Non-robustness of the Cash-in-advance Equilibrium in the Trading-Post Model

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Abstract

The main justification for cash-in-advance (CIA) equilibria when there are multiple assets is a Shapley-Shubik trading-post model where the agents coordinate on a particular medium of exchange. Of course, there are other equilibria. We introduce a refinement and show that the CIA equilibrium does not satisfy our refinement while there exist equilibria that do.

1 Introduction

The main rationale for cash-in-advance models when there are multiple assets seems to be a Shapley-Shubik trading-post model and an equilibrium in that model with no activity at the posts at which assets other than money can be traded for goods—or, at least, those goods labelled cash goods. The notion that money has value only because it is the generally agreed upon convention is old and is described by, for instance, Tobin [4]. Howitt [3] suggests that this can also justify cash-in-advance equilibria in the trading post model. In a static Cournot-type quantity game for the trading-post model, inactivity of any given post is a Nash equilibrium because a single agent has no incentive to place quantity orders on an inactive post. Such potential inactivity is the rationale for assuming that people cannot trade assets other than money directly for some goods.

However, the fact that inactivity of any given post is a Nash equilibrium also implies that no trade at all is a Nash equilibrium. In part to eliminate such equilibria, Dubey and Shubik [1], in a static quantity-game version of the trading-post model, introduce a refinement which eliminates no trade: they assume that there are small exogenous offers (given from the outside) at each of their posts and say that an equilibrium satisfies the refinement if it is a limit as those exogenous offers approach zero. Here, we apply a version of that refinement to a trading-post model with one perishable good per date, money, and a bond which dominates money in rate

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of return. Since analyzing the Cournot quantity game is difficult in an infinite-horizon setting, we follow Hayashi and Matsui [2] and assume that the agents in the model take prices as given.

We show that there is no equilibrium satisfying the refinement with activity at the post at which money trades for the good (the money post). In other words, the cash-in-advance equilibrium does not satisfy the refinement. To show that there can be active trade equilibria that satisfy the refinement, we produce such an equilibrium for an example.

2 The Model

Time is discrete and there is one perishable, non produced good at each date. There are N infinitely lived agents who maximize discounted utility. Agent i has a discount factor $\beta^i \in (0,1)$ and a period utility (of consumption) function, $u^i : \mathbb{R}_+ \to \mathbb{R}$, which is strictly increasing, strictly concave, and continuously differentiable, and also satisfies $u'(0) = \infty$. Agent i has an endowment of the date t good denoted ω_t^i and starts date 1, the initial date, with some money, denoted m_0^i. The only other asset is a one-period nominal discount bond offered by the government.[1] At the beginning of each date, each agent can buy bonds with money (only) at an exogenously determined price $q < 1$. (At the end of the period, the bonds "mature," which will be taken to mean that they automatically turn into money at a one-for-one rate.) The quantity of bonds bought by agent i at date t measured at maturity value in term of money is denoted b_t^i. After bond purchases, there is trade at Shapley-Shubik trading posts. There are two such posts: at the money post (denoted M) money trades for date t good; at the bond post (denoted B) bonds trade for the same good. Interest on bonds is financed by a proportional tax on end-of-period money holdings, a tax which is equivalent to financing interest by money creation. Our version of the Dubey-Shubik refinement is that there is an exogenous positive amount of the good, denoted ε, offered at each of the two trading posts at each date.

To define an equilibrium, we first define what an agent can afford. In the definition, we denote the sequence $(x_t)_1^\infty$ by x.

