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by

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Possibility Theorems for Aggregating Infinite Utility Streams Equitably*

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Abstract

The standard route for aggregating infinite utility streams using a real-valued representation runs into well-known difficulties as soon as we insist on the axiom of inter-generational equity. The aim of this paper is to explore what is feasible without abandoning this axiom. The paper focusses its attention on the Pareto axiom and domain restrictions. It turns out that once we weaken these requirements, real-valued aggregation becomes possible in a variety of ways (though, of course, impossibility results lurk everywhere). By establishing a series of results, this paper tries to chalk out the frontier between what is possible and what is not.

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Key Words: intergenerational equity, Pareto axiom, social welfare functions, domain restrictions, weak Pareto axioms.

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1 Introduction

The need to aggregate and evaluate infinite streams of returns or utility arises in several areas of economics, ranging from inter-generational welfare theory to environmental economics. The subject of inter-generational equity in the context of aggregating infinite utility streams has been of enduring interest to economists, starting with the work of Ramsey (1928), who had maintained that discounting one generation's utility or income vis-a-vis another's to be “ethically indefensible”, and something that “arises merely from the weakness of the imagination.” His conjecture about the difficulty of aggregating infinite streams, while respecting inter-generational equity, turned out to be compelling, as a large number of impossibility theorems were proved subsequently by a number of authors, starting with the seminal works of Koopmans (1960) and Diamond (1965).

Yet, it would be wrong to abandon the effort to search for a social welfare function that aggregates infinite streams of returns and satisfies intergenerational anonymity and some form of the Pareto criterion. In reality, we encounter this problem all the time. In deciding whether to build a dam on a river, which will help irrigation and generate electricity but damage fauna and flora, we clearly face a problem of choosing between long streams of utility, stretching far into the future. Even if we believe that the world has a finite future, since we do not know its termination date, we effectively face an infinite decision problem.

Moreover, every time we analyze an infinitely repeated game, we are forced to confront an infinite decision problem. And, if we are to pass judgement on which among a set of possible outcomes is superior, we are compelled to contend with precisely the problem that is the concern of this paper.

In Diamond's celebrated paper (1965) he had shown that there is no social welfare function that aggregates infinite utility streams while satisfying the Pareto condition, a weak form of anonymity and a continuity property[1]. In a recent paper (Basu and Mitra, 2003), we tried to show that the problem is more discouraging because the impossibility result survives even if we do not impose any continuity restriction on the social welfare function. Are we then completely into a cul de sac? This paper tries to answer this in the negative.

We can think of many routes to getting possibililty results. In an elegant

[1] The continuity property postulated by Diamond is with respect to the sup metric on $X = [0,1]^{\mathbb{N}}$.

paper, Svensson (1980) had shown that if, instead of seeking a (real-valued) social welfare function, we merely searched for the ability to rank infinite streams of utilities, then it is possible to prove that the requirements of equity and the Pareto principle are compatible. He does this, however, with the use of Szpilrajn's theorem, which implies a non-constructive proof. Related results have been obtained by Suzumura and Shinotsuka (2003) and Bossert, Sprumont and Suzumura (2004).

Though we delve briefly into this, our main aim in this paper is to look for possibility theorems that satisfy *representability*; that is, the existence of real-valued social welfare functions. More precisely our aim is to delineate the frontier of possibility and impossibility results for the existence of real-valued social welfare functions. We consider, in particular, weakening the Pareto axiom and exploring domain restrictions.

It does seem that in reality the domain of values that individual utilities can take is often quite limited. The simple assumption that an individual's utility can be represented by any real number may be mathematically convenient but is unrealistic. Given the limits of human perception, it is much more realistic to suppose that individual utilities can take a finite number of values or, at most, a countably infinite number of values. Thus, exploring the implications of such domain restrictions certainly seems worthwhile.

Of course, domain restrictions by themselves will not yield possibility theorems, given the general impossibility theorem of Basu and Mitra (2003, Theorem 1), which applies to all domains, however restrictive they may be.[2] But, we will try to show that, as soon as we combine domain restrictions with weaker versions of the Pareto axiom, the scope for the use of social welfare functions expands considerably (Theorem 3).

Our investigation also reveals that the particular nature of the domain restriction may be quite important for such possibility results. Under domain restrictions of other types, even the Weak Pareto axiom is seen to be incompatible with the requirement of an equitable social welfare function (Theorem 4). However, if the postulated version of Pareto is sufficiently weak, then it is possible to generate equitable and Paretian social welfare functions without any domain restrictions (Theorem 5).

