Modeling Small Change: A Review Article*

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Abstract

In *The Big Problem of Small Change*, Sargent and Velde apply a cash-in-advance model to the history of coinage and to contemporary thought about coinage. They assert that their model accounts for puzzling observations involving the depreciation and disappearance of small coins. I question its usefulness for that purpose and for other issues pertaining to coinage. My main concern is that their model does not depict the problems usually associated with full-bodied coinage systems—problems that stem from the technological difficulties of having a full-bodied coinage system in which money is portable, divisible, durable, and recognizable.

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1 Introduction

In *The Big Problem of Small Change* (Princeton University Press, Princeton and Oxford, 2002, xxi+405 pages), Thomas Sargent and François Velde present an ambitious treatise on the history of coinage and contemporary thought about coinage in Western Europe during six centuries, roughly from 1200 until 1800. As the title suggests, they focus on the seeming inability of governments to provide adequate small-denomination money. The main conclusions of the authors are even more narrowly focused and have to do with the usefulness of a two-money, nested, cash-in-advance model—nested in that one of the monies,

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labeled small change, can be used to purchase anything, while the other, labeled large denomination money, can be used only to purchase some goods.

The authors make two main assertions about this model. The first is that

> [o]ur model explains why the medieval money supply mechanism was prone to shortages of small coins. It shows why debasements or reinforcements of parts of the denomination structure would temporarily cure those shortages. The model also explains why shortages and depreciations of small coins happened *simultaneously* (page 8, their italics).

The second is that if governments had had access to the model, then they would have more quickly put into place a monetary system with an adequate supply of small-denomination money. In particular, they would sooner have adoped a system in which small-denomination coins are token coins (rather than full-bodied coins) that are exchangable on demand for large-denomination, full-bodied coins. Such a monetary system, which is called the *standard formula,* was not adopted until the 19th century. In their words,

> The theme of this book is that implementing the standard formula required several things: a good technology to make coins, an understanding of the relevant monetary theory, and a government wanting to convert token coins into full-bodied coins on demand (page 272).

And the relevant monetary theory is their cash-in-advance model: "Our historical account refers to our model so often that sometimes we may seem to be writing a history of how past monetary experts learned our model, piece by piece through a long process of trial and error (page 14)."

I was not persuaded about either assertion. As regards the first, the authors do not describe the equilibria to which they repeatedly allude. They appeal to *back-solving* without describing the processes for exogenous variables that produce those equilibria. Moreover, even if the model has such equilibria, the authors do not convincingly argue that it provides a good interpretation of the episodes on which they focus. Many of those episodes involve disruption of trade, and nothing in the model resembles disruption of trade. Finally, and this alone would lead me to question both assertions, their model does not depict any of the problems usually associated with full-bodied coinage systems.

Economists have long agreed that ideal monies must be portable, divisible, durable, and recognizable. The history of coinage is mainly about the technological difficulties of achieving a full-bodied coinage system that comes close to having those attributes. Although the authors describe such difficulties and the governmental commitment needed to implement the standard formula, neither the difficulties nor the inability to commit appear in their model.

To illustrate that there are models that include at least some of those difficulties and that the difficulties matter, I will briefly describe some models of indivisible money. Given the focus on small change, indivisibility is an obvious difficulty to try to study. After all, how could there be a small-change problem if the objects used to make large-denomination money were naturally divisible?

2 The Sargent-Velde Model

Sargent and Velde describe their model twice—informally near the beginning of their book and then, more completely, in the last part of their book which is entitled *A Formal Theory*. Here is their model, although in slightly different language.

There are two perishable consumption goods per discrete date, good 1 and good 2. The representative household, a shopper-producer pair, maximizes discounted utility with a period utility function that depends on its consumption of the two goods. There are two assets which the household can carry over from one date to the next: money 1 and money 2. The shopper is constrained by the quantities of the two monies the household brings into the period, the prices faced, and two cash-in-advance constraints: money 1, labeled small change, is required for purchases of good 1; either money can be used to purchase good 2. The producer has an endowment of resource that can be turned one-for-one into either of the two consumption goods, but not into goods that the producer's household consumes. The producer is a price-taker whose activities result in end-of-period holdings of the two monies which then get added to any unspent money of the shopper to give the household's money for the next period.

If this were a closed economy, then that would be the end of the story. Instead, it is a small open economy with proportional (iceberg) transport costs. The producer faces f.o.b. world prices of the resource, one price in terms of each money, and the transport costs. The world market can be used to turn one money into the other only by bearing double transport costs. For example, in order to turn money 2 into money 1 using the world market, the producer imports resource on the world market paying with money 2 and then exports the resource for money 1. Clearly, the world market puts bounds on domestic prices.