Definition 1. *Agent i can **afford** the (non-negative) tuple $(c^i, m^i, b^i, s_B^i, s_M^i, d_B^i, d_M^i)$ at prices (p_B, p_M) and tax rates π if*

$$\begin{aligned}
c_t^i &\leqslant \omega_t^i - (s_{tB}^i + s_{tM}^i) + \frac{d_{tB}^i}{p_{tB}} + \frac{d_{tM}^i}{p_{tM}}, \\
qb_t^i &\leqslant m_{t-1}, \\
s_{tB}^i + s_{tM}^i &\leqslant \omega_t^i, \\
d_{tM}^i &\leqslant m_{t-1}^i - qb_t^i, \\
d_{tB}^i &\leqslant b_t^i \quad \textit{and} \\
m_t^i &\leqslant (1-\pi_t)[(m_{t-1}^i - qb_t^i - d_{tM}^i) + (b_t^i - d_{tB}^i) + s_{tM}^i p_{tM} + s_{tB}^i p_{tB}],
\end{aligned}$$

[1] As this suggests, there is no private borrowing and lending. One rationale is that people are anonymous.

where c_t^i is consumption, s_{tB}^i (s_{tM}^i) is the offer of goods at the bond (money) post, d_{tB}^i (d_{tM}^i) is the offer of bonds (money) at the bond (money) post, and p_{tB} (p_{tM}) is the price at the bond (money) post.

Letting $S_{tj} \equiv \sum_i s_{tj}^i$, $D_{tj} \equiv \sum_i d_{tj}^i$, $M_t \equiv \sum_i m_t^i$, and $B_t \equiv \sum_i b_t^i$, an equilibrium can be defined as follows.

Definition 2. *A tuple $(c^i, m^i, b^i, s_B^i, s_N^i, d_B^i, d_M^i)$ for each i, (p_B, p_M), and π is an* ***equilibrium*** *if (i) c^i maximises i's utility from among all consumption sequences affordable at (p_B, p_M) and π, (ii) $p_{tB} = D_{tB}/(S_{tB} + \varepsilon)$ and $p_{tM} = D_{tM}/(S_{tM} + \varepsilon)$ and (iii) $M_{t-1} = (1 - \pi_t)[(M_{t-1} - qB_t) + B_t - \varepsilon(p_{tM} + p_{tB})]$.*

Condition (ii) is market clearing at each post and condition (iii) requires that the tax rate be such as to hold constant the quantity of money. We are interested in $\varepsilon = 0$ equilibria that are the limits of equilibria as $\varepsilon \to 0$.

Definition 3. *An $\varepsilon = 0$ equilibrium satisfies the refinement if it is a (point-wise) limit of ε_n equilibria for any sequence $(\varepsilon_n) \downarrow 0$.*

3 Results

The first result is that a cash-in-advance equilibrium does not satisfy the refinement.

Proposition 1. *If $q < 1$ and $\epsilon > 0$, then there is no equilibrium with $S_{tM} > 0$ (with some of the good offered at the money post).*

Proof. The proof is a simple arbitrage argument, one which is consistent with the short-sales constraints of the trading-post model and one which makes no appeal to the special assumptions of the model. Suppose to the contrary that there is an equilibrium with $S_{tM} > 0$. If so, then $p_{tM} \geqslant p_{tB}$ (if not, then it is better to sell the good at the bond post) and $p_{tM} > 0$ (if not, then it is better to consume rather than offer any of the good). The latter implies that $D_{tM} > 0$. But the former implies that any person whose offer of money contributes to making $D_{tM} > 0$ would do better by using that money to buy bonds and offering the bonds at the bond post. Hence, there is no such equilibrium. □

We now show that the above proposition is not vacuous by producing an example which has an active trade equilibrium that satisfies the refinement. To do that, we need an example in which there is a motivation for trade. A simple example is the alternating endowment economy with identical preferences.

Example: $N = 2$, $\beta^i = \beta$, $u^i = u$, $\omega^1 = (y_H, y_L, y_H, ...)$, $\omega^2 = (y_L, y_H, y_L, ...)$, where $y_H > y_L$, $u'(y_H)/[\beta u'(y_L)] < 1$, and $m_0^1 = 0$, $m_0^2 = 1$.

