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Herding and Bank Runs

by

Chao Gu

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Chao Gu*

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Abstract

Traditional models of bank runs do not allow for herding effects, because in these models withdrawal decisions are assumed to be made simultaneously. I extend the banking model to allow a depositor to choose his withdrawal time. When he withdraws depends on his liquidity type (patient or impatient), his private, noisy signal about the quality of the bank's portfolio, and the withdrawal histories of the other depositors. In some cases, the optimal banking contract permits herding runs. Some of these "runs" are efficient in that the bank is liquidated before the portfolio worsens. Others are not efficient; these are cases in which the herd is misled.

JEL Classification Numbers: C73, D82, E59, G21.

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1 Introduction

In the classic bank-runs model of Diamond and Dybvig (1983), individual withdrawal decisions are made simultaneously. The lack of detailed dynamics of withdrawals makes it difficult to explain some observed features of bank runs. In reality, at least some withdrawals are based on the information about the previous withdrawals of others.[1] During the 1994-1995 Argentine

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[1] Brunnermeier (2001) says that "...withdrawals by deposit holders occur sequentially in reality, [whereas] the literature typically models bank runs as a simultaneous move game."

banking crisis, large depositors were responsible for most of the deposit outflows at the beginning of the crisis. Small depositors began to make substantial withdrawals two months later.[2] In their analysis on the runs on Turkish special finance houses (SFHs)[3] in 2001, Starr and Yilmaz (2007) find that depositors made sequential withdrawals influenced by the history of the withdrawals of others. The authors argue that the "increased withdrawals by moderate-size accountholders tended to boost withdrawals by [their] small counterparts, suggesting that the latter viewed the former as informative with respect to the SFH's financial condition."

In the present paper, I build a model in which the timing of individual withdrawals is determined by the depositor's information about his consumption type (*patient*, which means he does not need to consume immediately, or *impatient*, which means he needs to consume immediately), his noisy signal about the quality of the bank's portfolio, and the observed withdrawal history of other depositors. In my model, the signals are received in an exogenously determined sequence, but the timing of withdrawal is determined endogenously.[4] Because one's simple withdraw-or-not action does not reveal perfectly to others the pair of private signals that the depositor receives, other depositors can only imperfectly extract the depositor's private signals from his action. They update their beliefs about the quality of the bank's portfolio accordingly.

This paper does not focus on the panic-based bank runs of Diamond and Dybvig (1983). (See also Peck and Shell [2003].) I focus instead on bank runs that occur as a result of depositors trying to extract information about bank portfolio quality from the withdrawal histories of others. Because signals about the fundamentals are imperfect, and because signal extraction from the observed withdrawal history is also imperfect, a bank run can occur when the bank fundamentals are strong. In particular, it can occur when "too many" depositors receive early liquidity shocks. A bank run due to imperfect signal extraction is unique to the model with non-simultaneous withdrawal decisions. Bank runs in this sense are not purely fundamental-based.[5]

I show that there is a perfect Bayesian equilibrium in which a depositor withdraws if his expected utility is below his threshold level, and otherwise he waits. A depositor's expected utility depends upon his beliefs about the quality of the bank's portfolio; these beliefs are updated recursively by the observed withdrawal history of the other depositors. Before a depositor's

[2] See Schumacher (2005).

[3] Special financial houses are like commercial banks, but their deposits are not insured.

[4] Chari and Kehoe (2003) were the first to introduce a model of herding in investment decisions with endogenous timing.

[5] See Allen and Gale (1994) and Goldstein and Pauzner (2005), etc. for fundamental-based bank runs.

beliefs become sufficiently favorable, he follows his private signals: If he is impatient or the portfolio signal is unfavorable, he withdraws; otherwise he waits. A bank run occurs as a result of a herd of withdrawals when all depositors withdraw due to unfavorable signals and/or unfavorable observations on withdrawals. If his belief is sufficiently favorable, the private signal received by the depositor will not be decisive: the depositor always waits to withdraw unless he is impatient. In this case, his private signal will not be revealed through his withdrawal behavior, so his withdrawal behavior does not affect others' beliefs or their expected utilities. A "no-bank-run" regime thus takes place as a result of a "herd of non-withdrawals."

Compared with herding in investment decisions (Banerjee, 1992; Bikhchandani et al., 1992; and more recently Chari and Kehoe, 2003, 2004), herding in bank runs has some special features that complicate the model and lead to interesting results. The most important difference lies in the payment interdependence and uncertainty. In the banking setup, a depositor's payoff depends not only on his own actions, but also on the actions of others. The uncertainty in future payoffs – in particular, whether a bank run occurs or not – adds additional risk to the depositor's decision-making. This uncertainty is not necessarily bad, because a run can force the bank to liquidate assets before low productivity is actually realized, i.e., before a higher welfare cost is incurred.

An interesting result of payment interdependence and uncertainty is the possibility that the expected utility is not monotone in the depositor's beliefs and thus the possibility that his threshold beliefs are not unique. If a bank run takes place when depositors' aggregate expected utility, or social welfare, would be lower if there would be no bank run due to the low probability of having a high return, then the bank run serves as a lower bound on social welfare. Information about production is valuable in this situation. Though a more favorable level of beliefs makes a depositor more confident in the quality of a bank's portfolio, it is also more likely to lead to a herd of non-withdrawals where no more information will be made available in the future. Hence, expected utilities might not be increasing in the probability that the portfolio is good. As a result, the uniqueness of the threshold beliefs in the traditional herding literature is not guaranteed.

Computed examples show that in some economies a run-admitting contract is optimal because it not only provides more liquidity to the depositors to ensure against liquidity shocks, but it also encourages depositors to reveal the signals they receive. In other economies, a run-proof contract is optimal because it protects the economy from costly undesirable bank runs. Herding runs are equilibrium phenomena when the risk of bank runs or cost is sufficiently small.[6]

This remainder of the paper is organized as follows: The model is introduced in Section 2. In Section 3, I describe the equilibrium for an arbitrary demand-deposit contract. A perfect Bayesian Nash equilibrium is shown to exist. In section 4, I calculate some examples of optimal demand-deposit contract. The final section offers some concluding observations.

2 Model Set-up

Time: There are three periods, $t = 0, 1, 2$. Period 0 is a planning period, which is *ex ante*. Periods 1 and 2 are *ex post*. Period 1 is divided into $N + 1$ stages. N is a finite integer.

Depositors: There is a measure 1 of depositors in the economy. Each depositor is endowed with 1 unit of the consumption good in period 0. Depositors are identical at $t = 0$, but they face consumption shocks at $t = 1$. If a depositor receives a consumption shock, he is called impatient and has to consume immediately. An impatient depositor's utility is given by $u(c_1)$, where c_1 is the consumption received at $t = 1$. If a depositor does not receive a consumption shock, his consumption type is patient. Patient depositors derive utility from the consumption in the last period. If a patient depositor receives consumption at $t = 1$, he can reinvest it in a storage technology privately and consume it at $t = 2$. Thus, a patient depositor's utility is described by $u(c_1 + c_2)$, where c_2 is the consumption received at $t = 2$. $u(x)$ is strictly increasing, strictly concave, and twice differentiable. The coefficient of relative risk aversion of the utility function, $-xu''(x)/u'(x)$, is greater than 1 for $x \geq 1$. The utility function is normalized to 0 at $x = 0$, i.e., $u(0) = 0$. Each depositor has probability α $(0 < \alpha < 1)$ to be impatient and probability $1 - \alpha$ to be patient. By law of large numbers, a proportion α of the depositors is impatient.

Storage: Depositors can store the consumption good at no cost.

The bank and its technology: The bank behaves competitively. It takes deposits from depositors and invests in a production project. Production is risky and rigid. The investment in production can be made only in the initial period. One unit of consumption good invested at $t = 0$ yields R units at $t = 2$. $R = \overline{R} > 1$ with probability p_0, and $R = \underline{R} \leq 1$ with probability $1 - p_0$. The production asset can be liquidated at $t = 1$. Either all or none must be liquidated. The project therefore can be treated as an "indivisible good" after it is started. I assume an

[6] See Peck and Shell (2003) for somewhat similar results on panic-based bank runs.

individual depositor cannot invest in production on his own.

The contract: For convenience, I assume that if a depositor decides to deposit at the bank, the minimum amount of the deposit is 1 unit of consumption good. A competitive bank offers a simple demand-deposit contract that describes the amount of consumption goods paid to the depositors who withdraw in periods 1 and 2, c^1 and c^2, respectively. c^1 is independent of the productivity state. c^2 is state contingent. The bank pays c^1 to the depositors at $t = 1$ until it is out of funds. If the amount of consumption good in storage cannot meet the withdrawal demand, the bank has to liquidate assets. The bank distributes the remaining resource plus or minus the return on the portfolio equally among the depositors who wait until the last period. Denote the fraction of deposits that the bank keeps in storage by λ, and the fraction of depositors who withdraw deposits in period 1 by β $(0 \leq \beta \leq 1)$. The payment to the depositors who withdraw in period 2 is

$$c^2 = \begin{cases} \dfrac{\lambda - \beta c^1 + (1-\lambda)R}{1-\beta} & \text{if } \beta c^1 \leq \lambda; \\ \dfrac{1-\beta c^1}{1-\beta} & \text{if } \lambda < \beta c^1 \leq 1; \\ 0 & \text{if } \beta c^1 > 1. \end{cases}$$

Because at least a fraction α of the depositors need to consume at $t = 1$, λ must at least be αc^1. In the situation that the bank cannot meet payment requirements at $t = 1$, the bank fails. Because c^2 is dependent on the choice of c^1 and λ, the demand-deposit contract can therefore be described by $\left(c^1, \lambda\right)$.

Withdrawal stages and information: In each of the first N stages of $t = 1$, only one depositor is informed of his consumption type. Information about consumption is precise. He also receives a signal about the productivity of the bank portfolio. The signal about production status is accurate with probability q, $q > 0.5$. That is,

$$\Pr(S_n = H | R = \overline{R}) = \Pr(S_n = L | R = \underline{R}) = q.$$

S_n denotes the signal about productivity obtained by the depositor who is informed at stage n. Given productivity status, the probability of receiving a correct signal is q. Receiving a signal, a depositor updates his belief about productivity by Bayes' rule. The common initial prior is p_0. At stage $N + 1$, all depositors who have not received signals are informed of their consumption types but not about productivity. An impatient depositor has to consume at the stage when he receives the consumption shock.

Depositors have equal opportunity to be informed at each stage. Because N is very small compared with the infinite number of depositors, the probability of getting informed in the first N stages is zero. Depositors do not communicate with each other about the signals they receive. However, a depositor's withdrawal action is observed by all others[7]. Once a depositor withdraws, he cannot reverse his decision. But if a depositor chooses to wait, he can withdraw at a later stage. The final deadline for depositors to withdraw at $t = 1$ is stage $N + 1$. Depositors are not allowed to change decisions after observing other depositors' decisions at stage $N + 1$.

There are four types of depositors at each of the first N stages. The first type is those who have already withdrawn their deposits from the bank. Those are inactive depositors who have no more decisions to make. The second type is the newly informed depositor who receives signals at the current stage. The third type is those who were previously informed but have been waiting. The remaining type is the uninformed depositors.

The rigidity in liquidation of long-term assets imposes difficulty for the bank to adjust its portfolio at $t = 1$ by varying the fraction of assets in production. The bank does not have private information about productivity. It is in the same position as an uninformed depositor in terms of information. The bank does not liquidate the assets unless it is forced to do so when a bank run occurs.

A finite number of stages is necessary because it imposes a deadline for the depositors to make decisions at $t = 1$, so the expected utility can be calculated by backward induction. The specification of a continuum of depositors tremendously simplifies calculation. Consider a model that has a finite number of depositors. Each depositor has a non-atomic share at the bank. Seeing a depositor withdraw his funds, the rest need to recalculate their payoffs in different productivity states because the amount of remaining resource at the bank has changed significantly. The description of the equilibrium will be dependent on the parameters of the economy, and there will be many more cases to discuss. In the appendix, I present a simple example of a two-stage, two-depositor economy. Similar results are obtained in the example.

The sequence of timing of the banking game is as follows.

$t = 0$:

Bank announces the contract;

[7] I consider the limit of large finite economies. I assume individual withdrawals are observable as in an economy with a large number of depositors, while the effects on the total resouces is negligible.

Depositors make deposit decision.

$t = 1$:

Stage 1:

One depositor receives signals about his consumption type and about productivity.

He decides whether to withdraw or not.

Other depositors decide whether to withdraw or not.

(repeat for N stages)

Stage $N + 1$:

Consumption types are revealed to those who are not informed.

Depositors decide whether to withdraw or not.

$t = 2$:

Bank allocates the remaining resource to the rest of the depositors.

The post-deposit game starts after depositors make deposits at the bank. An individual depositor decides when to withdraw from the bank. Knowing what depositors will be doing in the post-deposit game, the competitive bank offers a contract that maximizes the *ex-ante* expected utility of the depositors at $t = 0$. Depositors determine whether to deposit at the bank or stay in autarky. Starting at $t = 0$, the entire game is called the pre-deposit game. I start with the analysis in the post-deposit game. I first prove that in the post-deposit game, there exists a perfect Bayesian equilibrium given a contract. Then I will calculate some examples of the optimal contract that the bank offers in the pre-deposit game given the equilibrium strategies in the post-deposit game.

3 Post-Deposit Game

In Diamond and Dybvig (1983), a demand-deposit banking contract allows for a panic-based bank run in the post-deposit game given $c^1 > 1$. For convenience, the panic-based run is not considered in the present paper. A bank run occurs in my model solely due to the information about the productivity or the imperfect extraction of the information from the actions of other depositors.

Depositors observe the total number of withdrawals at each stage. Let X_n denote the total number of withdrawals at stage n. The public history of withdrawals records the total number of withdrawals at each stage up to stage n. A depositor's private history as of stage n differs from the public history only if he has received signals at stage r, $r \leq n$. Depositor i's strategy at stage n, x_{in}, is a function that maps his private history into zero-one withdrawal decision. Let $x_{in} = 0$ represent the decision to wait, and let $x_{in} = 1$ represent the decision to withdraw. Depositor i's belief at stage n, p_{in}, is a function that maps his private history into the probalility that the productivity is high.

