- and regression-equivariant. Let us recall that having denoted $M(n, p)$ the set of all matrices of type $(n \times p)$ and recalling that the estimator $\hat{\beta}$ can be considered as a mapping

$$\hat{\beta}(Y, X) : M(n, p + 1) \rightarrow \mathbb{R}^p,$$
the estimator \( \hat{\beta} \) of \( \beta^0 \) is called scale-equivariant, if for any \( c \in R^+ \), \( Y \in R^n \) and \( X \in M(n, p) \) we have

\[
\hat{\beta}(cY, X) = c\hat{\beta}(Y, X)
\]

and regression-equivariant if for any \( b \in R^p \), \( Y \in R^n \) and \( X \in M(n, p) \)

\[
\hat{\beta}(Y + Xb, X) = \hat{\beta}(Y, X) + b.
\]

But, there are a lot of nonlinear estimators which are scale- and regression-equivariant. In the regression framework, the estimators as the Least Median of Squares, the Least Trimmed Squares or the Least Weighted Squares can serve as examples (for an interesting discussion of this topic see again Hampel et al. (1986), and also Bickel (1975) or Jurečková and Sen (1993)).

Since LWS are also based on \( L_2 \)-metric, we guess that they are approximately optimal for finite sample sizes under the (approximative) normality of disturbances, for some hint consult Mašíček (2003). As the present proposal of robustified instrumental variables is based on the same metric (due to the normal equations (17)), we can expect that the estimate can be approximately optimal under (approximative) normality of disturbances. But then our assumptions seem to be quite acceptable.

The only assumption which deserve further discussion is the assumption (30). We are going to show that it is a restriction on the weight function \( v \). Let us return to (26) (or to (27)). We have

\[
\gamma_{\lambda,a} = F_{\beta'Z'X'X} \left( a \right) = \frac{1}{2} \left( \left( \beta_0'Z_1X'_1 \beta_0 \leq 0 \right) + \left( 0 < \beta_0'Z_1X'_1 \beta_0 \leq a \right) \right).
\]

If we assume for a while \( Z_j = X_j \), for any fix \( \lambda \in R^+ \) we have

\[
\lim_{a \to \infty} F_{\beta'Z'X'X} (a) = 0 \tag{31}
\]

but for \( \gamma_{\lambda,a} \) we have (again for fix \( \lambda \in R^+ \))

\[
\lim_{a \to \infty} F_{\beta'Z'X'X} (a) = \frac{1}{2} \left( \beta_0'Z_1X'_1 \beta_0 \leq 0 \right) \tag{32}
\]

On the other hand, for any \( a > 0 \) we have

\[
\gamma_{\lambda,a} < 1 \tag{33}
\]

Now let us turn to \( \tau_\lambda \). As

\[
\mathbb{E} \left| \beta'Z_1X'_1 \beta \right| \leq \| \beta \|^2 \mathbb{E} \{ \| Z_1 \| \| X_1 \| \} \leq \| \beta \|^2 \mathbb{E} \{ \| Z_1 \| \| X_1 \| \}^q < \infty,
\]

we have

\[
\limsup_{\| \beta \| \to 0} \left| \beta' \mathbb{E} \left[ Z_1X'_1 I \{ \beta'Z_1X'_1 \beta < 0 \} \right] \beta \right| = 0. \tag{34}
\]

In other words, \( \tau_\lambda \) can be done arbitrary small (just selecting \( \lambda \in R^+ \) so that \( \| \lambda \| \) is small). It says that if \( w(b) \equiv 1 \), there is \( b \in (0, 1) > \gamma_{\lambda,a} \) (even for any \( a > 0 \)). It means that (31), (32), (33) and (34) indicate that (30) can be always fulfilled but we may have restricted possibility to depress the influence of “bad” observations.

In what follows there are defined some constants inside the proofs of lemmas. They are assumed to be defined only inside the corresponding proof. Now we can prove:
Lemma 1 Let Conditions C1, C2 and C3 be fulfilled. Then for any $\varepsilon > 0$ and $\delta > 0$ there is $\theta > \delta$ and $\Delta > 0$ such that

$$P \left( \left\{ \omega \in \Omega : \inf_{\|\beta\| = \theta} - \frac{1}{n} \hat{\beta}^T \hat{N} E_{Y,Z,n}(\beta) > \Delta \right\} \right) > 1 - \varepsilon.$$  

In other words, any sequence $\{\hat{\beta}(I^W,n,w)\}_{n=1}^\infty$ of the solutions of the (sequence of) normal equations $\hat{N} E_{Z,n}(\hat{\beta}(I^W,n,w)) = 0$ (see (16)) is bounded in probability.

Proof: The plan of the proof is simple: We shall show that for any positive $\varepsilon$ there are positive $\kappa$ and $n_\varepsilon$ so that for any $n > n_\varepsilon$ with probability at least $1 - \varepsilon$, outside the ball of the diameter $\kappa$ the expression $-\frac{1}{n} \hat{\beta}^T \hat{N} E_{Y,Z,n}(\beta)$ is positive. The way how to demonstrate it is based on the idea to show that quadratic part of $-\frac{1}{n} \hat{\beta}^T \hat{N} E_{Y,Z,n}(\beta)$ is positive and hence for enough large $\beta$ it overcomes the linear one. In order to establish the positivity of quadratic part, we evaluate the number of terms in the corresponding sum which are negative and the number of terms which are positive and simultaneously having weight larger than a constant $c$ (of course, there are some other positive terms, contribution of which will be neglected, since their weights are smaller than $c$). Since the mean of sum of the negative terms is bounded from below in probability, we estimate from below the value of quadratic term.

First of all, denote the set of all indices $i = 1, 2, ..., n$ by $I_n$, for $b$ from Condition C3 the set of indices for which $F_{\beta}^{(n)}(\|Y_i(\beta)\|) \geq b$ by $I_b$ and finally, for any $\beta \in R^p$ denote the set of indices for which $\beta^T Z_i X_i^\beta < a$ by $I_a(\beta)$. Of course, the set of indices $I_b$ also depends on $\beta$ but due to the fact that we shall need only an upper estimate of number of elements of $I_b$ which doesn’t depend on $\beta$, we have omitted $\beta$ in notations. Returning to (20) or (21), we easy verify that the empirical d.f. overcomes $b$ at least at its $[nb] + 1$ jump, i.e. at least $[nb]$ of $n$ observations are in $I_b^C$. Hence

$$\#I_b \leq n \cdot (1 - b) + 1 \quad (35)$$

where $\#A$ stays for the number of elements of the set $A$. Denote $E \{\|e_1\| \cdot \|Z_1\|\} = \gamma^{(1)}$ and $E \{\|X_1\| \cdot \|Z_1\|\} = \gamma^{(2)}$ and fix a positive $\varepsilon$. Further, let $\lambda > 0$ be that from C3 and put (see (30))

$$\delta = a \cdot (b - \gamma_{\lambda,a}) \cdot w(b) - \tau_\lambda.$$  

Recalling that we have assumed that $\beta^0 = 0$, we shall consider for $\beta \in R^p$

$$-\frac{1}{n} \hat{\beta}^T \hat{N} E_{Y,Z,n}(\beta) = -\frac{1}{n} \sum_{i=1}^n w \left( F_{\beta}^{(n)}(\|Y_i(\beta)\|) \right) \beta^T Z_i \left( e_i - X_i^\beta \right)$$

$$= \frac{1}{n} \sum_{i=1}^n w \left( F_{\beta}^{(n)}(\|Y_i(\beta)\|) \right) \beta^T Z_i X_i^\beta - \frac{1}{n} \sum_{i=1}^n w \left( F_{\beta}^{(n)}(\|Y_i(\beta)\|) \right) e_i Z_i^\beta.$$

Let us start with the first term in (36) and put $\tau^{(1)} = \delta/(2L \cdot \gamma^{(2)} \cdot \lambda^2)$, for $L$ see C2. Due to Lemma A.2 we can find $n_1 \in N$ so that for any $n > n_1$ there is a set $B_n^{(1)}$ such that $P(B_n^{(1)}) > 1 - \varepsilon/5$ and for any $\omega \in B_n^{(1)}$

$$\sup_{\beta \in R^p} \sup_{r \in R^l} \left| F_{\beta}^{(n)}(r) - F_{\beta}(r) \right| \leq \tau^{(1)}.$$  

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Employing the law of large numbers, find \( n_2 \in \mathcal{N} \) so that for any \( n > n_2 \) there is a set \( B^{(2)}_n \) such that \( P(B^{(2)}_n) > 1 - \varepsilon/5 \) and for any \( \omega \in B^{(2)}_n \)

\[
\frac{1}{n} \sum_{i=1}^{n} \|Z_i\| \cdot \|X_i\| < 2\gamma^{(2)}.
\]

Since then for any \( n > \max \{n_1, n_2\} \) and any \( \omega \in B^{(1)}_n \cap B^{(2)}_n \) (of course \( P(B^{(1)}_n \cap B^{(2)}_n) > 1 - 2\varepsilon/5 \))

\[
\frac{1}{n} \sup_{\beta \in R^p} \left\| \frac{1}{n} \sum_{i=1}^{n} \left\{ w\left( F^{(n)}_{\beta}(|r_i(\beta)|) \right) - w\left( F_{\beta}(|r_i(\beta)|) \right) \right\} Z_iX_i' \right\|
\leq \frac{1}{n} \sum_{i=1}^{n} \|Z_i\| \cdot \|X_i\| \leq L \cdot \tau^{(1)} \cdot 2\gamma^{(2)} = \frac{\delta}{\lambda^2},
\]

we have for any \( n > \max \{n_1, n_2\} \), any \( \omega \in B^{(1)}_n \cap B^{(2)}_n \) and any \( \beta \in R^p \)

\[
\frac{1}{n} \sup_{\beta \in R^p} \left\| \frac{1}{n} \sum_{i=1}^{n} \left\{ w\left( F^{(n)}_{\beta}(|r_i(\beta)|) \right) - w\left( F_{\beta}(|r_i(\beta)|) \right) \right\} \beta Z_iX_i' \right\| \leq \frac{\delta \cdot \|\beta\|^2}{\lambda^2}. \tag{37}
\]