Proposition 2. *The example has an $\varepsilon = 0$ equilibrium that satisfies the refinement and that has $S_{tB} > 0$.*

Proof. The proof is constructive. And, as might be expected, a constant equilibrium is constructed. We start by constructing consumption. Let $(c_H(\varepsilon), c_L(\varepsilon))$ be the solution for (c_H, c_L) to

$$c_H + c_L = y_H + y_L + 2\varepsilon \tag{1}$$

and

$$\frac{u'(c_H)}{\beta u'(c_L)} = \frac{c_L - y_L}{y_H - c_H}. \tag{2}$$

(Notice that (1) is the resource constraint at equality and that (2) is the condition that a high endowment person makes an optimal two-date saving decision from a linear budget set.) It is obvious that $(c_H(\varepsilon), c_L(\varepsilon))$ exists, is unique, is continuous in ε and that $c_H(\varepsilon) < y_H$ and $c_L(\varepsilon) > y_L$. Moreover, $\lim_{\varepsilon \to 0}(c_H(\varepsilon), c_L(\varepsilon)) = (c_H(0), c_L(0))$, where $\frac{u'(c_H(0))}{\beta u'(c_L(0))} = 1$. Consequently, for sufficiently small ε, $\frac{u'(c_H(\varepsilon))}{\beta u'(c_L(\varepsilon))} < \frac{1}{\beta}$ and $c_H(\varepsilon) > c_L(\varepsilon)$.

A solution is depicted in Figure 1 below, for large ε. The thick line represents what the agent can afford. Note that there is no credit in the economy. The pair $(c_H(\varepsilon), c_L(\varepsilon))$ is determined by two conditions: an indifference curve is tangent at $(c_H(\varepsilon), c_L(\varepsilon))$ to a line through $(c_H(\varepsilon), c_L(\varepsilon))$ and (y_H, y_L) and $(c_H(\varepsilon), c_L(\varepsilon))$ satisfies the resource constraint with equality.