Rather than transport costs, the authors set out their model in terms of the minting of coins from *silver* and the melting of coins into *silver*, where *silver* is an object which differs from the coins and which is available at an exogenous world price in terms of the resource. A government mint stands ready to turn silver into each money at announced prices and for per unit fees which are interpreted to be a sum of taxes and real costs of producing coins. The taxes are rebated to people lump-sum. To be consistent with the taxes, my transport-cost description has to be supplemented with import and export taxes.

2.1 Equilibria with shortages

As noted above, the crucial and paradoxical observations for the authors are small coins losing value relative to large coins and, at the same time, reported decreases in the stock of small coins—and, sometimes, even the complete disappearance of such coins. For the authors, an explanation is an equilibrium of their model in which small coins are losing value (relative to large coins). However, as is well-known, such paths do not generally exist in their model. The authors' solution, as they emphasize, is to use back-solving: they posit the above kinds of sequences for endogenous variables and then claim that there are supporting sequences of the exogenous variables—the endowments and world prices. However, they never describe the supporting sequences. To see why existence of such sequences which are also plausible is not obvious, let's first consider two extreme cases of their model.

One extreme case is *infinite transport costs*. In that case, the economy is closed and has exogenous and constant stocks of two monies. The authors present the conditions that characterize equilibria for this case on page 351 in a section entitled *Equilibria with neither melting nor minting*. If the good-1 cash-in-advance constraint is binding (the second column of the display on page 351), which is necessary for an equilibrium with a falling relative value of money 1, and if the resource endowment is constant, then the equilibrium conditions which are equalities reduce to the following implicit autonomous first-order difference equation,

$$\frac{x_t}{y - x_t} r(x_t, y - x_t) = \frac{x_{t-1}}{y - x_{t-1}}. \tag{1}$$

Here, y denotes the endowment, $x_t \in [0, y]$ denotes date t consumption of good 1 (which is equal to the value of money 1 in a binding equilibrium), $y - x_t$ is consumption of good 2, and the function r is the ratio of the marginal utility of good 1 to that of good 2.

Under their assumptions, which include an infinite value of $r(0, y)$, the following can be said about solutions to (1). There is one steady state with $x_t = x^*$, where x^* is the unique solution to $r(x^*, y - x^*) = 1$. There may or may not be a second steady state at $x_t = 0$.[1] In this second steady state, money 1 has no value.[2] In addition, and as is familiar from closely related models, there is always at least a one-dimensional continuum of equilibria: for any $x_0 \in (0, x^*)$, there exists at least one sequence converging to 0 which is an equilibrium.[3]

Another extreme case is *zero transport costs*. With zero transport costs, prices are exogenous. The initial condition is arbitrary: there can be surplus amounts of both monies, deficient amounts of both, or a deficient amount of money 1 only—all relative to the amounts needed to support the consumption pair $(x^*, y - x^*)$. If the amount of money 1 is deficient, then the unique equilibrium has some of the resource for a time devoted to acquiring money 1.

Now consider the general case, *finite transport costs*. For a constant environment, the implied lower bound on the value of money 1 eliminates equilibria with a falling value of money 1. That is why the authors appeal to back-solving. However, if the sequences of the exogenous variables needed to get such equilibria are implausible, then so are the equilibria.

2.2 Steady-state neutrality

Occasionally, the authors allude to an equilibrium of their model with unchanging relative values of the two monies. Against the background of a constant environment, such an equilibrium has a constant allocation with consumption given by $(x^*, y - x^*)$. This constant allocation does not depend on—is neutral with respect to—the silver content of small and large coins and the world price of silver. Moreover, any such equilibrium of their model is an optimum subject only to the resource constraint. These are questionable predictions for the kinds of economies the authors claim to be analyzing—economies in which it is technologically impossible to provide full-bodied coinage that is portable, divisible, durable, and recognizable.

3 The History

Most of the book, Parts II-IV, is devoted to a discussion of episodes from the histories of Venice, Florence, England, France, Flanders, and Castile and, to a lesser extent, a discussion of contemporary monetary theory. A prelude to the history is given at the beginning of chapter 2:

[1] It exists if $\lim_{x \to 0} xr(x, y - x) = 0$.

[2] Both steady states satisfy the inequality requirements for equilibrium. The second steady state is somewhat awkward in their notation. It is not awkward if the resource is chosen as the numeraire.

[3] Such paths also satisfy the equilibrium conditions that are inequalities.

