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On the Existence of Paretian Social Welfare Relations for Infinite Utility Streams with Extended Anonymity

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Abstract

In this paper, we examine the restrictions that any concept of extended anonymity must satisfy in order to be compatible with the existence of a Paretian social welfare relation (SWR). We completely characterize the class of *permissible permutations* associated with any Paretian SWR; that is, those permutations with respect to which every utility stream is pronounced to be indifferent to the corresponding permuted utility stream, according to the Paretian SWR. Based on the characterization result, we propose a particular class of extensions of anonymity, which allows comparisons of utility streams that are related to each other by an infinite number of permutations of a specific type. The merits of this particular class of extensions are discussed.

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Key Words: intergenerational equity, extended anonymity, Pareto axiom, social welfare relations, cyclic permutations, permutation groups, grading principle.

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1 Introduction

In ranking social states which are specified by infinite utility streams, it is customary to use a social welfare relation (SWR), a reflexive and transitive binary relation on the social states, satisfying two widely accepted guiding principles. The equal treatment of all generations, proposed by Ramsey (1928), is formalized in the Anonymity Axiom. The positive sensitivity of the social preference structure to the well-being of each generation is reflected in the Pareto Axiom.

The Anonymity axiom says that if $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are infinite utility streams, and x can be obtained by applying a finite permutation to y, then x should be declared indifferent to y. Many authors have felt that a stronger notion than the Anonymity Axiom is needed to reflect intergenerational equity in intertemporal preferences.[1] This essentially means that in comparing infinite utility streams, indifference would be postulated for a larger class of permutations[2], which would include finite permutations as a special case.

The basic question that arises then is the following: how would one specify this larger class of permutations? An approach followed in the literature has been to specify a class of infinite permutations and to argue that society *should be* indifferent between utility streams when one stream can be obtained from another by applying such an infinite permutation to it.[3]

The approach taken in this paper is somewhat different. We wish to identify the class of permutations that *can be* allowed, given the very structure of the problem. That is, given that we seek a SWR which must satisfy the Pareto axiom, we wish to analyze the restrictions (if any) on the class of permutations with respect to which utility streams can be pronounced to be indifferent. What is involved here is a logical consistency check rather than any ethical principle.

[1] See, for example, Lauwers (1995, 1997), Liedekerke and Lauwers (1977), Fleurbaey and Michel (2003).

[2] In what follows, we will use the terms "permutations" and "permutation matrices" interchangeably. The connection between the two is the following. A permutation is a one to one map from the natural numbers onto the natural numbers. Any such permutation can be represented by a permutation matrix. See Section 2 for a discussion.

[3] See the papers by Lauwers (1995, 1998), where he considers a class of permutations π (which he calls "bounded permutations") which satisfy $(\pi(n)/n) \to 1$ as $n \to \infty$. It is not quite clear, though, why this class is of special interest from the point of view of intergenerational equity.

The problem with postulating indifference with respect to *arbitrary infinite permutations* is, of course, that preference relations with this feature would violate the Pareto axiom.[4] Thus, it is clear that the class of permutations, with respect to which indifference is postulated, would have to be restricted in some way if it is to be compatible with any given Paretian SWR. However, somewhat surprisingly, there is no systematic study in the literature of the class of permutations which are permissible, in the sense that every utility stream is pronounced to be indifferent to the corresponding permuted utility stream, according to the given Paretian SWR.[5]

Our analysis reveals two clear-cut restrictions. Given a Paretian SWR, and denoting the set of *permissible permutations* associated with it by Π, we see that the Pareto axiom implies that the permutations in Π must be *cyclic*. Further the transitivity property of the SWR implies that the set Π (together with the operation of matrix mutliplication of infinite permutation matrices) must be a *group*.[6]

These are significant restrictions, dictated entirely by the mathematical structure of the problem. They also exhaust *all* the restrictions imposed by the nature of the problem. That is, given any group $\mathcal{Q}$ of cyclic permutations, there is a Paretian SWR, such that the class Π of permissible permutations associated with it coincides exactly with $\mathcal{Q}$. Thus, we provide a complete characterization of permissible permutations that are consistent with the existence of a Paretian social welfare relation on infinite utility streams.[7]

As the proof of our (sufficiency) result shows, a social welfare relation that suffices for this purpose is exactly of the type known as the *Suppes-Sen grading principle*, except that it is defined with respect to all the permutations

[4]This point is well-recognized in the literature. See, for example, Lauwers (1997), and Asheim and Tungodden (2004).

[5]Fleurbaey and Michel (2003) have undertaken a very comprehensive study of anonymity with respect to infinite permutations. However, their approach is to specify a class of permutations (they consider fixed step, variable step and finite length permutations)and ask whether indifference with respect to this class is consistent with axioms like Pareto or Weak Pareto or Continuity. Our approach treats the class of permutations as a "choice variable" and seeks to characterize the class which is compatible with Paretian SWRs.

