Existence of Monetary Steady States in a Matching Model: Indivisible Money

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Abstract

Existence of a monetary steady state is established for a random matching model with divisible goods, indivisible money, and take-it-or-leave-it offers by consumers. There is no restriction on individual money holdings. The background environment is that in papers by Shi and by Trejos and Wright. The monetary steady state shown to exist has nice properties: the value function, defined on money holdings, is increasing and strictly concave, and the measure over money holdings has full support.

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1 Introduction

Shi (1995) and Trejos and Wright (1995) introduce a matching model of money with divisible goods. While the model builds on the indivisible goods model of Kiyotaki and Wright (1989), the introduction of divisible goods permits output and prices to be determined as part of an equilibrium. Trejos and Wright show that equilibrium under a bargaining rule is easily formulated for general individual money holdings. However, existence of a monetary equilibrium has been established only for special versions. Here, I give a general existence proof for indivisible money. In particular, under the bargaining rule that potential consumers make take-it-or-leave-it offers, I prove that there exists a steady state with a value function defined on money holdings that is increasing and strictly concave and with a measure over money holdings that has full support. The only assumptions are lower bounds on (a) the marginal utility of consumption at zero and (b) the ratio of the average stock of money to the size of the smallest unit of money.

Proving existence is difficult because the general model has endogenous heterogeneity of money holdings. Most researchers simplify or avoid the endogeneity of the distribution of money holdings by making special assumptions.[1] One exception is Molico (1997). He studies the model numerically and claims to find monetary steady states for divisible money and unbounded individual holdings. My results—and those in a companion paper on divisible money—provide a basis for interpreting his numerical results. Another exception is Taber and Wallace (1999), who study indivisible commodity money, money with a direct utility payoff, with a general finite bound on individual holdings. They establish existence of a steady state with a concave and strictly increasing value function. I extend their result in two respects. I allow individual money holdings to be unbounded and I consider fiat money. To deal with fiat money, I show that there exists a steady state for the corresponding commodity money version in which the value of money is bounded

[1] Green and Zhou (1998) and Zhou (1999) assume indivisible goods and divisible money. Green and Zhou (in print) assume divisible goods and divisible money, but make preference assumptions that effectively make goods indivisible. Camera and Corbae (1999) consider the same model as I study with a finite bound on individual money holdings. For a special region of the parameter space, they construct a steady state in which one unit of money is offered in every trade. Cavalcanti (2000) assumes a unit bound of money holdings and a large number of kinds of money. Shi (1997) and Lagos and Wright (2000) make special assumptions that produce a degenerate distribution of money holdings.

away from zero as the direct utility payoff approaches zero.[2]

The properties of the steady state shown to exist—monotonicity and strict concavity of the value function and full support of the measure—are important. One implication is a non-neutrality result. Two economies that have different ratios of average holdings of money to the smallest unit of money have different sets of steady states in terms of allocations. In fact, if the larger ratio is an integer multiple of the smaller ratio, then the set of steady states for the economy with more money is a strict superset of that for the economy with less money. As shown below, this is an immediate implication of the full-support property.

2 The Model

As noted above, the model is essentially that in Shi (1995) and Trejos and Wright (1995).

2.1 Environment

Time is discrete, dated as $t \geq 0$. There is a $[0, 1]$ continuum of each of $N \geq 3$ types of infinitely lived agents, and there are N distinct produced and perishable types of divisible goods at each date. A type n agent, $n \in \{1, 2, ..., N\}$, produces only type n good and consumes only type $n+1$ good (modulo N). Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. For a type n agent, utility in a period is $u(q_{n+1}) - q_n$, where $q_{n+1} \in \mathbb{R}_+$ is consumption of type $n+1$ good and $q_n \in \mathbb{R}_+$ is production of type n good. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, strictly concave, continuously differentiable, and satisfies $u(0) = 0$ and $u'(\infty) < 1$. In addition, there is a lower bound on $u'(0)$ which is specified later.

There exists a fixed stock of money which is perfectly durable. Money is symmetrically distributed among the N specialization types. Let the average money holding be denoted by $\overline{m}$ and let the (smallest) unit of money be denoted by $\Delta(> 0)$. I assume that Δ is small relative to $\overline{m}$ with a lower bound on $\overline{m}/\Delta$ that is specified later. Also, let the exogenous upper bound of individual money holdings be denoted by B. Although the focus of the paper is unbounded individual holdings ($B = \infty$), I also include the bounded

[2] The approach used here is likey to be applicable to models in which the source of heterogeneity in money holdings is preference shocks rather than random meetings.

case (B finite). If B is finite, then it is assumed to be large relative to $\overline{m}$ with a lower bound on $B/\overline{m}$ that is specified later. ($B > \overline{m}$ is necessary for trade to occur.) Also, if B is finite, then B/Δ is assumed to be an integer. Let $B_\Delta = \{0, \Delta, ..., B\}$ denote the set of possible individual holdings of money.

In each period, agents are randomly matched in pairs. A meeting between a type n agent and a type $n+1$ agent is called a single-coincidence meeting. Other meetings are not relevant. In meetings, the agents' types and money holdings are observable, but any other information about an agent's trading history is private.

2.2 Definition of Equilibrium

In single-coincidence meetings, the potential consumer makes a take-it-or-leave-it offer, (p, q), where p is the amount of money offered and q is the amount of production demanded. Let $w_t(x)$ be the expected discounted value of holding x amount of money at the start of period t, prior to date t matching, where $w_t : B_\Delta \to \mathbb{R}_+$ is nondecreasing. Consider a date t single-coincidence meeting between a consumer with x amount of money and a producer with m amount of money. Let

$$\Gamma(x, m) = \{p \in B_\Delta : p \leq \min\{x, B - m\}\}, \tag{1}$$

the set of feasible offers of money. (As a convention, $\infty - m = \infty$.) Assuming, as is standard, that the producer accepts all offers which leave him no worse off, an optimal offer satisfies $p \in \Gamma(x, m)$ and $q = \beta w_{t+1}(m+p) - \beta w_{t+1}(m)$, where the equality for q says that the lower bound on the producer's gain-from-trade, zero, is attained. Therefore, the consumer's problem reduces to $\max_{p\in\Gamma(x,m)}\{u[\beta w_{t+1}(m+p) - \beta w_{t+1}(m)] + \beta w_{t+1}(x-p)\}$. To express this objective function more succinctly, it is convenient to introduce a symbol for an increment in a function: for any function $g : \mathbb{R} \to \mathbb{R}$, let $g(x, y) \equiv g(x) - g(x - y)$. Using this shorthand and dropping the time subscript on the value function, for a nondecreasing $w : B_\Delta \to \mathbb{R}_+$ and $(x, m) \in B_\Delta^2$, let

$$f(x, m, w) = \max_{p\in\Gamma(x,m)} \{u[\beta w(m + p, p)] + \beta w(x - p)\}, \tag{2}$$

and

$$p(x, m, w) = \arg\max_{p\in\Gamma(x,m)} \{u[\beta w(m + p, p)] + \beta w(x - p)\}. \tag{3}$$

That is, when w is the value of money at the start of the next period, f is the payoff for a consumer with x (pre-trade) who meets a producer with m (pre-trade) while p is the set of optimal offers of money.

Because $p(x, m, w)$ is discrete, and, may, therefore, be multi-valued, it is important for existence to allow all possible randomizations over the elements of $p(x, m, w)$. In order to describe the law of motion for the distribution of money holdings, it is convenient to express randomizations over the post-trade money holdings of consumers. Therefore, I define the set of randomizations, a set of measures on B_Δ, as

$$\Lambda(y, m, w) = \{\lambda(.; y, m, w) : \lambda(x; y, m, w) = 0 \text{ if } x \notin y - p(y, m, w)\}, \quad (4)$$

where $\lambda(x; y, m, w)$ is the fraction of consumers with y (pre-trade) in meetings with producers with m (pre-trade) who end up with x.

