AUCO Czech Economic Review 2 (2008) 227–236
Acta Universitatis Carolinae Oeconomica

How Hierarchical Structures Impact on Competition

Alexsandr Galegov*, **Andrey Garnaev**†

Received 29 September 2008; Accepted 4 December 2008

Abstract Stackelberg models for hierarchical oligopolistic markets with a homogenous product were studied by researchers extensively. The goal of this paper is to extend the classical solution in closed form of the Stackelberg model for a general hierarchical structures composed by firms arranged into groups of different hierarchical levels.

Keywords Hierarchical structures, multi-level Stackelberg equilibrium, Nash-Cournot equilibrium

JEL classification C72

1. Introduction

Stackelberg models for hierarchical oligopolistic markets with a homogenous product were studied by researchers extensively. Mainly two types of the models were considered. One is a hierarchical Stackelberg game in which each firm chooses its output at a stage sequentially. This is formulated as a multi-stage game. The other is a standard two stage game in which multiple leaders choose outputs simultaneously and independently at first, and multiple followers decide outputs simultaneously and independently later, given the leader's total output.

Several researchers have tackled to investigate the existence and uniqueness of the hierarchical Stackelberg equilibrium. Under linear demand and cost functions Boyer and Moreax (1986), and Vives (1988) showed the existence of the unique Stackelberg equilibrium of the hierarchical Stackelberg game by directly computing its solution. Robson (1990) established the existence of the Stackelberg equilibrium under general conditions of demand and cost functions. For the Stackelberg models with many leaders and followers researchers tackled questions concerning the existence and uniqueness of the Stackelberg equilibrium. In duopoly case, Okamura, Futagarni, and Ohkawa (1998) proved that there exists a unique Stackelberg equilibrium under general demand and cost functions. The convexity of the follower's reaction function is essential for uniqueness of the Stackelberg equilibrium. In cases of a single leader and multiple followers, Sherall, Soyster and Murphy (1983) showed the existence and uniqueness of the Stackelberg equilibrium under general demand and cost functions,

* St Petersburg State University, Faculty of Applied Mathematics and Control Processes, Universitetskii prospekt 35, Peterhof, 198504 St Petersburg, Russia. Phone +7 (812) 4287159, E-mail: galegov@rambler.ru.
† St Petersburg State University, Faculty of Applied Mathematics and Control Processes, Universitetskii prospekt 35, Peterhof, 198504 St Petersburg, Russia. Phone +7 (812) 4284664, E-mail: agarnaev@rambler.ru.

and also that convexity of the reaction function of the follower's total output with respect to the leader's output is crucial for the uniqueness of the Stackelberg equilibrium.

This paper aims to obtain generalization of closed form solution for a general hierarchical structure of firms arranged by leaderships into groups which can be modelled by multi-stage game with perfect information in which sequentially level by level multiple players (firms) of each level choose outputs simultaneously and independently, and multiple followers (firms) of the next (lower) level of hierarchical structure decide outputs simultaneously and independently later, given the players's of the higher level their total output, and then after all these sequential setting the firms of the highest level assigns simultaneously their outputs.

It is worth to note that in the modern market a lot of hierarchical structures arise. For example, market of operation systems is split mainly between Windows (67.1%) and Linux (22.8%) meanwhile all the rest operations system takes together 10.1% of the market. So, in the operation systems markets presets three level hierarchical structure where the first and second levels are occupied by one OS (Windows and Linux) each meanwhile the third one is shared by all the rest OS. The world market of tobacco (except China) is split into four levels. The first level is shared by Altria (28%) and British American Tobacco (25%). Japan Tobacco holds the second one (16%). The third level is split among Imperial Tobacco (6%) and Altadis (3%). All the rest equal competitors share the fourth level.