[6]The terms "permissible permutations","cyclic permutations" and "group of permutation matrices" are formally defined in Section 2.

[7]The framework in which our result is established is, by now, the standard one, employed, for instance, in Diamond (1965), Svensson (1980) and Basu and Mitra (2003).

in the specified group, instead of the class of finite permutations.[8] Thus, the social welfare relations we propose can be viewed as extended Suppes-Sen grading principles.

In view of our characterization result, we introduce a notion of extended anonymity in which the rearrangements of utility streams allowed in any pairwise comparison are restricted to a sequence of permutations within blocks of time of *equal length*.[9] These blocks of time might be considered to be extended "time periods" and permutations within each block might be treated just like rearrangements in finite societies. We show that this class of permutations is a group of cyclic permutations (and hence consistent with the existence of a Paretian social welfare relation), which constitutes a strict extension of the class of finite permutations.

2 Preliminaries

2.1 Notation

Let $\mathbb{N}$ denote, as usual, the set of natural numbers $\{1, 2, 3, ...\}$, and let $\mathbb{R}$ denote the set of real numbers. Let Y denote the closed interval $[0, 1]$, and let the set $Y^{\mathbb{N}}$ be denoted by X . Then, X is the domain of utility sequences that we are interested in. Hence, $x \equiv (x_1, x_2, \ldots) \in X$ if and only if $x_n \in [0, 1]$ for all $n \in \mathbb{N}$.

For $y, z \in \mathbb{R}^{\mathbb{N}}$, we write $y \geq z$ if $y_i \geq z_i$ for all $i \in \mathbb{N}$; and, we write $y > z$ if $y \geq z$, and $y \neq z$.

2.2 Definitions

A *social welfare relation* (SWR) is a binary relation, $\succsim$, on X , which is reflexive and transitive (a quasi ordering). We associate with $\succsim$ its symmetric and asymmetric components in the usual way. Thus, we write $x \sim y$ when

[8] The grading principle is due to Suppes (1966). For a comprehensive analysis of it, see Sen (1971). Svensson (1980) provides a formal definition of the Suppes-Sen grading principle in the context of infinite utility streams. It can be characterized as the least restrictive SWR satisfying the Pareto and Anonymity axioms; see d'Aspremont (1985) and Asheim, Buchholz and Tungodden (2001).

[9] This is precisely the class of permutations, which are called fixed-step permutations in Fleurbaey and Michel (2003).

$x \succsim y$ and $y \succsim x$ both hold; and, we write $x \succ y$ when $x \succsim y$ holds, but $y \not\succsim x$ does not hold.

A SWR $\succsim_A$ is a *subrelation* to a SWR $\succsim_B$ if (a) $x, y \in X$ and $x \succsim_A y$ implies $x \succsim_B y$; and (b) $x, y \in X$ and $x \succ_A y$ implies $x \succ_B y$.

2.3 Permutations

A *permutation* π is a one-to-one map from $\mathbb{N}$ onto $\mathbb{N}$. Any $x \in X$ can be viewed as a map from $\mathbb{N}$ to Y, associating with each $n \in \mathbb{N}$ the element $x_n \in Y$. The composite map $x \circ \pi$ is then a map from $\mathbb{N}$ to Y, associating with each $n \in \mathbb{N}$ an element $\pi(n)$ through the map π, and then associating the element $x_{\pi(n)} \in Y$ through the map x. Thus, if x is written as the sequence $(x_1, x_2, \ldots) \in X$, then $x \circ \pi$ is written as the sequence $(x_{\pi(1)}, x_{\pi(2)}, \ldots) \in X$.

Any permutation π can be *represented* by a permutation matrix. A *permutation matrix* $P = (p_{ij})_{i \in \mathbb{N}, j \in \mathbb{N}}$, is defined as follows:

$$\begin{gathered}
(i)\, For\ each\ i \in \mathbb{N},\ there\ is\ j(i) \in \mathbb{N},\ such\ that\ p_{ij(i)} = 1 \\
and\ p_{ij} = 0\ for\ all\ j \neq j(i) \\
(ii)\, For\ each\ j \in \mathbb{N},\ there\ is\ i(j) \in \mathbb{N},\ such\ that\ p_{i(j)j} = 1 \\
and\ p_{ij} = 0\ for\ all\ i \neq i(j)
\end{gathered}$$

Given any permutation π, there is a permutation matrix, P, such that if $x \in X$, then $x \circ \pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots)$ can be written as Px in the usual sense of matrix multiplication. Notice that for any permutation matrix P and any $x \in X$, the matrix multiplication is well-defined, since each row of P has one non-zero entry. Conversely, given any permutation matrix P, there is a permutation π defined by $\pi = Pa$, where $a = (1, 2, 3, ...)$. We denote the set of all permutation matrices by $\mathcal{P}$.

A *finite permutation* π is a permutation, such that there is some $N \in \mathbb{N}$, with $\pi(n) = n$ for all $n > N$. The set of all finite permutations[10] is denoted by $\mathcal{F}$.