Notice please that for any \( \beta \in R^p \), for indices for which \( F^{(n)}_{\beta}(|r_i(\beta)|) \leq b \), we have \( w\left( F^{(n)}_{\beta}(|r_i(\beta)|) \right) \geq w(b) \). Now, let us consider for any \( \beta \in R^p \)

\[
\frac{1}{n} \sum_{i=1}^{n} w\left( F_{\beta}(|r_i(\beta)|) \right) \beta Z_iX_i' \beta = \frac{1}{n} \sum_{i=1}^{n} w\left( F_{\beta}(|r_i(\beta)|) \right) \beta Z_iX_i' \beta \cdot I\{\beta Z_iX_i' \beta < 0\}
\]

\[
+ \frac{1}{n} \sum_{i=1}^{n} w\left( F_{\beta}(|r_i(\beta)|) \right) \beta Z_iX_i' \beta \cdot I\{\beta Z_iX_i' \beta \geq 0\}
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} \beta Z_iX_i' \beta \cdot I\{\beta Z_iX_i' \beta < 0\} + \frac{1}{n} \sum_{i \in I \setminus \beta} w(b) \beta Z_iX_i' \beta \cdot I\{\beta Z_iX_i' \beta \geq 0\} \tag{38}
\]

where we have employed monotonicity of \( w(r) \). Notice please that (38) holds for any \( \beta \in R^p \). Utilizing Lemma A.8 find such \( n_3 \in \mathcal{N} \) that for all \( n > n_3 \) we have

\[
P \left( \left\{ \omega \in \Omega : \inf_{\|\beta\| \leq \lambda} \frac{1}{n} \sum_{i=1}^{n} \beta Z_iX_i' \beta \cdot I\{\beta Z_iX_i' \beta < 0\} > \tau_\lambda - \frac{\delta}{2} \right\} \right) > 1 - \frac{\varepsilon}{5} \tag{39}
\]

and denote the corresponding set by \( B^{(3)}_n \). Employing Lemma A.3 find \( n_4 \in \mathcal{N} \) so that for all \( n > n_4 \) we have

\[
P \left( \left\{ \omega \in \Omega : \sup_{\beta \in R^p} \sup_{u \in R} \left| F^{(n)}_{\beta}(Z_iX' \beta) - F_{\beta}(Z_iX' \beta) \right| \leq \frac{\delta}{2 \cdot a \cdot w(b)} \right\} \right) > 1 - \frac{\varepsilon}{5} \tag{40}
\]

and denote the corresponding set by \( B^{(4)}_n \). Recalling that, due to the fact how the empirical distribution function is defined, we have

\[
F^{(n)}_{\beta}(a) = \frac{\# \{ i : \beta Z_iX_i' \beta < a \}}{n} = \frac{\# I_a(\beta)}{n}
\]

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(where again \#A denotes the number of points of the set A), we conclude that (40) implies for any \( n > n_4 \) and \( \omega \in B_n^{(4)} \)

\[
\#I_n(\beta) < \left( F_{\beta'}ZX'\beta(a) + \frac{\delta}{2 \cdot a \cdot w(b)} \right) \cdot n \leq \left( \gamma_{\lambda,a} + \frac{\delta}{2 \cdot a \cdot w(b)} \right) \cdot n \quad (41)
\]

(for \( \gamma_{\lambda,a} \) see (26)). Finally, find \( n_5 \in \mathcal{N} \) so that for all \( n > n_5 \) we have

\[
\frac{a \cdot w(b)}{n} < \delta. \quad (42)
\]

Consider \( \omega \in B_n^* = B_n^{(3)} \cap B_n^{(4)} \) and \( n > \max \{n_3, n_4, n_5\} \). Let us recall once again that for any \( \beta \in R^p \), for indices for which \( F_{\beta}^{(n)}(|r_i(\beta)|) \leq b \), we have \( w(F_{\beta}^{(n)}(|r_i(\beta)|)) \geq w(b) \). Hence, (35) and (41) imply that the number of indices for which \( \beta'Z_iX'_i\beta \geq a \) and simultaneously \( w(F_{\beta}^{(n)}(|r_i(\beta)|)) \geq w(b) \) is at least

\[
n - n \cdot (1 - b) - 1 - n \cdot \left( \gamma_{\lambda,a} + \frac{\delta}{2 \cdot a \cdot w(b)} \right) = n \cdot \left( b - \gamma_{\lambda,a} - \frac{\delta}{2 \cdot a \cdot w(b)} \right) - 1.
\]

Now, taking into account (39) and (42) we have for any \( n > \max \{n_3, n_4, n_5\} \), any \( \omega \in B_n^* = B_n^{(3)} \cap B_n^{(4)} \) and any \( ||\beta|| = \lambda \)

\[
\frac{1}{n} \sum_{i=1}^{n} \beta'Z_iX'_i\beta \cdot I\{\beta'Z_iX'_i\beta < 0\} + \frac{1}{n} \sum_{I_n \setminus I_b} w(b)\beta'Z_iX'_i\beta \cdot I\{\beta'Z_iX'_i\beta \geq 0\}
\]

\[
\geq a \cdot \left( b - \gamma_{\lambda,a} - \frac{\delta}{2 \cdot a \cdot w(b)} - \frac{1}{n} \right) \cdot w(b) - \tau_\lambda - \frac{\delta}{2} = a \cdot \left( b - \gamma_{\lambda,a} - \frac{1}{n} \right) \cdot w(b) - \tau_\lambda - \delta > 3\delta.
\]

Consider now any \( \beta \in R^p \), \( ||\beta|| = \theta > \lambda \) and put \( \tilde{\beta} = \theta^{-1} \cdot \lambda \cdot \beta \). Notice please that for any \( \beta \in R^p \) for which \( \beta'Z_iX'_i\beta < 0 \), also \( \tilde{\beta}'Z_iX'_i\tilde{\beta} < 0 \) and similarly for the case when \( \beta'Z_iX'_i\beta \geq 0 \). Then \( ||\tilde{\beta}|| = \lambda \) and hence, again for any \( n > \max \{n_3, n_4, n_5\} \) and any \( \omega \in B_n^* = B_n^{(3)} \cap B_n^{(4)} \) (due to (38))

\[
\frac{1}{n} \sum_{i=1}^{n} w(F_{\beta}(|r_i(\tilde{\beta})|)) \beta'Z_iX'_i\beta
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} \beta'Z_iX'_i\beta \cdot I\{\beta'Z_iX'_i\beta < 0\} + \frac{1}{n} \sum_{I_n \setminus I_b} w(b)\beta'Z_iX'_i\beta \cdot I\{\beta'Z_iX'_i\beta \geq 0\}
\]

\[
= \left( \frac{\theta}{\lambda} \right)^2 \left\{ \frac{1}{n} \sum_{i=1}^{n} \tilde{\beta}'Z_iX'_i\tilde{\beta} \cdot I\{\tilde{\beta}'Z_iX'_i\tilde{\beta} < 0\}
\right.
\]

\[
+ \frac{1}{n} \sum_{I_n \setminus I_b} w(b)\tilde{\beta}'Z_iX'_i\tilde{\beta} \cdot I\{\tilde{\beta}'Z_iX'_i\tilde{\beta} \geq 0\} \right\} > 3 \left( \frac{||\beta||}{\lambda} \right)^2 \delta. \quad (43)
\]

Now, we shall consider the second term in (36). Recalling that we have denoted \( BE \{ |e_i| \cdot ||Z_1|| \} = \gamma^{(1)} \), we can find \( n_6 \in \mathcal{N} \) so that for any \( n > n_6 \) there is \( B_n^{(5)} \) so that \( P(B_n^{(5)}) > 1 - \varepsilon / 5 \) and for any \( \omega \in B_n^{(5)} \) we have

\[
\frac{1}{n} \sum_{i=1}^{n} w(F_{\beta}^{(n)}(|r_i(\tilde{\beta})|)) e_iZ_i^{}\tilde{\beta} \leq (\gamma^{(1)} + \delta) ||\beta||. \quad (44)
\]
Consider $n > \max \{n_1, n_2, n_3, n_4, n_5, n_6\}$ and $\omega \in B_n = \cap_{j=1}^5 B_n^j$. Of course, $P(B_n) > 1 - \varepsilon$ and (36), (37), (43) and (44) imply that for any $\beta \in R^p$, $\|\beta\| \geq \lambda$

$$-\frac{1}{n} \beta^I N E_{Y,Z,n}(\beta) \geq 2 \left(\frac{\|\beta\|}{\lambda}\right)^2 \delta - (\gamma^{(1)} + \delta)\|\beta\|.$$ 

Then there is a $\kappa > 0$ such that for any $\beta \in R^p$, $\|\beta\| > \kappa$ with probability at least $1 - \varepsilon$ we have

$$-\frac{1}{n} \beta^I N E_{Y,Z,n}(\beta) > \delta.$$ 

\[\Box\]

**Remark 6** The fact that for any $i$ and any $\omega \in \Omega$ the matrix $X_iX^t_i$ is positive semidefinite allows to prove the same assertion (i.e. that all solutions of the normal equations are bounded in probability) for the Least Weighted Squares in significantly simpler way, see Mašíček (2003).

**Lemma 2** Let Conditions C1, C2 and C3 be fulfilled. Then for any $\varepsilon > 0$, $\delta \in (0, 1)$ and $\zeta > 0$ there is $n_{\varepsilon, \delta, \zeta} \in \mathbb{N}$ so that for any $n > n_{\varepsilon, \delta, \zeta}$ we have

$$P \left( \omega \in \Omega : \sup_{\|\beta\| \leq \zeta} \frac{1}{n} \sum_{i=1}^n w \left( F^{(n)}_\beta(|r_i(\beta)|) \right) \beta^t Z_i \left( e_i - X'_i \beta \right) - \beta^t J E \left[ w \left( F^{(n)}_\beta(|r_1(\beta)|) \right) Z_1 \left( e_1 - X'_1 \beta \right) \right] < \delta \right) > 1 - \varepsilon.$$ 

**Proof:** Denoting $J E \{\|e_1\| \cdot \|Z_1\|\} = \gamma^{(1)}$ and $J E \{\|X_1\| \cdot \|Z_1\|\} = \gamma^{(2)}$, let us fix a positive $\varepsilon$, $\delta \in (0, 1)$ and $\zeta > 0$. Recalling that we have assumed that $\beta^0 = 0$, we shall consider for $\beta \in R^p$, $\|\beta\| \leq \zeta$

$$-\frac{1}{n} \beta^I N E_{Y,Z,n}(\beta) = -\frac{1}{n} \sum_{i=1}^n w \left( F^{(n)}_\beta(|r_i(\beta)|) \right) \beta^t Z_i \left( e_i - X'_i \beta \right) = -\frac{1}{n} \sum_{i=1}^n w \left( F^{(n)}_\beta(|r_i(\beta)|) \right) \beta^t Z_i X'_i \beta - \frac{1}{n} \sum_{i=1}^n w \left( F^{(n)}_\beta(|r_i(\beta)|) \right) e_i Z'_i \beta. \tag{45}$$