To simplify the notation, let x_n^U and p_n^U denote the strategy and belief, respectively, of an uninformed depositor at stage n. Let $x_n^{S_r}$ and $p_n^{S_r}$ denote the strategy and belief, respectively, of a depositor who is informed at stage r of a productivity signal S_r. If $r = n$, the depositor is newly informed. Otherwise, he is previously informed.

In order to show how withdrawals by some depositors affect the beliefs and actions of the others, I am interested in finding an equilibrium in which the newly informed depositors are willing to make decisions according to the signals that they receive under some conditions. I consider symmetric pure strategy perfect Bayesian equilibrium. At any stage, the strategies of the depositors are optimal given their beliefs. The beliefs of the depositors are updated by Bayes' rule whenever possible. Depositors with the same history adopt the same action at each stage.

For a contract that offers $c^1 < 1$, there does not exist a symmetric pure strategy run equilibrium, because given that all others withdraw from the bank, an individual depositor prefers to wait to get all the remaining resources, which is expected to be an infinite amount. Not withdrawing before stage $N+1$ is a patient depositor's dominant strategy regardless of all other depositors' actions and signals. Therefore, given $c^1 < 1$, I assume all patient depositors always wait until stage $N + 1$ to make decisions according to their beliefs and consumption types. Because no information can be inferred from the decisions of the newly informed depositors, and because the measure of depositors who are informed before the last stage is 0, a bank run does not occur. The analysis in the rest of this section is based on the assumption that $c^1 \geq 1$.

3.1 Bayesian Updates

A newly informed depositor at stage n Bayesian updates his belief by the productivity signal that he receives. His prior at stage n is his posterior belief at stage $n - 1$ when he was an uninformed depositor.

$$p_n^{S_n} = \begin{cases} P_H(p_{n-1}^U) = \dfrac{p_{n-1}^U q}{p_{n-1}^U q + (1-p_{n-1}^U)(1-q)}, & \text{if } S_n = H; \\ P_L(p_{n-1}^U) = \dfrac{p_{n-1}^U (1-q)}{p_{n-1}^U (1-q) + (1-p_{n-1}^U) q}, & \text{if } S_n = L. \end{cases}$$

P_H and P_L denote the rules of Bayesian updates when a high or a low signal is received, respectively. $p \leq P_H(p) \leq 1$ and $0 \leq P_L(p) \leq p$ for $p \in [0,1]$. $P_H(p)$ and $P_L(p)$ are strictly increasing in p.

The uninformed and previously informed depositors update their beliefs about the productivity being high by observing the decision made by the newly informed depositor. If the newly informed does not make decisions according to his signal about productivity, the uninformed and the previously informed depositors do not change their beliefs, because the decision of the newly informed carries no information about the productivity. Therefore, $p_n^U = p_{n-1}^U$, and $p_n^{S_r} = p_{n-1}^{S_r}$ for $r < n$. Suppose that the newly informed depositor waits if and only if a high signal is received and he is patient. The uninformed depositors then update their beliefs by

$$p_n^U = \begin{cases} P_H(p_{n-1}^U) = \dfrac{p_{n-1}^U q}{p_{n-1}^U q + (1-p_{n-1}^U)(1-q)}, & \text{if the newly informed waits;} \\ P_{\widetilde{L}}(p_{n-1}^U) = \dfrac{p_{n-1}^U (1-q+\alpha q)}{\alpha + (1-\alpha)\left[p_{n-1}^U (1-q) + \left(1-p_{n-1}^U\right) q\right]}, & \text{if the newly informed withdraws.} \end{cases}$$

$P_{\widetilde{L}}$ denotes the Bayesian update where the probability of observing an impatient depositor is taken into account. $0 \leq P_L(p) \leq P_{\widetilde{L}}(p) \leq p$ for $p \in [0,1]$. Note that $P_H^{n_1}\left(P_{\widetilde{L}}^{n_2}(p)\right) = P_{\widetilde{L}}^{n_2}(P_H^{n_1}(p))$, where the power on $P_{\widetilde{L}}$ (or P_H) denotes the number of updates by $P_{\widetilde{L}}$ (or P_H), given the prior. So long as depositors update their beliefs by the same numbers of P_H and $P_{\widetilde{L}}$, their beliefs are the same, no matter at which stages these updates have occurred. A previously informed depositor updates his belief in the same way.

3.2 A Perfect Bayesian Equilibrium

3.2.1 Beliefs and strategies

To simplify the notation, let $u_1 = u\left(c^1\right)$, $\overline{u}_2 = u\left(\frac{\lambda - \alpha c^1 + (1-\lambda)\overline{R}}{1-\alpha}\right)$, and $\underline{u}_2 = u\left(\frac{\lambda - \alpha c^1 + (1-\lambda)\underline{R}}{1-\alpha}\right)$. $\overline{u}_2$ and $\underline{u}_2$ represent a patient depositor's utility in $t = 2$, depending on the realization of production, if there is no bank run $t = 1$ (i.e., $\beta = \alpha$). I suppress $\left(c^1, \lambda\right)$ because c^1 and λ are given in the post-deposit game.

The construction of the perfect Bayesian equilibrium relies on finding a newly informed depositor's equilibrium strategies. The strategies of an uninformed or a previously informed depositor can be constructed accordingly. I will show that there exists a perfect Bayesian equilibrium in which a newly informed depositor makes his decision according to the following simple rule:

$$x_n^{S_n} = \begin{cases} 1, & \text{if impatient or } p_n^{S_n} < \hat{p}. \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

for $1 \leq n \leq N$, where $\hat{p}$ solves

$$u_1 = \hat{p}\overline{u}_2 + (1 - \hat{p})\,\underline{u}_2. \tag{2}$$

$\hat{p}$ is a function of $\left(c^1, \lambda\right)$. $\hat{p}$ is the cutoff belief with which a patient depositor is indifferent between withdrawing immediately and waiting until the last period if no information about productivity is available. Note that $\hat{p}$ is positive given $c^1 \geq 1$ and $\underline{R} \leq 1$. $\hat{p} = 0$ if and only if $c^1 = \underline{R} = 1$ or $c^1 = \lambda = 1$. Let $\overline{p}$ denote $P_H(\hat{p})$, and $\underline{p}$ denote $P_L(\hat{p})$.

The cutoff belief of a newly informed depositor is invariant to stages. A newly informed depositor makes his decision at the stage when he is informed as if it were a static game. I will show in proposition 1 that in the equilibrium, a newly informed depositor has no incentive to delay withdrawal if he receives a low signal before his prior belief exceeds $\overline{p}$.

A newly informed depositor share the same prior with the uninformed depositors. If no one else makes a withdrawal, the belief of a newly informed depositor at stage n, $1 \leq n \leq N$, is updated by the signal he receives

$$p_n^{S_n} = \begin{cases} P_L(p_{n-1}^U), & \text{if } S_n = L; \\ P_H(p_{n-1}^U), & \text{if } S_n = H, \end{cases} \tag{3}$$

with $p_0^U = p_0$. If anyone else makes a withdrawal, $p_n^{S_n} = 0$.

An uninformed depositor or a previously informed depositor updates his belief by watching the decision by the newly informed depositor. The belief of an uninformed depositor at stage n,

$1 \leq n \leq N$, is updated by

$$p_n^U = \begin{cases} 0, & \text{if } X_n > 1, \text{ or } (X_n = 0 \text{ and } p_{n-1}^U < \underline{p}); \\ P_{\widetilde{L}}(p_{n-1}^U), & \text{if } X_n = 1, \underline{p} \leq p_{n-1}^U < \overline{p}; \\ P_H(p_{n-1}^U), & \text{if } X_n = 0, \underline{p} \leq p_{n-1}^U < \overline{p}; \\ p_{n-1}^U, & \text{otherwise}; \end{cases} \tag{4}$$

with $p_0^U = p_0$.

Similarly, the belief of a previously informed depositor at stage n, $1 \leq n \leq N$, is updated by $(r < n)$

$$p_n^{S_r} = \begin{cases} 0, & \text{if } X_n > 1, \text{ or } (X_n = 0 \text{ and } p_{n-1}^U < \underline{p}); \\ P_{\widetilde{L}}(p_{n-1}^{S_r}), & \text{if } X_n = 1, \underline{p} \leq p_{n-1}^U < \overline{p}; \\ P_H(p_{n-1}^{S_r}), & \text{if } X_n = 0, \underline{p} \leq p_{n-1}^U < \overline{p}; \\ p_{n-1}^{S_r}, & \text{otherwise}. \end{cases} \tag{5}$$

At stage $N+1$, an active depositor's belief is equal to his belief at stage N. That is, $p_{N+1} = p_N$.

On the equilibrium path, depositors update their beliefs by the signals received or the information inferred. Off the equilibrium path, I assume the beliefs are zero. Between the end of the last stage and the beginning of the current stage, only the newly informed depositor receives new information. He would be the only one who would make a withdrawal at the beginning of a stage. If other depositors withdraw, the newly informed detects the deviation, and his belief falls to 0. He will withdraw if $\hat{p} > 0$. Thus, at least two withdrawals occur at the current stage. The belief of an uninformed depositor also falls to 0. If $\hat{p} = 0$, depositors prefer to wait regardless of the actions by other depositors as $u_1 = \underline{u}_2 = u(1)$. If $p_{n-1}^U < \underline{p}$, the newly informed at stage n is supposed to withdraw even if he receives a high signal (although in equilibrium, there is no active depositor with beliefs lower than $\underline{p}$). If he does not withdraw, the uninformed depositors detect the deviation, and their beliefs become 0.

Suppose that an uninformed depositor has the posterior belief p_N^U at the end of stage N. He will not get information about productivity at stage $N+1$. Therefore, p_N^U is his finalized belief. If $p_N^U \geq \hat{p}$, he will wait for period 2 unless he is told to be impatient at stage $N+1$. Otherwise, he will withdraw, regardless of the actions of the other depositors. By symmetric strategies, each depositor has a chance of $\frac{1}{c^1}$ to get paid given $c^1 \geq 1$. The expected utility of an uninformed depositor at the end of stage N is

$$w_N^U\left(p_N^U\right)=\begin{cases}\alpha u_1+(1-\alpha)\left[p_N^U\overline{u}_2+\left(1-p_N^U\right)\underline{u}_2\right], & \text{if } p_N^U\geq\hat{p};\\ \dfrac{1}{c^1}u_1, & \text{otherwise.}\end{cases} \tag{6}$$

Given an uninformed depositor's expected utility at stage N and the rules of belief updates, the expected utility of an uninformed depositor at stage n, $n<N$, can be constructed in a recursive way:

$$w_n^U\left(p_n^U\right)=\begin{cases}\alpha u_1+(1-\alpha)\left[p_n^U\overline{u}_2+\left(1-p_n^U\right)\underline{u}_2\right], & \text{if } p_n^U\geq\overline{p};\\ \pi\left(p_n^U\right)w_{n+1}^U\left(P_H\left(p_n^U\right)\right)+ & \text{if } \underline{p}\leq p_n^U<\overline{p} \text{ and}\\ +(1-\pi\left(p_n^U\right))w_{n+1}^U\left(P_{\widetilde{L}}\left(p_n^U\right)\right), & \pi\left(p_n^U\right)w_{n+1}^U\left(P_H\left(p_n^U\right)\right)+(1-\pi\left(p_n^U\right))\cdot\\ & \cdot\, w_{n+1}^U\left(P_{\widetilde{L}}\left(p_n^U\right)\right)\geq u_1;\\ \dfrac{1}{c^1}u_1, & \text{otherwise,}\end{cases} \tag{7}$$

where

$$\pi\left(p\right)=(1-\alpha)\left[(1-p)\left(1-q\right)+pq\right] \tag{8}$$

is the probability that the depositor informed at the next stage receives a high signal and is also patient, given the posterior belief of p at the current stage.

An uninformed depositor's strategy is

$$x_n^U=\begin{cases}1, & \text{if } w_n^U\left(p_n^U\right)<u_1.\\ 0, & \text{otherwise.}\end{cases} \tag{9}$$

for $1\leq n\leq N$.

If the prior at stage $n+1$ is very high (very low), i.e., $p_n^U\geq\overline{p}$ ($p_n^U<\underline{p}$), even though a low (high) signal is received, the newly informed depositor's posterior belief at stage $n+1$ is still above (below) the critical level of $\hat{p}$. So the newly informed depositor will not withdraw (wait). The newly informed depositor's action does not carry information about his signal, so the beliefs of the uninformed depositors will not change. From then on, no more information can be inferred from the decisions by the newly informed depositors at future stages. According to his current belief, the expected utility of an uninformed depositor in the last period is $\alpha u_1+(1-\alpha)\left[p_n^U\overline{u}_2+\left(1-p_n^U\right)\underline{u}_2\right]$, which is greater (lower) than u_1 as $p_n^U\geq\overline{p}$ ($p_n^U<\underline{p}$).

Suppose the newly informed depositor's prior is moderately high. If a low signal is received, the posterior belief falls below $\hat{p}$, whereas if a high signal is received, the posterior belief is above $\hat{p}$. When the newly informed waits, his decision fully reveals that he gets a high signal. The belief of the uninformed depositors will be updated to the same level as the newly informed depositor. However, if a withdrawal is observed, an uninformed depositor's belief will be updated by $P_{\tilde{L}}$. The expected utility of an uninformed depositor at the current stage is the weighted average of the possible expected utilities at next stage, where the weights are the probabilities that his current belief will be updated by either P_H or $P_{\tilde{L}}$ at next stage. Whether an uninformed depositor decides to withdraw depends on whether the weighted average exceeds u_1.