Let $\pi_t(x)$ denote the fraction of agents holding x amount of money at the start of period t, so that π_t is a measure on B_Δ. The law of motion for π_{t+1} can be expressed as

$$\begin{aligned} \pi_{t+1}(x) \quad &= \quad \frac{N-2}{N}\pi_t(x) + \frac{1}{N}\textstyle\sum_{y,m} \pi_t(y)\pi_t(m)[\lambda(x; y, m, w_{t+1}) \\ &\quad +\lambda(m + y - x; m, y, w_{t+1})] \end{aligned} \quad (5)$$

for some

$$\lambda(.; y, m, w_{t+1}) \in \Lambda(y, m, w_{t+1}). \quad (6)$$

Note that $\lambda(m + y - x; m, y, w_{t+1})$ is the fraction of producers with y (pre-trade) in meetings with consumers with m (pre-trade) who end up with x. The value function, $w_t(x)$, satisfies

$$w_t(x) = \frac{N-1}{N}\beta w_{t+1}(x) + \frac{1}{N}\textstyle\sum_m \pi_t(m) f(x, m, w_{t+1}). \quad (7)$$

This follows from the fact that the payoff to being a producer with x is $\beta w_{t+1}(x)$.

I can now state the relevant definitions.

Definition 1 *Given π_0, a sequence $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ is an equilibrium if it satisfies (1) − (7). A monetary equilibrium is an equilibrium with positive consumption and production. A pair (w, π) is a steady state if $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ with $w_t = w$ and $\pi_{t+1} = \pi$ for all t is an equilibrium for $\pi_0 = \pi$.*

3 Existence of a Monetary Steady State

To establish the existence of a monetary steady state, the following assumptions are maintained from now on.

$(A1) u'(0) > [2/(R\beta)]^2$, where $R \equiv [N-(N-1)\beta]^{-1}$.[3]

$(A2) B \geq 4\overline{m}$.

$(A3) \Delta \leq \overline{m}D/(\beta\overline{W})$, where D is the unique solution of $u'(D) = [2/(R\beta)]^2$ and $\overline{W}$ is the unique positive solution of $N(1-\beta)\overline{W} = u(\beta\overline{W}) + N$.[4]

In this model, existence always requires a lower bound on $u'(0)$ because a producer has to see a future reward from producing. Assumptions $(A2)$ and $(A3)$ say that the set of individual holdings is large enough (relative to the average holding). In what follows, except to the discussion of neutrality and non-neutrality at the end of the paper, it is convenient to normalize the exogenous nominal variables $\overline{m}$, Δ, and B by letting $\overline{m} = 1$.

I start by defining the main correspondences used. These are essentially implied by $(1)-(7)$. Let $\mathbf{W}$ be the set of concave and nondecreasing functions from B_Δ to $[0, \overline{W}]$. Let $\mathbf{\Pi}$ be the subset of measures on B_Δ satisfying the unit mean condition. Let both $\mathbf{W}$ and $\mathbf{\Pi}$ be equipped with the topology of pointwise convergence.

Let the single-valued map Φ_w on $\mathbf{W} \times \mathbf{\Pi}$ be defined by

$$\Phi_w(w,\pi)(x) = \frac{N-1}{N}\beta w(x) + \frac{1}{N}\sum\nolimits_m \pi(m) f(x,m,w). \tag{8}$$

Let the correspondence Φ_π on $\mathbf{W} \times \mathbf{\Pi}$ be defined by

$$\begin{aligned}\Phi_\pi(w,\pi) \;=\; & \{\nu : \nu(x) = \frac{N-2}{N}\pi(x) + \frac{1}{N}\sum\nolimits_{y,m} \pi(y)\pi(m)[\lambda(x;y,m,w) \\ & +\lambda(m+y-x;m,y,w)] \text{ for some } \lambda(.;y,m,w) \in \Lambda(y,m,w)\}.\end{aligned} \tag{9}$$

Finally, let $\Phi = (\Phi_w, \Phi_\pi)$.

In what follows, I deal directly with the unbounded case ($B = \infty$). The finite bound situation is a special case. The next lemma establishes important properties of $\mathbf{W} \times \mathbf{\Pi}$ and Φ.

[3] If (w,π) is a steady state, then (7) can be written as $w(x) = R\sum_m \pi(m) f(x,m,w)$. This expression for $w(x)$ is used repeatedly below. Also, note that $R < 1$.

[4] As will be shown in Lemma 3, $\overline{W} - 1$ can be taken to be an upper bound on steady state value functions. Also, note that $D/(\beta\overline{W}) < 1$ because $\beta[2/(R\beta)]^2 > N(1-\beta)$.

Lemma 1 *(i)*$\mathbf{W}\times\mathbf{\Pi}$ *is compact and metrizable.* *(ii)*$\Phi(w,\pi)\subset\mathbf{W}\times\mathbf{\Pi}$ *with* $\Phi_w(w,\pi)$ *bounded above by* $\overline{W}-1$. *(iii)*Φ *is convex-valued.* *(iv)*Φ *is upper hemicontinuous.*

Proof. By the Tychonoff Product Theorem (see Aliprantis and Border (1994, page 53)), both $\mathbf{W}$ and $\mathbf{\Pi}$ are compact. By 3.30 Theorem of Aliprantis and Border (1994, page 89), both $\mathbf{W}$ and $\mathbf{\Pi}$ are metrizable. By the definition of Φ_π in (9), there is no disposal of money. Hence $\Phi_\pi(w,\pi)\subset\mathbf{\Pi}$. Taber and Wallace (1999) show that $\Phi_w(w,\pi)$ preserves concavity and monotonicity. As regards the bound,

$$\begin{aligned} N\Phi_w(w,\pi)(x) &\leq (N-1)\beta w(x)+u[\beta w(x)]+\beta w(x) \\ &\leq N\beta\overline{W}+u(\beta\overline{W})=N\overline{W}-N, \end{aligned}$$

where the first inequality follows from (8) and the equality from the definition of $\overline{W}$. Because $\Lambda(y,m,w)$ is convex, it follows that Φ_π is convex-valued.

Now we consider part *(iv)*. We begin with three claims.

Claim 1 : $f(.,.,.)$ *is continuous on* $B_\Delta^2\times\mathbf{W}$ *and* $p(.,.,.)$ *is upper hemicontinuous on* $B_\Delta^2\times\mathbf{W}$. Let $A=\{(x,m,w,p):(x,m,w)\in B_\Delta^2\times\mathbf{W}$ and $p\in\Gamma(x,m)\}$. Let $g:A\to\mathbb{R}_+$ be defined by $g(x,m,w,p)=u[\beta w(m+p,p)]+\beta w(x-p)$. Because the value of $g(x,m,w,p)$ only depends on $x,m,w(0),w(\Delta),...,w(x+m)$, and p, it follows that g is continuous on A. Then claim 1 follows from Berge's Maximum Theorem (see Aliprantis and Border (1994, page 473)).

Claim 2 : *Let* $w_n,w\in\mathbf{W}$ *with* $w_n\to w$. *For all* x, $f(x,m,w_n)\to f(x,m,w)$ *uniformly in* m. Fix x and fix $\varepsilon>0$. Let m^* be such that $u(\beta x\overline{W}/m^*)<\varepsilon$ and let n be such that $\beta\left|w_n(y)-w(y)\right|<\varepsilon$ for all $y\leq x$. By Claim 1, for sufficiently large n, $|f(x,m,w)-f(x,m,w_n)|<\varepsilon$ for all $m\leq m^*$. So we only need to consider $m>m^*$. Let $m>m^*$. Because w is concave and bounded above by $\overline{W}$ and $w(0)\geq 0$, it follows that $\overline{W}/m^*>w(m+x,x)/x$ or $w(m+x,x)<x\overline{W}/m^*$. Let $p\in p(x,m,w)$. Because $g(x,m,w_n,p)\leq f(x,m,w_n)$ (for g, see Claim 1), it follows that

$$\begin{aligned} & f(x,m,w)-f(x,m,w_n) \\ \leq\ & u[\beta w(m+p,p)]-u[\beta w_n(m+p,p)]+\beta[w(x-p)-w_n(x-p)] \\ <\ & u[\beta w(m+p,p)]+\varepsilon\leq u[\beta w(m+x,x)]+\varepsilon<u(\beta x\overline{W}/m^*)+\varepsilon<2\varepsilon. \end{aligned}$$

By reversing the roles of $f(x,m,w)$ and $f(x,m,w_n)$, we have $f(x,m,w_n)-f(x,m,w)<2\varepsilon$. This establishes Claim 2.