Let us start with the first term in (45) and put $\tau^{(1)} = \delta / (16 \gamma^{(2)} \zeta^2 \cdot L)$, for $L$ see Condition C2. Due to Lemma A2 we can find $n_1 \in \mathbb{N}$ so that for any $n > n_1$ there is a set $B^{(1)}_n$ such that $P(B^{(1)}_n) > 1 - \varepsilon/8$ and for any $\omega \in B^{(1)}_n$

$$\sup_{\beta \in R^p} \sup_{r \in R} \left| F^{(n)}_\beta(r) - F_\beta(r) \right| \leq \tau^{(1)}. \tag{46}$$

Employing the law of large numbers, find $n_2 > n_1$ so that for any $n > n_2$ there is a set $B^{(2)}_n$ such that $P(B^{(2)}_n) > 1 - \varepsilon/8$ and for any $\omega \in B^{(2)}_n$

$$\frac{1}{n} \sum_{i=1}^n \|Z_i\| \cdot \|X_i\| < 2 \gamma^{(2)}. \tag{47}$$

Since then for any $n > n_2$ and any $\omega \in B^{(1)}_n \cap B^{(2)}_n$ (of course $P \left( B^{(1)}_n \cap B^{(2)}_n \right) > 1 - \frac{\varepsilon}{4}$)

$$\frac{1}{n} \sup_{\|\beta\| \leq \zeta} \left\| \sum_{i=1}^n \left\{ w \left( F^{(n)}_\beta(|r_i(\beta)|) \right) - w \left( F_\beta(|r_i(\beta)|) \right) \right\} Z_i X'_i \right\|$$

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we have for any \( n > n_2 \) and any \( \omega \in B_n^{(1)} \cap B_n^{(2)} \)

\[
\frac{1}{n} \sup_{\| \beta \| \leq \zeta} \left| \sum_{i=1}^{n} \left\{ w \left( F_{\beta}^{(1)}(|r_i(\beta)|) \right) - w \left( F_{\beta}^{(2)}(|r_i(\beta)|) \right) \right\} \beta_i' Z_i X_i' \right| \leq \frac{\delta}{8}. \tag{48}
\]

Employ Lemma A.1 and find for \( \Delta = \frac{\delta}{16 L \cdot 2 \cdot \gamma(2) \cdot \zeta^2} \) such \( \tau(2) > 0 \) that for

\[
T(\tau(2)) = \left\{ \| \beta(1) \| \leq \zeta, \| \beta(2) \| \leq \zeta, \| \beta(1) - \beta(2) \| < \tau(2) \right\}
\]

we have

\[
\sup_{(\beta(1), \beta(2)) \in T(\tau(2))} \sup_{r \in R} \left| F_{\beta(1)}(r) - F_{\beta(2)}(r) \right| < \Delta.
\]

Then for any \( n > n_2 \) and any \( \omega \in B_n^{(1)} \cap B_n^{(2)} \)

\[
\frac{1}{n} \sup_{(\beta(1), \beta(2)) \in T(\tau(2))} \left| \sum_{i=1}^{n} \left\{ w \left( F_{\beta(2)}(|r_i(\beta(2))|) \right) - w \left( F_{\beta(1)}(|r_i(\beta(2))|) \right) \right\} \beta_i' Z_i X_i' \beta(1) \right| \leq L \cdot \Delta \cdot \zeta^2 \cdot \frac{1}{n} \sum_{i=1}^{n} \| Z_i \| \cdot \| X_i \| \leq \frac{\delta}{8}. \tag{50}
\]

(notice that the in the previous inequality the subindices of the d.f.'s are \( \beta(1) \) and \( \beta(2) \) but the arguments are the same, namely \( r_i(\beta(2)) \)). Further denote \( \gamma(3) = \bar{E} \left\{ \| Z_i \| \cdot \| X_i \| \right\}^q \), \( \gamma(4) = \bar{E} \| X_i \| \) and applying the law of large numbers find \( n_3 > n_2 \) so that for any \( n > n_3 \) there is a set \( B_n^{(3)} \) such that \( P(B_n^{(3)}) > 1 - \varepsilon/8 \) and for any \( \omega \in B_n^{(3)} \) we have

\[
\frac{1}{n} \sum_{i=1}^{n} \| Z_i \| \cdot \| X_i \| < 2 \cdot \gamma(3) \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} \| X_i \| < 2 \cdot \gamma(4).
\]

Finally, let us recall that \( w(r) \in [0, 1] \), so that for any pair \( r_1, r_2 \in R \) we have \( |w(r_1) - w(r_2)| \leq 1 \) and hence for any \( q' > 1 \)

\[
|w(r_1) - w(r_2)|^{q'} \leq |w(r_1) - w(r_2)|. \tag{51}
\]

Then select a \( \tau(3) \in \left( 0, \min \left\{ \tau(2), \delta \cdot \left( 2q' \cdot 2q \cdot 8 \cdot U_e \cdot L \cdot \left[ \gamma(3)^{q'/q} \cdot \gamma(4) \cdot \zeta^2 q' \right]^{-1} \right) \right\} \) (for \( U_e \) see C1) and put

\[
T(\tau(3)) = \left\{ \| \beta(1) \| \leq \zeta, \| \beta(2) \| \leq \zeta, \| \beta(1) - \beta(2) \| < \tau(3) \right\}.
\]

Employing Hölder’s inequality we arrive at

\[
\sup_{(\beta(1), \beta(2)) \in T(\tau(3))} \left| \sum_{i=1}^{n} \left\{ \frac{w \left( F_{\beta(2)}(|r_i(\beta(2))|) \right) - w \left( F_{\beta(1)}(|r_i(\beta(2))|) \right)}{\| \beta(1) \|} \beta_i' Z_i X_i' \beta(1) \right| \leq \sup_{(\beta(1), \beta(2)) \in T(\tau(3))} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} \left( \frac{w \left( F_{\beta(2)}(|r_i(\beta(2))|) \right) - w \left( F_{\beta(1)}(|r_i(\beta(2))|) \right)}{\| \beta(1) \|} \right)^{q'} \right|^\frac{1}{q'} \times \right.
\]


\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \| \beta^{(1)} \| \cdot \| Z_i \| \cdot \| X_i \| \cdot \| \beta^{(1)} \| \right)^{\frac{1}{2}} \right\} \times \\sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\tau^{(3)})} \left\{ \left[ \frac{1}{n} \sum_{i=1}^{n} \left| w \left( F_{\beta^{(1)}}(r_i(\beta^{(2)})) \right) - w \left( F_{\beta^{(1)}}(r_i(\beta^{(1)})) \right) \right]^{\frac{1}{p}} \times \right. \\
\left. \times \zeta^2 \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \| Z_i \| \cdot \| X_i \| \right)^{\frac{1}{2}} \right] \right\} \leq \zeta^2 \cdot U_e^{\frac{1}{p}} \cdot L^{\frac{1}{p}} \cdot [\tau^{(3)}]^{\frac{1}{p}} \cdot \left[ \frac{1}{n} \sum_{i=1}^{n} \| X_i \| \right]^{\frac{1}{p}} \times \\
\times \zeta^2 \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \| Z_i \| \cdot \| X_i \| \right)^{\frac{1}{2}} \right] \leq \zeta^2 \cdot U_e^{\frac{1}{p}} \cdot L^{\frac{1}{p}} \cdot [\tau^{(3)}]^{\frac{1}{p}} \cdot [2\gamma^{(4)}]^{\frac{1}{p}} \cdot [2\gamma^{(3)}]^{\frac{1}{p}} \leq \frac{\delta}{8}. \quad (52)
\]

Finally, utilizing Lemma A.6 find \( \tau^{(4)} \in (0, \min \{\delta/8, \tau^{(3)}\}) \) so that for any pair \( \| \beta^{(1)} \| \leq \zeta, \| \beta^{(2)} \| \leq \zeta, \| \beta^{(1)} - \beta^{(2)} \| \leq \tau^{(4)} \), we have
\[
-\left| [\beta^{(1)}] \mathbb{E} \left[ w \left( F_{\beta^{(1)}}(r_1(\beta^{(1)})) \right) Z_1 \left( e_i - X_i^{(1)} \right) \right] \\
-\left| [\beta^{(2)}] \mathbb{E} \left[ w \left( F_{\beta^{(2)}}(r_1(\beta^{(2)})) \right) Z_1 \left( e_i - X_i^{(2)} \right) \right] \right| \leq \frac{\delta}{8}. \quad (53)
\]

Now find a minimal system of open balls of type \( B(\beta, \tau^{(4)}) \) covering the \( p \)-dimensional ball with center at zero and radius \( \zeta \), i.e. \( B(\zeta) = \{ \beta \in R^p : \| \beta \| \leq \zeta \} \). Of course, due to the compactness of \( B(\zeta) \) the system has finite number of balls, say \( K(\zeta) \), and denote this system by \( \{B(\beta^{(j)}, \tau^{(4)})\}_{j=1}^{K(\zeta)} \).