A previously informed patient depositor's expected utility can be constructed in a similar way:

$$w_N^{S_r}\left(p_N^{S_r}\right) = \begin{cases} \max\left\{p_N^{S_r}\overline{u}_2 + \left(1-p_N^{S_r}\right)\underline{u}_2,\ u_1\right\}, & \text{if } w_N^U\left(p_N^U\right) \geq u_1; \\ \frac{1}{c^1}u_1, & \text{otherwise.} \end{cases} \tag{10}$$

$$w_n^{S_r}\left(p_n^{S_r}\right) = \begin{cases} p_n^{S_r}\overline{u}_2 + \left(1-p_n^{S_r}\right)\underline{u}_2, & \text{if } p_n^U \geq \overline{p}; \\ \max\{\pi\left(p_n^{S_r}\right) w_{n+1}^{S_r}\left(P_H\left(p_n^{S_r}\right)\right) + & \text{if } \underline{p} \leq p_n^U < \overline{p} \text{ and} \\ +\left(1-\pi\left(p_n^{S_r}\right)\right) w_{n+1}^{S_r}\left(P_{\tilde{L}}\left(p_n^{S_r}\right)\right), u_1\}, & w_n^U\left(p_n^U\right) \geq u_1; \\ \frac{1}{c^1}u_1, & \text{otherwise.} \end{cases} \tag{11}$$

for $1 \leq n < N$, $r < n$. A previously informed depositor is patient, otherwise he would have withdrawn already. He knows the beliefs of the uninformed depositors, and he can predict whether the uninformed depositors will withdraw or not. Because the uninformed depositors are of measure 1, when they withdraw, a previously informed depositor should also do so, otherwise he will be left unpaid. Therefore, the expected utility of a previously informed depositor is conditional on whether the uninformed depositors withdraw or not. If $r = n$, $(10)-(11)$ defines the expected utility of a newly informed depositor if he is patient.

For $1 \leq n \leq N$, a previously informed depositor's strategy is $(r < n)$

$$x_n^{S_r} = \begin{cases} 1, & \text{if } w_n^{S_r}\left(p_n^{S_r}\right) < u_1. \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

At stage $N+1$, an active depositor's strategy is

$$x_{N+1} = \begin{cases} 1, & \text{if impatient or } p_{N+1} < \hat{p}. \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

where $p_{N+1} = p_N$.

3.2.2 Lemmas

Before the equilibrium is proved, I first introduce the definitions of a herd of withdrawals and a herd of non-withdrawals and the lemmas needed to prove the equilibrium.

Definition 1 *A herd of non-withdrawals begins when (1) the newly informed depositor does not withdraw deposits unless he is impatient even though a low productivity signal is received, and (2) all other depositors wait until their consumption types are revealed to be impatient.*

Definition 2 *A herd of withdrawals begins when all depositors withdraw deposits.*

The logic behind the proof of the equilibrium is similar to Chari and Kehoe (2003). However, due to the facts that the payoffs of the depositors are dependent on each other's actions, and that the liquidity types are private information, the following lemmas are needed to establish the properties of an active depositor's expected utility function. I will discuss the properties of an uninformed depositor's expected utility function according to whether the contract satisfies the "high cutoff probability" condition or the "low cutoff probability" condition. The meaning of the conditions will become clear at the end of this section. Lemmas 1-2 and Corollary 1 show that the uninformed depositors are willing to wait if high signals are inferred. So in the equilibrium, a newly informed depositor is willing to wait if a high signal is received. Lemma 3 shows that if a previously informed depositor and an uninformed depositor share the same belief, and the uninformed depositor is willing to wait, then a previously informed depositor is also willing to wait. In the equilibrium, a previously informed depositor will not change his decision of waiting unless the uninformed depositors decide to run on the bank.

Definition 3 *Define a cutoff probability of $w_n^U(p)$ as follows: $\tilde{p}_n$ is a cutoff probability if there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $w_n^U(p) \geq u_1$ for $p \in [\tilde{p}_n,\ \tilde{p}_n + \varepsilon_1]$, and $w_n^U(p) < u_1$ for $p \in [\tilde{p}_n - \varepsilon_2, \tilde{p}_n,)$.*

"High Cutoff Probability" Condition: $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\hat{p})\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}(\hat{p})\right)\underline{u}_2\right] > \frac{1}{c^1}u_1.$

"Low Cutoff Probability" Condition: $\alpha u_1 + (1-\alpha)\left[P_{\widetilde{L}}(\hat{p})\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}(\hat{p})\right)\underline{u}_2\right] \leq \frac{1}{c^1}u_1.$

The left-hand side of the "high (low) cutoff probability" condition is an uninformed depositor's expected utility with belief $P_{\widetilde{L}}(\hat{p})$ at stage N if no bank run occurs. The right-hand side is his expected utility when a bank run occurs. With the "high (low) cutoff probability" condition, the cutoff probabilities at stages before N are above (below) $\hat{p}$. The "high cutoff probability" condition is a sufficient condition for a bank run to be costly. As the uncertainty of having a bank run is resolved gradually, depositors become more willing to wait. The expected utility function is increasing in belief not only because the bank portfolio is more likely to be good, but also the chance of having a costly bank run is small. Lemma 1 states the properties of $w_n^U(p)$ when the "high cutoff probability" condition is satisfied.

Lemma 1 *Consider a contract that pays $c^1 \geq 1$ and satisfies the "high cutoff probability" condition. $w_n^U(p)$ is increasing in p for $1 \leq n \leq N$. There exists a unique cutoff probability $\tilde{p}_n$ such that $w_n^U(p) \geq u_1$ for $p \in [\tilde{p}_n, 1]$, and $w_n^U(p) = \frac{1}{c^1}u_1$ for $p \in [0, \tilde{p}_n)$. $\tilde{p}_n$ is decreasing in n. $w_n^U(p) \leq \alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\underline{u}_2\right]$ for $p \in [\tilde{p}_n, 1]$.*

Proof. Prove by induction. See appendix. ■

By lemma 1, if the "high cutoff probability" condition is satisfied, the expected utility is increasing in belief. Hence, there is a unique cutoff belief at each stage above which the uninformed depositors are willing to wait, and below which they will withdraw.

When a bank run occurs, the bank liquidates all its assets to meet the payment demands. If the liquidation helps mitigate future losses when the portfolio return is low, a bank run is not undesirable. The "low cutoff probability" condition is a necessary condition that a bank run can be desirable. With the "low cutoff probability" condition, the monotonicity of the expected utility function is not guaranteed, and there can be multiple cutoff probabilities at a stage. However, the cutoffs are always below $\hat{p}$, which ensures that the uninformed depositors are willing to wait if a high signal is inferred.

Lemma 2 *Consider a contract that pays $c^1 \geq 1$ and satisfies the "low cutoff probability" condition. $w_n^U(p) \geq u_1$ on $[\hat{p}, 1]$.*

Proof. See appendix. ■

Lemma 2 says that if the "low cutoff probability" condition holds, depositors are willing to wait if their beliefs are above $\hat{p}$. In other words, the cutoff probabilities of $\tilde{p}_n$ are lower than $\hat{p}$ for stages before N.

Given either of the "high/low cutoff probability" conditions, assume an uninformed depositor is willing to wait the stage before. He is also willing to wait at the current stage assuming a high signal is inferred.

Corollary 1 *Consider a contract that pays $c^1 \geq 1$. Given a posterior belief of p at stage n, if $w_n^U(p) \geq u_1$, then $w_{n+1}^U(P_H(p)) \geq u_1$.*

Proof. See appendix. ■

By Corollary 1, if a newly informed depositor's decision of waiting conveys a high signal to the uninformed depositors, his decision will not trigger a bank run.

Example 1:

Figure 1 shows an example of $w_n^U(p)$ where the "high cutoff probability" condition holds.

$u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$. $\overline{R} = 1.5$, $\underline{R} = 1$, $p_0 = 0.9$. $q = 0.999$. $\alpha = 0.01$.

Let $c^1 = 1.04$ and $\lambda = \alpha c^1 = 0.0104$. $\overline{u}_2 = 7.5568$, $\underline{u}_2 = 7.1525$, and $u_1 = 7.1921$.

In this example, $\widetilde{p}_N = \hat{p} = 0.0978$, $\widetilde{p}_n = 0.4383$ for $n = N-1, N-2, ...1$.

In all figures in this paper, a solid thin line represents $\alpha u_1 + (1-\alpha)[p_n \overline{u}_2 + (1-p_n)\underline{u}_2]$, a solid thick line represents w_n^U, and a dashed line represents u_1.

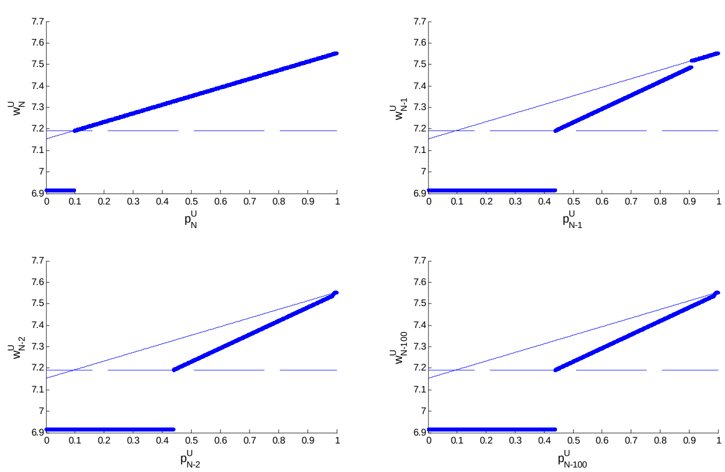

Figure 1: An example of $w_N^U(p)$, $w_{N-1}^U(p)$, $w_{N-2}^U(p)$, and $w_{N-100}^U(p)$.

Example 2:

Figure 2 shows an example of $w_n^U(p)$ where the "low cutoff probability" condition holds. $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$. $\overline{R} = 1.5$, $\underline{R} = 0.8$, $p_0 = 0.9$. $q = 0.9$. $\alpha = 0.01$. Let $c^1 = 1.011$, $\lambda = \alpha c^1 = 0.0101$. $\overline{u}_2 = 7.5571$, $\underline{u}_2 = 6.9297$, and $u_1 = 7.1629$.

In this example, there exist unique cutoff probabilities at stages N, $N-1$, $N-2$, and $N-100$, above which $w_n^U(p)$ is greater than u_1, and below which $w_n^U(p)$ is less than u_1. $\widetilde{p}_N = \hat{p} = 0.3716$, $\widetilde{p}_{N-1} = 0.2032$, $\widetilde{p}_{N-2} = 0.1971$, $\widetilde{p}_{N-100} = 0.1783$. However, the uniqueness of the cutoff probability is not guaranteed. We will see an example of non-uniqueness later. Also note that $w_n^U(p)$ is not necessarily increasing in p.

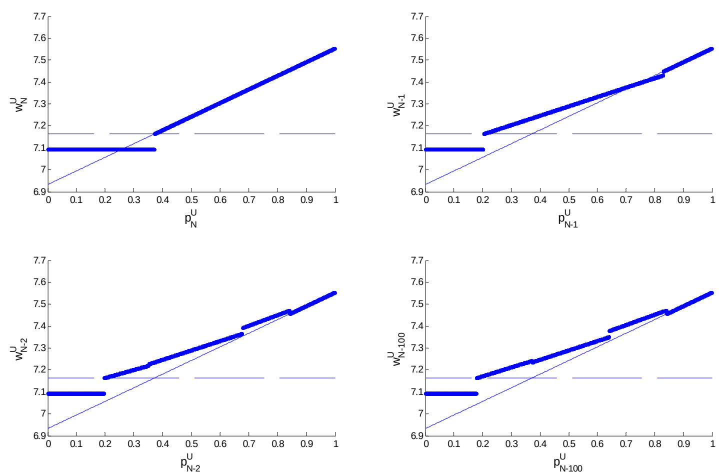

Figure 2: An example of $w_N^U(p)$, $w_{N-1}^U(p)$, $w_{N-2}^U(p)$, and $w_{N-100}^U(p)$.

Lemma 3 *If* $p_n^U = p_n^{S_r}$ *and* $w_n^U\left(p_n^U\right) \geq u_1$, *then* $w_n^{S_r}\left(p_n^{S_r}\right) \geq u_1$.

Proof. See appendix. ■

The intuition behind the lemma 3 is the following. Conditional on being impatient, a depositor prefers to withdraw immediately. If an uninformed depositor is willing to wait, it must be true that conditional on being patient, the expected utility from waiting is higher than that from withdrawing immediately.

3.2.3 Proof of the equilibrium

Proposition 1 *Given* $c^1 \geq 1$, *the beliefs and strategies in* $(1)-(13)$ *constitute a perfect Bayesian equilibrium in the post-deposit game.*

Proof. By construction, an active depositor's belief is updated by Bayes' rule whenever possible. The strategies of an uninformed or a previously informed depositor are constructed to be the equilibrium strategies given the strategies of a newly informed depositor. Hence, the proof of the equilibrium is reduced to illustrate that a newly informed depositor will follow the strategies described by $(1) - (2)$ given the strategies of the uninformed and the previously informed depositors.

A newly informed depositor's prior at stage n is higher than $\underline{p}$. That is, $p^U_{n-1} \geq \underline{p}$. If a herd of non-withdrawals has begun already, that is, $p^U_{n-1} \geq \overline{p}$, the newly informed depositor's action does not change the beliefs of other depositors, and he will not be able to infer any information in future. Even if he receives a low signal, his private belief is still above $\hat{p}$, so he will be waiting. In what follows, I will discuss cases according to the signal that the newly informed depositor gets at stage n, given that a herd of non-withdrawals has not begun yet, that is, $\underline{p} \leq p^U_{n-1} < \overline{p}$.

(1) The newly informed depositor gets a high signal. His belief now is higher than $\hat{p}$. If he waits, he conveys the high signal to all other depositors. By corollary 1, the uninformed depositors will be waiting. If the newly informed depositor waits, he will become a previously informed depositor and share the same belief with the uninformed depositors. By lemma 3, the newly informed depositor will wait.