> Our theory allows us to interpret a pervasive and persistent depreciation of small denomination coins, exhibited for example in the data shown in figure 2.1. The six panels record estimates of the (inverse of the) silver content of small denomination coins from 1200 to 1800 for six countries [those listed above]. Increases in exchange rates of large for small coins and recurrent shortages of small coins accompanied these persistent depreciations in the silver content of small coins. Our theory identifies the source of the upward drifts in figure 2.1 and explains how they related to the concurrent shortages (page 15).

Despite repeatedly referring to the model, the authors' exposition of historical episodes is not a straightforward comparison between observations and the implications of their model. Even after my first reading of most of the book, I thought that the authors' claims about the model's implications rested on computer files of solutions which were too complicated and messy to present. Then, quite late in my reading I came across the following statement: "Though the formal model in part V assumes rational expectations, our *narrative* assumes that policy makers frequently used incorrect or incomplete models (page 327, my italics)." To what does narrative refer? It is certainly not referring to data like that displayed in the authors' figure 2.1. I think it refers to their claims about what their theory implies. If so, then I must conclude that we have not been given a complete description of their model.

Even if the model has plausible equilibria with a falling value of money 1, I question whether it can explain most of the shortages in the historical record on which the authors focus. The shortages are reports by contemporaries concerning the difficulty of carrying out trade in the face of sudden disappearances of some kinds of coins (see, for example, the description of shortages in 16th century France on pages 202, 203.)

The kinds of shortages that can occur in the model are not consistent with such reports. A shortage in the model is a positive multiplier on the money-1 cash-in-advance constraint together with a decline in the total value of money 1. How would people in the model feel about an equilibrium in which this happens? Would they associate such a multiplier with a shortage? A positive multiplier on a cash-in-advance constraint is the usual equilibrium in such models. In the model, real balances in the form of money 1 are like refrigerator space that must accompany consumption of good 1 in a fixed proportion. The positive multiplier implies that the pecuniary rate-of-return on refrigerator space is lower than that on other assets. That does not translate into a shortage of refrigerator space.

The model's quantity implications are no better. The people in the model are freely choosing quantities while taking prices as given. Therefore, nothing in the model looks like a shortage or a disruption of trade.

So what is going on during these shortage episodes in the historical record? The data in the authors' figure 2.1 depict recoinages. Recoinages were sometimes accompanied by attempts to coerce the public into turning in their coins for reminting. And there were repeated attempts to prohibit the exporting of coins. My guess is that these sorts of actions and speculation about them gave rise to disruption of trade. Such policies do not appear in the authors' model. More generally, while the authors discuss the interplay between the public's actions and the minting and associated policies of governments, the interplay is not formally described and analyzed.

Although the model is mainly appealed to for explanations of episodes of changing relative values of small and large coins, it is occasionally invoked to deal with episodes of unchanging relative values. Here, it is used for 18th century Britain:

> In addition, the price of silver coins expressed in terms of gold stabilized after 1717. Persistent depreciation of small denomination coins, the telltale sign of shortages of small change found in figure 2.1 on page 16, is absent in eighteenth-century Britain. Thus, besides transforming the unit of account, eighteenth-century Britain effectively solved the problem of small change. Counterfeiters and other suppliers of tokens somehow produced enough small change to have allowed the exchange rate between large (now gold) coins and smaller silver coins to stabilize after 1720 (page 292).

Sargent and Velde cannot *see* a shortage of small coins without seeing such coins depreciating relative to large coins. That accounts for their conclusion that non government suppliers "somehow"—must have (?)—filled the well-documented denomination gap left by the government. As I now demonstrate, models which deal directly with the imperfect physical attributes of coins have different implications.

4 A Model of Indivisible Money

The imperfect physical attribute that I analyze is indivisibility. I do this primarily because it is hard to imagine that a small change problem could exist if the stuff out of which large coins were made were divisible in the sense in which goods in standard models are divisible. Indeed, Sargent and Velde seem to motivate their analysis by calling attention to the substantial size of coins. They begin Part II, *Ideas and Technologies*, with a description of the value of the smallest silver coins in the middle ages:

> Typically, the daily wage represented 1 to 3 silver coins, and thus daily necessities required smaller coins. Another way to appreciate its role is to estimate what the smallest silver coin could purchase. In Florence, in the second half of the 14th century, the smallest silver coin was the grosso (5s.): it could purchase 5 liters of the cheapest wine, 1 kg of mutton, 20 eggs, 1 kg of olive oil; or pay a month's rent for an unmarried manual laborer (page 48).