It is true that the exercise that we undertake in this paper is abstract and

[2] Of course, the case in which the period utility space is a singleton, and so the domain of the social welfare function is also a singleton, is ruled out in the framework of Basu and Mitra (2003, Theorem 1).

theoretical but it is motivated by the practical concern for shedding light on what is feasible once we reject the standard (inequitable) method of aggregating streams by discounting the returns that accrue to future generations.

2 Formal Setting and Basic Results

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ the set of positive integers, and $\mathbb{M}$ the set of non-negative integers. Suppose $Y \subset \mathbb{R}$ is the set of all possible utilities that any generation can achieve. Then $X = Y^{\mathbb{N}}$ is the set of all possible utility streams. If $\{x_t\} \in X$, then $\{x_t\} = (x_1, x_2, ...)$, where, for all $t \in \mathbb{N}$, $x_t \in Y$ represents the amount of utility that the generation of period t earns. For all $y, z \in X$, we write $y \geq z$ if $y_i \geq z_i$, for all $i \in \mathbb{N}$; we write $y > z$ if $y \geq z$ and $y \neq z$; and we write $y >> z$, if $y_i > z_i$, for all $i \in \mathbb{N}$.

If Y has only one element, then X is a singleton, and the problem of ranking or evaluating infinite utility streams is trivial. Thus, without further mention, the set Y will always be assumed to have at least two distinct elements.

A social welfare function (SWF) is a mapping $W : X \to \mathbb{R}$. Consider now the axioms that we may want the SWF to satisfy. The first axiom is the standard Pareto condition.

Pareto Axiom: For all $x, y \in X$, if $x > y$, then $W(x) > W(y)$.

The next axiom is the one that captures the notion of 'inter-generational equity'. We shall call it the 'anonymity axiom'.[3] It is equivalent to the notion of 'finite equitableness' (Svensson, 1980) or 'finite anonymity' (Basu, 1994).[4]

Anonymity Axiom: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$ such that $x_i = y_j$ and $x_j = y_i$, and for every $k \in \mathbb{N} - \{i, j\}$, $x_k = y_k$, then $W(x) = W(y)$.

We shall begin by stating the main impossibility theorem that was established in Basu and Mitra (2003, Theorem 1). This will be the setting in which we can then ask the question of what is possible.

[3] In informal discussions throughout the paper, the terms "equity" and "anonymity" are used interchangeably.

[4] The Anonymity Axiom figures prominently in the social choice theory literature, where it is stated as follows: the social ordering is invariant to the information regarding individual orderings as to who holds which preference ordering. Thus, interchanging individual preference profiles does not change the social preference profile. For discussions of this axiom and its acceptability see May (1952) and Sen (1970, 1977).

Theorem 1 *There does not exist any SWF satisfying the Pareto and Anonymity Axioms.*

It is the rather sparse requirement of this theorem that is at the root of the frustration that this field of inquiry has generated. Note, in particular, that the impossibility result does not depend on any continuity postulate on the SWF; and, it applies to all domains of the SWF.

Before exploring the routes out of this, it is useful to place the problem in perspective by recalling Svensson's (1980) important theorem. Let us suppose that we abandon the search for an SWF and instead look for a social welfare ordering[5] (SWO). We then have the result due to Svensson (1980) that there is an SWO which satisfies the (appropriate relational versions of the) Pareto and Anonymity axioms. For reasons of completeness we briefly review Svensson's result. We do this also because, the use of a variant of Szpilrajn's Theorem (due to Suzumura, 1983, Theorem A(5)) allows us to give a particularly easy proof of it. Furthermore, Svensson (1980) restricts his exercise to the case where Y is the closed interval $[0, 1]$; we state the version of his result which applies to any utility space Y. His proof, as well as ours, applies to this more general setting.

Formally, an SWO is a binary relation, $\succsim$, on X, which is complete and transitive. We use $\succ$ and $\sim$ to denote, respectively, the asymmetric and symmetric parts of $\succsim$. The properties of Pareto and Anonymity for an SWO are easy to define. We shall call these axioms $\succsim$-Pareto and $\succsim$-Anonymity to distinguish them from the axioms applied to an SWF.

$\succsim$-Pareto Axiom: For all x, $y \in X$, $x > y$ implies $x \succ y$.

$\succsim$-Anonymity Axiom: For all $x, y \in X$, if there exist $i, j \in \mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$ and for every $k \in \mathbb{N} - \{i, j\}$, $x_k = y_k$, then $x \sim y$.