When one deals with such hierarchical structures as a first approximation one could consider the produced product as a homogeneous one. Of course, products sold in both mentioned markets are differentiated. Sure, the importance of product differentiation is underscored by smokers brand loyalty in the market for tobacco products and by positive network externalities (stemming from the need of compatibility of an application software with an operating system) in the market for operating systems. But as a first and very rough approximation under very strong assumption about homogeneous nature of the products these markets could be described in frame of Cournot and Stackelberg models. When one starts studying Cournot model even for two firms presented on a market, the first two usual questions one has to answer are to find Cournot-Nash and Stackelberg equilibria and compare them (Gibbons, 1992). The goal of this paper is to extend the classical solution in closed form of the Stackelberg model for a general hierarchical structures composed by firms arranged into groups of different hierarchical levels acting sequentially level by level and simultaneously inside of a level.

2. Cournot model

In Cournot model of oligopoly there are M firms producing the same good. Each firm i, $i \in \{1,...M\}$ has a constant marginal cost of production c_i. Each firm simultaneously and independently sets the quantity q_i of the good its is going to produce. An inverse aggregate demand function of $p(q) = \max\{A-q,0\}$, where $q = q_1 + \ldots + q_M$, is given. The payoff to firm i, $i \in \{1,...,M\}$ is given as follows

$$\Pi_i(q_1,\ldots,q_M) = (A - \sum_{j=1}^{M} q_j)q_i - c_i q_i. \qquad (1)$$

Then the following result is a well known (see, for example Gibbons (1992) and we produce it here only for convenience of the readers.

Theorem 1. *In the Cournot model the equilibrium strategies are given as follows*

$$q_i = \frac{1}{M+1}\left(A + \sum_{j=1, j\neq i}^{M} c_j\right) - \frac{M}{M+1} c_i \text{ for } i \in \{1,\dots,M\} \tag{2}$$

with payoffs

$$\Pi_i^* = \frac{1}{(M+1)^2}\left(A + \overline{C} - (M+1)c_i\right)^2.$$

Aggregate output is given by

$$\sum_{i=1}^{M} q_i = \frac{1}{M+1}\left(MA - \overline{C}\right).$$

For the case with equal production cost $c_i = c$, $i \in \{1,\dots,M\}$ the equilibrium strategies are given as follows

$$q_i = \frac{1}{M+1}(A-c)$$

with payoff

$$\Pi_i^* = \frac{1}{(M+1)^2}(A-c)^2.$$

Aggregate output is given by

$$\sum_{i=1}^{M} q_i = \frac{M}{M+1}(A-c).$$

Of course, in Theorem 1 we deal only with conception of interior solution which exists under assumption that the parameters of the model are such that all the q_i given by (2) are positive, namely, if the following inequalities hold:

$$A + \sum_{j=1, j\neq i}^{M} c_j \geq Mc_i \text{ for } i \in \{1,\dots,M\}.$$

3. Stackelberg model

In this section we consider the strong linear hierarchical structure model Leader-Follower where the number of levels coincides with number of firms. This kind of Stackelberg model can be solved in the sense of the subgame perfect Nash equilibrium. Without loss of generality we can assume that the first level leader is firm 1, the second level leader is firm 2 and so on. Thus, firm M is the lowest firm in the hierarchical structure. The game is played in M stages. On the first stage firm M chooses its strategy to maximize Π_M assuming that all the other strategies are fixed. So, since

$\partial^2\Pi_M/\partial q_M^2 = -2$, the firm sets up its strategy as a root of the equation $\partial\Pi_M/\partial q_M = 0$ where

$$\Pi_M = (A - \sum_{j=1}^{M} q_j)q_M - c_M q_M. \tag{3}$$

Thus,

$$q_M = \frac{1}{2}\left(A - \sum_{j=1}^{M-1} q_j\right) - \frac{1}{2}c_M.$$

So, after substituting q_M into (1) for $i \in \{1,\dots,M-1\}$ we obtain that the payoff to firm i is given as follows:

$$\Pi_i = \frac{1}{2}\left(A - \sum_{j=1}^{M-1} q_j\right)q_i - \left(c_i - \frac{1}{2}c_M\right)q_i, \quad i \in \{1,\dots,M-1\}. \tag{4}$$