(2) The newly informed depositor gets a low signal. His belief is now $p^{S_n}_n = P_L\left(p^U_{n-1}\right) < \hat{p}$. According to the strategies, he should withdraw and get u_1. Suppose he waits. Then the belief of an uninformed depositor is misled to be updated to $p^U_n = P_H\left(p^U_{n-1}\right)$. From then on, the belief of an uninformed depositor is always two signals above that of the depositor informed at n, that is, $p^{S_n}_m = P^2_L\left(p^U_m\right)$ for $m \geq n$. By choosing to wait, the best outcome that the newly informed depositor can anticipate is a herd of non-withdrawals. (If he anticipates a herd of withdrawals to occur, he would withdraw immediately.) Suppose a herd of non-withdrawals occurs at a later stage $m < N$. The posterior belief of an uninformed depositor at stage m satisfies $p^U_m \geq \overline{p}$. It also must be true that $p^U_{m-1} < \overline{p}$, $P_L\left(p^U_{m-1}\right) < \hat{p}$, and $P_H\left(p^U_{m-1}\right) \geq \overline{p}$. Otherwise, the herd of non-withdrawals could have begun earlier. As $p^U_{m-1} < p^U_m$, we have $p^U_m = P_H\left(p^U_{m-1}\right)$. The belief of the depositor who has deviated is $p^{S_n}_m = P^2_L\left(p^U_m\right) = P_L\left(p^U_{m-1}\right) < \hat{p}$. Thus, at the stage that the herd of non-withdrawals begins, the expected utility of the depositor who has deviated is still lower than u_1. In the case when neither a herd of withdrawals nor a herd of non-withdrawals occurs before stage N, the uninformed depositors' belief satisfies $p^U_{N-1} < \overline{p}$, which implies the

deviator's belief at stage N is below $\hat{p}$. Therefore, the depositor informed at stage n does not benefit from deviation. A newly informed depositor weakly prefers to withdraw immediately if a low productivity signal is received.

In the equilibrium, the previously informed depositors who have been informed before a herd of non-withdrawals begins share the same belief with the uninformed depositors. By Lemma 3, the previously informed always wait unless the uninformed decide to run on the bank. Those who are informed after a herd of non-withdrawals begins always wait.

Because the consumption types are private information, deviations are undetectable to the uninformed and previously depositors unless more than one withdrawal is observed at a stage before a herd of withdrawals begins. However, the newly informed depositor can detect deviations if anyone else makes a withdrawal at the current stage, and he will withdraw if $\hat{p} > 0$ because his belief is 0 now. In this case, the beliefs of the uninformed and previously informed depositors also fall to 0 because at least two withdrawals at a stage are observed. Therefore, all depositors withdraw. If $\hat{p} = 0$, waiting is the dominant strategy even if all other depositors withdraw as $u_1 = \underline{u}_2 = u(1)$. ■

3.3 Discussion of the Equilibrium - the "High Cutoff Probability" Condition Holds

With the "high cutoff probability" condition, the sequence of $(\tilde{p}_0, \tilde{p}_1, ..., \tilde{p}_{N-1}, \hat{p}, \hat{p})$ comprises the threshold beliefs above which the uninformed depositors wait, and below which they withdraw, whereas $(\hat{p}, \hat{p}, ..\hat{p}, \hat{p}, \hat{p})$ is the sequence of the threshold beliefs above which the newly informed depositors wait, and below which they withdraw. A herd of non-withdrawals happens before stage n if $p_n^U \geq \overline{p}$. At stages N and $N+1$, if beliefs are above $\hat{p}$, depositors will wait unless they are impatient. Therefore, for all depositors $(\overline{p}, \overline{p}, ..., \overline{p}, \hat{p}, \hat{p})$ is the sequence of beliefs above which a herd of non-withdrawals occurs at a stage.

Because $\tilde{p}_n$ is unique and is decreasing in n, we can calculate the number of updates by $P_{\tilde{L}}$ that are needed to trigger a bank run at stage n starting with p_0. Let Z_n solve

$$P_{\tilde{L}}^{Z_n - 1}(p_0) \geq \tilde{p}_n, \text{ and } P_{\tilde{L}}^{Z_n}(p_0) < \tilde{p}_n.$$

If there have been Z_n number of withdrawals up to stage n, a bank run will take place. Because $\tilde{p}_n \geq \hat{p}$, a non-withdrawal will trigger a herd of non-withdrawals.

What we observe in the equilibrium is as follows: A newly informed depositor follows his productivity signal if his prior at the current stage is below $\overline{p}$. If the newly informed depositors keep withdrawing from the bank, the beliefs of the uninformed depositors will finally fall below the cutoff, and they will demand their deposits back. Before their beliefs drop below the cutoff, if one non-withdrawal is observed, the uninformed depositors will be convinced to wait. In a situation where the uninformed depositors observe consecutive withdrawals but the number of withdrawals is not too large, the uninformed depositors watch the line closely. Their beliefs will be updated by the decisions of the newly informed depositors.

Let us try to understand why the cutoff probabilities are higher before stage N if the "high cutoff probability" condition is satisfied. Given p_N^U in the interval of $[P_{\widetilde{L}}(\hat{p}), \hat{p})$, a bank run takes place at stage N. The social welfare, measured by the aggregate expected utility, falls to $\frac{1}{c^1}u_1$. However, with the "high cutoff probability" condition, if depositors do not withdraw, the social welfare would actually be higher than that in the bank run. From the view of social welfare, the bank run is undesirable. Nevertheless, it is in an individual depositor's own interest to withdraw early. To an individual depositor, due to the costly liquidation, his expected utility also experiences a sudden drop when there presents a possibility of bank runs. Aware of the possibility of having a bank run at the next stage, the depositors must be more optimistic to wait for more information at stage $N-1$. Hence, the cutoff belief at stage $N-1$ is higher than $\hat{p}$. Working backward, as the uncertainty of having a bank run gradually resolves, the cutoff beliefs decrease as time goes by. Depositors become more and more willing to wait.

3.4 Discussion of the Equilibrium - the "Low Cutoff Probability" Condition Holds

If the "low cutoff probability" condition is satisfied, when depositors withdraw with the belief of $P_{\widetilde{L}}(\hat{p})$ at stage N, the aggregate expected utility is $\frac{1}{c^1}u_1$. If they wait, however, the expected utility in the last period will be lower. Bank runs under such a circumstance are not undesirable because they mitigate future losses. Bank runs serve as a valuable "option," so the uninformed depositors with belief slightly lower than $\hat{p}$ are still willing to wait at stage $N-1$, even though they are aware of the positive probability of bank runs. The expected utility at stage $N-1$ given the posterior belief of $\hat{p}$ is thus raised above u_1. By backward induction, the cutoff probabilities are lower than $\hat{p}$ for any stage before N.

Two possible and interesting results associated with the "low cutoff probability" condition are (1) non-monotonicity of the expected utility in belief and (2) non-uniqueness of the cutoff probabilities.

Non-monotonicity of the expected utility in belief:

Because early liquidation can help mitigate future losses, the economy in which information has a chance to be revealed can do better than the economy without information. From Figure 2, we see that for some p, $w_n^U(p)$ is above $\alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$, which is the expected utility in an economy with no information about production.

Because information about production is valuable and a herd of non-withdrawals suppresses the inference of private information, a higher belief does not necessarily result in a higher expected utility. There are two opposite forces behind belief: A higher belief brings more confidence in production. However, an economy with a higher belief also reaches a herd of non-withdrawals faster, after which no information will be available. Whether the expected utility increases in belief depends on the strength of the two forces.[8]

The non-monotonicity of the expected utility function in herding has not been paid attention in the literature. In the literature, herding is usually treated as a partial equilibrium problem, in which the cutoffs are determined by the assumption of parameters. An agent's 0-1 decision either perfectly reveals the signal received, or both decisions carry the same amount of noises. Given an initial prior, only a few crucial probability levels (1 and 2 signals above and below the initial prior) are needed to prove the equilibrium. In the banking set-up with a one-side signal extraction problem, the belief updated by observing a non-withdrawal is not completely offset by a withdrawal. The number of possible posterior beliefs increases geometrically in each stage. A general description of the expected utility function on the full domain of beliefs thus becomes necessary. Also, the cutoff probabilities vary with the contract. In order to calculate the optimal contract, the value of the expected utility given any parameters (in particular, c^1 and λ) needs to be determined.

Then why is the expected utility function always increasing in beliefs when the "high cutoff probability" condition holds? Note that the backup option here is a bank run. Unlike a safe asset in an investment herding problem, a bank run is costly because some depositors are not paid. If the welfare cost is too high, a bank run is no longer a "safety net." The "high cutoff

[8] The monotonicity is guaranteed for w_N^U and w_{N-1}^U.

probability" condition is a sufficient condition for a bank run to be too costly. With such a condition, the uncertainty of having a bank run lowers the expected utility. A higher belief not only stands for a higher expected return, but it also means a lower probability of having a costly bank run. Because an earlier stage faces more future history paths and the paths are gradually ruled out throughout period 1, the uncertainty is smaller at a later stage than at an earlier stage. The cutoff belief is thus decreasing in n.

Note that the "high/low cutoff probability" condition relies on backward induction to decide whether the cutoff probabilities at stages before N are higher or lower than $\hat{p}$. It is not the necessary and sufficient condition for the monotonicity of the expected utility function.

Non-uniqueness of the cutoff probabilities:

Because the monotonicity of expected utility is not guaranteed, our next question is whether the cutoff probability $\tilde{p}_n$ is unique. In fact, the uniqueness of the cutoff probabilities is no longer assured.[9] Figure 3 shows an example.

Example 3: An example of non-uniqueness of the cutoff probabilities:

$u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.01$, $\gamma = 1.5$. $\overline{R} = 2.07$, $\underline{R} = 0$, $p_0 = 0.9$. $q = 0.7$. $\alpha = 0.25$.

Let $c^1 = 1.011$ and $\lambda = \alpha c^1 = 0.2528$. $\overline{u}_2 = 18.6107$, $\underline{u}_2 = 0$, $u_1 = 18.0207$. $\hat{p} = 0.9683$. $\overline{p} = 0.9862$.

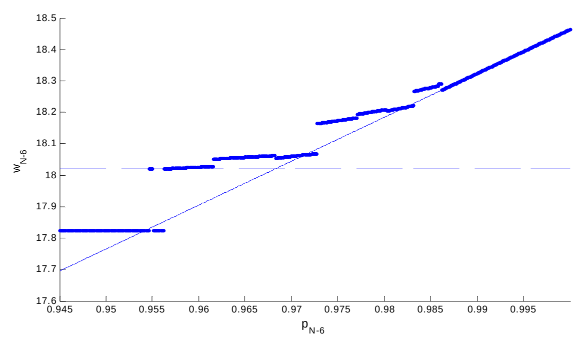

Figure 3: An example of non-uniqueness of the cutoff probabilities

Figure 3 shows the expected utility of an uninformed depositor at the stage of $N - 6$. There are two cutoffs at stage $N - 6$, 0.9546 and 0.9562. If the posterior belief at stage $N - 6$ falls below (including) 0.9546 or between (including) 0.9551 and (excluding) 0.9562, the uninformed depositors will run on the bank.

[9] It is guaranteed for w^U_N, w^U_{N-1}, and w^U_{N-2}.

Non-uniqueness of the cutoff beliefs results from payment interdependence. In an investment herding problem with no payment dependence, an investor's expected utility is always higher than the return on a safe asset because the safe asset is always available and its value is constant. Therefore, the cutoff belief is the lowest level of belief given which information is still able to be revealed. It is always unique. Here in the banking setup, the value of the option to withdraw decreases when all depositors exercise it. An individual depositor compares his expected utility with u_1, whereas his expected utility in a bank run is actually $\frac{1}{c}u_1$. The cutoff level of his expected utility is higher than the realized value of his option to withdraw. When the expected utility is low, an individual depositor prefers to use his option to withdraw before all others do so (although all others do the same thing) rather than waiting for more information. Because the expected utility does not necessarily increase in belief, there can be more than one cutoff belief. A bank run can happen given a relatively higher belief instead of a lower one.

Non-uniqueness of the cutoff beliefs implies the following: Given the same contract, an economy that starts with higher initial prior p_0 can be more vulnerable to bank runs than the one with lower initial prior. A bank run may be triggered by fewer withdrawals in the economy with a higher initial belief than with a lower initial belief. This is because an economy with higher initial prior has higher probability to reach a herd of non-withdrawals and thus has less chance to reveal information. In example 3, uninformed depositors with belief of $p_{N-7}^U = 0.9727$ ($P_{\tilde{L}}(0.9727) = 0.9562$) run on the bank if a withdrawal is observed at stage $N-6$, whereas if their belief is $p_{N-7}^U = 0.9717$ ($P_{\tilde{L}}(0.9717) = 0.9547$), they prefer to wait.

A question associated with non-uniqueness is whether it is possible that a shorter queue can encourage a bank run more than a longer queue given the same parameters of the economy but different sequences of signals. To formalize the question, suppose $w_n^U\left(p^1\right) \geq u_1$, whereas $w_n^U\left(p^2\right) < u_1$, and $\underline{p} \leq p^1 < p^2 < \hat{p}$. Is it possible that p^1 results from more observed withdrawals than p^2? The answer is no. Suppose the economy observes m withdrawals up to stage n to reach p^1, whereas it takes $m-1$ withdrawals up to stage n to reach p^2. We have $p^1 = P_L P_{\tilde{L}}\left(p^2\right)$. Because $p^2 < \hat{p}$, $p^1 < P_L\left(\hat{p}\right) = \underline{p}$. It contradicts the assumption that p^1 is above $\underline{p}$. Therefore, in the equilibrium, a longer queue always implies that low productivity is more likely, and it encourages people to run on the bank.

Without the uniqueness of the cutoffs, it is difficult to describe generally the sequence of actions that can trigger a herd. Two non-withdrawals in a row will definitely trigger a herd of non-withdrawals. Because a decision of withdrawal conveys noisy information about the signal received, it does not offset a decision of non-withdrawal completely. For example, if $p_0 < \hat{p}$, $P_H^2 P_{\tilde{L}}(p_0) \geq \overline{p}$, and $P_H P_{\tilde{L}}^2(p_0) < \underline{p}$, then a sequence of $(0,1,0)$ can trigger a herd of non-withdrawals, whereas a sequence of $(0,1,1)$ can trigger a herd of withdrawals.