Claim 3 : *Let the correspondence* J *on* $\mathbf{W} \times \mathbf{\Pi}$ *be defined as*

$$\begin{aligned} J(w,\pi) &= \{\mu : \mu(x) = \textstyle\sum_{y,m} \pi(y)\pi(m)[\lambda(x;y,m,w) \\ &\quad +\lambda(m+y-x;m,y,w)] \textit{ for some } \lambda(.;y,m,w) \in \Lambda(y,m,w)\}. \end{aligned}$$

Let $(w_n,\pi_n),(w,\pi) \in \mathbf{W} \times \mathbf{\Pi}$ *with* $(w_n,\pi_n) \to (w,\pi)$. *There exists a subsequence of* n, *denoted by* j, *such that there exist* $\mu_j \in J(w_j,\pi_j)$ *and* $\mu \in J(w,\pi)$ *with* $\mu_j \to \mu$. Fix $\varepsilon > 0$. Let $x^* > 1/\varepsilon$ and let n be such that $|\pi_n(x) - \pi(x)| < \varepsilon$ for all $x \le x^*$. By Claim 1, $p(y,m,.)$ is upper hemicontinuous on $\mathbf{W}$ for all (y,m). Hence there exists a subsequence of n, denoted by j, such that for large j, an element of $p(y,m,w_j)$ coincides with an element of $p \in p(y,m,w)$ for all $y,m \le x^*$. It follows that there exist $\lambda(.;y,m,w_j) \in \Lambda(y,m,w_j)$ and $\lambda(.;y,m,w) \in \Lambda(y,m,w)$ such that for large j, $\lambda(.;y,m,w_j) = \lambda(.;y,m,w)$ for all $y,m \le x^*$. Let $\lambda(x;y,m,w_j)+\lambda(m+y-x;m,y,w_j)$ be denoted by $c^{x,j}_{y,m}$ and let $\lambda(x;y,m,w) + \lambda(m+y-x;m,y,w)$ be denoted by $c^x_{y,m}$. Now let $\mu_j(x) = \sum_{y,m} \pi_j(y)\pi_j(m)c^{x,j}_{y,m}$ and $\mu(x) = \sum_{y,m} \pi(y)\pi(m)c^x_{y,m}$. Because $x^* > 1/\varepsilon$ and the average holding is unity, it follows that both $\mu_j(x)$ and $\mu(x)$ are bounded above by ε for all $x > x^*$. So we only need to consider $x \le x^*$. For $x \le x^*$, we have

$$\begin{aligned} \mu_j(x) - \mu(x) &= \textstyle\sum_{y,m \le x^*} [\pi_j(y)\pi_j(m) - \pi(y)\pi(m)]c^x_{y,m} \\ &\quad + \textstyle\sum_{\text{either } y>x^* \text{ or } m>x^*} [\pi_j(y)\pi_j(m)c^{x,j}_{y,m} - \pi(y)\pi(m)c^x_{y,m}] \\ &< 2\textstyle\sum_{y,m \le x^*} \{[\pi(y)+\varepsilon][\pi(m)+\varepsilon] - \pi(y)\pi(m)\} \\ &\quad +2\textstyle\sum_{y>x^*} \pi_j(y)\pi_j(m) + 2\textstyle\sum_{m>x^*} \pi_j(y)\pi_j(m) \\ &< 8\varepsilon + 2\varepsilon^2, \end{aligned}$$

where the last inequality follows from $\pi_j(m) < \varepsilon$ for all $m > x^*$. By reversing the roles of $\mu_j(x)$ and $\mu(x)$, we have $\mu(x) - \mu_j(x) < 8\varepsilon + 2\varepsilon^2$. This establishes Claim 3.

By 12.6 Corollary of Aliprantis and Border (1994, page 417), Claim 2 implies that for all x, $(w,\pi) \mapsto \sum_m \pi(m)f(x,m,w)$ is continuous. It follows that Φ_w is continuous. By Claim 3, J is upper hemicontinuous. It follows that Φ_π is upper hemicontinuous. ■

Next, I introduce a perturbation of the mapping Φ, which can be interpreted as assigning some direct utility to money. Let the real function h on B_Δ be defined by

$$h(x) = x/4 \text{ if } x \le 4, \quad h(x) = 1 \text{ if } x > 4.$$

Let $\mathbf{K}$ be the set of concave and nondecreasing functions from B_Δ to $[0, \overline{W} - 1]$. (Note that $\mathbf{K} \subset \mathbf{W}$.) For a positive integer n, let the correspondence $\Phi_n = (\Phi_{w,n}, \Phi_{\pi,n})$ on $\mathbf{K} \times \mathbf{\Pi}$ be defined by

$$\Phi_n = \Phi(w + h/n, \pi). \tag{10}$$

Lemma 2 *Φ_n has a fixed point.*

Proof. Because $w \in \mathbf{K}$ implies $w + h/n \in \mathbf{W}$ and because $(w, \pi) \mapsto (w + h/n, \pi)$ is continuous, by Lemma 1 (*iii*) and (*iv*), Φ_n is convex-valued and upper hemicontinuous. By Lemma 1 (*ii*), $\Phi_n(w, \pi) \subset \mathbf{K} \times \mathbf{\Pi}$. Because $\mathbf{K} \subset \mathbf{W}$ is closed, it follows that $\mathbf{K}$ is compact, and, hence, that $\mathbf{K} \times \mathbf{\Pi}$ is compact. Then by Kakutani's fixed point theorem, Φ_n has a fixed point. ■

The next lemma, the main ingredient in the existence proof, establishes a uniform (with respect to n in (10)) lower bound on the value functions of the fixed points of Φ_n.

Lemma 3 *If (w, π) is a fixed point of Φ_n, then $w(4) \geq D/\beta - 1/n$.*

Proof. Assume by contradiction that $w(4) < D/\beta - 1/n$. Let $w + h/n$ be denoted by φ. The proof is split into two steps. In the first step, we calculate a lower bound on $f(4, m, \varphi) - f(4 - \Delta, m, \varphi)$ for $m \leq 2$. In the second step, we draw contradictions based on this bound. In this and subsequent proofs, we suppress the dependence of f and p on φ or w. Also, for a measure μ on B_Δ and an interval I, we denote $\mu(I \cap B_\Delta)$ by μI.

Step 1. To get the lower bound, consider two possibilities for $p(4 - \Delta, m)$ for each $m \leq 2$, according to whether an element of $p(4 - \Delta, m)$ does or does not exceed 2. First, assume $p(4 - \Delta, m) \ni p \geq 2$. Because the consumer with money holding 4 can make the same offer as the consumer with $4 - \Delta$ does, and, hence, get the same amount of the consumption good, it follows that

$$f(4, m) - f(4 - \Delta, m) \geq \beta\varphi(4 - p, \Delta) \geq \beta\varphi(2, \Delta) > \beta w(2, \Delta),$$

where the second inequality follows from concavity of φ. Next, assume $p(4 - \Delta, m) \ni p < 2$. Because $m \leq 2$, we have $p + \Delta + m \leq 4$. Hence the consumer with 4 can make the offer $p + \Delta$ to the producer with m and end up with the same money holding as the consumer with $4-\Delta$. It follows that

$$\begin{aligned} & f(4,m)-f(4-\Delta,m) \\ \geq\ & u[\beta\varphi(m+p+\Delta,p+\Delta)]-u[\beta\varphi(m+p,p)] \\ >\ & u'[\beta\varphi(m+p+\Delta,p+\Delta)]\beta\varphi(m+p+\Delta,\Delta) \\ >\ & u'(D)\beta\varphi(m+p+\Delta,\Delta)\geq u'(D)\beta\varphi(4,\Delta)>u'(D)\beta w(4,\Delta), \end{aligned} \quad (11)$$

where the second inequality follows from the mean value theorem and strict concavity of u, the third from $\beta\varphi(m+p+\Delta,p+\Delta)<\beta\varphi(4)=\beta[w(4)+1/n]<D$ and strict concavity of u, and the fourth from concavity of φ. Let $l=\min\{\beta w(2,\Delta),\, u'(D)\beta w(4,\Delta)\}$. Then for $m\leq 2$, $f(4,m)-f(4-\Delta,m)>l$.