On the second stage, since $\partial^2\Pi_{M-1}/\partial q_{M-1}^2 = -1$, firm $M-1$ chooses its strategy as a root of the equation $\partial\Pi_{M-1}/\partial q_{M-1} = 0$. Thus,

$$q_{M-1} = \frac{1}{2}\left(A - \sum_{j=1}^{M-2} q_j\right) - \frac{1}{2}\left(2c_{M-1} - c_M\right).$$

After substituting q_{M-1} into (4) for $i \in \{1,\dots,M-2\}$ we obtain that the payoff to firm i is given as follows:

$$\Pi_i = \frac{1}{4}\left(A - \sum_{j=1}^{M-2} q_j\right)q_i - \left(c_i - \frac{1}{4}(2c_{M-1} + c_M)\right)q_i, \quad i \in \{1,\dots,M-2\}.$$

Thus,

$$q_{M-2} = \frac{1}{2}\left(A - \sum_{j=1}^{M-3} q_j\right) - \frac{1}{2}(4c_{M-2} - 2c_{M-1} - c_M).$$

and so on. Then, step by step firm $M-k$, $k \in \{1,\dots M-2\}$ recursively sets its strategy as a root of the equation $\partial\Pi_{M-k}/\partial q_{M-k} = 0$. Thus,

$$q_{M-k} = \frac{1}{2}\left(A - \sum_{j=1}^{M-k-1} q_j\right) - \frac{1}{2}(2^k c_{M-k} - \sum_{j=0}^{k-1} 2^j c_{M-j})$$

and payoffs on step $k+1$ are given as follows

$$\Pi_i = \frac{1}{2^{k+1}}\left(A - \sum_{j=1}^{M-k-1} q_j\right)q_i - \left(c_i - \frac{1}{2^{k+1}}\sum_{j=0}^{k} 2^j c_{M-j}\right)q_i, \quad i \in \{1,\dots,M-k-1\}.$$

Hence,

$$q_1 = \frac{1}{2}\left(A - 2^{M-1}c_1 + \sum_{j=0}^{M-2} 2^j c_{M-j}\right),$$

and hence, moving backward we have that on the level i the firm has the following optimal strategy:

$$q_i = \frac{1}{2^i}\left(A + \sum_{j=0}^{M-1} 2^j c_{M-j} - 2^M c_i\right), \quad i \in \{1,\dots,M\}$$

Thus, we proved the following result.

Theorem 2. *In the Stackelberg model the equilibrium strategies are given as follows*

$$q_i = \frac{1}{2^i}\left(A + \sum_{j=0}^{M-1} 2^j c_{M-j} - 2^M c_i\right), \quad i \in \{1,\dots,M\} \tag{5}$$

with payoffs

$$\Pi_i = \frac{1}{2^{M+i}}\left(A - 2^M c_i + \sum_{j=0}^{M-1} 2^j c_{M-j}\right)^2 \text{ for } i \in \{1,\dots,M\}.$$

Aggregate output is given by

$$\sum_{i=1}^{M} q_i = \left(1 - \frac{1}{2^M}\right) A - \frac{1}{2^M} \sum_{j=0}^{M-1} 2^j c_{M-j}.$$

Of course, in Theorem 2 we deal only with conception of interior solution which exists under assumption that the parameters of the model are such that all the q_i given by (5) are positive, namely, if the following inequalities hold:

$$A + \sum_{j=0}^{M-1} 2^j c_{M-j} \geq 2^M c_i \text{ for } i \in \{1,\dots,M\}.$$

It is clear that a firm increases own production if production cost of its rival is increasing and it reduces own production if its own production cost arises. Namely, q_i is increasing in each c_j where $j \neq i$ and q_i is decreasing in each c_i.

For a particular case with equal production cost $c_i = c$, $i \in \{1,\dots,M\}$ from Theorem 2 we have the following result.