In summary, the following will be observed in the equilibrium: A newly informed depositor follows his productivity signals until his belief is above $\overline{p}$. If many informed depositors do not withdraw, the beliefs of the uninformed depositors will be raised above $\overline{p}$, and a herd of non-withdrawals will start. In the opposite case, if many people withdraw, all other depositors will demand their deposits back. In a situation where the uninformed depositors observe neither too many withdrawals nor too many non-withdrawals, they will watch the line closely. Their beliefs will be updated by the decisions of the newly informed depositor.

The equilibrium proved in proposition 1 is not unique. For example, there can be equilibria in which at the first few stages, the newly informed depositors adopt the strategies described in proposition 1. But from stage m $(1 < m \leq N)$ on, the newly informed depositors always wait until the last stage to make their decisions. Because $w_n^U\left(p_n^U\right)$ changes with the strategies adopted, it is difficult to exhaust all possible equilibria. However, because the purpose of this paper is to illustrate how people make withdrawal decisions based on the observed withdrawals of others, I assume that depositors only play the equilibrium strategies in proposition 1 in the post-deposit game.

4 Pre-deposit Game

Once the equilibrium in the post-deposit game is proved, the probability of having a bank run given a contract is determined. Questions remaining are: (1) Knowing the probability of bank runs in any possible situation, what is the optimal contract that a competitive bank will provide? (2) Is the optimal contract individually rational (is it better than autarky and accepted by the depositors *ex ante)*? Peck and Shell (2003) show that the *ex-ante* acceptable optimal contract can tolerate panic-based bank runs if the probability of runs is small enough, and that bank runs are equilibrium phenomena. In this section, I will follow their logic to illustrate that the optimal demand-deposit contract can permit herding runs.

In the static bank-runs model, a feasible contract should at least satisfy the participation incentive compatibility constraint, which says that given all other patient depositors do not withdraw the deposits, an individual patient depositor prefers to wait. In the dynamic setup, a bank run can happen at any stage, but a feasible contract should at least give depositors the incentive to wait before anyone gets a signal. The participation incentive compatibility constraint is

$$w_0^U(p_0) \geq u_1. \tag{14}$$

The participation incentive compatibility constraint in the traditional Diamond-Dybvig model is a special case here, with $N = 0$ and $p = 1$.

The bank chooses a contract to offer. There are two types of contracts available to the bank: run-proof contracts and run-admitting contracts. A run-proof contract guarantees that whichever signals are sent in the post-deposit game, the expected utility of the uninformed depositors never falls below the threshold at any stage.

4.1 Run-Proof Contracts

A run-proof contract is in any one of the three cases in my model:

Case 1: A contract that provides $c^1 < 1$. All patient depositors wait until stage $N+1$ to make decisions according to their beliefs and consumption types. No information can be inferred from the decision of a newly informed depositor. The belief of an uninformed depositor is p_0 at all stages. The expected utility of an uninformed depositor at each stage is

$$w_n^U(p_0) = \alpha u_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\underline{u}_2\right].$$

for $0 \leq n \leq N$.

Case 2: $c^1 \geq 1$, and

$$P_L(p_0)\,\overline{u}_2 + (1 - P_L(p_0))\,\underline{u}_2 \geq u_1. \tag{15}$$

That is, the initial belief is already above $\overline{p}$. A herd of non-withdrawals has already begun before anyone gets signals. The uninformed depositors never update their beliefs by the observed actions. If (15) is satisfied, we have $p_n^U = p_0 > \widetilde{p}_0 > \widetilde{p}_n$ for all n.

Case 3: $c^1 \geq 1$, and

$$P_L(p_0)\,\overline{u}_2 + (1 - P_L(p_0))\,\underline{u}_2 < u_1, \tag{16}$$

$$w_n^U\left(P_{\widetilde{L}}^n(p_0)\right) \geq u_1 \;\forall 0 \leq n \leq N. \tag{17}$$

That is, the newly informed depositors withdraw if low signals are received. However, because there are too few stages and/or because the probability of being impatient is high, even though the beliefs are updated by $P_{\widetilde{L}}(\cdot)$ at every stage, the beliefs of the uninformed depositors are still above the thresholds. Note that if (17) holds, $w_n^U\left(p_n^U\right) = \alpha u_1 + (1-\alpha)\left[p_n \overline{u}_2 + (1-p_n)\underline{u}_2\right]$ for any $0 \leq n \leq N$ and for any p_n derived from p_0. Therefore, (17) can be rewritten as

$$P_{\widetilde{L}}^N(p_0) \geq \hat{p}. \tag{17'}$$

Given a run-proof contract, $w_0^U(p_0) = \alpha u_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\underline{u}_2\right]$. The best run-proof contract solves

$$\begin{aligned}
&\max_{c^1,\lambda} w_0^U(p_0) = \alpha u_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\underline{u}_2\right] \\
&s.t. \quad c^1 < 1, \qquad\qquad\qquad\qquad \text{or} \\
&\qquad\quad c^1 \geq 1 \text{ and } (14)-(15), \qquad \text{or} \\
&\qquad\quad c^1 \geq 1,\ (14),\ \text{and } (16)-(17).
\end{aligned}$$

4.2 Run-Admitting Contracts ($N = 2$)

A run-admitting contract admits a herd of withdrawals because $w_n^U\left(p_n^U\right) < u_1$ at at least one stage for some realization of p_n^U derived from p_0. The *ex-ante* probability of having a bank run given a contract can be calculated by checking the probability that $w_n^U\left(p_n^U\right)$ will be lower than u_1 at each stage. The probability of having a bank run at a stage depends on the contract and other parameters. The realization of a bank run relies on the random process in which the signals are sent. If a contract satisfies the "high cutoff probability" condition, the probability of bank runs is determined by the probability of getting Z_n number of withdrawals in a row up to stage n. If a contract satisfies the "low cutoff probability" condition, it is difficult to write out the general rules of calculating the probability of bank runs. In this section, a pre-deposit game of $N = 2$ is calculated. A more general case can be calculated in the same way. There are five cases for a run-admitting contract for $N = 2$, depending on the conditions with which a herd of withdrawals starts. The conditions for each case and the objective function of $w_0^U(p_0)$ of each case are listed in the appendix.

Case I: A herd of non-withdrawals begins if the first informed depositor waits. If the first informed depositor withdraws and the second also withdraws, then a bank run occurs. If the first withdraws and the second waits, the uninformed depositors wait. The probability of bank runs is

$$\sigma_1 = (1 - \pi(p_0))\left(1 - \pi\left(P_{\widetilde{L}}(p_0)\right)\right).$$

Case II: A herd of non-withdrawals does not occur if the first informed depositor waits. The second depositor follows his signals, but the uninformed depositors do not withdraw regardless of the second depositor's decision. A herd of withdrawals does not occur after the first depositor withdraws. If both the first and the second informed depositors withdraw, then a bank run occurs. The probability of bank runs is σ_1.

Case III: A herd of withdrawals begins if the first informed depositor withdraws. If the first informed depositor waits, a herd of non-withdrawals begins. The probability of bank runs is

$$\sigma_2 = 1 - \pi(p_0).$$

Case IV: A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor follows his signal. However, the uninformed depositors do not withdraw regardless of the second depositor's decision. The probability of bank runs is σ_2.

Case V: A herd of withdrawals starts if the first informed depositor withdraws. If the first informed depositor waits, the second depositor still follows his signal. The uninformed depositors wait if the second depositor waits, and they withdraw if the second depositor withdraws. The probability of bank runs is

$$\sigma_3 = 1 - \pi(p_0) + \pi(p_0)(1 - \pi(P_H(p_0))).$$

A competitive bank chooses the optimal contract from the classes of run-proof and run-admitting contracts. A run-proof contract is usually associated with lower c^1. The bank keeps more asset in storage so that the difference between payments in different periods and in different production state is small. A run-admitting contract usually provides higher c^1. Although c^2 in a run-admitting contract varies more between different production states, when the probability of low productivity is small, investing more in production is more desirable. There are three factors concerning which type of contract to offer. First, because a run-admitting contract usually provides more liquidity to early withdrawals, and the bank invests more in production though it is risky, the contract helps smooth consumptions and allows for higher return in the last period when productivity is high. This is a positive side of providing a run-admitting contract. Second, a run-admitting contract allows depositors to reveal their private information by their decisions. A herding run is partly fundamental driven. It is not necessarily undesirable in an economy with weak fundamentals because it mitigates future losses. It is again a positive side of a run-admitting contract. Third, because the signals and the information extracted from a depositor's action are not perfect, a bank run can happen when fundamentals are strong. This is a negative side of a run-admitting contract. Which contract to provide depends on the overall effects of the three.

The choice among run-admitting contracts also depends on several factors. First, a higher c^1 helps smooth consumptions across types, but it is usually associated with higher probability of bank runs and lower social welfare in bank runs. The second factor is unique to a sequential-move game. The optimal run-admitting contract should allow as much information as possible to be sensed publicly before any type of herd begins. The first N depositors can be treated as experiments. The result of each experiment can only be read before herds begin. A careful choice of contract should prolong the effective experiment process as much as possible. High c^1 and low c^2's can encourage people to run on the bank, and a bank run can happen too soon.

I compute two examples to illustrate that in some economies a run-admitting contract is optimal, whereas in other economies a run-proof contract is optimal. I compute the best contract in each of the three run-proof cases and the five run-admitting cases. The optimal contract is "the best of the best."

In the economy without signals about production, the bank chooses c^1 and λ to maximize $\alpha u_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\,\underline{u}_2\right]$, subject to the incentive compatibility constraint $p_0 \overline{u}_2 + (1-p_0)\,\underline{u}_2 \geq u_1$. If even given the optimal demand deposit contract herding runs are undesirable, the bank may want to use a "curtain" to prevent depositors from seeing each others' actions. From the examples below, we will see that information can improve *ex-ante* welfare.

An individual depositor's expected utility in autarky is $u(1)$. If the optimal banking contract is accepted *ex ante*, $w_0^U(p_0)$ must be at least equal to $u(1)$.

4.3 Computed Examples

Parameters and functions used in examples 4 and 5 are $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$. $\overline{R} = 1.5$, $\underline{R} = 0.2$, $p_0 = 0.99$. $q = 0.99$.

Example 4: $\alpha = 0.01$.

Table 1: Optimal Contract - Example 4

	σ	c^1	λ	$w_0\,(p_0)$
Autarky	0	1.0000	1	7.1529
Banking economy without info	0	1.0001	0.0100	7.5332
Best run-proof contract in case 1	0	1.0000	0.0100	7.5332
Best run-proof contract in case 2	0	1.0000	1.0000	7.1529
Best run-proof contract in case 3	0	1.0000	1.0000	7.1529
Best run-admitting contract in case I	0.0102	1.0000	0.0100	7.5487*
Best run-admitting contract in case II	0.0102	1.0000	1.0000	7.1529
Best run-admitting contract in case III	0.0296	1.0876	0.0109	7.5263
Best run-admitting contract in case IV	0.0296	1.0000	1.0000	7.1529
Best run-admitting contract in case V	0.0490	1.4868	0.0149	7.4310

Note that the best run-proof contract in case 1 provides c^1 that is less than, but very close to, 1. A run-proof contract is not the best in this example mainly because it does not induce depositors to reveal the signals they receive. The economy cannot benefit from the available information about productivity. This is also the reason why the economy with information about production can achieve higher *ex-ante* welfare than the economy without information can.

Example 5: $\alpha = 0.2$.

Table 2: Optimal Contract - Example 5

	σ	c^1	λ	$w_0\,(p_0)$
Autarky	0	1	1	7.1529
Banking economy without info	0	1.0028	0.2006	7.4602
Best run-proof contract in case 1	0	1.0000	0.2000	7.4602*
Best run-proof contract in case 2	0	1.0000	1.0000	7.1529
Best run-proof contract in case 3	0	1.0028	0.2006	7.4602*
Best run-admitting contract in case I	0.0527	1.0213	0.2043	7.4523
Best run-admitting contract in case II	0.0527	1.0000	1.0000	7.1529
Best run-admitting contract in case III	0.2158	1.1047	0.2209	7.2785
Best run-admitting contract in case IV	0.2158	1.0000	1.0000	7.1529
Best run-admitting contract in case V	0.3790	1.0000	1.0000	7.1529

In this example, a run-proof contract is optimal. The increase in α adds more noise to the informed depositors' withdrawal decisions. If it is a run-admitting contract, the probability of bank runs is increased because the probability of observing informed depositors withdraw is raised. In addition, because there are more impatient depositors in the economy, the payments to the depositors in period 1 are decreased due to the resource constraint, which leaves more room for using a run-proof contract. In this example, a run-admitting contract is not desirable as bank runs happen too frequently when the fundamentals are strong.

Green and Lin (2000, 2003) provide a model in which depositors make decisions whether to withdraw in sequence, although the depositors do not observe the line nor the decisions by others. They show that there exists an optimal banking contract that completely eliminates panic-based bank runs. My paper discusses bank runs given a demand-deposit contract. It does not seek a banking mechanism that eliminates herding runs. A demand-deposit contract with sequential service is widely used in the banking industry.[10] It is worthwhile as the first attempt to explain the queuing process given a contract in a narrow class of banking mechanism such as a simple demand-deposit contract.

A crucial difference between Green and Lin's economy and my economy is that there is no production uncertainty in Green and Lin's economy. Their mechanism induces the depositors to tell their private information – their consumption type – truthfully by their decisions. In my model, however, there are two dimensions of uncertainty. The 0-1 withdrawal decision cannot fully reveal the private information that a depositor has. Thus, there exists information asymmetry between the bank and depositors. Even if the bank is allowed to provide a contract that offers payments contingent on withdrawal history, it may not be able to eliminate bank runs. In a different paper, I show that in a two-depositor, two-stage economy with partial suspension of convertibility in the sense of Wallace (1988, 1990), a run-admitting contract can be optimal. However, it is still an open question whether there exists an optimal banking mechanism that eliminates both panic-runs and fundamental-runs.