First, let us give a statement of Suzumura's result. Let Ω be a set of alternatives. If R is a binary relation on Ω and R^* an ordering on Ω, we shall say that R^* is an *ordering extension* of R if, for all $x, y \in \Omega$, $x \, R \, y$ implies $x \, R^* \, y$. We say that R is *consistent* if, for all $t \in \mathbb{N}$, and for all x^1, $x^2, ..., x^t \in \Omega$, $[x^1 R \, x^2$ and not $x^2 R x^1$, and for all $k \in \{2, 3, ..., t-1\}$, $x^k R x^{k+1}]$ implies not $x^t R x^1$.

Lemma 1 (Szpilrajn's Corollary [Suzumura, 1983]) *: A binary relation R on Ω has an ordering extension if and only if it is consistent.*

[5] An ordering is a binary relation which is complete and transitive.

Before proving the next theorem it is useful to introduce some new notation. If $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation and there exists $t \in \mathbb{N}$ such that for all $k > t$, $\sigma(k) = k$, then we shall call σ a *finite permutation.* Given a finite permutation, σ, we shall use $n(\sigma)$ to denote the smallest integer t which has the property that, for all $k > t$, $\sigma(k) = k$.

Given a finite permutation, σ, and $x \in X$, we shall use $x(\sigma)$ to denote $y \in X$, where y is obtained by permuting the elements of x using σ.

In contrast to Theorem 1, we now have:

Theorem 2 (Svensson, 1980) *: There exists a social welfare ordering satisfying the $\succsim$-Pareto and $\succsim$-Anonymity Axioms.*

Proof. Define two binary relations, P and I, on X, as follows. For all $x, y \in X$, if $x > y$ then xPy. And if there exists i, j such that $x_i = y_j$ and $x_j = y_i$, and $x_k = y_k$ for all $k \neq i, j$, then xIy. Now define the binary relation R as follows: $xR\,y \Longleftrightarrow xPy$ or xIy.

To see that R is consistent, suppose $t \in \mathbb{N}$ and $x^1, x^2, ..., x^t \in X$ such that

(A) x^1Rx^2 and not x^2Rx^1, and

(B) x^kRx^{k+1}, for all $k \in \{2, 3, ...t-1\}$.

We have to show that not x^tRx^1.

Note that (A) and (B) can be written equivalently as

(A') x^1Px^2, and

(B') x^kPx^{k+1} or x^kIx^{k+1}, for all $k \in \{2, 3, ...t-1\}$.

Note that (A') and $[x^2Px^3$ or $x^2Ix^3]$ imply that there exists a finite permutation, σ_3, such that:

(A") $x^1Px^3(\sigma_3)$.

Next note that (A") and $[x^3Px^4$ or $x^3Ix^4]$ imply that there exists a finite permutation, σ_4, such that:

$$x^1Px^4(\sigma_4).$$

Continuing in the same way we get the result that there exists a finite permutation, σ_t, such that:

$$x^1Px^t(\sigma_t).$$

This implies not $[x^tPx^1$ or $x^tIx^1]$. Therefore, not x^tRx^1.

Hence, by Szpilrajn's Corollary, R has an ordering extension $\succsim$. Clearly $\succsim$ satisfies the $\succsim$-Pareto Axiom and the $\succsim$-Anonymity Axiom. ∎

For a long time, researchers have conjectured that the impossibility of having a social welfare function satisfying Pareto and anonymity was a problem of *representability*; that is, of there not being "enough real numbers" to do the job. Since Diamond's theorem (1965) showed that the requirements of Pareto, anonymity *and continuity* were inconsistent, the conjecture remained an open one. But in the light of Theorem 1 above we can state a corollary which (a) confirms the conjecture, and (b) clarifies the relation between Theorems 1 and 2 in a way that is especially useful. Toward this end, define:

Representability: A SWO, $\succsim$, is *representable* if there exists a mapping, $f : X \to \mathbb{R}$, such that, for all $x, y \in X$, $x \succsim y \iff f(x) \geq f(y)$.

In the light of Svensson's result, Theorem 1 can be restated as follows.

Corollary 1 *There does not exist a SWO satisfying the $\succsim$-Pareto Axiom, the $\succsim$-Anonymity Axiom and representability.*

Proof. If a representable SWO satisfies the $\succsim$-Pareto Axiom and the $\succsim$-Anonymity Axiom, the real-valued function, $f : X \to \mathbb{R}$, that represents the SWO, must satisfy the Pareto and Anonymity Axioms. But we know from Theorem 1 that no such f exists. This establishes the result. ■

This result makes the nature of the impossibility clear. If we are looking for an equitable SWO (that is, one satisfying the anonymity principle) to evaluate infinite streams of returns, we have to be prepared to weaken the Pareto axiom or to give up the representability requirement. There is a case for exploring both these avenues. In a recent paper Bossert, Sprumont and Suzumura (2004) have looked at the possibilities that emerge when one does not require representability.[6] In what follows, we explore what is possible by relaxing the Pareto axiom.