Theorem 3. *For the case with equal production cost $c_i = c$, $i \in \{1,\dots,M\}$ the equilibrium strategies are given as follows*

$$q_i = \frac{1}{2^i}(A - c), \quad i \in \{1,\dots,M\}$$

with payoffs

$$\Pi_i = \frac{1}{2^{M+i}}(A - c)^2, \quad i \in \{1,\dots,M\}.$$

Aggregate output is given by

$$\sum_{i=1}^{M} q_i = (A - c)\left(1 - \frac{1}{2^M}\right).$$

If the number of firms with equal production cost c increases then the aggregate output tends to $A - c$.

4. General case

As a general case we consider a hierarchical structure composed by M firms arranged into N groups of firms $\Gamma_1, \dots, \Gamma_N$ of different hierarchical level such that the groups Γ_i composes ith level and consists of M_i firms. Let $\overline{\Gamma}_i = \cup_{j=1}^{i} \Gamma_i$, $i \in \{1, \dots N\}$ and $\overline{M}_i = \sum_{j=1}^{i} M_i$ is the number of firms which are on levels from 1 to i. Then $\overline{M}_N = M$. Also, let $\overline{M}_0 = 0$. Thus, the payoff of firm i in the new notations is given as follows:

$$\Pi_i = \left(A - \sum_{j \in \overline{\Gamma}_N} q_j \right) q_i - c_i q_i, \quad i \in \overline{\Gamma}_N. \tag{6}$$

Let start stage by stage, level by level from the level N (first stage) which is the lowest one and it is composed by firms of group Γ_N. Since $\partial^2 \Pi_i / \partial q_i^2 = -2$ these firms set up their strategies as a solution of the system of equations $\partial \Pi_i / \partial q_i = 0$, $i \in \Gamma_N$ or

$$-2q_i + A - \sum_{j \in \overline{\Gamma}_N \setminus \{i\}} q_j - c_i = 0, \quad i \in \Gamma_N.$$

Thus,

$$q_i = \frac{1}{M_N + 1} \left(A - \sum_{j \in \overline{\Gamma}_{N-1}} q_j \right) - \left(c_i - \frac{\overline{C}_N}{M_N + 1} \right), \quad i \in \Gamma_N, \tag{7}$$

where

$$\overline{C}_k = \sum_{j \in \Gamma_k} c_j, \quad k \in \{1, \dots, N\}.$$

So, after substituting (7) into (6) for $i \in \overline{\Gamma}_{N-1}$ we obtain that the payoff to firm i is given as follows:

$$\Pi_i = \frac{1}{M_N + 1} \left(A - \sum_{j \in \overline{\Gamma}_{N-1}} q_j \right) q_i - \left(c_i - \frac{1}{M_N + 1} \overline{C}_N \right) q_i, \quad i \in \overline{\Gamma}_{N-1}. \tag{8}$$

Pass on to the next level (the second stage), namely, to the level $N-1$ composed by firms from group Γ_{N-1}. Since $\partial^2 \Pi_i / \partial q_i^2 = -2/(M_N + 1)$ these firms set up their strategies as a solution of the system of equations $\partial \Pi_i / \partial q_i = 0$, $i \in \Gamma_{N-1}$ where Π_i are given by (8). Then

$$-2q_i + A - \sum_{j \in \overline{\Gamma}_{N-1} \setminus \{i\}} q_j - (M_N + 1) c_i + \overline{C}_N = 0, \quad i \in \Gamma_{N-1}.$$

Thus,

$$q_i = \frac{1}{M_{N-1} + 1} \left(A - \sum_{j \in \overline{\Gamma}_{N-2}} q_j \right) - \frac{1}{M_{N-1} + 1} \left(P_{N-1}^{N} c_i - P_N^N \overline{C}_{N-1} - \overline{C}_N \right) \text{ for } i \in \Gamma_{N-1}, \tag{9}$$

where

$$P_s^r = \prod_{k=r}^{s} (M_k + 1) \text{ for } 1 \le s \le r \le N$$

and

$$P_s^r = 1 \text{ for } s > r.$$

Thus, substituting q_i from (9) into (8) we obtain the payoffs of the firms from group $\overline{\Gamma}_{N-2}$ given as follows:

$$\Pi_i = \frac{1}{P_{N-1}^N}\left(A - \sum_{j\in\overline{\Gamma}_{N-2}} q_j\right)q_i - \left(c_i - \frac{1}{P_{N-1}^N}\left(P_N^N\overline{C}_{N-1} + \overline{C}_N\right)\right)q_i, \quad i \in \overline{\Gamma}_{N-2}.$$