5 Conclusion

This paper provides a model for studying detailed dynamics in bank runs. In an economy with uncertainty in production, a line in front of a bank carries information about the production status. The formation of a line outside a bank can persuade others to join the line. In my model, a depositor makes withdrawal decisions according to his observation of the withdrawal histories of the others as well as his private information about the bank fundamentals. Given a simple demand-deposit contract, there is a perfect Bayesian equilibrium in which depositors withdraw deposits if too many withdrawals are observed, and wait otherwise. In some economies, the simple demand-deposit contract allowing for herding runs is optimal because it achieves

[10] Calomiris and Kahn (1991) show that demand-deposit contract is efficient if a bank's moral hazard problem potentially exists. Because bank runs are costly, depositors are motivated to monitor the bank and the moral hazard problem will be reduced.

higher risk-sharing among depositors and/or allows private information about production to be revealed.

There is some literature on bank runs that is closely related to this paper. Goldstein and Pauzner (2005) construct a model in which depositors receive i.i.d. signals on fundamentals and determine whether to run on the bank simultaneously. Chen (1999) explains contagious bank runs using information externality. Chari and Jagannathan (1988) analyze an economy with random productivity. Some depositors are informed of the productivity status and others are not. The uninformed depositors infer information about productivity by observing the aggregate withdrawals rate. There is a rational expectation equilibrium in the model, which allows for bank runs. However, Chari and Jagannathan adopt a static equilibrium concept. The bank in their model does not have an intrinsic role in the economy. The cost of bank runs is imposed exogenously. Long-run payments do not depend on whether bank runs occur in the short run. My paper addresses these problems and emphasizes the welfare aspect of herding runs.

In the present paper, the bank has no information advantage over the majority depositors, which is not quite true in reality. In a more complicated model in which the bank receives signals about productivity, there arise problems such as how to eliminate the bank's moral hazard problem due to the information asymmetry between the bank and the depositors, and how the bank reduces the probability of bank runs due to the misleading signals. These can be extensions to the paper.

Allowing payments to vary with the evolution of history will give the bank more flexibility and will achieve higher social welfare (Wallace,1988, 1990). Is there a more general banking mechanism, for example, a mechanism that induces people to report truthfully about the signals, achieving a more efficient allocation? An efficient banking mechanism should not only allow the bank to provide a contract depending on the withdrawal history but also eliminate asymmetric information between the bank and the depositors as much as possible. To find a more efficient mechanism in the economy with uncertainties in both production and consumption is another extension of this paper, and more policy implications can be derived from the finding of such a mechanism.

References

[1] Allen, Franklin, and Douglas Gale, "Limited Market Participation and Volatility of Asset Prices," *American Economic Review*, 84 (4), (1994): 933-955.

[2] Banerjee, Abhijit V., "A Simple Model of Herd Behavior," *Quarterly Journal of Economics*, 107 (1992): 797-817.

[3] Bikhchandani, Sushil, David Hirshleifer, and Ivo Welch, "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades," *Journal of Political Economy,* 100 (5), (1992): 992-1026.

[4] Brunnermeier, Markus, "Asset Pricing under Asymmetric Information-Bubbles, Crashes, Technical Analysis, and Herding," *Oxford University Press*, 2001.

[5] Calomiris, Charles, and Charles Kahn, "The Role of Demandable Debt in Structuring Optimal Banking Arrangements," *American Economic Review*, 81 (3), (1991): 497-513.

[6] Chari, V.V., and Ravi Jagannathan, "Banking Panics, Information, and Rational Expectations Equilibrium," *Journal of Finance*, XLIII (3), (1988): 749-760.

[7] Chari, V.V., and Patrick Kekoe, "Hot Money," *Journal of Political Economy*, 11 (6), (2003): 1262-1292.

[8] ——. "Financial Crises and Herds: Overturning the Critiques," *Journal of Economic Theory*, 119, (2004): 128-150.

[9] Chen, Yehning, "Banking Panics: The Role of the First-Come, First-Served Rule and Information Externalities," *Journal of Political Economy*, 107 (5), (1999): 946-968.

[10] Diamond, Douglas, and Philip Dybvig, "Bank Runs, Deposit Insurance, and Liquidity", *Journal of Political Economy*, 91, (1983): 401-419.

[11] Goldstein, Itay, and Ady Pauzner, "Demand-Deposit Contracts and the Probability of Bank Runs", *Journal of Finance*, LX (3), (2005): 1293-1327.

[12] Green, Edward J., and Ping Lin, "Diamond and Dybvig's Classic Theory of Financial Intermediation: What's Missing?" *Federal Reserve Bank Minneapolis Quarterly Review,* 24, (Winter 2000): 3-13.

[13] ——. “Implementing Efficient Allocations in a Model of Financial Intermediation,” *Journal of Economic Theory,* 109, (2003): 1-23.

[14] Peck, James, and Karl Shell, “Equilibrium Bank Runs,” *Journal of Political Economy*, 111 (1), (2003): 103-123.

[15] Schumacher, Liliana, “Bank Runs and Currency Run in a System without a Safety Net: Argentina and the ‘Tequila’ Shock,” *Journal of Monetary Economics,* 46, (2000): 257-277.

[16] Starr, Martha, and Rasim Yilmaz, “Bank Runs in Emerging-Market Economies: Evidence from Turkey’s Special Finance Houses,” *Southern Economic Journal*, 73 (4), (April 2007): 1112-1132.

[17] Wallace, Neil, “Another Attempt to Explain an Illiquid Banking System: The Diamond and Dybvig Model with Sequential Service Taken Seriously,” *Federal Reserve Bank Minneapolis Quarterly Review,* 12, (Fall 1988): 3-16.

[18] ——. “A Banking Model in Which Partial Suspension is Best,” *Federal Reserve Bank Minneapolis Quarterly Review,* 14, (Fall 1990): 3-16.

6 Appendix

6.1 Proofs of Lemmas 1- 3 and Corollary 1

Lemma 1 *Consider a contract that pays* $c^1 \geq 1$ *and satisfies the "high cutoff probability" condition.* $w_n^U(p)$ *is increasing in* p *for* $0 \leq n \leq N$. *There exists a unique cutoff probability* $\tilde{p}_n$ *such that* $w_n^U(p) \geq u_1$ *for* $p \in [\tilde{p}_n, 1]$, *and* $w_n^U(p) = \frac{1}{c^1}u_1$ *for* $p \in [0, \tilde{p}_n)$. $\tilde{p}_n$ *is decreasing in* n. $w_n^U(p) \leq \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ *for* $p \in [\tilde{p}_n, 1]$.

Proof. If $\hat{p} \leq 0$, $p\overline{u}_2 + (1-p)\underline{u}_2 \geq u_1$ for any $p \in [0,1]$. Hence, $w_n^U(p_n) = \alpha u_1 + (1-\alpha)[p_n\overline{u}_2 + (1-p_n)\underline{u}_2] \geq u_1$ for $p_n \in [0,1]$. $\tilde{p}_n = 0$ for all n.

Same argument applies to $\hat{p} \geq 1$. $w_n^U(p) = \frac{1}{c^1}u_1$ on $p_n \in [0,1]$. $\tilde{p}_n = 1$ for all n.

Discuss the case in which $\hat{p} \in (0,1)$.

$w_N^U(p)$ is increasing in p by its definition. It has a unique cutoff probability $\hat{p}$. For stage $N-1$, $w_{N-1}^U(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \geq \overline{p}$ by definition. Check $\pi(p) w_N^U(P_H(p)) + (1-\pi(p)) w_N^U\left(P_{\tilde{L}}(p)\right)$ for $p < \overline{p}$.

Because $w_N^U(p)$ is increasing in p, $\pi(p) w_N^U(P_H(p)) + (1-\pi(p)) w_N^U\left(P_{\tilde{L}}(p)\right)$ is also increasing in p for $p < \overline{p}$. Because $\lim_{p \to \overline{p}} \pi(p) w_N^U(P_H(p)) + (1-\pi(p)) w_N^U\left(P_{\tilde{L}}(p)\right) = \alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\underline{u}_2]$, $w_{N-1}^U(p)$ is increasing on the entire domain of $[0,1]$. Hence, a unique cutoff probability $\tilde{p}_{N-1}$ can be found.

Let $P_{\tilde{H}}(p)$ be the inverse function of $P_{\tilde{L}}(p)$. $w_{N-1}^U(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \geq P_{\tilde{H}}(\hat{p})$. If

$$
\begin{aligned}
& \pi\left(P_{\tilde{H}}(\hat{p})\right)\left\{\alpha u_1 + (1-\alpha)\left[P_H\left(P_{\tilde{H}}(\hat{p})\right)\overline{u}_2 + \left(1 - P_H\left(P_{\tilde{H}}(\hat{p})\right)\right)\underline{u}_2\right]\right\} + \\
& + \left(1 - \pi\left(P_{\tilde{H}}(\hat{p})\right)\right)\frac{1}{c^1}u_1 \\
< \; & u_1,
\end{aligned}
$$

then $\hat{p} < \tilde{p}_{N-1} = P_{\tilde{H}}(\hat{p}) < \overline{p}$. $w_{N-1}^U(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2] > u_1$ for $p \geq \tilde{p}_{N-1}$. $w_{N-1}^U(p) = \frac{1}{c^1}u_1$ for $p < \tilde{p}_{N-1}$.

If the inequality does not hold, a unique cutoff $\tilde{p}_{N-1} < P_{\tilde{H}}(\hat{p})$ can be found to solve

$$\begin{aligned} & \pi(\tilde{p}_{N-1})\{\alpha u_1 + (1-\alpha)[P_H(\tilde{p}_{N-1})\overline{u}_2 + (1-P_H(\tilde{p}_{N-1}))\underline{u}_2]\} + \\ & +(1-\pi(\tilde{p}_{N-1}))\frac{1}{c^1}u_1 \\ = \ & u_1, \end{aligned}$$

by the continuity and the monotonicity of the above function in p. By the "high cutoff probability" condition, $\hat{p} < \tilde{p}_{N-1} < P_{\tilde{H}}(\hat{p}) < \overline{p}$. Also by the "high cutoff probability" condition, $w^U_{N-1}\left(P_{\tilde{L}}(p)\right) = \frac{1}{c^1}u_1 < \alpha u_1 + (1-\alpha)\left[P_{\tilde{L}}(p)\overline{u}_2 + \left(1-P_{\tilde{L}}(p)\right)\underline{u}_2\right]$ for $p \in [\tilde{p}_{N-1}, P_{\tilde{H}}(\hat{p}))$. Therefore, $w^U_{N-1}(p) \leq \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ on $[\tilde{p}_{N-1}, 1]$.

Prove the rest by induction.

Suppose it is true for every stage up to stage $n+1$ that (1) $w^U_{n+1}(p)$ is increasing in p. (2) $\hat{p} < \tilde{p}_{n+2} \leq \tilde{p}_{n+1} \leq \overline{p}$. If $w^U_{n+1}(\tilde{p}_{n+1}) > u_1$, $\tilde{p}_{n+1} = \min\left\{P^{N-(n+1)}_{\tilde{H}}(\hat{p}), \overline{p}\right\}$, $w^U_{n+1}(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \geq \tilde{p}_{n+1}$. If $w^U_{n+1}(\tilde{p}_{n+1}) = u_1$, $\tilde{p}_{n+1} < \min\left\{P^{N-(n+1)}_{\tilde{H}}(\hat{p}), \overline{p}\right\}$. $w^U_{n+2}\left(P_{\tilde{L}}(\tilde{p}_{n+1})\right) = \frac{1}{c^1}u_1$; (3) $w^U_{n+1}(p) \leq \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \in [\tilde{p}_{n+1}, 1]$.

Check the properties of $w^U_n(p)$:

(i) monotonicity.

$w^U_n(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2]$ for $p \geq \overline{p}$. For $p < \overline{p}$, as $w^U_{n+1}(p)$ is increasing in p, $\pi(p)w^U_{n+1}(P_H(p)) + (1-\pi(p))w^U_{n+1}\left(P_{\tilde{L}}(p)\right)$ is also increasing. Check $\pi(\overline{p})w^U_{n+1}(P_H(\overline{p})) + (1-\pi(\overline{p}))w^U_{n+1}\left(P_{\tilde{L}}(\overline{p})\right)$.

If $P_{\tilde{L}}(\overline{p}) \geq \tilde{p}_{n+1}$,

$$u_1 \leq w^U_{n+1}\left(P_{\tilde{L}}(\overline{p})\right) \leq \alpha u_1 + (1-\alpha)\left[P_{\tilde{L}}(\overline{p})\overline{u}_2 + \left(1-P_{\tilde{L}}(\overline{p})\right)\underline{u}_2\right].$$

If $P_{\tilde{L}}(\overline{p}) < \tilde{p}_{n+1}$,

$$w^U_{n+1}\left(P_{\tilde{L}}(\overline{p})\right) = \frac{1}{c^1}u_1 < \alpha u_1 + (1-\alpha)\left[P_{\tilde{L}}(\overline{p})\overline{u}_2 + \left(1-P_{\tilde{L}}(\overline{p})\right)\underline{u}_2\right].$$

Therefore,

$$\pi(\overline{p})w^U_{n+1}(P_H(\overline{p})) + (1-\pi(\overline{p}))w^U_{n+1}\left(P_{\tilde{L}}(\overline{p})\right) \leq \alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\underline{u}_2].$$

$w_n^U(p)$ is increasing on $[0,1]$, and there exists a unique cutoff probability $\tilde{p}_n$.

(ii) $\hat{p} < \tilde{p}_{n+1} \leq \tilde{p}_n \leq \overline{p}$.