It is useful to recall some basic properties of permutation matrices.[11] (i) If $P, Q \in \mathcal{P}$, then $PQ \in \mathcal{P}$. (ii) The infinite identity matrix, I, belongs to $\mathcal{P}$, and for each $P \in \mathcal{P}$, we have $PI = IP = P$. (iii) Given any $P \in \mathcal{P}$, the transpose of P, denoted by P', belongs to $\mathcal{P}$, and $PP' = P'P = I$, so that

[10] For basic properties of finite permutations, see, for example, Hohn (1973).

[11] Some of the basic properties of infinite permutation matrices can be found in Cooke (1950).

P' is the inverse of P. (iv) Finally, for $P, Q, R \in \mathcal{P}$, we have:

$$P(QR) = (PQ)R$$

Thus, $\mathcal{P}$ is a *group* under the usual matrix multiplication operation.[12]

The n-th unit vector in X is the sequence in X with 1 in the n-th place and 0 elsewhere, and is denoted by e^n for each $n \in \mathbb{N}$. The set of unit vectors $\{e^1, e^2, ...\}$ is denoted by U.

If $x \in X$, then x can be written as:

$$x = \sum_{n=1}^{\infty} x_n e^n$$

where the infinite sum is interpreted as the co-ordinate wise convergence limit of the finite sum $\sum_{n=1}^{N} x_n e^n$ as $N \to \infty$.

If $P \in \mathcal{P}$, and $x \in X$, then (in view of the above representation of x) properties of the rearranged sequence Px can be studied by seeing how the permutation P acts on the unit vectors of X. Note that given any $e^n \in U$, the permutation P transforms the unit vector e^n to a unit vector (possibly different from e^n). Thus, the permutation matrix P maps U to U. We can, therefore, consider repeated applications of P to U, and these iterates would also remain in U. Given any $n \in \mathbb{N}$, we can consider the sequence:

$$(Pe^n, P^2 e^n, ...)$$

generated by iterates of P applied to the unit vector e^n. The sequence is called *non-wandering* if there exist $i, j \in \mathbb{N}$, with $i < j$, such that $P^i e^n = P^j e^n$. Otherwise, it is called *wandering*. As the name suggests, a wandering sequence never revisits a point.

Non-wandering sequences can be characterized more simply as follows. Denoting $(j - i)$ by k, we see that by applying P' repeatedly (i times) to the equation $P^i e^n = P^j e^n$, we would get $e^n = P^k e^n$. Thus, a non-wandering sequence returns to e^n after a finite number of iterations. Its structure therefore is of the form of an infinitely repeated *cycle* $(Pe^n, P^2 e^n, ..., P^k e^n (=$

[12]A *group* is a set of objects, $\mathcal{G}$, together with a binary operation $\otimes$ on $\mathcal{G}$ such that:
(i) If $A, B \in \mathcal{G}$, then $A \otimes B \in \mathcal{G}$.
(ii) An identity element, $I \in \mathcal{G}$, such that for every $A \in \mathcal{G}$, $I \otimes A = A \otimes I = A$.
(iii) For every $A \in \mathcal{G}$, there is $A' \in \mathcal{G}$, such that $A \otimes A' = A' \otimes A = I$.
(iv) If $A, B, C \in \mathcal{G}$, then $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

$e^n), Pe^n, P^2e^n, ..., e^n, ...)$. If m is the smallest integer for which $P^m e^n = e^n$, then m is called the *period* of the cycle.

If P is a permutation such that for each unit vector $e^n \in U$, its iterates generate a non-wandering sequence, then P is called *cyclic*. Thus, a cyclic permutation P generates an infinitely repeated cycle, starting with every unit vector. [Notice that, in general, the period of the cycle generated might be different for different unit vectors].

A useful property of a cyclic permutation P is that its inverse is also cyclic. To see this, consider an arbitrary unit vector $e^k \in U$. Since P is cyclic, there is some $m \in \mathbb{N}$ such that $P^m e^k = e^k$. Applying $Q = P' = P^{-1}$ to this equation repeatedly (m times), we get $e^k = Q^m e^k$. Thus, Q is cyclic.

3 On Paretian SWRs with Extended Anonymity: Necessary Conditions

Given a social welfare relation $\succsim$ on X, the set of its *permissible permutations* is defined to be:

$$\Pi(\succsim) = \{P \in \mathcal{P} : Px \sim x \text{ for all } x \in X\}$$

That is, it is the class of permutations with respect to which every utility stream is pronounced to be indifferent to the corresponding permuted utility stream. Note that since the infinite identity matrix, I, belongs to $\mathcal{P}$, and $\succsim$ is reflexive, I belongs to $\Pi(\succsim)$, so that $\Pi(\succsim)$ is always non-empty.

The standard anonymity axiom may be stated as follows.