Step 2. Because (w,π) is a fixed point of Φ^n, by (10) and (8), we have

$$w(x,\Delta)=R(N-1)\beta h(x,\Delta)/n+R\textstyle\sum_m \pi(m)[f(x,m)-f(x-\Delta,m)].$$

Because $f(x,m)\geq f(x-\Delta,m)$ for all m, it follows that for $m^*<\infty$,

$$w(x,\Delta)\geq R\textstyle\sum_{m=0}^{m^*}\pi(m)[f(x,m)-f(x-\Delta,m)]. \quad (12)$$

Because the average holding is 1, $\pi[0,2]\geq 1/2$. Then by (12) and Step 1,

$$w(4,\Delta)>Rl/2. \quad (13)$$

Now consider the two possible values of l. If $l=u'(D)\beta w(4,\Delta)$, then by (13),

$$w(4,\Delta)>Rl/2=(R\beta/2)u'(D)w(4,\Delta)=[2/(R\beta)]w(4,\Delta)>w(4,\Delta),$$

a contradiction. So it must be that $l=\beta w(2,\Delta)$. Then by (13),

$$w(4,\Delta)>Rl/2=(R\beta/2)w(2,\Delta). \quad (14)$$

To rule this out, we calculate a lower bound on $f(2,m)-f(2-\Delta,m)$ for $m\leq 2$. Let $p\in p(2-\Delta,m)$. Because $p\leq 2-\Delta$ and $m\leq 2$, we have $p+\Delta+m\leq 4$. Hence the consumer with 2 can offer $p+\Delta$ to the producer with m. It follows that

$$\begin{aligned} & f(2,m)-f(2-\Delta,m) \\ \geq\ & u[\beta\varphi(m+p+\Delta,p+\Delta)]-u[\beta\varphi(m+p,p)] \\ >\ & u'(D)\beta w(4,\Delta)>u'(D)\beta(R\beta/2)w(2,\Delta), \end{aligned}$$

where the second inequality follows exactly the logic used in (11) and the last from (14). Let $l' = u'(D)\beta(R\beta/2)w(2,\Delta)$. Then by (12) and $\pi[0,2] \geq 1/2$, we have

$$w(2,\Delta) > Rl'/2 = (R\beta/2)^2 u'(D)w(2,\Delta) = w(2,\Delta),$$

a contradiction.[5] ■

In some respects, the ingredients in the proof of Lemma 3 have analogues in the simple case with $B = \Delta = 1$. In my proof, I require that there be a set of "poor" agents with positive measure. This set plays the role of the agents with 0 when $B = \Delta = 1$. In the proof, the "poor" set is $[0,2] \cap B_\Delta$ because there is an obvious lower bound on the measure of this set — namely, 1/2. However, other sets would also work. The agents with 4 are like those with holdings of 1 when $B = \Delta = 1$. Of course, when $B = \Delta = 1$, the monetary steady state can be computed directly because the distribution of money holdings and the offers in trades are fixed. The argument here is complicated because very little is known either about the distribution or the offers that agents make.

Now I show that there is a monetary steady state by taking a limit as the direct utility payoff of money approaches zero.

Lemma 4 *Let $\{(w_n, \pi_n)\}$ be a sequence such that (w_n, π_n) is a fixed point of Φ_n. (i)$\{(w_n, \pi_n)\}$ has at least one limit (accumulation) point, denoted (w, π). (ii)(w, π) is a fixed point of Φ. (iii)$w(0) = 0$ and $w(4) \geq D/\beta$.*

Proof. Because $\mathbf{W} \times \mathbf{\Pi}$ is compact, there is a subsequence of $\{(w_n, \pi_n)\}$ that converges to some $(w, \pi) \in \mathbf{W} \times \mathbf{\Pi}$. To simplify the notation, let $\{(w_n, \pi_n)\}$ represent the subsequence whose limit is (w, π). Because (w_n, π_n) is a fixed point of Φ, it follows from (10) that $(w_n, \pi_n) \in \Phi(w_n + h/n, \pi_n)$. Because $(w_n, \pi_n) \to (w, \pi)$, it follows that $(w_n + h/n, \pi_n) \to (w, \pi)$. Because Φ is upper hemicontinuous, $(w_n + h/n, \pi_n) \to (w, \pi)$ and $(w_n, \pi_n) \in \Phi(w_n + h/n, \pi_n)$ imply that there is a subsequence of $\{(w_n, \pi_n)\}$ converging to an element of $\Phi(w, \pi)$. Because $\{(w_n, \pi_n)\}$ itself converges to (w, π), it follows that $(w, \pi) \in \Phi(w, \pi)$. Part (*iii*) is obvious. ■

[5] In this proof, Δ is simply required to be no greater than unity, the average holding. If the average holding $\overline{m}$ is not unity, then we require $\Delta \leq \overline{m}$ and we redefine h as $h(x) = x/(4\overline{m})$ for $x \leq 4\overline{m}$ and $h(x) = 1$ for $x > 4\overline{m}$. It follows that $w_n(4\overline{m})$ is bounded below by $D/\beta - 1/n$.

Any Lemma 4 limit point (w, π) is a monetary steady state according to Definition 1. The next lemma establishes some of the properties of (w, π).

Lemma 5 *Let* (w, π) *be a Lemma* 4 *limit point.* $(i)w$ *is concave and strictly increasing.* $(ii)\pi(0) > 0$.

Proof. (i)Concavity is obvious. Assume by contradiction that w is not strictly increasing. By concavity of w, there exists $a > 0$ such that $w(x) = w(a)$ if $x \geq a$ and $w(x) < w(a)$ if $x < a$. (That is, by concavity, the flat portion of w must occur over a set of the form $\{a, a + \Delta, ...\}$.) It follows that $w(a) > 0$, and, hence, that there must be a positive probability that the consumer with a makes an offer $p \geq \Delta$ to some producers. The consumer with $a + \Delta$ has the same probability of meeting those producers and can also make the offer p. If so, he ends up with $a + \Delta - p$ and the consumer with a ends up with $a - p$. But then $a - p < a$ implies $w(a + \Delta - p) > w(a - p)$. This, in turn, implies $w(a + \Delta) > w(a)$, a contradiction.

(ii)Assume by contradiction that $\pi(0) = 0$ and let $a = \min\{x : \pi(x) > 0\}$. It follows that $w(a) > 0$, and, hence, that there must be a positive probability that the consumer with a makes an offer $p \geq \Delta$ to some producers. That is, for some m with $\pi(m) > 0$, $p \in p(a, m)$ with $p \geq \Delta$ occurs with positive possibility. But then $\pi(a)\pi(m) > 0$ implies $\pi(x - p) > 0$, a contradiction. ■

Now I turn to establishing that the steady state measure has full support. In what follows, let (w, π) be a Lemma 4 limit point and let *supp* π denote the support of π. The full support result relies on some facts about the optimal offers of money, $p(x, m, w)$, and their dependence on x and m.

Lemma 6 (i)*If* $p_1 \in p(x, m, w)$ *and* $p_2 \in p(x + \Delta, m, w)$*, then* $p_2 - p_1 \in \{0, \Delta\}$. (*The consumer's marginal propensity to spend on a given producer is between* 0 *and* 1.)

(ii)*If* $x_1 < x_2$*, then* $x_1 + \max p(m, x_1, w) \leq x_2 + \min p(m, x_2, w)$. (*For a given consumer, the producer's post-trade money holding is weakly increasing in his pre-trade holding.*)

(iii)*If* $x_2 \geq x_1$ *and* $m_2 < m_1$*, then* $\max p(x_2, m_2, w) \leq \max\{x_2 - x_1, m_1 - m_2\} + \min p(x_1, m_1, w)$. (*If the consumer is richer and the producer is poorer, then the change in spending is bounded above by the maximum of differences in the consumer's and the producer's holdings.*)

(iv)Assume $\min p(x_1, m_1, w) = 0$ *and* $m_2 \geq m_1$. *If* $x_2 > x_1$, *then* $\max p(x_2, m_2, w) \leq x_2 - x_1$. *If* $x_2 = x_1$, *then* $\min p(x_2, m_2, w) = 0$. *(If a consumer and a producer do not trade, then a richer consumer who meets a richer producer offers at most the consumer's increment.)*

(v)If $x > m$, *then* $0 \notin p(x, m, w)$. *(If the consumer is richer than the producer, then there is trade.)*

Proof. See the Appendix. ■

The next lemma shows that there is no endogenous bound.