3 Weakening Pareto

It is arguable that for certain philosophical and even policy purposes we do not need the full power of the Pareto condition (even if we are committed Paretians) simply because all the possibilities that are technically allowed in our specification of the domain may not arise under any eventuality. Indeed for certain ethical discourses involving the comparison of the moral worth of individual actions and universalizable rules (see Basu, 1994) it may be enough to be armed with some weaker forms of Paretianism.

[6] In this connection, see also the papers by Suzumura and Shinotsuka (2003), and Xu (2005).

One idea that may be of interest is to restrict the analysis to cases where one state is obtained from another through changes in a finite number of periods. For such cases it is enough to use the following weakening of Pareto that we shall call "weak dominance."

Weak Dominance Axiom: For all $x, y \in X$, if for some $j \in \mathbb{N}$, $x_j > y_j$, while, for all $k \neq j$, $x_k = y_k$, then $W(x) > W(y)$.

Another version of Pareto–this one has been widely used in the literature (see Arrow, 1963; Sen, 1977)–is the "Weak Pareto" axiom, as defined below.[7]

Weak Pareto Axiom: For all $x, y \in X$, if $x >> y$, then $W(x) > W(y)$.

A natural next step is to consider an axiom that combines the two above axioms. That is precisely what the next axiom does.

Partial Pareto Axiom: The SWF, W, satisfies the Weak Dominance axiom and the Weak Pareto Axiom.

The Partial Pareto Axiom demands that the SWF be positively sensitive to an increase in utility of a single generation, the utilities of other generations being unchanged (and therefore that it be positively sensitive to increases in utilities of any finite number of generations, the utilities of other generations being unchanged), and also that the SWF be positively sensitive to an increase in utilities of all generations. However, it need not be positively sensitive to an increase in utilities of an infinite number of generations, when the utilities of a (non-empty) set of generations is unchanged. This is the principal difference between the Partial Pareto axiom and the Pareto axiom.

3.1 Possibility Results for Restricted Domains

Note that if we recognize that human perception or cognition is not endlessly fine, so that sufficiently small changes in well-being go unperceived, it seems reasonable to suppose that the set of feasible utilities will be a discrete set.[8] The same is true if the benefits are measured in money and there is a well-defined smallest unit, as is true for all currencies (Segerberg, 1976). Thus, it seems worthwhile to explore whether, with $Y \subset \mathbb{M}$ (which captures this very reasonable possibility), there is a social welfare function (on X) respecting

[7] In fact, in some of the literature, what we are calling "Weak Pareto" is often called "Pareto", with the suffix "strong" added to what we have called simply the "Pareto axiom".

[8] The idea of setting a limit to the fineness of human perception has been used in a different context by Armstrong(1939) to argue that it is unreasonable to suppose that indifference is a transitive relation. For a discussion of this issue in individual choice theory, see Majumdar (1962).

Anonymity and one of the weaker versions of the Pareto axiom, introduced above.[9] It is interesting to note that the domain restriction allows us to establish the existence of an equitable SWF, which satisfies the strongest of these versions of Pareto, namely the Partial Pareto axiom.

Proposition 1 *Assume $Y \subset \mathbb{M}$. There exists an SWF satisfying the Partial Pareto and Anonymity Axioms.*

Proof. For each $x \in X$, let $E(x) = \{y \in X :$ there is some $N \in \mathbb{N}$, such that $y_k = x_k$ for all $k \in \mathbb{N}$, which are $\geq N\}$. Let $\Im$ be the collection $\{E : E = E(x)$ for some $x \in X\}$. Then $\Im$ is a partition of X. That is, if E and F belong to $\Im$, then either $E = F$, or E is disjoint from F; further, $\cup_{E \in \Im} E = X$.

Define a function, $f : X \to \mathbb{M}$ as follows. Given any $x \in X$, let $f(x) = \min\{x_1, x_2, ...\}$. Since $x_i \in \mathbb{M}$ for all $i \in \mathbb{N}$, the set $\{x_1, x_2, ...\}$ is a non-empty subset of the set of non-negative integers and therefore has a smallest element [Munkres, 1975, p. 32]. Thus, f is well-defined.

By the axiom of choice, there is a function, $g : \Im \to X$, such that $g(E) \in E$ for each $E \in \Im$.