Utilizing the law of large numbers find for any \( j \in \{1, 2, ..., K(\zeta)\} \) some \( n_j^{*} \in \mathcal{N} \) so that for all \( n > n_j^{*} \) the set
\[
B_{n_j}^{(4)} = \left\{ \omega \in \Omega : \frac{1}{n} \sum_{i=1}^{n} \left| w \left( F_{\beta^{(j)}}(r_i(\beta^{(j)})) \right) X_i X_i' \right| \leq \frac{\delta}{8\zeta^2} \right\}
\]
has probability at least \( 1 - \frac{\varepsilon}{8K(\zeta)} \). Finally put \( n^{(1)}_{\epsilon, \delta, \zeta} = \max \{ n_3, n_1^{*}, n_2^{*}, ..., n_{K(\zeta)}^{*} \} \) and \( B_n = B_{n_1}^{(1)} \cap B_{n_2}^{(2)} \cap B_{n_3}^{(3)} \cap B_{n_4}^{(4)} \). We have \( P(B_n) > 1 - \frac{\varepsilon}{2} \). Since for any \( n > n^{(1)}_{\epsilon, \delta, \zeta} \) and any \( \beta \in R^p, \| \beta \| \leq \zeta \) there is \( j \in \{1, 2, ..., K(\zeta)\} \) so that \( \| \beta - \beta^{(j)} \| \leq \tau^{(4)} \), taking into account (48), (50), (52), (53) and (54) we have for for any \( \omega \in B_n 
\]
\[
\sup_{\| \beta \| \leq \zeta} \frac{1}{n} \left| \beta^T \sum_{i=1}^{n} \left( w \left( F_{\beta}(r_i(\beta)) \right) Z_i X_i' - \mathbb{E} \left[ w \left( F_{\beta}(r_1(\beta)) \right) Z_1 X_1' \right] \right) \beta \right| < \frac{\delta}{2} \quad (55)
\]
Now, we shall consider the second term in (45). Along similar lines as in the first part of the proof, we can find \( n_{e,\delta} \in \mathcal{N} \) so that for any \( n > n_{e,\delta} \) there is \( C_n \subset \Omega \) so that \( P(C_n) > 1 - \varepsilon / 2 \) and for any \( \omega \in C_n \) we have

\[
\sup_{\|\beta\| \leq \varepsilon} \frac{1}{n} \left| \sum_{i=1}^{n} \left\{ w \left( F_{\beta}^{(n)}(|r_i(\beta)|) \right) e_i Z_i \beta - \beta^0 \right\} \right| < \frac{\delta}{2}.
\]

(56)

Taking into account (55) and (56), we conclude the proof. □

C4 The vector equation

\[
\beta^0 \left[ w \left( F_\beta(|r_1(\beta)|) \right) Z_1 \left( e_1 - X'_1 \beta \right) \right] = 0
\]

in the variable \( \beta \in R^p \) has unique solution \( \beta^0 = 0 \).

Lemma 3 Let Conditions C1, C2, C3 and C4 be fulfilled. Then any sequence \( \{\hat{\beta}(IW,n,w)\}^\infty_{n=1} \) of the solutions of normal equations \( \mathcal{N}E_{Z,n}(\hat{\beta}(IW,n,w)) = 0 \) is weakly consistent.

Proof: To prove the consistency of \( \{\hat{\beta}(IW,n,w)\}^\infty_{n=1} \), we have to show that for any \( \varepsilon > 0 \) and \( \delta > 0 \) there is \( n_{e,\delta} \in \mathcal{N} \) such that for all \( n > n_{e,\delta} \)

\[
P \left( \left\{ \omega \in \Omega : \left\| \hat{\beta}(IW,n,w) - \beta^0 \right\| < \delta \right\} \right) > 1 - \varepsilon.
\]

(57)

So fix \( \varepsilon > 0 \) and \( \delta > 0 \).

According to Lemma 1 there are \( \Delta_1 > 0 \) and \( \theta_1 > \delta_1 \) so that for \( \varepsilon > 0 \) there is \( n_{\Delta_1,\varepsilon_1} \in \mathcal{N} \) so that for any \( n > n_{\Delta_1,\varepsilon_1} \)

\[
P \left( \left\{ \omega \in \Omega : \inf_{\|\beta\| \geq \theta_1} \frac{1}{n} \beta^0 \mathcal{N}E_{Y,Z,n}(\beta) > \Delta_1 \right\} \right) > 1 - \frac{\varepsilon_1}{2}
\]

(denote the corresponding set by \( B_n \)). It means that for all \( n > n_{\Delta_1,\varepsilon_1} \) all solutions of the normal equations \( \mathcal{N}E_{Y,Z,n}(\beta) = 0 \) are inside the ball \( B(0,\theta_1) \) with probability at least \( 1 - \frac{\varepsilon_1}{2} \).

Now, utilizing Lemma 2 we may find for \( \varepsilon_1, \delta = \min \{ \Delta_1, \delta_1 \} \) and \( \theta_1 \) such \( n_{\varepsilon_1,\delta,\theta_1} \in \mathcal{N}, n_{\varepsilon_1,\delta,\theta_1} \geq n_{\Delta_1,\varepsilon_1} \) so that for any \( n > n_{\varepsilon_1,\delta,\theta_1} \) there is a set \( C_n \) (with \( P(C_n) > 1 - \frac{\delta_1}{2} \)) such that for any \( \omega \in C_n \)

\[
\sup_{\|\beta\| \leq \theta_1} \left| \frac{1}{n} \sum_{i=1}^{n} w \left( F_{\beta}^{(n)}(|r_i(\beta)|) \right) \beta^0 Z_i \left( e_i - X'_i \beta \right) \right| < \delta.
\]

But it means that

\[
\inf_{\|\beta\| = \theta_1} \left\{ -\beta^0 \mathcal{N}E \left[ w \left( F_\beta(|r_1(\beta)|) \right) Z_1 \left( e_1 - X'_1 \beta \right) \right] \right\} > \frac{\Delta_1}{2} > 0.
\]

(58)

Further consider the compact set \( C = \{ \beta \in R^p : \delta_1 \leq \|\beta\| \leq \theta_1 \} \) and find \( \tau_C = \inf_{\beta \in C} \left\{ -\beta^0 \mathcal{N}E \left[ w \left( F_\beta(|r_1(\beta)|) \right) Z_1 \left( e_1 - X'_1 \beta \right) \right] \right\} \).

(59)
Then there is a \( \{\beta_k\}_{k=1}^{\infty} \) such that
\[
\lim_{k \to \infty} \beta_k \mathbb{E} \left[ w(F_{\beta_k}(|r_1(\beta_k)|)) Z_1 \left( e_i - X_i' \beta_k \right) \right] = -\tau_C.
\]
On the other hand, due to compactness of \( C \) there is a \( \beta^* \) and a subsequence \( \{\beta_{k_j}\}_{j=1}^{\infty} \) such that
\[
\lim_{j \to \infty} \beta_{k_j} = \beta^*
\]
and due to the continuity of \( \beta \mathbb{E} \left[ w(F_{\beta}(|r_1(\beta)|)) Z_1 \left( e_i - X_i' \beta \right) \right] \) (see Lemma A.6) we have
\[
- [\beta^*]' \mathbb{E} \left[ w(F_{\beta^*}(|r_1(\beta^*)|)) Z_1 \left( e_i - X_i' \beta^* \right) \right] = \tau_C. \tag{61}
\]
Then the continuity of \( \beta \mathbb{E} \left[ w(F_{\beta}(|r_1(\beta)|)) Z_1 \left( e_i - X_i' \beta \right) \right] \) together with Condition C4 and (59) imply that \( \tau_C > 0 \) (otherwise there has to be a solution of (57) inside the compact \( C \)).

Now, utilizing Lemma 2 once again we may find for \( \varepsilon_1, \delta_1, \theta_1 \) and \( \tau_C \) \( n_{\varepsilon_1, \delta_1, \theta_1, \tau_C} \in \mathcal{N} \) \( n_{\varepsilon_1, \delta_1, \theta_1, \tau_C} \geq n_{\varepsilon_1, \delta, \theta} \) so that for any \( n > n_{\varepsilon_1, \delta_1, \theta_1, \tau_C} \) there is a set \( D_n \) (with \( P(D_n) > 1 - \frac{\varepsilon}{2} \)) such that for any \( \omega \in D_n \)
\[
\sup_{\|\beta\| \leq \delta_1} \left| \frac{1}{n} \sum_{i=1}^{n} w \left( F_{\beta}^{(n)}(|r_i(\beta)|) \right) \beta' Z_i \left( e_i - X_i' \beta \right) \right| < \frac{\tau_C}{2}. \tag{62}
\]
But (60) and (62) imply that for any \( n > n_{\varepsilon_1, \delta_1, \theta_1, \tau_C} \) and any \( \omega \in B_n \cap D_n \) we have
\[
\inf_{\|\beta\| > \delta_1} \frac{1}{n} \beta \mathbb{E} Y_{Z,n}(\beta) > \frac{\tau_C}{2}. \tag{63}
\]
Of course, \( P(B_n \cap D_n) > 1 - \varepsilon_1 \). But it means that all solutions of normal equations (57) are inside the ball of radius \( \delta_1 \) with probability at least \( 1 - \varepsilon_1 \), i.e. in other words, \( \hat{\beta}(\text{IWV},n,w) \) is weakly consistent.

\[\square\]

CONCLUDING REMARKS

We have added a small pebble (of mosaic) to equip the Least Weighted Squares by additional (or alternative, if you want) methods (similarly as the classical (Ordinary) Least Squares are equipped) to be able to build up the regression model in the situations when the basic assumptions are broken or when the “main” method is not suitable. We have discussed the situation when orthogonality condition is broken and hence the (Ordinary) Least Squares are biased. That is why we have proposed the robustified version of the classical instrumental variables. The other situation, e.g. discrete or limited response variable, will require also modifications of the Least Weighted Variables.

The lack of such tools and of course the lack of easy available and reliable implementations of robust methods hamper a wide (or at least wider than the present) employment of robust methods. We have at present at hand already a reliable algorithm for the Instrumental Weighted Variables which is based on the same idea as the algorithm which for the Least Trimmed Squares was tested in Višek (1996b, 2000a). The algorithm appeared to be reliable, we have referred
Lemma A.1 Under Conditions C1 the distribution function $F_{\beta}(r)$ is, uniformly with respect to $r \in R$, uniformly continuous in $\beta$, i.e. for any $\delta > 0$ there is $\varepsilon \in (0, 1)$ so that for any pair $\beta^{(1)}$ and $\beta^{(2)}$ such that $\|\beta^{(1)} - \beta^{(2)}\| < \varepsilon$ we have

$$\sup_{r \in R} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq \delta.$$ 

Proof: Let us recall that - see (18) (remember we have assumed that $\beta^{0} = 0$)

$$F_{\beta}(r) = P \left( |e_1 - X_1 \beta| < r \right) = \int I \{ s - x' \beta | < r \} dF_{X,e}(x, s).$$

Then

$$\sup_{r \in R} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq \sup_{r \in R} \int \left| I \{ s - x' \beta^{(1)} | < r \} - I \{ s - x' \beta^{(2)} | < r \} \right| dF_{X,e}(x, s)$$

$$\leq \sup_{r \in R} \int \left| I \{ s - x' \beta^{(1)} | < r \} - I \{ s - x' \beta^{(2)} | < r \} \right| dF_{e|X}(s | X = x) dF_X(x).$$