Axiom 1 ***(Anonymity)*** *If $x, y \in X$, and there exist i, j in $\mathbb{N}$, such that $x_i = y_j$ and $x_j = y_i$, while $x_k = y_k$ for all $k \in \mathbb{N}$, such that $k \neq i, j$, then $x \sim y$.*

It is easy to see that a SWR $\succsim$ satisfies the Anonymity axiom if and only if for every *finite* permutation $P \in \mathcal{F}$, and every $x \in X$, we have $Px \sim x$. That is, $\succsim$ satisfies the Anonymity axiom if $\mathcal{F} \subset \Pi(\succsim)$.

The standard anonymity axiom suggests that we can write an extended anonymity axiom in the following way, with respect to a class $\mathcal{Q} \subset \mathcal{P}$ of permutations, where $\mathcal{F} \subset \mathcal{Q}$.

Axiom 2 ***($\mathcal{Q}-$Anonymity)*** *If $\mathcal{F} \subset \mathcal{Q} \subset \mathcal{P}$, then for every $x \in X$, we have $Px \sim x$ if $P \in \mathcal{Q}$.*

That is, $\succsim$ satisfies $\mathcal{Q}$-Anonymity (where $\mathcal{F} \subset \mathcal{Q} \subset \mathcal{P}$) if $\mathcal{Q} \subset \Pi(\succsim)$.

We are interested in SWRs on X, which satisfy the well-known Pareto Axiom.

Axiom 3 ***(Pareto)*** *If $x, y \in X$, and there is some $j \in \mathbb{N}$, such that $x_j > y_j$, while $x_k \geq y_k$ for all $k \neq j$, then $x \succ y$.*

SWRs, satisfying the Pareto axiom, are called *Paretian* SWRs.

3.1 Permissible Extensions of Anonymity: Two Results

The question we seek to address in this subsection is the following. Given a Paretian SWR $\succsim$, what properties are satisfied by the set of its *permissible permutations*, $\Pi(\succsim)$? Unlike the literature, we do not *postulate* any form of anonymity axiom, but rather seek to *identify* the class of permutations under which the given relation pronounces every utility stream to be indifferent to the corresponding permuted utility stream.

We obtain two restrictions that Π must satisfy.[13] First, every $P \in \Pi$ must be cyclic; second the set Π (together with the usual operation of matrix multiplication) must constitute a group.[14] We take up each of these results in turn.

For the first result, we provide, in fact, a complete characterization of cyclic permutations which might be of independent interest.[15]

[13]When there is no danger of confusion, we will denote $\Pi(\succsim)$ by Π, it being understood that Π is associated with the SWR $\succsim$ given in the relevant context. This simplifies the notation.

[14]Infinite permutation matrices have not been systematically studied in the mathematics literature, which focuses almost exclusively on one problem: what is the class of rearrangements which will preserve the sum of a conditionally convergent series? See Schaefer (1981) and the references cited in his paper. This problem arose from a famous result of Riemann that a rearrangement of a conditionally convergent series can be convergent to any pre-specified number or even divergent.

[15]Finite permutations are always cyclic. Thus, the characterization result in Lemma 1 distinguishes the class of infinite permutation matrices from finite permutation matrices, once one recognizes that there are infinite permutation matrices which are not cyclic (see Example 1 discussed in the next subsection). For non-cyclic permutation matrices, the well-known (homogeneous version of the) Farkas Lemma (see, for example, Gale (1960, Theorem 2.9, p.48)) fails. There are, of course, infinite dimensional versions of the Farkas Lemma (see, for example, Braunschweiger and Clark (1962)), but the conditions for their validity rule out (as they must) non-cyclic permutation matrices.

Lemma 1 *A permutation $P \in \mathcal{P}$ is cyclic if and only if there is no $x \in X$ satisfying $Px > x$.*

Proof. Suppose $P \in \mathcal{P}$ is cyclic, but there is some $x \in X$ satisfying $Px > x$. Then we can find a unit vector e^k and a positive real number ε, such that:

$$Px - x \geq \varepsilon e^k \tag{1}$$

This yields the sequence of inequalities:

$$\begin{gathered} Px - x \geq \varepsilon e^k \\ P^2x - Px \geq \varepsilon P e^k \\ P^3x - P^2x \geq \varepsilon P^2 e^k \\ \cdots \end{gathered} \tag{2}$$

Let m be the period of the cycle of P. Summing the inequalities in (2) for $N = sm$, where $s \in \mathbb{N}$,

$$P^N x - x = \sum_{n=1}^{N} [P^n x - P^{n-1} x] \geq \varepsilon s [\sum_{n=1}^{m} P^n e^k] \tag{3}$$

Denoting the sequence $(1, 1, 1, ...)$ by e, we have from (3),

$$(e/s) \geq \varepsilon [\sum_{n=1}^{m} P^n e^k] \;\; for \; all \; s \in \mathbb{N} \tag{4}$$

But the vector on the left-hand side of (4) goes to zero as $s \to \infty$, while the right-hand side of (4) is a non-negative non-zero vector independent of s. This contradiction establishes the necessity part of the result.