And one need not go back to the middle ages for reports of the absence of small change: such shortages appeared during the colonial period in America (see Hanson, 1979) and in Australia in the early 1800's (see Butlin, 1968).

I will discuss the implications of indivisible money using versions of a somewhat well-known model—a matching model due to Trejos and Wright (1995) and Shi (1995), which, itself, is built on ideas in Kiyotaki and Wright (1989). According to this model, trade takes place in meetings between pairs of people, rather than in centralized markets. This feature is attractive for thinking about small-change problems. After all, it is natural to motivate the need for small change by describing a buyer and a seller and the problem they face if neither has small change.

4.1 The environment

Time is discrete. There are $N > 2$ perishable types of goods at each date and a $[0, 1]$ continuum of each of N specialization types of people. For $n \in \{1, 2, ..., N\}$, a type n person consumes only good n and is able to produce only good $n + 1$ (modulo N). Each person maximizes expected discounted utility and the period utility function is $u(x) - y$, where x is consumption of the relevant good and y is production of the relevant good. The function u is strictly concave and increasing, and satisfies $u(0) = 0$. In addition, there exists $y' > 0$ that satisfies $u(y') = y'$.

At each date, each person meets one person at random. There is an exogenous stock of money, which is uniform, exists in indivisible units, and is perfectly durable. Each person sees the money holdings of his or her trading partner and individual histories are private except as reflected in current money holdings. In addition, people cannot commit to future actions.

4.2 Non-neutrality and steady-state shortages

Zhu (2002) studies a version of this model with no restrictions on individual holdings of money except a general and sufficiently large upper bound (a technical assumption) and with deterministic take-it-or-leave-it offers by potential consumers in single-coincidence meetings. (Under the assumptions made, there are no double-coincidence meetings and no-coincidence meetings are not relevant.) He shows that if $u'(0)$ is sufficiently large, then there is a *nice* steady state which is symmetric across specialization types—nice in that it has an increasing and strictly concave value function defined on money holdings and has a distribution over money holdings that has full support.

Would the people in Zhu's steady state be experiencing a shortage of small coins? Yes. Very generally, they would be making different and from their point of view more desirable trades if they had smaller units of money.[4]

As Zhu shows, his existence result also implies non-neutrality of the following sort. The model has three exogenous nominal quantities, $(\Delta, \overline{m}, B) \equiv z$, where Δ is the size of the smallest unit, $\overline{m}$ is average holding per specialization type (in effect, the total amount of money), and B is the upper bound on individual holdings. Let k be an integer that exceeds unity and consider the following three alternative vectors of exogenous nominal quantities: $z' = kz$ and $z'' = (\Delta, k\overline{m}, kB)$ and $z''' = (\Delta, k\overline{m}, B)$. Notice that z''' relative to z corresponds to a large exogenous increase in the stock of money, one way of viewing the 16th century specie discoveries in the Americas.

As might be expected, neutrality holds for the comparison between z and for z', neutrality in the sense of real allocations. As regards z and z'', z'' has strictly more steady states in terms of allocations. Any steady state for z has an equivalent (in terms of real allocations) steady state for z'': the equivalent steady state is produced by replacing the trade of m units of money under z by a trade of km units under z''. There are more steady states for z'' because no nice steady state for z'' has the same real allocation as any steady state for z. Finally, as regards z and z''', if the bound B is large enough, then the nice steady states for $z''$and for z''' should be similar.

Although the model allows for the possibility of observing the same real allocation under z and z'', such neutrality requires that a change in the money supply be accompanied by a proportional change in the size of the smallest unit of money *in use*. Such a change in units seems far-fetched for an economy which starts with an indivisibility-of-money problem. Therefore, we should focus on

[4] All of the results carry over to a version in which consumers are permitted to randomize their offers. Then, more divisible money would allow for less individual uncertainty.

the nice steady states for z and z''. Aside from being different, a good bet is that a nice steady state for z'' has more trade and output than a nice steady state for z'. Thus, indivisibility alone may be able to explain the purported expansionary real effects in Europe of 16th century specie discoveries in the Americas. Moreover, because a comparison between z and z'' is equivalent to a comparison between z and $(\Delta/k, \overline{m}, B)$, the same claim about expansionary effects applies to having a more divisible money.