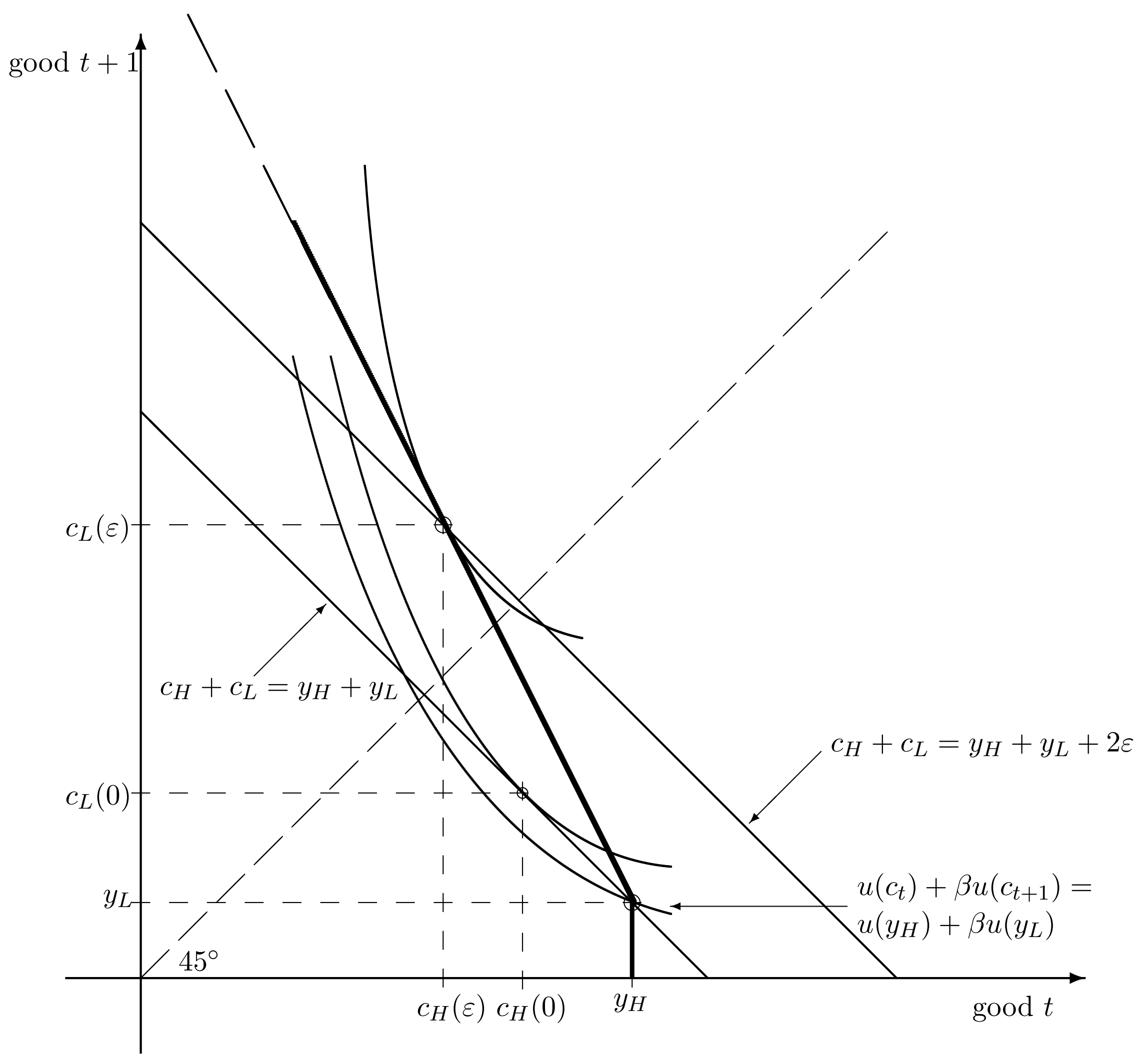

Figure 1: Solution for $c_H(\varepsilon), c_L(\varepsilon)$

We now construct prices, portfolios and offers. For any $\varepsilon > 0$, both D_{tB} and D_{tM} must be positive. That implies that $p_{tM} = qp_{tB}$. Using the fact that no goods are offered at the money post and the conjecture that the low endowment person does not save, we propose

$$p_{tM} = \frac{1 - qB_t}{\varepsilon} = q\frac{B_t}{y_H - c_H(\varepsilon) + \varepsilon} = qp_{tB}. \tag{3}$$

Notice that the second equality is a linear equation in one unknown, B_t, which is to be interpreted as the bond purchases at each date of the person with endowment y_L. This gives us candidates for all the equilibrium objects except the tax rate: it is obtained directly from equilibrium condition (iii) and is constant.

By construction, the candidate satisfies equilibrium conditions (ii) and (iii). It remains to verify that it satisfies individual optimization. The main step in doing that involves showing that the gross real rate of return implied by (3) and the tax rate is equal to the righthand side of (2). Goods can be sold for an after-tax price of $(1 - \pi)p_{tB}$ and are purchased for qp_{tB}. Therefore, the gross real rate of return is $(1 - \pi)/q$. From equilibrium condition (iii),

$$\begin{aligned} \frac{q}{1-\pi} &= qB_t - q\varepsilon p_{tB} \\ &= \frac{qB_t}{y_H - c_H(\varepsilon) + \varepsilon}(y_H - c_H(\varepsilon)) \\ &= \frac{y_H - c_H(\varepsilon)}{c_L(\varepsilon) - y_L}, \end{aligned} \tag{4}$$

where the first equality follows from the first equality in (3), the second from the last equality in (3), and the third from solving the second equality in (3) for qB and using (1). This implies that the proposed prices and tax rates imply that people choose consumption facing a constant gross real rate of return given by the right-hand side of (2), which for small enough ε is less than $1/\beta$. It follows that the low endowment person wants to save 0 and that the high endowment person wants to save $y_H - c_H(\varepsilon)$, exactly as proposed. $\square$

Notice that the real aspects of the equilibrium constructed for the example happen to be the same as for a cash-in-advance equilibrium for $q \in (\beta, 1]$. That is, if one simply shuts down the bond post and if the discount on bonds is not too large, then no one buys bonds, the tax rate is zero, and people face a gross real rate of return of unity. Needless to say, that is not a justification for shutting down the bond post. It should also be noted that this equilibrium construction is essentially the same as the equilibrium of the so-called "turnpike model" in Townsend [5].

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