Plug $\tilde{p}_{n+1}$ into $\pi(p)\, w_{n+1}^U(P_H(p)) + (1-\pi(p)) w_{n+1}^U\left(P_{\tilde{L}}(p)\right)$, and we have

$$\begin{aligned} & \pi(\tilde{p}_{n+1})\, w_{n+1}^U(P_H(\tilde{p}_{n+1})) + (1-\pi(\tilde{p}_{n+1})) w_{n+1}^U\left(P_{\tilde{L}}(\tilde{p}_{n+1})\right) \\ = \;& \pi(\tilde{p}_{n+1})\left\{\alpha u_1 + (1-\alpha)\left[P_H(\tilde{p}_{n+1})\,\overline{u}_2 + (1-P_H(\tilde{p}_{n+1}))\,\underline{u}_2\right]\right\} + \\ & (1-\pi(\tilde{p}_{n+1}))\frac{1}{c^1}u_1. \end{aligned}$$

(a) If $w_{n+1}^U(\tilde{p}_{n+1}) = u_1$, $w_n^U(\tilde{p}_{n+1}) = u_1$ because $\tilde{p}_{n+1}$ solves the same problem. Hence, we have $\tilde{p}_n = \tilde{p}_{n+1} < \min\left\{P_{\tilde{H}}^{N-(n+1)}(\hat{p}), \overline{p}\right\} \leq \min\left\{P_{\tilde{H}}^{N-n}(\hat{p}), \overline{p}\right\}$, $w_{n+1}^U\left(P_{\tilde{L}}(\tilde{p}_n)\right) = w_{n+2}^U\left(P_{\tilde{L}}(\tilde{p}_{n+1})\right) = \frac{1}{c^1}u_1$.

(b) If $w_{n+1}^U(\tilde{p}_{n+1}) > u_1$, $\tilde{p}_{n+1} = \min\left\{P_{\tilde{H}}^{N-(n+1)}(\hat{p}), \overline{p}\right\}$. It must be true that

$$\begin{aligned} & \pi(\tilde{p}_{n+1})\left\{\alpha u_1 + (1-\alpha)\left[P_H(\tilde{p}_{n+1})\,\overline{u}_2 + (1-P_H(\tilde{p}_{n+1}))\,\underline{u}_2\right]\right\} + \\ & (1-\pi(\tilde{p}_{n+1}))\frac{1}{c^1}u_1 \leq u_1. \end{aligned}$$

If not, we could have found a cutoff that is less than $\tilde{p}_{n+1}$ for stage $n+1$. Therefore, $w_n^U(\tilde{p}_{n+1}) \leq u_1$, and $\tilde{p}_n \geq \tilde{p}_{n+1}$ by the monotonicity of $w_{n+1}^U(p)$.

Discuss $\tilde{p}_n$ in case (b). At $p_n = \min\left\{P_{\tilde{H}}^{N-n}(\hat{p}), \overline{p}\right\}$, $w_n^U(p_n) = \alpha u_1 + (1-\alpha)\left[p_n\overline{u}_2 + (1-p_n)\,\underline{u}_2\right] > u_1$. Check

$$\pi(p_n)\left\{\alpha u_1 + (1-\alpha)\left[P_H(p_n)\,\overline{u}_2 + (1-P_H(p_n))\,\underline{u}_2\right]\right\} + (1-\pi(p_n))\frac{1}{c^1}u_1.$$

If it is greater than u_1, we can find a cutoff of $\tilde{p}_n$ between $\left(\tilde{p}_{n+1}, \min\left\{P_{\tilde{H}}^{N-n}(\hat{p}), \overline{p}\right\}\right)$ to satisfy $w_n^U(\tilde{p}_n) = u_1$. If it is less than or equal to u_1, $\tilde{p}_n = \min\left\{P_{\tilde{H}}^{N-n}(\hat{p}), \overline{p}\right\}$.

(iii) $w_n^U(p) \leq \alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\,\underline{u}_2\right]$ for $p \in [\tilde{p}_n, 1]$.

$w_n^U(p) = \alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\,\underline{u}_2\right]$ for $p \in [\bar{p}, 1]$. For $p \in [\tilde{p}_n, \bar{p})$, by the "high cutoff probability" condition,

$$
\begin{aligned}
w_n^U(p) &= \pi(p)\, w_{n+1}^U(P_H(p)) + (1-\pi(\tilde{p}_n)) w_{n+1}^U\left(P_{\tilde{L}}(p)\right) \\
&= \pi(p)\{\alpha v_1 + (1-\alpha)[P_H(p)\,\overline{u}_2 + (1-P_H(p))\,\underline{u}_2]\} + \\
&\quad +(1-\pi(p_n)) w_{n+1}^U\left(P_{\tilde{L}}(p)\right) \\
&\leq \pi(p)\{\alpha u_1 + (1-\alpha)[P_H(p)\,\overline{u}_2 + (1-P_H(p))\,\underline{u}_2]\} + \\
&\quad +(1-\pi(p_n))\left\{\alpha u_1 + (1-\alpha)\left[P_{\tilde{L}}(p)\,\overline{u}_2 + \left(1-P_{\tilde{L}}(p)\right)\underline{u}_2\right]\right\} \\
&= \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\,\underline{u}_2].
\end{aligned}
$$

■

Lemma 2 *Consider a contract that satisfies "low cutoff probability" condition.* $w_n^U(p) \geq u_1$ *on* $[\hat{p}, 1]$.

Proof. $w_n^U(p)$ is increasing on $[\overline{p}, 1]$ by definition. $w_n^U(p) > u_1$ on $[\overline{p}, 1]$. For $p \in [\hat{p}, \overline{p})$, we have $P_H(p) \geq \overline{p}$. Check $\pi(p)\, w_{n+1}^U(P_H(p)) + (1-\pi(\tilde{p}_n)) w_{n+1}^U\left(P_{\tilde{L}}(p)\right)$.

(I) If $w_{n+1}^U\left(P_{\tilde{L}}(p)\right) \geq u_1$,

$$
\pi(p)\, w_{n+1}^U(P_H(p)) + (1-\pi(p)) w_{n+1}^U\left(P_{\tilde{L}}(p)\right) > u_1.
$$

(II) If $w_{n+1}^U\left(P_{\tilde{L}}(p)\right) = \frac{1}{c^1} u_1$,

$$
\begin{aligned}
&\pi(p)\, w_{n+1}^U(P_H(p)) + (1-\pi(p)) w_{n+1}^U\left(P_{\tilde{L}}(p)\right) \\
= \;& \pi(p)\{\alpha u_1 + (1-\alpha)[P_H(p)\,\overline{u}_2 + (1-P_H(p))\,\underline{u}_2]\} + \\
& +(1-\pi(p))\tfrac{1}{c^1} u_1
\end{aligned}
$$

is strictly increasing in p in this case. Because

$$
\begin{aligned}
& \pi(\hat{p})\{\alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\,\underline{u}_2]\} + (1-\pi(\hat{p}))\tfrac{1}{c^1} u_1 \\
\geq \;& \pi(\hat{p})\{\alpha u_1 + (1-\alpha)[\overline{p}\overline{u}_2 + (1-\overline{p})\,\underline{u}_2]\} + (1-\pi(\hat{p}))\left\{\alpha u_1 + (1-\alpha)\left[P_{\tilde{L}}(\hat{p})\,\overline{u}_2 + \left(1-P_{\tilde{L}}(\hat{p})\right)\underline{u}_2\right]\right\} \\
= \;& \alpha u_1 + (1-\alpha)[\hat{p}\overline{u}_2 + (1-\hat{p})\,\underline{u}_2] \\
\geq \;& u_1
\end{aligned}
$$

by the "low cutoff probability" condition, $\pi(p) w_{n+1}^{U}(P_H(p)) + (1-\pi(\tilde{p}_n)) w_{n+1}^{U}\left(P_{\widetilde{L}}(p)\right) \geq u_1$ for $p \in [\hat{p}, \overline{p})$. In both cases, $w_n^U(p) \geq u_1$ on $p \in [\hat{p}, 1]$. ■

Corollary 1 *Consider a contract that pays $c^1 \geq 1$. Given a posterior belief of p at stage n, if $w_n^U(p) \geq u_1$, then $w_{n+1}^U(P_H(p)) \geq u_1$.*

Proof. It is obvious that Corollary 1 is true if the "high cutoff probability condition" is satisfied. If the "low cutoff probability condition" holds, p must be greater than or equal to $\underline{p}$ as $w_n^U(p) \geq u_1$.

If $p \in [\hat{p}, 1]$, $P_H(p) \geq \overline{p}$.

$$w_n^U(P_H(p)) = \alpha u_1 + (1-\alpha)\left[P_H(p)\,\overline{u}_2 + (1-P_H(p))\,\underline{u}_2\right] \geq u_1.$$

If $p \in [\underline{p}, \hat{p}), \hat{p} \leq P_H(p) < \overline{p}$. By lemma 2, $w_n^U(P_H(p)) \geq u_1$. ■

Lemma 3 If $p_n^U = p_n^{S_r}$ and $w_n^U\left(p_n^U\right) \geq u_1$, then $w_n^{S_r}\left(p_n^{S_r}\right) \geq u_1$.

Proof. Prove by induction. Let $p_n^U = p_n^{S_r} = p$. Show that at each stage, if $w_n^U(p) \geq u_1$, $w_n^U(p)$ can be written as

$$w_n^U(p) = \alpha\left[\rho_n(p)\,u_1 + (1-\rho_n(p))\tfrac{1}{c^1}u_1\right] + (1-\alpha)\,w_n^{S_r}(p),$$

where $\rho_n(p) \in [0,1]$, and $w_n^{S_r}(p) \geq u_1$.

Begin with stage N, if $w_N^U(p) \geq u_1$,

$$\begin{aligned} w_N^U(p) &= \alpha u_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\,\underline{u}_2\right] \\ &= \alpha u_1 + (1-\alpha)\,w_N^{S_r}(p) \\ &\geq u_1, \text{ so} \\ w_N^{S_r}(p) &= p\overline{u}_2 + (1-p)\,\underline{u}_2 \geq u_1, \text{ and} \\ \rho_N &= 1. \end{aligned}$$

Suppose it is true for every stage up to stage $n+1$. If $w_{n+1}^U(p) \geq u_1$, we have

$$w_{n+1}^U(p) = \alpha\left[\rho_{n+1}(p)\,u_1 + (1-\rho_{n+1}(p))\tfrac{1}{c^1}u_1\right] + (1-\alpha)\,w_{n+1}^{S_r}(p),$$

where $\rho_{n+1}(p) \in [0,1]$, and $w_{n+1}^{S_r}(p) \geq u_1$.

At stage n, suppose $w_n^U(p) \geq u_1$.

If $p \geq \overline{p}$, $w_n^U(p) = \alpha u_1 + (1-\alpha)[p\overline{u}_2 + (1-p)\underline{u}_2] \geq u_1$. $w_n^{S_r}(p) = p\overline{u}_2 + (1-p)\underline{u}_2 \geq u_1$. $\rho_n = 1$.

If $p < \overline{p}$, $w_n^U(p) = \pi(p) w_{n+1}^U(P_H(p)) + (1-\pi(p))w_{n+1}^U\left(P_{\widetilde{L}}(p)\right) \geq u_1$. By corollary 1, $w_{n+1}^U(P_H(p)) \geq u_1$. Suppose $w_{n+1}^U\left(P_{\widetilde{L}}(p)\right) \geq u_1$. By the assumption at stage n, we have $w_{n+1}^{S_r}(P_H(p)) \geq u_1$ and $w_{n+1}^{S_r}\left(P_{\widetilde{L}}(p)\right) \geq u_1$. So $w_n^{S_r}(p) = \pi(p) w_{n+1}^{S_r}(P_H(p)) + (1-\pi(p))w_{n+1}^{S_r}\left(P_{\widetilde{L}}(p)\right) \geq u_1$.

$$
\begin{aligned}
w_n^U(p) &= \pi(p) w_{n+1}^U(P_H(p)) + (1-\pi(p))w_{n+1}^U\left(P_{\widetilde{L}}(p)\right) \\
&= \pi(p)\left\{\alpha\left[\rho_{n+1}(P_H(p))u_1 + (1-\rho_{n+1}(P_H(p)))\tfrac{1}{c^1}u_1\right] + (1-\alpha)w_{n+1}^{S_r}(P_H(p))\right\} + \\
&\quad +(1-\pi(p))\left\{\alpha\left[\rho_{n+1}\left(P_{\widetilde{L}}(p)\right)u_1 + (1-\rho_{n+1}\left(P_{\widetilde{L}}(p)\right))\tfrac{1}{c^1}u_1\right] + (1-\alpha)w_{n+1}^{S_r}\left(P_{\widetilde{L}}(p)\right)\right\} \\
&= \alpha\left[\rho_n(p)u_1 + (1-\rho_n(p))\tfrac{1}{c^1}u_1\right] + (1-\alpha)w_{n+1}^{S_r}(p)
\end{aligned}
$$

and $\rho_n = \pi(p)\rho_{n+1}(P_H(p)) + (1-\pi(p))\rho_{n+1}\left(P_{\widetilde{L}}(p)\right)$.