To establish sufficiency, suppose that $P \in \mathcal{P}$ is not cyclic. Then, denoting the inverse of P by Q, we know that Q cannot be cyclic. Thus, there is some unit vector $e^k \in U$, for which the sequence $(Qe^k, Q^2e^k, ...)$ is wandering. Each vector in this sequence is a unit vector. Since the sequence is wandering, any unit vector occurs at most once in the sequence. Thus, the sequence $(x(1), x(2), ...)$ defined by:

$$x(N) = \sum_{n=1}^{N} Q^n e^k \;\; for \; N \in \mathbb{N} \tag{5}$$

is a monotonic non-decreasing sequence in X, bounded above by $e = (1, 1, 1, ...)$. Consequently, $x(N)$ has a (coordinatewise convergence) limit as $N \to \infty$. Define this limit by x; then $x \in X$.

Multiplying through in (5) by Q, we have:

$$Qx(N) = \sum_{n=1}^{N} Q^{n+1} e^k \quad for\ N \in \mathbb{N} \tag{6}$$

Subtracting (6) from (5), for each $N \in \mathbb{N}$,

$$x(N) - Qx(N) = Qe^k - Q^{N+1} e^k \tag{7}$$

Taking coordinatewise convergence limits in (7), we obtain:

$$x - Qx = Qe^k \tag{8}$$

Multiplying through in (8) by P, we get:

$$Px - x = e^k > 0$$

This completes the sufficiency part of the proof. ■

We now note a principal implication of this characterization of cyclic permutations.

Proposition 1 *Suppose $\succsim$ is a Paretian SWR. Then, every $P \in \Pi(\succsim)$ must be cyclic.*

Proof. Given $\succsim$, denote $\Pi(\succsim)$ by Π. Suppose, contrary to the proposition, there is some $P \in \Pi$, which is not cyclic. Then, by Lemma 1, there is $x \in X$ such that $Px > x$. Since $P \in \Pi$, we must have $Px \sim x$. But, since $Px \in X$ and $Px > x$, we must have $Px \succ x$ because $\succsim$ is Paretian. This, contradiction establishes the result. ■

The second result, while fairly straightforward to establish, provides a restriction which is more involved and therefore harder to check.

Proposition 2 *Suppose $\succsim$ is a Paretian SWR. Then, $\Pi(\succsim)$ is a group with respect to the operation of matrix multiplication.*

Proof. Given $\succsim$, denote $\Pi(\succsim)$ by Π. We check the four properties which define a group. First, let P, Q belong to Π. Define $R = PQ$; we know that $R \in \mathcal{P}$. We have to show that $R \in \Pi$. Let $x \in X$ be arbitrarily specified. Then, since $Q \in \Pi$, we have $Qx \sim x$. Denoting Qx by y, we note that $y \in X$, and since $P \in \Pi$, we also have $Py \sim y$. Thus, denoting Py by z, we note that $z \in X$, and $z \sim y$ while $y \sim x$, so that $z \sim x$ since $\succsim$ is transitive. Thus, $PQx = Py = z$ is indifferent to x. Thus, $R = PQ$ must belong to Π.

Second, the identity matrix $I \in \Pi$ (by definition of $\Pi(\succsim)$, since $\succsim$ is reflexive) and given any $P \in \Pi$, we have $PI = IP = P$, since $P \in \mathcal{P}$.

Third, if $P \in \Pi$, then $P' \in \mathcal{P}$, and we have to show that $P' \in \Pi$. Let x be an arbitrary point in X. Then, defining $y = P'x$, we see that $y \in X$. Further, multiplying both sides of this equation by P, we see that $Py = PP'x = x$ (since P' is the inverse of P). Since $P \in \Pi$, we must have $Py \sim y$; this means that $x \sim P'x$. Since $x \in X$ was arbitrary, this shows that $P' \in \Pi$.

Finally, if $P, Q, R \in \Pi$, then $P, Q, R \in \mathcal{P}$, and so $(PQ)R = P(QR)$. ■

3.2 Permissible Extensions of Anonymity: Two Examples

The restrictions imposed by the above propositions on the set of permissible permutations of Paretian SWRs are significant ones. The set of all permutations $\mathcal{P}$ is clearly a group, but not all elements of $\mathcal{P}$ are cyclic. Consider the following example.

Example 1:

Let π be the permutation which maps $\mathbb{N}$ onto $\mathbb{N}$ as follows:

$$\left.\begin{array}{l} \pi(n) = n + 2 \;\; for\ n\ even \\ \pi(n) = n - 2 \;\; for\ n > 1\ and\ odd \\ \pi(1) = 2 \end{array}\right\} \tag{9}$$

Note that if P is the permutation matrix associated with π then the iterates of P, when applied to the first unit vector, e^1, will generate the sequence $(e^2, e^4, e^6, ...)$, clearly a wandering sequence. Thus, P is not cyclic, and consequently P cannot belong to $\Pi(\succsim)$ if $\succsim$ is any Paretian SWR.