Now, let us pass on to the level $M-k$ composed by firms of group Γ_{N-k}. Since $\partial^2\Pi_i/\partial q_i^2 = -2/P_{N-1}^N$ these firms set up their strategies as a solution of the system of equations $\partial\Pi_i/\partial q_i = 0$, $i \in \Gamma_{N-k}$. Thus,

$$q_i = \frac{1}{M_{N-k}+1}\left(A - \sum_{j\in\overline{\Gamma}_{N-k-1}} q_j\right) - \frac{1}{M_{N-k}+1}\left(P_{N-k}^N c_i - \sum_{j=0}^{k} P_{N-j+1}^N \overline{C}_{N-j}\right) \text{ for } i \in \Gamma_{N-k}$$

and

$$\Pi_i = \frac{1}{P_{N-k}^N}\left(A - \sum_{j\in\overline{\Gamma}_{N-k-1}} q_j\right)q_i - \left(c_i - \frac{1}{P_{N-k}^N}\sum_{j=0}^{k} P_{N-j+1}^N\overline{C}_{N-j}\right)q_i \text{ for } i \in \Gamma_{N-k-1}.$$

So, for the highest (the first) level firms we have the following optimal strategy

$$q_i = \frac{1}{M_1+1}\left(A + \sum_{j=0}^{N-1} P_{N-j+1}^N \overline{C}_{N-j}\right) - \frac{P_1^N}{M_1+1}c_i$$

and the joint goods produced by firm of the first level is

$$\sum_{i\in\Gamma_1} q_i = \frac{M_1}{M_1+1}\left(A + \sum_{j=1}^{N} P_{j+1}^N \overline{C}_j\right) - P_2^N\overline{C}_1.$$

Moving backward we have that on the level k, $k \in \{1,\ldots,N\}$ the firms have the following optimal strategies

$$q_i = \frac{1}{P_1^k}\left(A + \sum_{j=0}^{N-1} P_{N-j+1}^N \overline{C}_{N-j}\right) - \frac{P_k^N}{M_k+1}c_i, \quad i \in \Gamma_k$$

and the joint goods produced by firm of k-th level is

$$\sum_{i\in\Gamma_k} q_i = \frac{M_k}{P_1^k}\left(A + \sum_{j=1}^{N} P_{j+1}^N \overline{C}_j\right) - P_{k+1}^N\overline{C}_k.$$

Thus, we proved the following result.

Theorem 4. *In the Stackelberg model with N groups of firms the equilibrium strategies are given as follows:*

$$q_i = \frac{1}{P_1^k}\left(A + \sum_{j=1}^{N} P_{j+1}^N \overline{C}_j - P_1^N c_i\right), \quad i \in \Gamma_k \tag{10}$$

with payoffs

$$\Pi_i = \frac{\left(A + \sum_{j=1}^{N} P_{j+1}^N \overline{C}_j - P_1^N c_i\right)^2}{P_1^k P_1^N}, \quad i \in \Gamma_k.$$

Aggregate output is given by

$$\sum_{i=1}^{M} q_i = \left(1 - \frac{1}{P_1^N}\right)A - \frac{1}{P_1^N}\sum_{i=1}^{N} P_{i+1}^N \overline{C}_i.$$

Of course, in Theorem 4 we deal only with conception of interior solution which exists under assumption that the parameters of the model are such that all the q_i given by (10) are positive, namely, if the following inequalities hold:

$$A + \sum_{j=1}^{N} P_{j+1}^N \overline{C}_j \geq P_1^N c_i \text{ for } i \in \Gamma_k,\ k \in \{1,\ldots,N\}$$

For a particular case with equal marginal cost $c_i = c$, $i \in \{1,\ldots,M\}$ from Theorem 2 we have the following result.