Since $f_{e|X}$ is bounded by $U_e$

$$\int \left| I \left. \left( |s - x' \beta^{(1)}| < r \right) \right\} - I \left. \left( |s - x' \beta^{(2)}| < r \right) \right\{ f_{e|X}(s | X = x) ds \right|$$

$$\leq \int_{\min \{ -r + x' \beta^{(1)}, -r + x' \beta^{(2)} \}}^{\max \{ -r + x' \beta^{(1)}, -r + x' \beta^{(2)} \}} f_{e|X}(s | X = x) ds + \int_{\min \{ r + x' \beta^{(1)}, r + x' \beta^{(2)} \}}^{\max \{ r + x' \beta^{(1)}, r + x' \beta^{(2)} \}} f_{e|X}(s | X = x) ds$$

$$\leq 2 \cdot U_e \cdot \| x' \beta^{(1)} - x' \beta^{(2)} \|.$$ 

Hence

$$\sup_{r \in R} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq 2 \cdot U_e \int \left| x' \beta^{(1)} - x' \beta^{(2)} \right| dF_X(x)$$

$$\leq 2 \cdot U_e \cdot \| X_1 \| \cdot \| \beta^{(1)} - \beta^{(2)} \|.$$
So, for any \( \delta > 0 \), putting \( \zeta = \frac{1}{2} \mathbf{d} \cdot U^{-1} \cdot \mathbf{E}^{-1} \| X_1 \| \), for any \( \beta^{(1)}, \beta^{(2)} \in \mathbb{R}^p, \| \beta^{(1)} - \beta^{(2)} \| \leq \zeta \) we have

\[
\sup_{r \in \mathbb{R}} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq \delta.
\]

**Lemma A.2** Let Conditions C1 hold and fix arbitrary \( \varepsilon > 0 \). Then there are \( K < \infty \) and \( n_\varepsilon \in \mathbb{N} \) so that for all \( n > n_\varepsilon \)

\[
P \left( \left\{ \omega \in \Omega : \sup_{v \in \mathbb{R}^n} \frac{1}{n} \left| F_{\beta}(v) - F_{\beta}(v) \right| < K \right\} \right) > 1 - \varepsilon.
\]

(A.64)

For the proof of lemma see Višek (2006a).

**Remark A.1** At the first glance it may seem strange that the previous lemma holds for the supremum over all \( \beta \)'s. Nevertheless, taking into account that the residual for \( \beta \in \mathbb{R}^p \) and for \( \zeta \beta, \zeta \in \mathbb{R}^+ \) are given as (remember that we have assumed \( \beta^0 = 0 \))

\[
r_i(\beta) = e_i - X_i^\prime \beta \quad \text{and} \quad r_i(\zeta \cdot \beta) = e_i - \zeta X_i^\prime \beta = \zeta \left( \frac{e_i}{\zeta} - X_i^\prime \beta \right),
\]

we learn that they differ nearly only in the scale (at least for “large” \( \zeta \)). On the other hand, for any \( \zeta \in \mathbb{R}^+ \) and any sequence of i.i.d. r.v.’s, say \( \{\eta_i\}_{i=1}^\infty \) distributed according to a d.f. \( H_\eta \) (say), the corresponding empirical d.f.’s \( H^{(n)}_\eta(z) \) converge for \( n \to \infty \) to the d.f. \( H_\eta(z) \) precisely in the rate as empirical d.f.’s \( H^{(n)}_{\zeta \eta}(z) \) of the sequence of i.i.d. r.v.’s \( \{\zeta \cdot \eta_i\}_{i=1}^\infty \) converge to the d.f. \( H_{\zeta \eta}(z) \), just due to the fact that

\[
H^{(n)}_\eta(z) = H_{\zeta \eta}(\zeta \cdot z) \quad \text{and} \quad H^{(n)}_{\zeta \eta}(z) = H^{(n)}_{\zeta \eta}(\zeta \cdot z)
\]

and hence

\[
\sup_{\zeta \in \mathbb{R}^+} \sup_{z \in \mathbb{R}} \left| H^{(n)}_{\zeta \eta}(z) - H_{\zeta \eta}(z) \right| = \sup_{z \in \mathbb{R}} \left| H^{(n)}_{\eta}(z) - H_{\eta}(z) \right|.
\]

Let us recall that we have denoted for any \( \beta \in \mathbb{R}^p \) by \( F_{\beta} Z X^\prime \beta(u) \) the distribution of the product \( \beta Z X^\prime \beta \) (see (24)), i.e.

\[
F_{\beta} Z X^\prime \beta(u) = P(\beta Z X^\prime \beta < u)
\]

and the corresponding empirical distribution by \( F^{(n)}_{\beta} Z X^\prime \beta(u) \) (see (25)), i.e.

\[
F^{(n)}_{\beta} Z X^\prime \beta(u) = \frac{1}{n} \sum_{j=1}^n I \left\{ \beta Z_j X^\prime_j \beta < u \right\}.
\]

**Lemma A.3** Let Condition C3 hold and fix arbitrary \( \varepsilon > 0 \). Then there are \( K < \infty \) and \( n_\varepsilon \in \mathbb{N} \) so that for all \( n > n_\varepsilon \)

\[
P \left( \left\{ \omega \in \Omega : \sup_{\beta \in \mathbb{R}^p} \sup_{u \in \mathbb{R}} \sqrt{n} \left| F^{(n)}_{\beta} Z X^\prime \beta(u) - F_{\beta} Z X^\prime \beta(u) \right| \leq K \right\} \right) > 1 - \varepsilon.
\]

**Proof** runs along the same lines as the proof of previous lemma.
Lemma A.4 Let Condition C3 hold and fix arbitrary $\varepsilon > 0$. Then there is $K_\varepsilon < \infty$ and $n_\varepsilon \in \mathbb{N}$ so that for all $n > n_\varepsilon$

\[
P \left( \left\{ \omega \in \Omega : \sup_{\beta^{(1)}, \beta^{(2)} \in R^p} \sqrt{n} \left| \frac{1}{n} \sum_{i=1}^{n} I \left\{ [\beta^{(1)}]' Z_i X_1' \beta^{(1)} < 0, [\beta^{(2)}]' Z_i X_1' \beta^{(2)} \geq 0 \right\} \right| < K_\varepsilon \right) > 1 - \varepsilon. \]

**Proof** runs again along the same lines as the proof of Lemma A.2.

Lemma A.5 Let Condition C3 hold and fix arbitrary $\varepsilon > 0$ and $\zeta > 0$. Then there is $\Delta > 0$ so that

\[
\sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\zeta, \Delta)} P \left( [\beta^{(1)}]' Z X' \beta^{(1)} < 0, [\beta^{(2)}]' Z X' \beta^{(2)} \geq 0 \right) < \varepsilon.
\]

**Proof.** First of all, let us find $K$ so that

\[
P \left( \{ \omega \in \Omega : \max \{ \|X\|, \|Z\| \} > K \} \right) < \frac{\varepsilon}{2} \tag{A.65}
\]

and denote the corresponding set by $B$. Further, for any fix $\beta \in R^p$ we have

\[
\lim_{a \to 0_+} P \left( \{ \omega \in \Omega : -a < \beta' Z X' \beta < 0 \} \right) = 0.
\]

and it means that for any $\beta \in R^p$ there is an $\alpha_{\beta} > 0$ so that

\[
P \left( \{ \omega \in \Omega : -\alpha_{\beta} < \beta' Z X' \beta < 0 \} \right) < \frac{\varepsilon}{2}.
\]

Consider

\[
a = \inf_{\|\beta\| \leq \zeta} \alpha_{\beta}.
\]

Of course, $a \geq 0$. We shall show that $a > 0$. Since $a$ is the infimum of a set, there is a sequence of the elements of the set, say $\{\alpha_{\beta(k)}\}_{k=1}^{\infty}$ such, that $a = \lim_{k \to \infty} \alpha_{\beta(k)}$. Since the ball with center at 0 and with radius $\zeta$ is compact, there is a $\beta^{(0)}$ such that $\beta^{(0)} = \lim_{j \to \infty} \beta^{(k_j)}$ where $\{k_j\}_{j=1}^{\infty}$ is an appropriate subsequence of $\{k\}_{k=1}^{\infty}$. But then

\[
a = a_{\beta^{(0)}} > 0.
\]

So, we can establish $a > 0$ so that

\[
\sup_{\|\beta\| \leq \zeta} P \left( \{ \omega \in \Omega : -a < \beta' Z X' \beta < 0 \} \right) < \frac{\varepsilon}{2} \tag{A.66}
\]

and put $\Delta = a \left( 2 \cdot K^2 \cdot \zeta + 1 \right)^{-1}$. Then $2 \cdot K^2 \cdot \zeta \cdot \Delta < a$. Now, consider an $\omega \in B^c$ (see (A.65) and the first line under it). Then

\[
\left| [\beta^{(1)}]' Z X' \beta^{(1)} - [\beta^{(2)}]' Z X' \beta^{(2)} \right|
\]

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Finally, we have

\[
\leq \left\| \left[ \beta^{(1)} \right]' Z \beta^{(1)} - \left[ \beta^{(1)} \right]' Z \beta^{(2)} \right\| + \left\| \left[ \beta^{(1)} \right]' Z \beta^{(2)} - \left[ \beta^{(2)} \right]' Z \beta^{(2)} \right\|
\]

\[
\leq \left\| \left[ \beta^{(1)} \right]' Z \left( \beta^{(1)} - \beta^{(2)} \right) \right\| + \left\| \left( \beta^{(1)} - \beta^{(2)} \right)' Z \beta^{(2)} \right\|
\]

\[
\leq \left\| \beta^{(1)} \right\| \cdot \left\| Z \right\| \cdot \left\| X \right\| \cdot \left\| \beta^{(1)} - \beta^{(2)} \right\| + \left\| \beta^{(1)} - \beta^{(2)} \right\| \cdot \left\| Z \right\| \cdot \left\| X \right\| \cdot \left\| \beta^{(2)} \right\|
\]

\[
\leq 2 \cdot K^2 \cdot \zeta \cdot \Delta < a.
\]

So, if \( \left[ \beta^{(2)} \right]' Z X' \beta^{(2)} \geq 0 \), we have \(-a < \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} \). In other words, for any \( \beta^{(2)} \in R^p, \| \beta^{(2)} \| \leq \zeta \), we have for any \( \beta^{(1)} \in R^p, \| \beta^{(1)} \| \leq \zeta, \| \beta^{(1)} - \beta^{(2)} \| < \Delta \) and \( \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0 \), necessarily \(-a < \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0 \). It means that

\[
\sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\zeta, \Delta)} P \left( \left\{ \omega \in \Omega : \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z X' \beta^{(2)} \geq 0 \right\} \cap B^c \right)
\]

\[
\leq \sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\zeta, \Delta)} P \left( \left\{ \omega \in \Omega : -a < \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z X' \beta^{(2)} \geq 0 \right\} \cap B^c \right)
\]

\[
\leq \sup_{\| \beta \| \leq \zeta} P \left( \left\{ \omega \in \Omega : -a < \beta' Z X' \beta < 0 \right\} \right) \leq \frac{\varepsilon}{2}.
\]

Finally, we have

\[
\sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\zeta, \Delta)} P \left( \left\{ \omega \in \Omega : \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z X' \beta^{(2)} \geq 0 \right\} \right)
\]

\[
\leq \sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\zeta, \Delta)} P \left( \left\{ \omega \in \Omega : \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z X' \beta^{(2)} \geq 0 \right\} \cap B \right)
\]

\[
+ \sup_{(\beta^{(1)}, \beta^{(2)}) \in T(\zeta, \Delta)} P \left( \left\{ \omega \in \Omega : \left[ \beta^{(1)} \right]' Z X' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z X' \beta^{(2)} \geq 0 \right\} \cap B^c \right)
\]

\[
\leq P(B) + \sup_{\| \beta \| \leq \zeta} P \left( \left\{ \omega \in \Omega : -a < \beta' Z X' \beta < 0 \right\} \right) < \varepsilon. \Box
\]

**Lemma A.6** Let Conditions C1, C2 and C3 hold. Then for any positive \( \zeta \)

\[
\beta' IE \left[ w(F(\| r_1(\beta) \|)) Z_1 \left( e_1 - X_1 \beta \right) \right]
\]

is uniformly continuous in \( \beta \) on \( B = \{ \beta \in R^p : \| \beta \| \leq \zeta \} \).