Lemma 7 *There is no* $x \in B_\Delta$ *such that* $\pi(m) = 0$ *for* $m > x$.

Proof. Assume by contradiction that $\exists\ x = \max\{m : \pi(m) > 0\} < B$. Because w is concave and bounded above by $\overline{W}$ and $w(0) = 0$, it follows that $w(x + \Delta, \Delta) < \Delta\overline{W}/x$. Because the average holding is 1 and $\pi(0) > 0$, it follows that $x > 1$. Then by assumption ($A3$), we have $w(x + \Delta, \Delta) < \Delta\overline{W} \leq D/\beta$. By the definition of x , $0 \in p(x, x)$. It follows that

$$\beta w(x, \Delta) \geq u[\beta w(x + \Delta, \Delta)] > u'(D)\beta w(x + \Delta, \Delta). \tag{15}$$

Also, because $0 \in p(x, x)$ and $\Delta \in \Gamma(x + \Delta, x)$, it follows that

$$f(x + \Delta, x) - f(x, x) \geq u[\beta w(x + \Delta, \Delta)] > u'(D)\beta w(x + \Delta, \Delta), \tag{16}$$

where the last inequality follows from comparing the second and last terms in (15). Now, either $\pi[0, x) \geq 1/2$ or $\pi(x) \geq 1/2$. If the latter, then

$$\begin{aligned} w(x + \Delta, \Delta) &= R \textstyle\sum_m \pi(m)[f(x + \Delta, m) - f(x, m)] \\ &> R\pi(x)u'(D)\beta w(x + \Delta, \Delta) > w(x + \Delta, \Delta), \end{aligned} \tag{17}$$

a contradiction. (Here, the first inequality follows from (16) and $f(x + \Delta, m) > f(x, m)$ for all m. For the equality, see footnote 3.) So $\pi[0, x) \geq 1/2$. Fix $m < x$. By Lemma 6 (v), $\min p(x, m) > 0$. Because $p(x, m) \subset \Gamma(x + \Delta, m)$, it follows that $f(x + \Delta, m) - f(x, m) \geq \beta w(x, \Delta)$. Then by the logic used in (17), we have $w(x + \Delta, \Delta) > R\pi[0, x)\beta w(x, \Delta) > (R/2)u'(D)\beta w(x + \Delta, \Delta) > w(x + \Delta, \Delta)$, a contradiction. (Here, the second inequality follows from (15).) ■

The next lemma shows that *supp* π is periodic.

Lemma 8 *supp* $\pi = \{0, b\Delta, 2b\Delta, ...\}$, *where* b *is an integer.*

Proof. Let $a = \min\{x > 0 : \pi(x) > 0\}$. The following proof is written as if $a > \Delta$. It also applies if $a = \Delta$, which is a simple special case. In this proof, we let $i, j \in \mathbb{Z}_+$. Let $n = \max\{i : \min p(a, ia) \geq \Delta\}$.

Claim $1 : n \geq 1$. Assume by contradiction that $n = 0$. By Lemma 6 (iv), this implies $0 \in p(a, m)$ for $m > a$. Hence, letting $\rho = \pi(0)$, we have

$$w(a) = R\rho u[\beta w(a)] + R(1-\rho)\beta w(a). \tag{18}$$

We also have $w(2a) \geq R\rho\{u[\beta w(a)] + \beta w(a)\} + R(1-\rho)\beta w(2a)$. Comparing this with (18), we have

$$w(2a, a) \geq R\rho\beta w(a) + R(1-\rho)\beta w(2a, a). \tag{19}$$

Now let $c = [1 - R(1-\rho)\beta]/(R\rho)$. (Note that $c > 1$.) By (18) and (19), we have $cw(a) = u[\beta w(a)]$ and $cw(2a, a) \geq \beta w(a)$. Let g satisfy $cg = \beta w(a)$. (Note that $w(2a, a) \geq g$.) By $g < w(a)$ and $cw(a) = u[\beta w(a)]$, we have $cg < u(\beta g)$. Then $u[\beta w(2a, a)] \geq u(\beta g) > cg = \beta w(a)$. But $n = 0$ implies $\beta w(a) \geq u[\beta w(2a, a)]$, a contradiction.

Claim $2 : a \in p(a, ja)$ *for* $j = 1, ..., n$ *and* $\pi(ja) > 0$ *for* $j = 1, ..., n+1$. We proceed by induction: for $j = 1, ..., n$, $\pi(ja) > 0$ implies $a \in p(a, ja)$ and $\pi(ja + a) > 0$. By the definition of a, we only need to establish the induction step. By Lemma 6 (iv) and the definition of n, $\min p(a, ja) \geq \Delta$. If $p \in p(a, ja)$ with $p \in (0, a)$ occurs with positive probability, then $\pi(a)\pi(ja) > 0$ implies $\pi(a - p) > 0$, which contradicts the definition of a. It follows that $a \in p(a, ja)$ occurs with probability 1, and, hence, that $\pi(ja + a) > 0$.

Claim $3 : \pi(x) = 0$ *for* $x \neq ia$ *if* $x \leq na + a$. Suppose otherwise. We first establish the following induction argument: for $j = 2, ..., n$, $x \in (ja - a, ja)$ with $\pi(x) > 0$ implies $\pi(x + a) > 0$. To see this, assume that x and j satisfy the conditions. By Claim 2, $a \in p(a, ja - a)$. By Lemma 6 (ii), this implies $0 \notin p(a, x)$. It follows that $a \in p(a, x)$ occurs with probability 1, and, hence, that $\pi(x + a) > 0$. By the contradicting assumption and the induction argument, $\exists\, x \in (na, na + a)$ with $\pi(x) > 0$. Because $a \in p(a, na)$, by Lemma 6 (ii), min $p(a, x) \geq na + a - x$. Because $0 \in p(a, na + a)$, by Lemma 6 (ii), $\max p(a, x) \leq na + a - x$. Hence $p(a, x) = \{na + a - x\}$. But then $\pi(x - na) > 0$, a contradiction.

Claim $4 : \pi(x) = 0$ *for* $x \neq ia$ *if* $x > na + a$. We proceed by induction: for $j \geq 1$, $\pi(x) = 0$ for $x \neq ia$ if $x \leq na + ja$ implies $\pi(x) = 0$ for $x \neq ia$ if

$x \leq na + ja + a$. By Claim 3, the hypothesis holds for $j = 1$. So it suffices to establish the induction step. Assume by contradiction that $\pi(x) = 0$ for some $x \in (na + ja, na + ja + a)$. By Lemma 6 ($v$), $\min p(x, 0) > 0$. By Lemma 6 (iii), $\max p(x, 0) \leq na + ja + \min p(a, na + ja) = na + ja$. Because x is not a multiple of a, any feasible value of $p(x, 0)$ makes $\pi(y) > 0$ for some $y \leq na + ja$ where $y \neq ia$, a contradiction.