4.3 The disappearance of coins

Motivated in part by the supposed shortages of money in colonial America and in Australia in the early 18th century and in part by governmental attempts to prohibit the export of money, Wallace and Zhou (1997) study a version of the above model with permanent differences between people. They let the period utility function be $u(x) - \alpha y$, where α is a parameter that differs across two groups of people. (High α people are less productive.) For a version of the model in which individual money holdings are restricted to be either zero or one unit, they show that there are equilibria in which money ends up only in the hands of the low α people. Moreover, these can be such that an effective prohibition on trade between high α people and low α people, one way to interpret a prohibition on the export of money, would benefit high α people. It is obvious that this result does not depend on the unit upper bound on individual money holdings. It does depend on scarcity of money—on a small enough ratio of the per capita stock to the smallest unit.

4.4 Coin production

The above matching model can also be amended to include two kinds of money and an endogenous determination of the mix between the two. Let money exist in two potential forms, large coins and small coins, and let there be technologies that allow each person to produce one from the other: in particular, let j small coins be "meltable" into 1 large coin and let 1 large coin be "mintable" into j small coins.[5] (In the Sargent-Velde terminology, this is the special case of no melting or minting costs.) For the economy as a whole, the per capita amount of money (the raw material for making coins) is given in the following sense. Let $\pi(x_1, x_2)$ denote the fraction of each specialization type who hold x_1 small coins and x_2 large coins. The given amount of money, denoted M and measured

[5] These technologies define *small* and *large*.

in units of large coins per specialization type, constrains π to satisfy,

$$\sum_{x_1=0}^{\infty} \sum_{x_2=0}^{\infty} \pi(x_1, x_2)(\frac{x_1}{j} + x_2) = M. \tag{2}$$

In addition, some timing must be specified. I would be inclined to separate melting and minting from meetings. Moreover, if the model is to have a steady state with both large and small coins, then there must be some advantage to melting—to producing large coins from small coins. Following Kiyotaki and Wright (1989), it could be assumed that there is a utility cost to carrying coins from one period to the next that is proportional to the amount carried and that the cost of carrying j small coins exceeds that of one large coin.

This structure bears some resemblance to the Sargent-Velde model. Coins are the only assets and individuals have access to coin production. The model can be used to assess how the set of steady states depends on the size of small coins, j. Or, for a given j, it can be used to assess how the set of steady states depends on the total amount of money, M. In either case, we do not expect neutrality to hold. As noted above, the Sargent-Velde model implies neutrality for such experiments.

This two-coin model gives rise to a "demand" for small coins. Moreover, as already suggested, that demand would not disappear even if small coins have a lower rate-of-return than large coins. However, it is not obvious that the model can be made to generate a declining value of small coins relative to large coins—the kind of paradoxical observation on which Sargent and Velde focus. In that regard, this model is on a par with their model. In other respects, it seems better as a model of small-change problems.

5 Concluding Remarks

The most surprising thing about the Sargent-Velde book is the degree to which it ignores existing and long-standing ideas about the difficulties that government faced in providing money with desirable attributes: portability, divisibility, durability, and recognizability. In particular, it seemed to be infeasible to produce from the same underlying material coins that varied sufficiently in size and were also portable, durable, and recognizable. Bimetallism, making small coins out of less valuable objects and large coins out of more valuable objects, was one response; another was not to make small coins (see Glassman and Redish 1988). Bimetallism, as is well-known, gives rise to the following problem: as the relative values of the underlying objects change, "small" and "large" coins do not continue to exchange in convenient and fixed ratios; that is, the property called the *aliquot* property fails to hold.[6]

The standard formula goes a long way toward solving the bimetallism problem, but only at a cost. Under full-bodied coinage, there is no need for commitment on the part of those making coins and counterfeiting is no more severe a problem than it is for any other commodity. (That is what full-bodied means.) In contrast, as noted by Sargent and Velde, the standard formula involves commitment: it is banking with the token coins serving as liabilities and the objects into which they are convertible serving as the reserve. And because the small coins are token coins, counterfeiting must be prevented.

The authors' model deals with none of this. Is that because all of it is already understood, thereby leaving us free to go on to matters like the paradoxical observations concerning declining relative values of small coins? I do not think so. For example, the non-neutrality that stems from indivisibility seems to be new. And, where is a model that depicts the benefits of satisfying the *aliquot* property? And, if there were such a model, wouldn't we want to use it to deal with the subtler questions on which the authors focus? I agree with the authors that the history of coinage provides theoretical challenges. I do not agree that their cash-in-advance model provides useful insights regarding that history.

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[6] Kaplan (1999) describes the origin of number systems in a way that makes evident the difficulties that arise if the aliquot property does not hold. Although such difficulties seem less important now that we have electronic calculators, even we would not like to cope with worn and almost new $20's trading at different prices.

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