**Proof:** Fix a positive \( \zeta \) and \( \varepsilon \). Under C1 and C3

\[
\vartheta^{(1)} = \sup_{\| \beta \| \leq \zeta} IE \left\| Z_1 \left( e_1 - X_1 \beta \right) \right\| < \infty.
\]

Similarly

\[
\vartheta^{(2)} = \sup_{\| \beta \| \leq \zeta} IE \left\| X_1 \left( e_1 - X_1 \beta \right) \right\| < \infty.
\]
Finally, using Lemma A.1, let us find $\delta^{(1)} > 0$ so that for any $\|\beta^{(1)} - \beta^{(2)}\| < \delta^{(1)}$, $\|\beta^{(1)}\| \leq \zeta, \|\beta^{(2)}\| \leq \zeta$ we have

$$\sup_{r \in R} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq \frac{\varepsilon}{4 \cdot L \cdot \zeta \cdot \vartheta^{(1)}}. \tag{A.67}$$

Now, put

$$\delta_{\varepsilon, \zeta} = \min \left\{ 4 \cdot \vartheta^{(1)} \varepsilon, \zeta^{-1} \cdot L^{-1} \cdot U_e^{-1} \cdot 4 \cdot \vartheta^{(2)} \varepsilon, \delta^{(1)}, \varepsilon \left[ 4 \cdot \zeta \mathbb{E} \left\{ \|Z_1 \| \cdot \|X_1\| \right\} \right]^{-1} \right\}.$$

We are going to show that then for any pair of $\beta^{(1)}, \beta^{(2)}$ such that $\|\beta^{(1)}\| \leq \zeta, \|\beta^{(2)}\| \leq \zeta$ and $\|\beta^{(1)} - \beta^{(2)}\| < \delta_{\varepsilon, \zeta}$ we have

$$\left| \left[ \beta^{(1)} \right]' \mathbb{E} \left[ w \left( F_{\beta^{(1)}}(\{r_1(\beta^{(1)})\}) \right) Z_1 \left( e_1 - X_1' \beta^{(1)} \right) \right] \right| - \left| \left[ \beta^{(2)} \right]' \mathbb{E} \left[ w \left( F_{\beta^{(2)}}(\{r_1(\beta^{(2)})\}) \right) Z_1 \left( e_1 - X_1' \beta^{(2)} \right) \right] \right| \leq \varepsilon.$$

The absolute value of the last difference can be bounded by

$$\left| \left[ \beta^{(1)} \right]' - \left[ \beta^{(2)} \right]' \right| \mathbb{E} \left[ w \left( F_{\beta^{(1)}}(\{r_1(\beta^{(1)})\}) \right) Z_1 \left( e_1 - X_1' \beta^{(1)} \right) \right] \tag{A.68}$$

$$+ \left| \left[ \beta^{(2)} \right]' \mathbb{E} \left\{ w \left( F_{\beta^{(1)}}(\{r_1(\beta^{(1)})\}) \right) - w \left( F_{\beta^{(2)}}(\{r_1(\beta^{(2)})\}) \right) \right\} Z_1 \left( e_1 - X_1' \beta^{(1)} \right) \right| \tag{A.69}$$

$$+ \left| \left[ \beta^{(2)} \right]' \mathbb{E} \left\{ w \left( F_{\beta^{(1)}}(\{r_1(\beta^{(1)})\}) \right) - w \left( F_{\beta^{(2)}}(\{r_1(\beta^{(2)})\}) \right) \right\} Z_1 \left( e_1 - X_1' \beta^{(1)} \right) \right| \tag{A.70}$$

$$+ \left| \left[ \beta^{(2)} \right]' \mathbb{E} \left\{ w \left( F_{\beta^{(2)}}(\{r_1(\beta^{(2)})\}) \right) Z_1 X_1' \left( \beta^{(2)} - \beta^{(1)} \right) \right\} \right|. \tag{A.71}$$

Since

$$\sup_{\{\|\beta\| \leq \zeta\}} \mathbb{E} \left\{ \left| w \left( F_{\beta}(|r_1(\beta)|) \right) Z_1 \left( e_1 - X_1' \beta \right) \right| \right\} \leq \sup_{\{\|\beta\| \leq \zeta\}} \mathbb{E} \left\{ \left| w \left( F_{\beta}(|r_1(\beta)|) \right) \cdot \|Z_1 \| \cdot \|e_1 - X_1' \beta\| \right| \right\} \leq \vartheta^{(1)},$$

(A.68) is for $\|\beta^{(1)} - \beta^{(2)}\| < \delta_{\varepsilon, \zeta} \leq 4\vartheta^{(1)}^{-1} \varepsilon$ bounded by $\varepsilon/4$. Further (for $L$ see C2 and for $U_e$ see C1), since $|a| - |b| \leq |a - b|

$$\left| w \left( F_{\beta^{(1)}}(|r_1(\beta^{(1)})|) \right) - w \left( F_{\beta^{(2)}}(|r_1(\beta^{(2)})|) \right) \right| \leq L \cdot \left| F_{\beta^{(1)}}(|r_1(\beta^{(1)})|) - F_{\beta^{(1)}}(|r_1(\beta^{(2)})|) \right|$$

$$\leq L \cdot U_e \cdot \|r_1(\beta^{(1)}) - r_1(\beta^{(2)})\| \leq L \cdot U_e \cdot \|X_1' \beta^{(1)} - \beta^{(2)}\| \leq L \cdot U_e \cdot \|X_1\| \cdot \|\beta^{(1)} - \beta^{(2)}\|$$

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and hence for \(\|\beta^{(1)} - \beta^{(2)}\| < \delta_{e,\zeta} \leq \zeta^{-1} \cdot L^{-1} \cdot U^{-1}_e \cdot [4 \cdot \tilde{\vartheta}(2)]^{-1} \varepsilon\) we have found for (A.69)

\[
\left| \left[ \beta^{(2)} \right]^{\prime} \right|_{\mathcal{I} \mathcal{E}} \left\{ \left[ w \left( F_{\beta^{(1)}}(r_1(\beta^{(1)})) \right) - w \left( F_{\beta^{(2)}}(r_1(\beta^{(2)})) \right) \right] \right|_{Z_1} \left[ e_1 - X_1^\prime \beta^{(1)} \right] \right|
\leq \|\beta^{(2)}\| \cdot L \cdot U_e \cdot \|\beta^{(1)}(1) - \beta^{(2)}(1)\| \cdot \mathcal{I} \mathcal{E} \left\{ \|X_1\| \cdot \|Z_1 \cdot (e_1 - X_1' \beta^{(1)})\| \right\} \leq \varepsilon/4.
\]

So we have bounded by \(\varepsilon/4\) also (A.69). For (A.70) we have due to (A.67) for \(\|\beta^{(1)}(1) - \beta^{(2)}(1)\| < \delta_{e,\zeta} \leq \delta^{(1)}\) and \(\|\beta^{(1)} \| \leq \zeta, \|\beta^{(2)} \| \leq \zeta\)

\[
\left| \left[ \beta^{(2)} \right]^{\prime} \right|_{\mathcal{I} \mathcal{E}} \left\{ \left[ w \left( F_{\beta^{(1)}}(r_1(\beta^{(2)})) \right) - w \left( F_{\beta^{(2)}}(r_1(\beta^{(2)})) \right) \right] \right|_{Z_1} \left[ e_1 - X_1^\prime \beta^{(1)} \right] \right|
\leq \frac{\varepsilon}{4 \cdot \zeta \cdot \tilde{\vartheta}(1)} \|\beta^{(2)}\| \cdot \mathcal{I} \mathcal{E} \left\{ \|Z_1\| \cdot \|X_1\| \right\} \leq \frac{\varepsilon}{4}.
\]

Finally, for (A.71) we have for \(\|\beta^{(1)}(1) - \beta^{(2)}(1)\| < \delta_{e,\zeta} \leq \delta^{(1)}\) and \(\|\beta^{(1)} \| \leq \zeta, \|\beta^{(2)} \| \leq \zeta\)

\[
\left| \left[ \beta^{(2)} \right]^{\prime} \right|_{\mathcal{I} \mathcal{E}} \left\{ w \left( F_{\beta}(r_1(\beta^{(1)})) \right) \left[ Z_1 X'_1 \left( \beta^{(2)} - \beta^{(1)} \right) \right] \right|_{Z_1} \left[ e_1 - X_1^\prime \beta^{(1)} \right] \right|
\leq \|\beta^{(2)}\| \cdot \|\beta^{(2)}(2) - \beta^{(1)}(2)\| \cdot \mathcal{I} \mathcal{E} \left\{ \|Z_1\| \cdot \|X_1\| \right\} \leq \frac{\varepsilon}{4}
\]

which concludes the proof. \(\boxdot\)

**Lemma A.7** Let Condition C3 hold. Then for any positive \(\zeta\)

\[
\beta \mathcal{I} \mathcal{E} \left[ Z_1 X_1' \cdot I \left\{ \beta' Z_1 X_1' \beta < 0 \right\} \right] \beta
\]

is uniformly continuous in \(\beta\) on \(\mathcal{B} = \left\{ \beta \in \mathbb{R}^p : \|\beta\| \leq \zeta \right\}\).