Perhaps a more transparent way to look at the permutation defined above is to see the effect of it on a particular utility sequence $x \in X$:

$$\begin{array}{rl} x = & (0, 1, 0, 1, 0, 1, 0, ...) \\ Px = & (1, 1, 0, 1, 0, 1, 0, ...) \end{array}$$

Clearly, what the permutation effectively does is to produce a Pareto superior utility sequence.

Example 1 shows that the class $\mathcal{C}$ of cyclic permutations is a strict subset of $\mathcal{P}$, and Proposition 1 shows that for every Paretian SWR $\succsim$, $\Pi(\succsim)$ must be a subset of the class $\mathcal{C}$ of cyclic permutations. Thus, for every Paretian SWR $\succsim$, $\Pi(\succsim)$ must be a strict subset of $\mathcal{P}$. One might wonder whether it is possible to have a Paretian SWR $\succsim$, for which $\Pi(\succsim)$ is $\mathcal{C}$. Unfortunately, $\mathcal{C}$ is not a group, as the following example shows. Thus, for every Paretian SWR $\succsim$, the set of permissible permutations $\Pi(\succsim)$ must exclude some cyclic permutation, and $\Pi(\succsim)$ must be a strict subset of $\mathcal{C}$.

Example 2:

Let π_1 be a permutation, defined as follows:

$$\left.\begin{array}{l}\pi_1(n) = n+1 \;\; if\ n\ is\ odd \\ \pi_1(n) = n-1 \;\; if\ n\ is\ even\end{array}\right\} \tag{10}$$

Clearly the permutation matrix P_1 associated with π_1 is cyclic, with a cycle of period 2 for each unit vector.

Let π_2 be the permutation, defined as follows:

$$\left.\begin{array}{l}\pi_2(1) = 1 \\ \pi_2(n) = n+1 \;\; if\ n\ is\ even \\ \pi_2(n) = n-1 \;\; if\ n > 1\ and\ odd\end{array}\right\} \tag{11}$$

Clearly, the permutation matrix P_2 associated with π_2 is cyclic, with a cycle of period 2 for each unit vector, starting with the second one; it has a cycle of period 1 for the first unit vector.

While P_1 and P_2 belong to the set $\mathcal{C}$ of cyclic permutations, it is easy to check that the composite permutation $\pi_2 \circ \pi_1$ is precisely the permutation π of Example 1, so that $P_2P_1 = P$ is not cyclic.

Again, it is instructive to look at the effect of these permutations on a specific utility sequence $x \in X$:

$$\begin{array}{ll} x = & (0,1,0,1,0,1,0,...) \\ P_1x = & (1,0,1,0,1,0,1,...) \\ P_2P_1x = & (1,1,0,1,0,1,0,...) \end{array}$$

If P_1 and P_2 both belong to $\Pi(\succsim)$, for some Paretian SWR $\succsim$, then $P_1x \sim x$ and $P_2(P_1x) \sim (P_1x)$, and one might not find either of these binary comparisons to be unacceptable. However, since the SWR $\succsim$ is transitive, we must then have $P_2(P_1x) \sim x$, and this is clearly unacceptable since $P_2(P_1x)$ is Pareto superior to x.

4 On Paretian SWRs with Extended Anonymity: Sufficient Conditions

We have noted above that the very structure of our problem imposes significant restrictions on the class of permissible permutations associated with any Paretian SWR. Now, we ask whether the restrictions obtained in Propositions 1 and 2 exhaust all the restrictions on the class of permissible permutations associated with any Paretian SWR. In other words, if $\mathcal{Q}$ is an *arbitrary* group of cyclic permutations, can we always define a Paretian SWR $\succsim$, for which the class of permissible permutations $\Pi(\succsim)$ coincides exactly with $\mathcal{Q}$? The answer to this question is (somewhat surprisingly) in the affirmative, so that we have, in fact, a complete characterization of the class of permissible permutations associated with any Paretian SWR.

Our demonstration of the above result consists in writing down a binary relation $\succsim$ and checking that (i) it is a Paretian SWR, and that (ii) $\Pi(\succsim) = \mathcal{Q}$. The particular binary relation we use is exactly of the form of the Suppes-Sen grading principle, but with the set of finite permutations replaced by the given group of cyclic permutations, $\mathcal{Q}$. Thus, for every specification of a group of cyclic permutations, we have a corresponding *extended Suppes-Sen grading principle.*

Proposition 3 *Let $\mathcal{Q}$ be a group of cyclic permutations. Then, there is a Paretian social welfare relation $\succsim_E$ such that $\Pi(\succsim_E) = \mathcal{Q}$.*

Proof. Define a binary relation $\succsim_E$ as follows: if $x, y \in X$, then $x \succsim_E y$ if and only if there is some $P \in \mathcal{Q}$ such that $Px \geq y$. The symmetric $(\sim_E)$ and asymmetric $(\succ_E)$ parts of $\succsim_E$ are defined in the usual way.