Claim 5 : $\pi(na + ja) > 0$ *for* $j > 1$. We proceed by induction: for $j \geq 1$, $\pi(na + ja) > 0$ implies $\pi(na + ja + a) > 0$. By Claim 2, the hypothesis holds for $j = 1$. So it suffices to establish the induction step. Let $k = \min\{i : \min p(ia, na + ja) \geq \Delta\}$. First assume $k > n + j$. Now assume by contradiction that $\pi(na + ja + a) = 0$ and let $l = \min\{i : \pi(ia) > 0, i \geq n + j + 2\}$. Note that $p(la, na + ja - a)$ only contains multiples of a. By Lemma 6 (iii), $\max p(la, na + ja - a) \leq (l - n - j)a + \min p(na + ja, na + ja)$ $= (l - n - j)a$, where the equality comes from the definition of k and $k > n + j$. By Lemma 6 (v), $\min p(la, na + ja - a) > 0$. But then any feasible value of $p(la, na + ja - a)$ makes $\pi(ia) > 0$ for some $n + j < i < l$, a contradiction. So $\pi(na + ja + a) > 0$. Next assume $k \leq n + j$. Note that $p(ka, na + ja)$ only contains positive multiples of a. By the definition of k, $0 \in p(ka - a, na + ja)$. By Lemma 6 (iv), this implies $p(ka, na + ja) = \{a\}$. Then by the induction assumption, $\pi(na + ja + a) > 0$.[6] ■

Now I can prove that π has full support. The proof is by contradiction. If b (see Lemma 8) exceeds unity, then I can construct a mapping that is concave and strictly increasing and has more than one positive fixed point. However, this mapping can have at most one positive fixed point.[7]

Lemma 9 *supp* $\pi = B_\Delta$.

Proof. By Lemma 8, it suffices to prove that $b = 1$. So assume by contradiction that $b \geq 2$. In this proof, we let $i \in \mathbb{N}$ and $j \in \mathbb{Z}_+$.

First, we introduce some notation. Let $\pi(jb\Delta)$ be denoted by π_j and $w(jb\Delta)$ by w_j. Also, let

$$k_i = w_i - w_{i-1} \text{ and } h_i = w(ib\Delta - \Delta) - w_{i-1}.$$

[6] For finite B, we first prove Claims 1, 2, and 3. It is clear that B must be at least $na + a$. If $B = na + a$, then the proof is complete. Otherwise we continue to Claims 4 and 5. It is clear that B must be equal to $na + ja + a$ for some $j > 0$.

[7] The proof that the mapping has at most one positive fixed point resembles the proof of Corollary 7.45 of Zeidler (1985, page 309).

(Note that if $b = 1$, then $h_i = 0$.) Let $k = (k_1, k_2, ...)$ and $h = (h_1, h_2, ...)$. Let $f(ib\Delta, jb\Delta)$ be denoted by $f_{i,j}$. Now consider $p(ib\Delta, jb\Delta)$. By Lemma 6 (i), if $p_1, p_2 \in p(x, m)$, then $|p_2 - p_1| \in \{0, \Delta\}$. Because $b \geq 2$, this implies that there is at most one element of $p(ib\Delta, jb\Delta)$ that is equal to $nb\Delta$ for some $n \in \mathbb{Z}_+$. By Lemma 8, any element of $p(ib\Delta, jb\Delta)$ that occurs with positive probability is equal to $nb\Delta$ for some $n \in \mathbb{Z}_+$. Hence, there exists a unique element of $p(ib\Delta, jb\Delta)$ that is equal to $nb\Delta$ for some $n \in \mathbb{Z}_+$ and occurs with probability 1. Let this element be denoted by $\overline{p}(i,j)b\Delta$. Finally, let

$$A_{i0} = \{j : \overline{p}(i,j) = \overline{p}(i-1,j)\} \text{ and } A_{i1} = \{j : \overline{p}(i,j) = \overline{p}(i-1,j)+1\}.$$

By Lemma 6 (i), $A_{i0} \cup A_{i1} = \mathbb{Z}_+$. (Also note that $A_{i0} \cap A_{i1}$ is empty.)

Next, for each pair of (i,j), we define mappings $\phi_{i,j}$ and $\sigma_{i,j}$ according to whether $j \in A_{i0}$ or $j \in A_{i1}$. If $j \in A_{i0}$, then let the mapping $\phi_{i,j} : \mathbb{R}_+^\infty \to \mathbb{R}_+$ be defined by

$$\phi_{i,j}(x) = \beta(x_{i-\overline{p}(i,j)} + w_{i-\overline{p}(i,j)-1}) + u[\beta(w_{j+\overline{p}(i,j)} - w_j)]. \tag{20}$$

Note that

$$\phi_{i,j}(k) = f_{i,j}. \tag{21}$$

By Lemma 6 (i), $j \in A_{i0}$ implies $\overline{p}(i,j)b\Delta \in p(ib\Delta - \Delta, jb\Delta)$. Hence,

$$\phi_{i,j}(h) = f(ib\Delta - \Delta, jb\Delta). \tag{22}$$

If $j \in A_{i1}$, then let the mapping $\sigma_{i,j} : \mathbb{R}_+^\infty \to \mathbb{R}_+$ be defined by

$$\sigma_{i,j}(x) = \beta w_{i-\overline{p}(i,j)} + u[\beta(x_{j+\overline{p}(i,j)} + w_{j+\overline{p}(i,j)-1} - w_j)].$$

Note that

$$\sigma_{i,j}(k) = f_{i,j}. \tag{23}$$

By Lemma 6 (i), $j \in A_{i1}$ implies $\overline{p}(i,j)b\Delta - \Delta \in p(ib\Delta - \Delta, jb\Delta)$. Hence,

$$\sigma_{i,j}(h) = f(ib\Delta - \Delta, jb\Delta). \tag{24}$$

Next, for each i, let the mapping $\theta_i : \mathbb{R}_+^\infty \to \mathbb{R}_+$ be defined by

$$\begin{aligned} \theta_i(x) \quad = \quad & \frac{N-1}{N}\beta(x_i + w_{i-1}) + \frac{1}{N}[\textstyle\sum_{j \in A_{i0}} \pi_j \phi_{i,j}(x) + \sum_{j \in A_{i1}} \pi_j \sigma_{i,j}(x)] \\ & -w_{i-1}. \end{aligned} \tag{25}$$

Let $\theta = (\theta_1, \theta_2, ...)$. By (21) and (23), we have

$$\theta_i(k) = w_i - w_{i-1} = k_i, \tag{26}$$

By (22) and (24), we have

$$\theta_i(h) = w(ib\Delta - \Delta) - w_{i-1} = h_i. \tag{27}$$

(Hence the mapping θ has multiple positive fixed points.) By substituting (26) and (27) into (25), we have

$$k_i = R[\textstyle\sum_{j\in A_{i0}} \pi_j \phi_{i,j}(k) + \sum_{j\in A_{i1}} \pi_j \sigma_{i,j}(k)] - w_{i-1}, \tag{28}$$
$$h_i = R[\textstyle\sum_{j\in A_{i0}} \pi_j \phi_{i,j}(h) + \sum_{j\in A_{i1}} \pi_j \sigma_{i,j}(h)] - w_{i-1}. \tag{29}$$

Next we make some claims.

Claim 1 : *There exists $s \in (0, 1)$ such that $h \geq sk$ with $h_i = sk_i$ for some i.* By Cone Lemma 8.31 (i) of Zeidler (1985, page 292), there exists $s > 0$ such that $h \geq sk$ with $h_i = sk_i$ for some i. By monotonicity of w, $s < 1$.

Claim 2 : *$\theta_i(h) \geq \theta_i(sk)$.* Because $h \geq sk$, this is obvious.

Claim 3 : *$\phi_{i,j}(sk) = s\phi_{i,j}(k)$ and $\sigma_{i,j}(sk) > s\sigma_{i,j}(k)$.* The equality is obvious. Now let $c \geq 0$. Because $s \in (0, 1)$, by strict concavity of u, $u(\beta sk + c) > su(\beta k + c) + (1 - s)u(c) \geq su(\beta k + c)$. Then the inequality follows.

Claim 4 : *$\theta_i(sk) \geq s\theta_i(k)$, and, strictly if A_{i1} is nonempty.* This follows from Claim 3.

Claim 5 : *If $h_i = sk_i$, then A_{i1} is empty.* Assume that A_{i1} is nonempty. But then $h_i = \theta_i(h) \geq \theta_i(sk) > s\theta_i(k) = sk_i$, a contradiction. (Here, the first equality follows from (27) and the second from (26). The first inequality follows from Claim 2 and the second from Claim 4 and the contradicting assumption.)

Claim 6 : *$h_1 > sk_1$.* Because (w, π) is a steady state, A_{11} is nonempty. Then the result follows from Claim 5.