**Proof** runs along the same lines as the proof of previous lemma.

Let us recall that for any \(\zeta \in \mathbb{R}^+\) we have denoted

\[
\tau_\zeta = - \inf_{\|\beta\| \leq \zeta} \beta \mathcal{I} \mathcal{E} \left[ Z_1 X_1' \cdot I \{ \beta' Z_1 X_1' \beta < 0 \} \right] \beta.
\]

**Lemma A.8** Let Condition C3 be fulfilled. Then for any \(\varepsilon > 0, \delta \in (0, 1)\) and \(\zeta \geq 1\) there is \(n_{e,\delta,\zeta} \in \mathcal{N}\) so that for any \(n > n_{e,\delta,\zeta}\) we have

\[
P \left( \left\{ \omega \in \Omega : \inf_{\|\beta\| \leq \zeta} 1/\sqrt{n} \sum_{i=1}^{n} \beta' Z_i X_1' \beta \cdot I \{ \beta' Z_i X_1' \beta < 0 \} > -\tau_\zeta - \delta \right\} \right) > 1 - \varepsilon.
\]
Proof: Fix $\varepsilon > 0$, $\delta \in (0, 1)$ and $\zeta \geq 1$. Due to the assumption of existence $IE \{ \| Z_1 \| \cdot \| X_1 \| \}^q < \infty$ for $q > 1$, we have $IE \| Z_1 \| \cdot \| X_1 \| < \infty$. Then there is $n_1 \in N$ so that for all $n > n_1$ we have

$$ P \left( \left\{ \omega \in \Omega : \left| \frac{1}{n} \sum_{i=1}^{n} \| Z_i \| \cdot \| X_i \| - IE \| Z_1 \| \cdot \| X_1 \| \right| < \frac{\delta}{3} \right\} \right) > 1 - \frac{\varepsilon}{8}. \quad (A.72) $$

Denote the corresponding set by $D_n^{(1)}$. Then for any $n > n_1$ and any $\omega \in D_n^{(1)}$ we have

$$ \frac{1}{n} \sum_{i=1}^{n} \| Z_i \| \cdot \| X_i \| \leq IE \| Z_1 \| \cdot \| X_1 \| + \frac{\delta}{3}. $$

Then denote $IE \| Z_1 \| \cdot \| X_1 \| + \frac{\delta}{3}$ by $\tau(0)$ and put $\Delta(0) = \min \left\{ 1, \delta \left( 18 \cdot \zeta \cdot \tau(0) \right)^{-1} \right\}$. Then $\Delta(0) > 0$.

Let us write

$$ \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0 \right\} \right| - \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \cdot I \left\{ \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} \right| $$

$$ = \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} \right| $$

$$ + \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \geq 0 \right\} $$

$$ - \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] \geq 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} $$

$$ - \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} \right| $$

$$ \leq \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(1) \right]^T Z_i X_i \left\{ \left( \beta(1) - \beta(2) \right) \right\} \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} \right| $$

$$ - \frac{1}{n} \sum_{i=1}^{n} \left\{ \left[ \beta(2) \right] - \left[ \beta(1) \right] \right\}^T Z_i X_i \left[ \beta(2) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} \right| $$

$$ + \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \geq 0 \right\} \right| $$

$$ + \frac{1}{n} \sum_{i=1}^{n} \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] \geq 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] < 0 \right\} \right| $$

$$ \leq \frac{1}{n} \sum_{i=1}^{n} \left\{ \left[ \beta(1) \right] \cdot \| Z_i \| \cdot \| X_i \| \left\| \left[ \beta(1) \right] - \left[ \beta(2) \right] \right\| + \left[ \beta(2) \right] \cdot \| Z_i \| \cdot \| X_i \| \left\| \left[ \beta(1) \right] - \left[ \beta(2) \right] \right\| \right\} \right| $$

$$ + \frac{1}{n} \sum_{i=1}^{n} \left\| \left[ \beta(1) \right] \right\|^2 \cdot \| Z_i \| \cdot \| X_i \| \cdot I \left\{ \left[ \beta(1) \right]^T Z_i X_i \left[ \beta(1) \right] < 0, \left[ \beta(2) \right]^T Z_i X_i \left[ \beta(2) \right] \geq 0 \right\} \right| $$

(A.73)
applying H"older's inequality for
for which max
for
Further, due to
Then according to Lemma A.5, for
Now, let us turn to (A.74). Let us denote for
we can find
\[
\Delta = \sup_{n} \left\{ \left\| \beta(1) \right\| \cdot \left\| Z_i \right\| \cdot \left\| X_i \right\| \left\| \beta(1) - \beta(2) \right\| \right\} \leq \frac{\delta}{9}. \tag{A.76}
\]
Now, let us turn to (A.74). Let us denote for \( q \) given in \( C3 \) \( \varrho^{(1)} = \mathbb{E} \frac{1}{q} \left\{ \left\| Z_i \right\| \cdot \left\| X_i \right\| \right\}^q \) and for \( q' = q(q - 1)^{-1} \varrho^{(2)} \left( \beta(1), \beta(2) \right) = \mathbb{P}^{\frac{1}{q'}} \left( \left[ \beta(1) \right]' Z_i X_i \beta(1) < 0, \left[ \beta(2) \right]' Z_i X_i \beta(2) \geq 0 \right) \). Further, applying H"older's inequality for \( q \) and \( q' \), we have (due to the fact that \( I^q \{ A \} = I \{ A \} \))
\[
\frac{1}{n} \sum_{i=1}^{n} \left\| \beta(1) \right\|^2 \cdot \left\| Z_i \right\| \cdot \left\| X_i \right\| \cdot I \left\{ \left[ \beta(1) \right]' Z_i X_i \beta(1) < 0, \left[ \beta(2) \right]' Z_i X_i \beta(2) \geq 0 \right\}
\leq \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( \left\| \beta(1) \right\|^2 \cdot \left\| Z_i \right\| \cdot \left\| X_i \right\| \right)^q \right\}^{\frac{1}{q}} \times
\frac{1}{n} \sum_{i=1}^{n} I \left\{ \left[ \beta(1) \right]' Z_i X_i \beta(1) < 0, \left[ \beta(2) \right]' Z_i X_i \beta(2) \geq 0 \right\} \right\} \tag{A.77}
\leq \left\| \beta(1) \right\|^2 \cdot \left\{ \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| Z_i \right\| \cdot \left\| X_i \right\| \right]^q \right\}^{\frac{1}{q}} - \varrho^{(1)} \right\} \times
\left\{ \left[ \frac{1}{n} \sum_{i=1}^{n} \left\{ \left[ \beta(1) \right]' Z_i X_i \beta(1) < 0, \left[ \beta(2) \right]' Z_i X_i \beta(2) \geq 0 \right\} \right]^q - \varrho^{(2)} \left( \beta(1), \beta(2) \right) \right\}
+ \varrho^{(2)} \left( \beta(1), \beta(2) \right) \right\} \tag{A.77}
\]
Let us recall that we have denoted for any \( \tau_1 > 0 \) and \( \tau_2 > 0 \) (see (49))
\[
T(\tau_1, \tau_2) = \left\{ \left\| \beta(1) \right\| \leq \tau_1, \left\| \beta(2) \right\| \leq \tau_1, \left\| \beta(1) - \beta(2) \right\| \leq \tau_2 \right\}
\]
Then according to Lemma A.5, for \( \delta \) and \( \zeta \) (we have fixed them at the beginning of the proof), we can find \( \Delta(1) > 0 \) so that
\[
\varrho^{(3)} = \sup_{(\beta(1), \beta(2)) \in T(\zeta, \Delta(1))} P^{\frac{1}{q'}} \left( \left[ \beta(1) \right]' Z_i X_i \beta(1) < 0, \left[ \beta(2) \right]' Z_i X_i \beta(2) \geq 0 \right)
\leq \sup_{(\beta(1), \beta(2)) \in T(\zeta, \Delta(1))} \varrho^{(2)} \left( \beta(1), \beta(2) \right) < \frac{\delta}{36} \cdot \min \left\{ \left[ \varrho^{(1)} \right]^{-1}, 1 \right\} \zeta^{-2}. \tag{A.78}
\]
Further, due to \( C3 \) we can find \( n_2 \in \mathcal{N} \) so that for any \( n > n_2 \) and for
\[
D^{(2)} = \left\{ \omega \in \Omega : \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ \left\| Z_i \right\| \cdot \left\| X_i \right\| \right]^q \right\}^{\frac{1}{q}} - \varrho^{(1)} \right\} < \varrho^{(1)} \right\}, \tag{A.79}
\]
we have
\[ P \left( D_n^{(2)} \right) > 1 - \frac{\varepsilon}{8}. \]
Moreover, employing Lemma A.4, we can find \( n_3 \in \mathcal{N} \) so that for any \( n > n_3 \) and for
\[
 D_n^{(3)} = \left\{ \omega \in \Omega : \sup_{(\beta^{(1)}, \beta^{(2)}) \in \mathcal{F}(\zeta, \Delta^{(1)})} \left| \frac{1}{n} \sum_{i=1}^{n} I \left\{ \left[ \beta^{(1)} \right]' Z_i X_i' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z_i X_i' \beta^{(2)} \geq 0 \right\} \right| - \vartheta^{(2)} \left( \beta^{(1)}, \beta^{(2)} \right) \right| < \frac{\delta}{36} \min \left\{ \left[ \vartheta^{(1)} \right]^{-1}, 1 \right\} \zeta^{-2} \right\} (A.80)
\]
we have \( P \left( D_n^{(3)} \right) > 1 - \frac{\varepsilon}{8} \). So, we have for \( n > \max \{ n_1, n_2, n_3 \} \) and any \( \omega \in D_n^{(2)} \cap D_n^{(3)} \)
\[
\sup_{(\beta^{(1)}, \beta^{(2)}) \in \mathcal{F}(\zeta, \Delta^{(1)})} \left| \frac{1}{n} \sum_{i=1}^{n} \left\| \beta^{(1)} \right\|^2 \cdot \| Z_i \| \cdot \| X_i \| \times \right.
\]
\[
\times I \left\{ \left[ \beta^{(1)} \right]' Z_i X_i' \beta^{(1)} < 0, \left[ \beta^{(2)} \right]' Z_i X_i' \beta^{(2)} \geq 0 \right\} \left| < \frac{\delta}{9} \right.
\]
The same is of course true about the expression in (A.75) with some \( n_4 \in \mathcal{N} \) and \( D^{(4)} \). Then, putting \( \Delta^* = \min \{ \Delta^{(0)}, \Delta^{(1)} \} \), we have \( P \left( D_n^{(0)} \cap D_n^{(1)} \cap D_n^{(2)} \cap D_n^{(4)} \right) > 1 - \frac{\varepsilon}{2} \) and for any \( n > \max \{ n_1, n_2, n_3, n_4 \} \) and \( \omega \in D_n^{(0)} \cap D_n^{(1)} \cap D_n^{(2)} \cap D_n^{(4)} \)
\[
\sup_{(\beta^{(1)}, \beta^{(2)}) \in \mathcal{F}(\zeta, \Delta^{(1)})} \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \beta^{(1)} \right]' Z_i X_i' \beta^{(1)} \cdot I \left\{ \left[ \beta^{(1)} \right]' Z_i X_i' \beta^{(1)} < 0 \right\} \right. - \frac{1}{n} \sum_{i=1}^{n} \left[ \beta^{(2)} \right]' Z_i X_i' \beta^{(2)} \cdot I \left\{ \left[ \beta^{(2)} \right]' Z_i X_i' \beta^{(2)} < 0 \right\} \left| < \frac{\delta}{3}. \right. (A.81)
\]
Finally, due to the fact that
\[
\beta \mathcal{E} \left[ Z_0 \cdot X_1 \cdot I \{ \beta' Z_0 X_0' \beta < 0 \} \right] \psi
\]
is on the ball \( \mathcal{B}_\zeta = \{ \beta \in R^p, \| \beta \| \leq \zeta \} \) uniformly continuous in \( \beta \) (see Lemma A.7), there is \( \Delta^{(2)} > 0 \) so that for any pair \( \beta^{(1)}, \beta^{(2)} \in \mathcal{R}^{p}, \| \beta^{(1)} \| \leq \zeta, \| \beta^{(2)} \| \leq \zeta \) and \( \| \beta^{(1)} - \beta^{(2)} \| < \Delta^{(2)} \) we have
\[
\left| \left[ \beta^{(1)} \right]' \mathcal{E} \left[ Z_0 \cdot X_1 \cdot I \{ [\beta^{(1)}]' Z_0 X_0' \beta^{(1)} < 0 \} \right] \beta^{(1)} \right| - \left| \left[ \beta^{(2)} \right]' \mathcal{E} \left[ Z_0 \cdot X_1 \cdot I \{ [\beta^{(2)}]' Z_0 X_0' \beta^{(2)} < 0 \} \right] \beta^{(2)} \right| < \frac{\delta}{3}.
\]
Put \( \Delta = \min \{ \Delta^{(0)}, \Delta^{(1)}, \Delta^{(2)} \} \) and \( n_0 = \max \{ n_1, n_2, n_3, n_4 \} \). Denote by \( \mathcal{B}_\zeta \) the ball with center at zero and with radius \( \zeta \). Now, the system \( \left\{ \{ \beta : \| \beta - \beta_0 \| < \Delta \} : \beta \in \mathcal{B}_\zeta \right\} \) covers by open balls the ball \( \mathcal{B}_\zeta \) and hence there is a finite system of balls, say
\[
\left\{ \mathcal{B}_{\beta^{(j)}, \Delta} = \left\{ \beta : \| \beta - \beta^{(j)} \| < \Delta \right\} \right\}_{j=1}^{K(\zeta)}
\]
which also covers the ball $B_{\zeta}$. For any $j = 1, 2, \ldots, K(\zeta)$ let us find $n_j \in \mathcal{N}$ so that for any $n > n_j$ we have