We check first that the binary relation is reflexive and transitive, so that it constitutes a social welfare relation.

Let $x \in X$. Since the identity matrix $I \in \mathcal{Q}$, and $Ix \geq x$, we have $x \succsim_E x$, verifying that $\succsim_E$ is reflexive.

Let $x, y, z \in X$ with $x \succsim_E y$ and $y \succsim_E z$. Then, there exist $P \in \mathcal{Q}$ and $Q \in \mathcal{Q}$ such that $Px \geq y$ and $Qy \geq z$. Since $P, Q \in \mathcal{Q}$ and $\mathcal{Q}$ is a group, $R \equiv QP \in \mathcal{Q}$. Applying the permutation Q to the inequality $Px \geq y$, we get $QPx \geq Qy$, and using the inequality $Qy \geq z$, we get $QPx \geq z$. Thus, we have $R \in \mathcal{Q}$ and $Rx \geq z$, so that $x \succsim_E z$, establishing transitivity of $\succsim_E$.

We now show that $\succsim_E$ is Paretian. Let $x, y \in X$ with $x > y$. Then since the identity matrix $I \in \mathcal{Q}$, and $Ix = x > y$, we certainly have $x \succsim_E y$. We claim now that $y \succsim_E x$ does not hold. For, if $y \succsim_E x$, then there is some $P \in \mathcal{Q}$ such that $Py \geq x$. But, since $x > y$, we must then have $Py > y$. But, by Lemma 1, this contradicts the fact that P is cyclic. Thus, $x \succsim_E y$ holds and $y \succsim_E x$ does not hold, and so $x \succ_E y$.

Finally, we show that $\Pi(\succsim_E)$, the set of permissible permutations associated with $\succsim_E$, is equal to $\mathcal{Q}$. This part of the proof can be split up into two steps: (i) If $P \in \mathcal{Q}$, then $Px \sim_E x$ for all $x \in X$; (ii) If $P \in \mathcal{P}$, and $Px \sim_E x$ for all $x \in X$, then $P \in \mathcal{Q}$.

To prove (i), let $P \in \mathcal{Q}$, and let x be an arbitrary point in X. Define $y \equiv Px$. Since $Px = y$, we clearly have $x \succsim_E y$. Also, multiplying the equation $Px = y$ by P', we have $x = P'Px = P'y$. Since $\mathcal{Q}$ is a group, $P' \in \mathcal{Q}$, so we must have $y \succsim_E x$. Thus, $y \sim_E x$; that is, $Px \sim_E x$.

To prove (ii) let $P \in \mathcal{P}$ and suppose that $Px \sim_E x$ for all $x \in X$. Choose $\bar{x} = (\bar{x}_n)$, where $\bar{x}_n = 1/2^{n-1}$ for all $n \in \mathbb{N}$. Clearly $\bar{x} \in X$, and $P\bar{x} \sim_E \bar{x}$. Define $\bar{y} = P\bar{x}$; then $\bar{y} \in X$. Since $\bar{y} \sim_E \bar{x}$, there is $Q \in \mathcal{Q}$ and $R \in \mathcal{Q}$ such that $Q\bar{x} \geq \bar{y}$ and $R\bar{y} \geq \bar{x}$. Multiplying the latter inequality by $R' \in \mathcal{Q}$, we have $\bar{y} \geq R'\bar{x}$. Summarizing, we have:

$$z \equiv Q\bar{x} \geq \bar{y} \geq R'\bar{x} \equiv z' \tag{12}$$

We can write:

$$z_n = z'_n + (z_n - z'_n) \;\; for\ all\ n \in \mathbb{N} \tag{13}$$

and sum (13) from $n = 1$ to $n = N$, where $N \in \mathbb{N}$, to obtain:

$$\sum_{n=1}^{N} z_n = \sum_{n=1}^{N} z'_n + \sum_{n=1}^{N}(z_n - z'_n) \tag{14}$$

Note that z and z' are rearrangements of the sequence $\bar{x}$ and since $\sum_{n=1}^{N} \bar{x}_n$ is absolutely convergent (as $N \to \infty$) with a sum equal to 1, $\sum_{n=1}^{N} z_n$ and $\sum_{n=1}^{N} z'_n$ must both converge to 1 as $N \to \infty$.[16] Using (12), $\sum_{n=1}^{N}(z_n - z'_n)$ is a monotonically non-decreasing sequence (in N) bounded above by 1, and must converge. Taking limits in (14), we must have $\sum_{n=1}^{N}(z_n - z'_n)$ converging to zero as $N \to \infty$. But, since $(z_n - z'_n) \geq 0$ for each $n \in \mathbb{N}$, this is only possible if $(z_n - z'_n) = 0$ for each $n \in \mathbb{N}$. Thus, $z = z'$, and so by (12) we have:

$$z \equiv Q\bar{x} = \bar{y} = R'\bar{x} = z' \tag{15}$$

[16] See, for example, Rudin (1976, p.78).