Theorem 5. *For the case with equal production cost $c_i = c$, $i \in \{1,\ldots,M\}$ in the Stackelberg model with N group of firms the equilibrium strategies are given as follows:*

$$q_i = \frac{1}{P_1^k}(A - c), \quad i \in \Gamma_k$$

with payoffs

$$\Pi_i = \frac{(A-c)^2}{P_1^k P_1^N}, \quad i \in \Gamma_k.$$

Aggregate output is given by

$$\sum_{i=1}^{M} q_i = \left(1 - \frac{1}{P_1^N}\right)(A - c).$$

5. Conclusions

In this work we considered the hierarchical structures in general form in the frame of Cournot-Stackelberg model and constructed the optimal strategies in closed form. We can apply this closed form solutions to estimate which impact they produce on the market. As a criteria of such impact we can consider the market price p or the quantity of the goods ($Q = A - p$) produced by all the firms. Then Q is given as follows:

(i) In the case of the absent of the hierarchial structures among M firms:

$$Q_{\{1,2,\ldots,M\}} = \frac{1}{M+1}\left(MA-\overline{C}\right)$$

(ii) In the case of the linear hierarchial structure where each of the M firms occupies per one level:

$$Q_{\{1\},\{2\},\ldots,\{M\}} = \left(1-\frac{1}{2^M}\right)A-\frac{1}{2^M}\sum_{j=1}^{M}2^{M-j}c_j$$

(iii) In the general case where the hierarchical structure is composed by M firms arranged into N groups:

$$Q_{\{1,\ldots,M_1\},\{M_1+1,\ldots,M_2\},\ldots,\{M_{N-1}+1,\ldots,M_N\}} = \left(1-\frac{1}{P_1^N}\right)A-\frac{1}{P_1^N}\sum_{i=1}^{N}P_{i+1}^N\overline{C}_i$$

For example if there are three firms ($M=3$) with marginal cost of production c_i, $i=1,2,3$ equals 1, 2 and 3, and $A=10$. Then, $Q_{\{1,2,3\}}=6$, $Q_{\{1,2\},\{3\}}=6.833$, $Q_{\{1\},\{2,3\}}=7$ and $Q_{\{1\},\{2\},\{3\}}=7.375$ and the market prices are $p_{\{1,2,3\}}=4$, $p_{\{1,2\},\{3\}}=3.167$, $p_{\{1\},\{2,3\}}=3$ and $p_{\{1\},\{2\},\{3\}}=2.625$.

Acknowledgment Authors acknowledge the support of joint Russian Foundation for Basic Research and State Natural Science Foundation (China) Grant No. 06-01-39005.

References

Gibbons, R. (1992). *Game Theory for Applied Economists*. Princeton, Princeton University Press.

British American Tobacco - Annual Report and Accounts. http://www.bat.com/group/sites/uk__3mnfen.nsf/vwPagesWebLive/DO52AK34/$FILE/medMD7D9KKN.pdf?openelement. Retrieved in April 2008.

Japan Tabacco International Annual reports. http://www.altria.com/AnnualReport/ar2006/2006ar_05_0200.aspx. Retrieved in April 2008.

Anderson, S. P., Engers, M. (1992). Stackelberg versus Cournot Oligopoly Equilibrium. *International Journal of Industrial Organization*, 10, 127–135.

Boyer, M., Moreaux, M. (1986). Perfect Competition as the Limit of a Hierarchical Market Game. *Economics Letters*, 22, 115–118.

Okamura, M., Futagami, K. and Ohkawa, T. (1998). On the Stackelberg Equilibrium Existence. Uniqueness. and Stability. *Indian Economic Journal*, 45, 87–100.

Robson, A. (1990). Stackelberg and Marshall. *American Economic Review*, 80, 69–82.

Sherali, H. D., Soyster, A. L. and Murphy, F. H. (1983). Stackelberg-Nash-Cournot Equilibria: Characterizations and Computations. *Operations Research*, 31, 253–276.

Vives, X. (1988). Sequential Entry. Industry Structure and Welfare. *European Economic Review*, 32, 1671–1687.