Given any $x \in X$, we can denote for each $N \geq 1$, $(x_1, ..., x_N)$ by $x(N)$, and $(x_1 + ... + x_N)$ by $I(x(N))$. Next, given any x, y in $E \in \Im$, define $h(x, y) = \lim_{N \to \infty}[I(x(N)) - I(y(N))]$. Notice that h is well-defined, since given any x, y in $E \in \Im$, there is some $M \in \mathbb{N}$, such that $[\ I(x(N)) - I(y(N))]$ is a constant for all $N \geq M$. Now, given any x, y in $E \in \Im$, define $H(x, y) = 0.5[h(x, y)/[1 + |h(x, y)|]]$. Then $H(x, y) \in (-0.5, 0.5)$.

We now define $W : X \to \mathbb{R}$ as follows. Given any $x \in X$, we associate with it its equivalence class, $E(x)$. Then, using the function g, we get $g(E(x)) \in E(x)$. Next, using the functions, h and H, we obtain $h(x, g(E(x)))$ and $H(x, g(E(x)))$. Finally, define $W(x) = f(x) + H(x, g(E(x)))$.

The Anonymity Axiom can be verified as follows. If x, y are in X, and there exist i, j in $\mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i, j$, then $E(x) = E(y)$. Furthermore, denoting this common set by E, we see that $h(x, g(E)) = h(y, g(E))$, and so $H(x, g(E)) =$

[9] While our choice of Y as a subset of the set of non-negative integers is motivated by the imprecision of human perception, the mathematical technique used to obtain our possibility result applies also to the case where $Y = \{(1/n) : n \in \mathbb{N}\}$, where clearly human perception has to be considered to be sufficiently refined.

$H(y, g(E))$. Further, the set $\{x_1, x_2, ...\}$ is the same as the set $\{y_1, y_2, ...\}$, so that $f(x) = f(y)$. Thus, we obtain: $W(x) = W(y)$.

The Partial Pareto Axiom can be verified as follows. If x, y are in X, and there exists $i \in \mathbb{N}$, such that $x_i > y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i$, then $E(x) = E(y)$. Furthermore, denoting the common set by E, we see that $h(x, g(E)) > h(y, g(E))$. This implies $H(x, g(E)) > H(y, g(E))$. Further, the smallest element of the set $\{x_1, x_2, ...\}$ is at least as large as the smallest element of the set $\{y_1, y_2, ...\}$, so that we have $f(x) \geq f(y)$. Thus, we obtain the desired inequality: $W(x) > W(y)$.

If $x, y \in X$, and $x >> y$, then $E(x) \neq E(y)$. Thus, we will not be able to compare $H(x, g(E(X)))$ with $H(y, g(E(y)))$. However, we do know that $H(x, g(E(x))) > -0.5$, and $H(y, g(E(y))) < 0.5$. Further, since $x >> y$, we have $f(x) \geq f(y) + 1$. Thus, we obtain:

$$W(x) = f(x) + H(x, g(E)) > f(y) + 1 - 0.5 > f(y) + H(y, g(E)) = W(y).$$

■

Proposition 1 has two shortcomings. First, it is a possibility result for a social welfare function, but we do not know how to construct the social welfare function whose existence is asserted, since our proof uses the Axiom of Choice.[10]The possible policy use of Proposition 1 is therefore limited.

The second shortcoming can be seen by considering the set-up, where $Y = \{0, 1\}$, so that we have the strongest possible domain restriction. Theorem 1 implies that there is no SWF respecting the Pareto and Anonymity Axioms. And, Proposition 1 implies that there is an SWF satisfying the Partial Pareto and Anonymity Axioms. It follows that any social welfare function, W, so obtained, must violate the Pareto principle in a way that is particularly disturbing; that is, it must be the case that there exist alternatives $x, y \in X$ such that $x > y$, but $W(x) < W(y)$.

To see this, suppose on the contrary that there is an SWF, W, satisfying the Anonymity and Partial Pareto axioms, and the "monotonicity condition":

$$\text{For all } x, y \in X, \text{ if } x > y, \text{ then } W(x) \geq W(y) \qquad \text{(M)}$$

We claim then that W must, in fact, satisfy the Pareto Axiom. To see this, let $x, y \in X$ with $x > y$. There are three possibilities: (i) $x >> y$, (ii) $x_i > y_i$

[10]The use of the Axiom of Choice in proving *impossibility results* is, perhaps, less objectionable.