In particular, we get $\bar{y} = Q\bar{x}$ from (15). But by definition $\bar{y} = P\bar{x}$. Thus, we must have:

$$P\bar{x} = Q\bar{x} \tag{16}$$

Since $\bar{x}_i \neq \bar{x}_j$ whenever $i, j \in \mathbb{N}$ with $i \neq j$, (16) can hold only if $Q = P$. Thus, $P \in \mathcal{Q}$, finishing the proof of (ii), and hence of the Proposition. ■

Remark:

Given a group of cyclic permutations $\mathcal{Q}$, the extended Suppes-Sen grading principle $\succsim_E$, defined in the proof of Proposition 3, is a Paretian SWR, which satisfies $\mathcal{Q}$-Anonymity. It can be shown that a SWR $\succsim$ satisfies the Pareto axiom and the $\mathcal{Q}$-Anonymity axiom if and only if $\succsim_E$ is a subrelation to $\succsim$. That is, the extended Suppes-Sen grading principle is the least restrictive SWR satisfying the Pareto axiom and the $\mathcal{Q}$-Anonymity axiom. This result has been obtained by Banerjee (2005).

5 On A Group of Cyclic Permutations

Our characterization of possible extensions of anonymity, consistent with a Paretian SWR, has not addressed one central question. Is there a group of cyclic permutations which is a strict extension of the class of finite permutations? In this section, we address this question by specifying a group of cyclic permutations, which has several attractive properties. First, it includes the class of finite permutations. Second, it strictly extends the class of finite permutations by allowing infinite permutations which can essentially be written as a sequence of finite permutations over blocks of time of equal length. Third, it includes the class of infinite permutations that has most commonly been proposed in extensions of the standard anonymity axiom.

Our class of permutations has to be carefully chosen in view of the restrictions imposed by Propositions 1 and 2. While the restriction of being cyclic is relatively easy to check, the restriction of being a group is more subtle, since it pertains to compositions of permutations. This difference between the two (independent) restrictions is most clearly displayed in Examples 1 and 2. Note that Example 2 shows that even if we choose the class of permutations $\mathcal{Q}$ to be the subset of $\mathcal{C}$, consisting only of cyclic permutations with the period of cycles uniformly bounded above (independent of the unit vector chosen), it would not satisfy the second restriction.

We now proceed to define formally our class of permutations as follows. Given a permutation matrix, $P \in \mathcal{P}$, and $n \in \mathbb{N}$, we denote the $n \times n$ matrix

$(p_{ij})_{i,j\in\{1,\ldots,n\}}$ by $P(n)$. Let $\mathcal{S} = \{P \in \mathcal{P}$: there is some $k \in \mathbb{N}$, such that for each $n \in \mathbb{N}$, $P(nk)$ is a finite dimensional permutation matrix $\}$.

If $P, Q \in \mathcal{S}$, then there are $k \in \mathbb{N}, k' \in \mathbb{N}$, such that for each $n \in \mathbb{N}$, $P(nk)$ and $Q(nk')$ are finite dimensional permutation matrices. Define $R = PQ$. Then $R \in \mathcal{P}$. Further, defining $k'' = kk'$, we can check that for each $n \in \mathbb{N}$, $R(nk'')$ is a finite dimensional permutation matrix. Thus, $R \in \mathcal{S}$. Now, it is easy to check that $\mathcal{S}$ is also a group.

If $P \in \mathcal{S}$, then P is clearly cyclic since the iterates of P acting on any unit vector will return to the unit vector in at most k iterations. Thus, $\mathcal{S}$ is a group of cyclic permutations.

If P represents a permutation in $\mathcal{F}$, then there is some $k \in \mathbb{N}$ such that $P(k)$ is a finite dimensional permutation matrix and $p_{ii} = 1$ for all $i > k$. Thus, we ceratinly have $P(nk)$ to be a finite dimensional permutation matrix for each $n \in \mathbb{N}$. Thus, $\mathcal{S}$ includes the class $\mathcal{F}$ of finite permutations.

One of the most common examples considered in proposing an extension of the Anonymity axiom is the following:

$$\begin{aligned} x = &\quad (0,1,0,1,0,1,0,1,....) \\ y = &\quad (1,0,1,0,1,0,1,0,....) \end{aligned}$$

Although x cannot be obtained from y (nor y from x) by applying a finite permutation, it has been felt that x should be declared indifferent to y. That is, at least this class of (infinite) permutation should be allowed in any extended notion of Anonymity. We see that for the (infinite) permutation P involved here, $P(2n)$ is a finite dimensional permutation matrix for each $n \in \mathbb{N}$, and so P belongs to $\mathcal{S}$.

We do not know whether $\mathcal{S}$ is a *maximal* group of cyclic permutations. In fact, it would be useful to know whether there are other groups of cyclic permutations, which have all the three properties stated above. If not, there is a strong case for focusing exclusively on the group of cyclic permutations proposed in this section, in discussions of extended anonymity.

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