$$P \left( \left\{ \omega \in \Omega : \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \beta^{(i)} \right]' Z_i X_i' \beta^{(j)} : \mathbb{1} \left\{ \left[ \beta^{(j)} \right]' Z_i X_i' \beta^{(j)} < 0 \right\} \right| \right. \right. \\
\left. \left. - |\beta^{(j)}|' \mathbb{E} \left[ Z_1 \cdot X_1' \cdot \mathbb{1} \left\{ |\beta^{(j)}|' Z_1 X_1' \beta^{(j)} < 0 \right\} \right] \right| < \frac{\delta}{3} \right) > 1 - \frac{\varepsilon}{2 \cdot K(\zeta)}$$

and denote the corresponding sets by $B_{n_j}^{(j)}$. Finally put $n_{\varepsilon, \delta, \zeta} = \max_{0 \leq j \leq K(\zeta)} n_j$ and denote

$$B_n = \cap_{j=1}^{K(\zeta)} B^{(j)} \cap D_n^{(0)} \cap D_n^{(1)} \cap D_n^{(2)} \cap D_n^{(4)}.$$

Then $P(B_n) > 1 - \varepsilon$. Moreover, for any $\beta \in B_{\zeta}$ there is $j_0 \in \{1, 2, \ldots, K(\zeta)\}$ so that $\beta \in B_{\beta^{(j_0)}}$, and hence for any $n > n_{\varepsilon, \delta, \zeta}$ and any $\omega \in B_n$ we have

$$\left| \frac{1}{n} \sum_{i=1}^{n} \beta' Z_i X_i' \cdot \mathbb{1} \{ \beta' Z_i X_i' \beta < 0 \} \cdot \beta \right. \\
\left. \left. - \beta' \mathbb{E} \left[ Z_1 \cdot X_1' \cdot \mathbb{1} \{ \beta' Z_1 X_1' \beta < 0 \} \right] \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} \beta' Z_i X_i' \cdot \mathbb{1} \{ \beta' Z_i X_i' \beta < 0 \} \cdot \beta \right. \\
\left. \left. - \frac{1}{n} \sum_{i=1}^{n} \left[ \beta^{(j_0)} \right]' Z_i X_i' \cdot \mathbb{1} \left\{ \left[ \beta^{(j_0)} \right]' Z_i X_i' \beta^{(j_0)} < 0 \right\} \cdot \beta^{(j_0)} \right| \\
\left. \left. + \left| \frac{1}{n} \sum_{i=1}^{n} \left[ \beta^{(j_0)} \right]' Z_i X_i' \cdot \mathbb{1} \left\{ \left[ \beta^{(j_0)} \right]' Z_i X_i' \beta^{(j_0)} < 0 \right\} \cdot \beta^{(j_0)} \right| \\
\left. \left. \left. - \left[ \beta^{(j_0)} \right]' \mathbb{E} \left[ Z_1 \cdot X_1' \cdot \mathbb{1} \left\{ \left[ \beta^{(j_0)} \right]' Z_1 X_1' \beta^{(j_0)} < 0 \right\} \right] \right| \cdot \beta^{(j_0)} \right| \right. \\
\left. \left. + \left| \beta^{(j_0)} \right]' \mathbb{E} \left[ Z_1 \cdot X_1' \cdot \mathbb{1} \left\{ \left[ \beta^{(j_0)} \right]' Z_1 X_1' \beta^{(j_0)} < 0 \right\} \right] \right| \cdot \beta^{(j_0)} \right. \\
\left. \left. \left. - \beta' \mathbb{E} \left[ Z_1 \cdot X_1' \cdot \mathbb{1} \{ \beta' Z_1 X_1' \beta < 0 \} \right] \right| \cdot \beta \right| < \delta.$$  

It means that for any $\beta \in B_{\zeta}$, any $n > n_{\varepsilon, \delta, \zeta}$ and any $\omega \in B_n$ we have

$$-\tau_{\zeta} - \delta \leq -\beta' \mathbb{E} \left[ Z_1 \cdot X_1' \cdot \mathbb{1} \{ \beta' Z_1 X_1' \beta < 0 \} \right] \beta - \delta \leq \frac{1}{n} \sum_{i=1}^{n} \beta' Z_i X_i' \cdot \mathbb{1} \{ \beta' Z_i X_i' \beta < 0 \} \cdot \beta \leq 0,$$

i. e. for any $n > n_{\varepsilon, \delta, \zeta}$ and any $\omega \in B_n$ we have

$$-\tau_{\zeta} - \delta \leq \inf_{\|\beta\| \leq \zeta} \frac{1}{n} \sum_{i=1}^{n} \beta' Z_i X_i' \cdot \mathbb{1} \{ \beta' Z_i X_i' \beta < 0 \} \cdot \beta.$$  

\[\square\]
Lemma A.9 Let Conditions C1 hold. Then for any $\varepsilon > 0$ and $\delta \in (0,1)$ there is $\zeta > 0$ and $n_{\varepsilon,\delta} \in \mathbb{N}$ so that for all $n > n_{\varepsilon,\delta}$

$$P \left( \left\{ \omega \in \Omega : \sup_{r \in R} \sup_{\| \beta^{(1)} - \beta^{(2)} \| < \zeta} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| < \delta \right\} \right) > 1 - \varepsilon. \quad (A.82)$$

**Proof:** Fix $\varepsilon > 0$ and $\delta \in (0,1)$ and according to Lemma A.1 find $\zeta > 0$ so that for any pair $\| \beta^{(1)} - \beta^{(2)} \| < \zeta$ we have

$$\sup_{r \in R} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq \frac{\delta}{3}. \quad (A.83)$$

Employing Lemma A.2 find $K < \infty$ and $n_{\varepsilon,\delta} \in \mathbb{N}$ so that (notice that we can increase $n_{\varepsilon,\delta}$ so that the next inequality (A.84) holds and of course (A.85) holds, too)

$$\frac{K}{\sqrt{n_{\varepsilon,\delta}}} < \frac{\delta}{3} \quad (A.84)$$

and for any $n > n_{\varepsilon,\delta}$ and $B_n = \left\{ \omega \in \Omega : \sup_{r \in R^+} \sup_{\beta \in R^p} \sqrt{n} \left| F_{\beta}(r) - F_{\beta}(r) \right| < K \right\}$

$$P (B_n) > 1 - \varepsilon. \quad (A.85)$$

Then, due to (A.83), (A.84) and (A.85), for any $n > n_{\varepsilon,\delta}$ and $\omega \in B_n$ we have

$$\sup_{r \in R} \sup_{\| \beta^{(1)} - \beta^{(2)} \| < \zeta} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(2)}}(r) \right| \leq \sup_{r \in R} \sup_{\beta^{(1)} \in R^p} \left| F_{\beta^{(1)}}(r) - F_{\beta^{(1)}}(r) \right|

+ \sup_{r \in R} \sup_{\beta^{(2)} \in R^p} \left| F_{\beta^{(2)}}(r) - F_{\beta^{(2)}}(r) \right| + \delta. \quad \square$$

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