for $i \in \mathbb{F}$, where $\mathbb{F}$ is a finite subset of $\mathbb{N}$, and $x_i = y_i$ for all $i \in \mathbb{N} \sim \mathbb{F}$, (iii) $x_i > y_i$ for $i \in \mathbb{I}$, where $\mathbb{I}$ is an infinite strict subset of $\mathbb{N}$, and $x_i = y_i$ for all $i \in \mathbb{N} \sim \mathbb{I}$. In cases (i) and (ii), by the Partial Pareto axiom, we must have $W(x) > W(y)$. In case (iii), let j be the smallest index in $\mathbb{I}$, and define z by $z_j = y_j$ and $z_i = x_i$ for all $i \neq j$. Then, $z \in X$, and $z > y$, so that by (M), $W(z) \geq W(y)$. Also, comparing x and z, we see that they differ in only the $j-th$ index, and $x_j > y_j = z_j$, so that the Partial Pareto axiom implies that $W(x) > W(z)$. Thus, $W(x) > W(y)$, and our claim is established. But, by Theorem 1, there is no SWF satisfying the Pareto and Anonymity axioms. Consequently, any SWF, W, satisfying the Anonymity and Partial Pareto axioms, must violate the "monotonicity condition" (M).[11]

Both the shortcomings of Proposition 1 arise from the fact that we are trying to define a social welfare function, which is sensitive to the utility of a single generation, when the utilities of all other generations are unchanged. If we give up this sensitivity, and weaken our Partial Pareto requirement to the Weak Pareto one, we get a particularly satisfying possibility result on all domains X, when $Y \subset \mathbb{M}$.

Theorem 3 *Assume $Y \subset \mathbb{M}$. Then the SWF, $W : X \to \mathbb{M}$, given by:*

$$W(x) = \min\{x_1, x_2, ...\} \text{ for all } x \in X \tag{MIN}$$

satisfies the Weak Pareto and Anonymity Axioms. Further, the SWF, defined by (MIN), satisfies the monotonicity condition (M).

Proof. The function, $W : X \to \mathbb{M}$, given by (MIN) is well-defined (as already noted in the proof of Proposition 1). If $x, y \in X$ and $x >> y$, then denoting an index, for which $\min\{x_1, x_2, ...\}$ is attained, by $k \in \mathbb{N}$, we have:

$$W(y) = \min\{y_1, y_2, ...\} \leq y_k < x_k = \min\{x_1, x_2, ...\} = W(x)$$

so that the Weak Pareto axiom is satisfied.

If $x, y \in X$, and there exist $i, j \in \mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i, j$, then the set $\{x_1, x_2, ...\}$ is the same as the set $\{y_1, y_2, ...\}$, so that $W(x) = W(y)$. Thus, the Anonymity axiom is satisfied.

[11] A weak version of Pareto, which requires that the "monotonicity condition" (M), together with what we have called Weak Pareto axiom, be satisfied, is quite appealing, and has been proposed and examined by Diamond (1965).

Finally, if $x, y \in X$ and $x > y$, then denoting an index, for which $\min\{x_1, x_2, ...\}$ is attained, by $k \in \mathbb{N}$, we have:

$$W(y) = \min\{y_1, y_2, ...\} \leq y_k \leq x_k = \min\{x_1, x_2, ...\} = W(x)$$

so that the monotonicity condition (M) is satisfied. ■

The social welfare function in Theorem 3 can be explicitly written down (as in (MIN)), and this makes the possibility result especially useful for policy purposes.

3.2 Weakening Domain Restrictions

The above possibility results are obtained by weakening the Pareto axiom (to Partial Pareto or to Weak Pareto)and also considering a discrete domain. How would a change in the latter affect the results? It is especially useful to ask this question in the context where $Y = [0, 1]$, since this is the standard framework used by Koopmans (1960), Diamond (1965), Svensson (1980) and others.

As it turns out, we run again into impossibility results, which means that with $Y = [0, 1]$, the weakening of Pareto to Partial Pareto or to Weak Pareto does not help to reverse the impossibility result of Theorem 1. To establish the first of these impossibility results, which follows directly from the result of Basu and Mitra (2003, Theorem 2), it is useful to introduce a new axiom, the interest in which is purely constructive, so as to be able to explain the next result clearly.

Dominance Axiom: For all $x, y \in X$, if there exists $j \in \mathbb{N}$ such that $x_j > y_j$, and, for all $k \neq j$, $x_k = y_k$, then $W(x) > W(y)$. For all $x, y \in X$, if $x >> y$, then $W(x) \geq W(y)$.

Note that the last inequality in the statement of this axiom is a weak inequality, unlike in the definition of the Partial Pareto Axiom. Hence, Partial Pareto is stronger than Dominance (which in turn is stronger than Weak Dominance).[12]

Proposition 2 *Assume* $Y \supset [0, 1]$. *There is no SWF satisfying the Partial Pareto and Anonymity Axioms.*

[12]It is also worth noting that between Dominance and Weak Pareto, neither is stronger than the other. They are in fact non-comparable in terms of strength. The same is true between Weak Dominance and Weak Pareto.