Now let $n = \min\{i : h_i = sk_i\}$. By Claim 6, $n > 1$. By Claim 5, A_{n1} is empty. Let $Q = R\sum_j \pi_j\{\beta w_{n-\overline{p}(n,j)-1} + u[\beta(w_{j+\overline{p}(n,j)} - w_j)]\}$. Then by (29) and (28), we have

$$\begin{aligned} h_n - sk_n &= R\textstyle\sum_j \pi_j[\phi_{n,j}(h) - s\phi_{n,j}(k)] - (1 - s)w_{n-1} \\ &= R\textstyle\sum_j \pi_j\beta(h_{n-\overline{p}(n,j)} - sk_{n-\overline{p}(n,j)}) + (1 - s)(Q - w_{n-1}). \end{aligned}$$

Because $j \in A_{n0}$ for all j, we have

$$Q = R\textstyle\sum_j \pi_j\{\beta w_{n-\overline{p}(n-1,j)-1} + u[\beta(w_{j+\overline{p}(n-1,j)} - w_j)]\} = w_{n-1}.$$

Hence, we have

$$h_n - sk_n = R\sum_j \pi_j \beta(h_{n-\overline{p}(n,j)} - sk_{n-\overline{p}(n,j)}) \geq R\pi_0\beta(h_{n-\overline{p}(n,0)} - sk_{n-\overline{p}(n,0)}),$$

where the inequality follows from $h \geq sk$. By Lemma 6 (v), $\overline{p}(n,0) > 0$. By $0 \in A_{n0}$, $\overline{p}(n,0) < n$. But then $h_i > sk_i$ for $1 \leq i < n$ implies $h_n > sk_n$, a contradiction. ■

Full support allows us to establish strict concavity of the value function.

Lemma 10 *w is strictly concave.*

Proof. See the Appendix. ■

Therefore, I have proved the following proposition.

Proposition 1 *Under assumptions $(A1)-(A3)$, which allow for unbounded individual holdings of money, there exists a steady state (w,π) where w is increasing and strictly concave and π has full support.*

One implication of this proposition is related to non-neutrality. To state the result, I first define a notion of equivalence between steady states.

Definition 2 *Let (w,π) and (w',π') be steady states. We say that (w',π') is equivalent to (w,π) if there exists a bijection γ from supp π to supp π' such that if $x \in$ supp π, then $w(x) = w'(\gamma(x))$ and $\pi(x) = \pi'(\gamma(x))$. Let $e \equiv (\overline{m},\Delta,B)$, the vector of exogenous nominal objects, and let $S(e)$ denote the set of* ***all*** *steady states associated with e. We say that $S(e) \subset S(e')$ if $(w,\pi) \in S(e)$ implies that there exists $(w',\pi') \in S(e')$ with (w',π') equivalent to (w,π). We say that $S(e)$ and $S(e')$ are equivalent if $S(e) \subset S(e')$ and $S(e') \subset S(e)$.*

It follows from this definition that equivalence between steady states is symmetric and transitive. Using this definition, I can state the obvious neutrality result. If two economies differ only in their vectors of exogenous nominal objects, e and e', and $e = \theta e'$ for some $\theta \in \mathbb{R}_+$ (as a convention, $\infty = \theta\infty$), then $S(e)$ and $S(e')$ are equivalent.[8] However, equivalence does not hold if e and e' differ but not proportional.

[8]Let $e' = (\overline{m},\Delta,B)$ and $e = \theta e'$. Let $(w',\pi') \in S(e')$. Under the vector e, let (w,π) be defined as follows. For $n \geq 0$, let $w(n\theta\Delta) = w'(n\Delta)$ and $\pi(n\theta\Delta) = \pi'(n\Delta)$. Notice from (3) that $p \in p(x,m,w')$ implies $\theta p \in p(\theta x, \theta m, w)$. Hence, $(w,\pi) \in S(e)$. It is clear that (w,π) is equivalent to (w',π').

Corollary 1 *If $e \neq \theta e'$ for any $\theta \in \mathbb{R}_+$, then $S(e)$ and $S(e')$ are not equivalent. Moreover, if $e = (k\overline{m}, \Delta, kB)$ and $e' = (\overline{m}, \Delta, B)$ where $k \geq 2$ and is an integer, then $S(e') \subset S(e)$.*

Proof. We begin with the first assertion. Without loss of generality, let $e = (\overline{m}, \Delta, B)$ and $e' = (\overline{m}', \Delta, B')$. By the hypothesis, either $\overline{m} \neq \overline{m}'$ or $B \neq B'$. First consider $\overline{m} \neq \overline{m}'$ and assume by contradiction that $S(e)$ and $S(e')$ are equivalent. Without loss of generality, assume that $\overline{m} > \overline{m}'$. Let $(w, \pi) \in S(e)$ be a Proposition 1 steady state and let $(w', \pi') \in S(e')$ be equivalent to (w, π). Let *supp* $\pi' = \{a_0, a_1, ...\}$ with $a_i < a_{i+1}$ for all i. (Note that $a_i \geq i\Delta$.) Because w is strictly increasing and w' is nondecreasing, the bijection γ in Definition 2 from *supp* π to *supp* π' is strictly increasing. That is, $\gamma(i\Delta) < \gamma(j\Delta)$ if $i < j$. Because *supp* $\pi = B_\Delta$, it follows that $\gamma(i\Delta) = a_i$. Hence $\pi(i\Delta) = \pi'(a_i)$. But because $i\Delta \leq a_i$, this implies $\overline{m} \leq \overline{m}'$, a contradiction. Next consider $B \neq B'$. Without loss of generality, assume that B' is finite and $B' < B$. Let $(w, \pi) \in S(e)$ be a Proposition 1 steady state. But because *supp* $\pi = B_\Delta$ and because B'_Δ is a strict subset of B_Δ, no $(w', \pi') \in S(e')$ is equivalent to (w, π).

For the second assertion, let $(w', \pi') \in S(e')$. The following construction of $(w, \pi) \in S(e)$, which is similar to that used to prove neutrality, is well known. For $n \geq 0$ and $0 \leq j \leq k-1$, let $w(nk\Delta + j\Delta) = w'(n\Delta)$, $\pi(nk\Delta) = \pi'(n\Delta)$, and $\pi(nk\Delta + j\Delta) = 0$. It is clear that (w, π) is equivalent to (w', π'). ■

A surmise is that if $\overline{m} > \overline{m}'$, then some $s \in S(\overline{m}, \Delta, \infty)$ has more trade and higher average welfare than any $s' \in S(\overline{m}', \Delta, \infty)$, but that remains to be established.

Appendix

Proof of Lemma 6

Proof. (i)See Taber and Wallace (1999, page 967).

(ii)It suffices to prove that $x + \max p(m, x) \leq x + \Delta + \min p(m, x + \Delta)$. Assume by contradiction that $p \in p(m, x)$, $p' \in p(m, x+\Delta)$, and $x+p'+\Delta < x+p$. Then $p'+\Delta < p$. Let $a_1 = \beta w(x+p'+\Delta, p')$, $a_2 = \beta w(x+p'+2\Delta, p'+\Delta)$, $b_1 = \beta w(x + p - \Delta, p - \Delta)$, and $b_2 = \beta w(x + p, p)$.

Because $a_2 - a_1 = \beta w(x+p'+2\Delta, \Delta) > 0$ and $b_2 - b_1 = \beta w(x+p, \Delta) > 0$, by the definitions of p and p', we have

$$\frac{u(a_2)-u(a_1)}{a_2-a_1}w(x+p'+2\Delta,\Delta) \leq w(m-p',\Delta), \tag{30}$$

$$\frac{u(b_2)-u(b_1)}{b_2-b_1}w(x+p,\Delta) \geq w(m-p+\Delta,\Delta). \tag{31}$$

By the definitions of a_i and b_i, we have

$$\begin{aligned} b_1-a_1 &= \beta[w(x+p-\Delta)-w(x+p'+\Delta)+w(x+\Delta,\Delta)] \\ b_2-a_2 &= \beta[w(x+p)-w(x+p'+2\Delta)+w(x+\Delta,\Delta)]. \end{aligned}$$

But, $p'+\Delta<p$ implies $p'+\Delta\leq p-\Delta$ and $p'+2\Delta\leq p$. Hence, $b_1>a_1$ and $b_2>a_2$. Then strict concavity of u implies $\frac{u(a_2)-u(a_1)}{a_2-a_1}>\frac{u(b_2)-u(b_1)}{b_2-b_1}$. This inequality, $p'+2\Delta\leq p$, and concavity of w imply that the left side of (30) exceeds the left side of (31). Then by (30) and (31), $w(m-p',\Delta)>w(m-p+\Delta,\Delta)$. But this contradicts $p'<p-\Delta$ and concavity of w.