Proof. By Theorem 2 of Basu and Mitra (2003), we know that there is no SWF satisfying the Dominance and Anonymity axioms. The result is proved by noting that the Partial Pareto axiom is stronger than the Dominance axiom. ■

When we weaken the Partial Pareto axiom (of Proposition 2) to Weak Pareto, the impossibility result persists, but it is a more subtle result, since the sensitivity of the SWF to a change in a single generation's utility (when the utilities of all other generations are unchanged) is not being imposed. The proof of it is, likewise, more intricate, combining the methods used by Basu and Mitra (2003, Theorem 2) and by Fleurbaey and Michel (2003).

Theorem 4 *Assume* $Y \supset [0,1]$. *There is no SWF satisfying the Weak Pareto Axiom and the Anonymity Axiom.*

Proof. To establish the theorem, assume that there exists a social welfare function, $W : X \to \mathbb{R}$, which satisfies the Weak Pareto and Anonymity Axioms.

Denote the vector $(1, 1, 1, ...)$ in X by e. Define the sequences x and y in X as follows:

$$x = (\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, ..., \frac{1}{4^k}, ..., \frac{4^k - 1}{4^k}, ...) \tag{1}$$

$$y = (\frac{1}{4} + \frac{1}{16}, \frac{2}{4} + \frac{1}{16}, \frac{3}{4} + \frac{1}{16}, ..., \frac{1}{4^k} + \frac{1}{4^{k+1}}, ..., \frac{4^k - 1}{4^k} + \frac{1}{4^{k+1}}, ...) \tag{2}$$

For $s \in I \equiv (-0.5, 0.5)$, define:

$$y(s) = 0.5y + 0.25(1 + s)e \tag{3}$$

Then $(1/8)e \leq y(s) \leq (7/8)e$, and so $y(s) \in X$ for each $s \in I$.

Define the function, $f : I \to \mathbb{R}$ by: $f(s) = W(y(s))$. By the Weak Pareto Axiom, f is monotonic increasing in s on I. Thus f has only a countable number of points of discontinuity in I. Let $a \in I$ be a point of continuity of the function f.

Define the sequence $x(a)$ as follows:

$$x(a) = 0.5x + 0.25(1 + a)e \tag{4}$$

Clearly, $x(a) \in X$ and $y(a) >> x(a)$. By the Weak Pareto Axiom, $W(y(a)) > W(x(a))$. We denote $[W(y(a)) - W(x(a))]$ by θ; then $\theta > 0$.

Denote max$(0.5-a,\ 0.5+a)$ by $\triangle$; then, $\triangle > 0$. Since f is continuous at a, given the θ defined above, there exists $\delta \in (0, \triangle)$, such that: $0 < |s-a| < \delta$ implies $|f(s) - f(a)| < \theta$. Note that for $0 < |s-a| < \delta$, we always have $s \in I$.

For $p \in \mathbb{N}$, let $r(p)$ denote the first non-zero remainder of the successive divisions of p by 4, and $q(p)$ the number of divisions with a zero remainder. [For example, $r(52) = 1$ and $q(52) = 1$].

Define (following Fleurbaey and Michel (2003, page 796)), for each $k \in \mathbb{N}$, a sequence x^k as follows:

$$\begin{aligned} x^k \quad = \quad & (\frac{1}{4} + \frac{1}{16}, \frac{2}{4} + \frac{1}{16}, \frac{3}{4} + \frac{1}{16}, ..., \frac{1}{4^k} + \frac{1}{4^{k+1}}, ..., \frac{4^k-1}{4^k} + \frac{1}{4^{k+1}}, \\ & \frac{1}{4^{k+1}}, ..., \frac{4p}{4^{k+1}}, ..., \frac{4p}{4^{k+1}}, \frac{4p+2}{4^{k+1}}, \frac{4p+3}{4^{k+1}}, ..., \frac{4^{k+1}-1}{4^{k+1}}, \frac{1}{4^{k+2}}, \\ & \frac{2}{4^{k+2}}, ..., \frac{4^{k+2}-1}{4^{k+2}}, ...) \end{aligned} \tag{5}$$

where p runs from 1 to $4^k - 1$, and the term $[4p/(4^{k+1})]$ is repeated $q(4p)$ times if $r(4p) = 1$, and $q(4p) + 1$ times otherwise. Now, for each $k \in \mathbb{N}$, we use x^k to define $x^k(a)$ as follows:

$$x^k(a) = 0.5x^k + 0.25(1+a)e \tag{6}$$

Clearly, $x^k(a) \in X$ for each $k \in \mathbb{N}$. Comparing the expressions for $x(a)$ and $x^1(a)$ in (4) and (6) respectively, we see that $x^1(a)$ is obtained from $x(a)$ by a finite permutation, and that for all $k > 1$, $x^k(a)$ is obtained from $x^{k-1}(a)$ by a finite permutation. Thus, for every $k \in \mathbb{N}$, $x^k(a)$ is obtained from $x(a)$ by a finite permutation, and the Anonymity Axiom yields:

$$W(x^k(a)) = W(x(a)) \qquad \textit{for all } k \in \mathbb{N} \tag{7}$$

Choose $K \in \mathbb{N}$ with $K \geq 2$ such that $(1/4^{K-2}) < \delta$, and define $S = (a - (1/4^{K-2}))$. We note that $0 < (a - S) < \delta$, and so $S \in I$, and:

$$W(y(S)) = f(S) > f(a) - \theta = W(y(a)) - \theta \tag{8}$$

We now compare the welfare levels associated with $x^K(a)$ and $y(S)$ as follows. Notice that:

$$\begin{aligned}
x^K(a) &= 0.5x^K + 0.25(1+a)e = 0.5y + 0.25(1+a)e - 0.5(y - x^K) \\
&= y(a) - 0.5(y - x^K) \\
&\geq y(a) - 0.5(1/4^K)e \\
&= 0.5y + 0.25(1+a)e - 0.5(1/4^K)e \\
&> 0.5y + 0.25(1 + a - (1/4^{K-1}))e \\
&= 0.5y + 0.25(1 + a - (1/4^{K-2}))e + 0.25(3/4^{K-1})e \\
&> \, > 0.5y + 0.25(1+S)e = y(S)
\end{aligned}$$

Thus, by the Weak Pareto Axiom, we have:

$$W(x^K(a)) > W(y(S)) \tag{9}$$

Using (7), (8) and (9), we obtain:

$$\begin{aligned}
W(y(a)) - \theta &= W(x(a)) \\
&= W(x^K(a)) \\
&> W(y(S)) \\
&> W(y(a)) - \theta
\end{aligned}$$

a contradiction, which establishes our result. ■

It is worth noting that, with the domain restriction $Y \subset \mathbb{M}$, weakening the Pareto axiom to the Weak Pareto axiom led to a reversal of the impossibility result of Theorem 1 to the possibility result of Theorem 3. When $Y = [0, 1]$, a similar weakening of the Pareto axiom (to the Weak Pareto axiom) does *not* produce such a reversal.

This suggests that to recover possibility when $Y = [0, 1]$, we need to go to a weaker form of Pareto. In fact, Weak Dominance is not weaker than Weak Pareto, but we *can* establish the existence of an equitable SWF, which satisfies Weak Dominance. In fact, this possibility result holds with *no* domain restriction. Our proof employs the idea, already used in the proof of Proposition 1, of partitioning X into sets such that the members of each set differ from each other in only a finite number of indices. The proof of the possibility result then crucially hinges on (i) the use of the Axiom of Choice, and (ii) the fact that Weak Dominance never requires one to compare the welfare of members in two different sets of the partition.

Theorem 5 *There exists an SWF satisfying the Weak Dominance and Anonymity Axioms.*

Proof. For each $x \in X$, let $E(x) = \{y \in X :$ there is some $N \in \mathbb{N}$, such that $y_k = x_k$ for all $k \in \mathbb{N}$, which are $\geq N\}$. Let $\Im$ be the collection $\{E : E = E(x)$ for some $x \in X\}$. Then, $\Im$ is a partition of X. By the axiom of choice, there is a function, $g : \Im \rightarrow X$, such that $g(E) \in E$, for each $E \in \Im$.

Given any x, y in $E \in \Im$, define $h(x, y) = \lim_{N\rightarrow\infty}[I(x(N)) - I(y(N))]$.We now define $W : X \rightarrow \mathbb{R}$ as follows. Given any $x \in X$, we associate with it its equivalence class, $E(x)$. Then, using g, we get $g(E(x)) \in E(x)$, and, using h, we obtain $h(x, g(E(x)))$. Now, define $W(x) = h(x, g(E(x)))$. The Anonymity Axiom and the Weak Dominance Axioms are easily verified. ■

4 Concluding Remarks

We wanted to demarcate the boundary between what is possible and what is not and the set of results established in this paper tries to do that vis-a-vis variations of the Pareto Axiom and the domain restriction for utilities. In setting out to write this paper we had wanted to display the positive side of this field, namely, the possibility theorems. We have done so. But now, at paper's end, we find that in the process we have also highlighted the robustness of the impossibility theorems of the literature. This is probably a reminder that we have no option but to play the hand that we are dealt.

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