(iii)Let $a=\max\{x_2-x_1,m_1-m_2\}$ and let $p_1=\min p(x_1,m_1)$. Let $p_2=a+p_1$. We assume that $x_2>p_2$ and $m_2+p_2<B$; otherwise the result follows immediately. By $m_2<m_1$ and $p_2>p_1$, we have $w(m_2+p_2,p_2)>w(m_1+p_1,p_1)$. By $m_2+p_2\geq m_1+p_1$, we have $w(m_2+p_2+\Delta,\Delta)\leq w(m_1+p_1+\Delta,\Delta)$. Then we have

$$\begin{aligned} \beta w(x_2-p_2,\Delta) &\geq \beta w(x_1-p_1,\Delta) \\ &\geq u[\beta w(m_1+p_1+\Delta,p_1+\Delta)]-u[\beta w(m_1+p_1,p_1)] \\ &> u[\beta w(m_2+p_2+\Delta,p_2+\Delta)]-u[\beta w(m_2+p_2,p_2)], \end{aligned} \tag{32}$$

where the second inequality follows from $p_1\in p(x_1,m_1)$ and the third from strict concavity of u. Note that $u[\beta w(m+p,p)]+\beta w(x-p)$, viewed as a function of p, is concave, and, hence, strictly increasing on $[0,\min p(x,m)]$ and strictly decreasing on $[\max p(x,m),\ \min\{x,B-m\}]$. Then by (32), $\max p(x_2,m_2)\leq p_2$.

(iv)First consider $x_2>x_1$. Let $p=x_2-x_1$. We assume that $x_1>0$ and $m_2+p<B$; otherwise the result follows immediately. We have $\beta w(x_2-p,\Delta)=\beta w(x_1,\Delta)\geq u[\beta w(m_1+\Delta,\Delta)]-u(0)>u[\beta w(m_2+p+\Delta,p+\Delta)]-u[\beta w(m_2+p,p)]$, where the first inequality follows from $0\in p(x_1,m_1)$ and $u(0)=0$ and the second from strict concavity of u. By the logic used in the proof of part (iii), $\max p(x_2,m_2)\leq p$. Next consider $x_2=x_1$. We assume that $x_1>0$ and $m_2<B$; otherwise the result follows immediately.

We have $\beta w(x_2, \Delta) = \beta w(x_1, \Delta) \geq u[\beta w(m_1 + \Delta, \Delta)] \geq u[\beta w(m_2 + \Delta, \Delta)]$, where the first inequality follows from $0 \in p(x_1, m_1)$. By the logic used in the proof of part (iii), $\min p(x_2, m_2) = 0$.

(v)We have $u[\beta w(\Delta)] > \beta w(\Delta)$; otherwise, by concavity of w, it follows that $u[\beta w(m + \Delta, \Delta)] \leq \beta w(\Delta)$ for all $m \geq \Delta$, and, hence, that $w(\Delta) = 0$. By concavity of u and w, this implies that for $x > m$, $u[\beta w(m + \Delta, \Delta)] > \beta w(m + \Delta, \Delta) \geq \beta w(x, \Delta)$. So $0 \notin p(x, m)$. ■

Proof of Lemma 10

Proof. We first prove the following.

Claim : *If for each $x > 0$, there exists m such that $p(x, m)$ is a positive singleton, then w is strictly concave.*

The proof of the claim is by induction on the set satisfying strict concavity. That is, we show that w is strict concave on $\{0, \Delta, 2\Delta\}$ and then show strict concavity on $\{0, \Delta, ..., x\}$ implies strict concavity on $\{0, \Delta, ..., x, x+\Delta\}$. First, we prove that $2w(\Delta) > w(0) + w(2\Delta) = w(2\Delta)$. Taber and Wallace (1999) show that $2f(x, m) \geq f(x - \Delta, m) + f(x + \Delta, m)$. Because $\pi(0) > 0$ and $f(0, 0) = 0$, it suffices to show that $2f(\Delta, 0) > f(2\Delta, 0)$. By Lemma 6 (v), $f(\Delta, 0) = u[\beta w(\Delta)] > \beta w(\Delta)$. There are two possibilities for $p(2\Delta, 0)$. (i)If $\Delta \in p(2\Delta, 0)$, then $f(2\Delta, 0) = u[\beta w(\Delta)] + \beta w(\Delta) < 2u[\beta w(\Delta)] = 2f(\Delta, 0)$. (ii)If $2\Delta \in p(2\Delta, 0)$, then $f(2\Delta, 0) = u[\beta w(2\Delta)] \leq u[2\beta w(\Delta)] < 2u[\beta w(\Delta)] = 2f(\Delta, 0)$, where the first inequality follows from concavity of w and the second from strict concavity of u. Next for the induction step. Let m be such that $p(x, m)$ is a positive singleton. As above, because $\pi(m) > 0$, it suffices to show that $2f(x, m) > f(x - \Delta, m) + f(x + \Delta, m)$. Let $\min p(x - \Delta, m) = p$. By Lemma 6 (i), there are three possibilities for $\min p(x+\Delta, m)$. $(i)$$\min p(x+\Delta, m) = p+\Delta$. Because $p, p+\Delta \in \Gamma(x, m)$ and because $p(x, m)$ is a singleton, it follows that $2f(x, m) > u[\beta w(m+p+\Delta, p+\Delta)]+\beta w(x-p-\Delta)+u[\beta w(m+p, p)]+\beta w(x-p) = f(x-\Delta, m)+f(x+\Delta, m)$. $(ii)$$\min p(x + \Delta, m) = p$. By Lemma 6 (i), $\min p(x + \Delta, m) \geq \max p(x, m) \geq \min p(x - \Delta, m)$. It follows that $p(x, m) = \{p\}$ and $p \geq \Delta$. Therefore, $2f(x, m) - f(x - \Delta, m) - f(x + \Delta, m) = 2\beta w(x - p) - \beta w(x - \Delta - p) - \beta w(x + \Delta - p) > 0$, where the last inequality follows from $p \geq \Delta$ and the induction assumption. $(iii)$$\min p(x + \Delta, m) = p + 2\Delta$. Because $p + \Delta \in \Gamma(x, m)$, it follows that $2f(x, m) - f(x - \Delta, m) - f(x + \Delta, m) \geq 2u[\beta w(m+p+\Delta, p+\Delta)] - u[\beta w(m+p, p)] - u[\beta w(m+p+2\Delta, p+2\Delta)] > 0$, where the last inequality follows from strict concavity of u and concavity of w.

Now we can finish the proof of this lemma. By the claim, it suffices to prove that $\forall\ x > 0$, $\exists\ m^*$ such that $p(x, m^*)$ is a positive singleton. Note that $\beta w(m + \Delta, \Delta) < u^{-1}[\beta w(x, \Delta)]$ implies $p(x, m) = \{0\}$. Also note that concavity of w implies $w(m + \Delta, \Delta)/\Delta < \overline{W}/m$. Hence $m > \beta\overline{W}\Delta/u^{-1}[\beta w(x, \Delta)]$ implies $p(x, m) = \{0\}$. By Lemma 6 (v), $0 \notin p(x, 0)$. Then $\exists\ y = \max\{m : 0 \notin p(x, m)\}$. By Lemma 6 (ii), $p(x, y) = \{\Delta\}$. Then $m^* = y$.[9] ■

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[9] For finite B, we can find m^* as follows. If $p(x, B - \Delta) = \{\Delta\}$, then $m^* = B - \Delta$ is as required. Hence, assume $0 \in p(x, B - \Delta)$. Then $\exists\ y = \max\{m : 0 \notin p(x, m)\}$ and $m^* = y$.

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