Suppose $w_{n+1}^U\left(P_{\widetilde{L}}(p)\right) < u_1$, so

$$
w_{n+1}^U\left(P_{\widetilde{L}}(p)\right) = \tfrac{1}{c^1}u_1,
$$

and $w_{n+1}^{S_r}\left(P_{\widetilde{L}}(p)\right) = \frac{1}{c^1}u_1$ by definition.

$$
\begin{aligned}
w_n^U(p) &= \pi(p) w_{n+1}^U(P_H(p)) + (1-\pi(p))w_{n+1}^U\left(P_{\widetilde{L}}(p)\right) \\
&= \pi(p) w_{n+1}^U(P_H(p)) + (1-\pi(p))\tfrac{1}{c^1}u_1 \\
&= \pi(p)\left\{\begin{array}{c}\alpha\left[\rho_{n+1}(p_H(p))u_1 + (1-\rho_{n+1}(p_H(p)))\tfrac{1}{c^1}u_1\right] + \\ (1-\alpha)w_{n+1}^{S_r}(P_H(p))\end{array}\right\} + (1-\pi(p))\tfrac{1}{c^1}u_1 \\
&= \alpha\left[\rho_n(p)u_1 + (1-\rho_n(p))\tfrac{1}{c^1}u_1\right] + \\
&\quad +(1-\alpha)\left[\pi(p)w_{n+1}^{S_r}(P_H(p)) + (1-\pi(p))w_{n+1}^{S_r}\left(P_{\widetilde{L}}(p)\right)\right] \\
&\geq u_1, \text{ where } \rho_n(p) = \pi(p)\rho_{n+1}(P_H(p)) \text{ so} \\
w_n^{S_r}(p) &= \pi(p) w_{n+1}^{S_r}(P_H(p)) + (1-\pi(p))w_{n+1}^{S_r}\left(P_{\widetilde{L}}(p)\right) \\
&\geq u_1.
\end{aligned}
$$

■

6.2 Conditions and Objective Functions of Run-Admitting Contracts $(N = 2)$

A run-admitting contract should at least satisfy (14) and the following:

$$P_{\widetilde{L}}^{2}(p_0)\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}^{2}(p_0)\right)\underline{u}_2 \leq u_1, \tag{18}$$

$$P_{H}^{2}(p_0)\,\overline{u}_2 + \left(1 - P_{H}^{2}(p_0)\right)\underline{u}_2 > u_1, \tag{19}$$

(18) and (19) imply

$$\begin{aligned} w_2^U\left(P_{\widetilde{L}}^{2}(p_0)\right) &\leq u_1, \text{ and} \\ w_2^U\left(P_{H}^{2}(p_0)\right) &> u_1. \end{aligned}$$

The feasible contract also implies $w_1^U\left(P_H(p_0)\right) > u_1$ by corollary 1. I first list the conditions for all of the possible outcomes after each newly informed depositor's decision is observed.

1. If the first informed depositor waits, a herd of non-withdrawals occurs.

$$P_L P_H(p_0)\,\overline{u}_2 + \left(1 - P_L P_H(p_0)\right)\underline{u}_2 = p_0\overline{u}_2 + (1 - p_0)\,\underline{u}_2 \geq u_1. \tag{20}$$

2. If the first informed depositor withdraws, a herd of withdrawals occurs.

$$w_1\left(P_{\widetilde{L}}(p_0)\right) < u_1.$$

3. If the first informed depositor withdraws, a herd of withdrawals does not occur. The second depositor follows the signal as $P_L P_{\widetilde{L}}(p_0)\,\overline{u}_2 + \left(1 - P_L P_{\widetilde{L}}(p_0)\right)\underline{u}_2 < u_1$, guaranteed by (18). The uninformed depositors withdraw if the second depositor withdraws (by (18)), and they wait if the second depositor waits.

$$\begin{aligned} w_1\left(P_{\widetilde{L}}(p_0)\right) &\geq u_1 \\ w_2\left(P_H P_{\widetilde{L}}(p_0)\right) &= \alpha u_1 + (1-\alpha)\left[P_H P_{\widetilde{L}}(p_0)\,\overline{u}_2 + \left(1 - P_H P_{\widetilde{L}}(p_0)\right)\underline{u}_2\right] \geq u_1 \end{aligned} \tag{21}$$

4. If the first informed depositor waits, a herd of non-withdrawals does not occur. The second

depositor follows the signal. The uninformed depositors withdraw if the second depositor withdraws, and they wait if the second depositor waits.

$$p_0 \overline{u}_2 + (1 - p_0)\, \underline{u}_2 \quad < \quad u_1, \text{ and} \tag{22}$$

$$\alpha u_1 + (1 - \alpha) \left[P_{\widetilde{L}} P_H (p_0)\, \overline{u}_2 + \left(1 - P_{\widetilde{L}} P_H (p_0)\right) \underline{u}_2 \right] \quad < \quad u_1. \tag{23}$$

5. If the first informed depositor waits, a herd of non-withdrawals does not occur. The second depositor follows the signal. The uninformed depositors wait regardless of the second depositor's decision. i.e. $(21) - (22)$.

The combinations of the above five outcomes constitute descriptions of equilibrium outcomes given the contract.

Case I: Combine 1 and 3.

The probability of bank runs is

$$\sigma_1 = (1 - \pi (p_0)) \left(1 - \pi \left(P_{\widetilde{L}} (p_0)\right)\right).$$

Equations $(18) - (20)$ are necessarily required for the outcome. The participation incentive constraint is

$$w_0^U (p_0) = \pi (p_0)\, w_1^U (P_H (p_0)) + (1 - \pi (p_0))\, w_1^U \left(P_{\widetilde{L}} (p_0)\right) \geq u_1 \tag{24}$$

where

$$\begin{aligned} w_1^U \left(P_{\widetilde{L}} (p_0)\right) &= \pi \left(P_{\widetilde{L}} (p_0)\right) \left\{\alpha u_1 + (1 - \alpha) \left[P_H P_{\widetilde{L}} (p_0)\, \overline{u}_2 + \left(1 - P_H P_{\widetilde{L}} (p_0)\right) \underline{u}_2\right]\right\} + \\ &\quad \left(1 - \pi \left(P_{\widetilde{L}} (p_0)\right)\right) \tfrac{1}{c^1} u_1 \\ &\geq u_1, \end{aligned} \tag{25}$$

and

$$w_1^U (P_H (p_0)) = \alpha u_1 + (1 - \alpha) \left[P_H (p_0)\, \overline{u}_2 + (1 - P_H (p_0))\, \underline{u}_2\right] \geq u_1, \tag{26}$$

which is guaranteed by (20).

The *ex-ante* expected utility maximization problem is

$$\max_{c^1,\lambda} w_0^U(p_0)$$
$$s.t.\ c^1 \geq 1,\ \ (18)-(20),(24)-(26).$$

Case II: Combine 3 and 5.

The probability of bank runs is σ_1.

The conditions for the outcome are $(18)-(19),(21)-(22)$, and $(24)-(26)$, where (26) is guaranteed by (21) in this case.

The *ex-ante* expected utility maximization problem is

$$\max_{c^1,\lambda} w_0^U(p_0)$$
$$s.t.\ c^1 \geq 1,\ \ (18)-(19),\ \ (21)-(22),\ \text{and}\ \ (24)-(26).$$

Case III: Combine 1 and 2.

The probability of bank runs is

$$\sigma_2 = 1 - \pi(p_0).$$

Equations $(18)-(20)$, and (24) are necessarily required for the outcome. In addition, the participation incentive constraint requires

$$w_0^U(p_0) = \pi(p_0)\, w_1^U(P_H(p_0)) + (1-\pi(p_0))\, w_1^U\left(P_{\widetilde{L}}(p_0)\right) \geq u_1$$

where

$$w_1^U(P_H(p_0)) = \alpha u_1 + (1-\alpha)\left[P_H(p_0)\,\overline{u}_2 + (1-P_H(p_0))\,\underline{u}_2\right] \geq u_1, \tag{27}$$

is guaranteed by (20), and

$$w_1^U\left(P_{\widetilde{L}}(p_0)\right) = \frac{1}{c^1}u_1,$$

requires

$$\pi\left(P_{\widetilde{L}}(p_0)\right)\left\{\alpha u_1 + (1-\alpha)\left[P_H P_{\widetilde{L}}(p_0)\,\overline{u}_2 + \left(1-P_H P_{\widetilde{L}}(p_0)\right)\underline{u}_2\right]\right\} + \left(1-\pi\left(P_{\widetilde{L}}(p_0)\right)\right)\frac{1}{c^1}u_1 < u_1. \tag{28}$$

The *ex-ante* expected utility maximization problem is

$$\max_{c^1,\lambda} w_0^U(p_0)$$
$$s.t. c^1 \geq 1,\ \ (18)-(20),\ \ (24),\ \text{and}\ \ (27)-(28).$$

Case IV: Combine 2 and 5.

The probability of bank runs is σ_2.

The conditions for the outcome are $(18)-(19)$, $(21)-(22)$, (24) and $(27)-(28)$, where (27) is guaranteed by (21).

The *ex-ante* expected utility maximization problem is

$$\max_{c^1,\lambda} w_0^U(p_0)$$
$$s.t. c^1 \geq 1,\ \ (18)-(19),\ \ (21)-(22),\ \ (24),\ \ (27)-(28).$$

Case V: Combine 2 and 4.

The probability of bank runs is

$$\sigma_3 = 1-\pi(p_0)+\pi(p_0)\left(1-\pi\left(P_H(p_0)\right)\right)$$

Equations (14), $(18)-(19)$, and $(22)-(24)$ are necessarily required for the outcome. The participation incentive constraint requires:

$$w_0^U(p_0) = \pi(p_0)\, w_1^U\left(P_H(p_0)\right) + \left(1-\pi(p_0)\right) w_1^U\left(P_{\widetilde{L}}(p_0)\right) \geq u_1$$

where

$$\begin{aligned} w_1^U\left(P_H(p_0)\right) &= \pi\left(P_H(p_0)\right)\left\{\alpha v_1 + (1-\alpha)\left[P_H^2(p_0)\,\overline{u}_2 + \left(1-P_H^2(p_0)\right)\underline{u}_2\right]\right\} + \\ &\quad \left(1-\pi\left(P_H(p_0)\right)\right)\frac{1}{c^1}u_1 \\ &\geq u_1. \end{aligned} \tag{29}$$

Also,

$$w_1^U\left(P_{\widetilde{L}}(p_0)\right) = \frac{1}{c^1}u_1,$$

which is guaranteed by (23).

The *ex-ante* expected utility maximization problem is

$$\max_{c^1,\lambda} w_0^U(p_0)$$
$$s.t. c^1 \geq 1,\ (18)-(19), (22)-(24), (29).$$

6.3 An Example of an Economy with Two Depositors

In this section, I present the model in a two-depositor, two-stage version. One of the two depositors will be informed about his consumption type as well as the productivity status at the beginning of stage 1, and the other will be informed only about his consumption type at stage 2. Both depositors have equal probability to be the first informed depositor *ex ante*. The two-depositor, two-stage setup is the simplest case that allows for herding runs. The deadline for the decision in period 1 is the end of stage 2. Depositor 1 (the depositor who is informed at stage 1) does not have the chance to revise his decision after observing the decision of the other. But he can delay his decision until stage 2. If both depositors are active at stage 2, they will make decisions simultaneously. $\overline{R} > 1$ and $\underline{R} < 1$. For convenience, the signal about production is assumed to be perfect ($q = 1$). Because there are only two depositors, there is no need for depositor 2 to make a decision before he receives his signal about consumption.

The bank announces the demand-deposit contract, which describes the payment to the depositor who withdraws in period 1, c^1, and the amount of resource kept in storage, λ. The bank liquidates either all or none of the assets in production and liquidates the assets only when it cannot meet the payment demands. If $c^1 > 1$, the depositor who withdraws second will not receive the full amount of c^1. So let $c^1(1)$ and $c^2(2)$ denote the payment received by depositors who withdraw first and second in period 1, respectively. Let $c^2(x_1 + x_2, R)$ denote the payment in period 2 conditional on the total withdrawals in period 1 and the realization of production.

To comply with the assumption in section 3, I assume that given $c^1 < 1$, depositor 1 always delays his decision until stage 2 and that depositor 2 cannot obtain any information from depositor 1's action at stage 1. Depositors play a simultaneous-move game if both are active at stage 2. I first illustrate the equilibrium given $c^1 \geq 1$, then the one given $c^1 < 1$.

6.3.1 Equilibrium given $c^1 \geq 1$

When $c^1 \geq 1$, the equilibrium strategies include: (1) depositor 1's strategy when he receives signals at stage 1; (2) depositor 2's strategy contingent on depositor 1's decision at stage 2.

I begin with depositor 2's strategy at stage 2. At stage 2, depositor 2 has an updated belief p_2. If he waits, he expects $p_2 u\left(c^2\left(x_1, \overline{R}\right)\right) + (1-p_2)\, u\left(c^2\left(x_1, \underline{R}\right)\right)$, whereas if he withdraws, he will get $u\left(c^1\left(x_1+1\right)\right)$ .It is easy to see that there exists a cutoff belief $\hat{p}_2\left(x_1\right)$ above which the depositor waits, below which he withdraws. $\hat{p}_2$ is contingent on x_1, as depositor 2's expected payoffs vary with depositor 1's decision.

If a contract specifies $c^1 = 1$ and $\lambda = 2$, depositor 1 does not have the incentive to withdraw if a low signal is received. Except for such a contract, withdrawing immediately is depositor 1's best response regardless of the decision of other depositors if a low signal is received, given $c^1 > 1$,

Given $c^1 \geq 1$, an acceptable contract must satisfy the following condition: If the productivity is known to be high, both depositors are willing to wait *ex ante.* That is,

$$\begin{aligned} &\alpha^2\left(0.5u\left(c^1\left(1\right)\right) + 0.5u\left(c^1\left(2\right)\right)\right) + (1-\alpha)^2 u\left(c^2\left(0, \overline{R}\right)\right) + \\ &+2\alpha\left(1-\alpha\right)\left[0.5u\left(c^1\left(1\right)\right) + u\left(0.5c^2\left(1, \overline{R}\right)\right)\right] \geq u\left(1\right). \end{aligned} \tag{30}$$

If a high signal is received, depositor 1 will always have the incentive to wait if he can convey the high signal to depositor 2 because

$$\alpha u\left(c^2\left(1, \overline{R}\right)\right) + (1-\alpha)\, u\left(c^2\left(0, \overline{R}\right)\right) \geq u\left(1\right)$$

by (30).

In this simplest setup, there is a perfect Bayesian equilibrium in the post-deposit game, given any contract that provides $c^1 \geq 1$. That is,

1. If $c^1 = 1$ and $\lambda = 2$, depositors 1 and 2 withdraw if and only if they are impatient. Depositor 1's belief is updated by the signal received. Depositor 2's belief does not change. This contract results in the same welfare level as in autarky.

2. If $c^1 > 1$ or $\lambda \neq 2$, depositor 1 withdraws if he is impatient and/or a low signal is received and does not withdraw otherwise. Depositor 2 has the updated belief $P_{\tilde{L}}\left(p_0\right)$ $\left(P_H\left(p_0 = 1\right)\right)$

if depositor 1 withdraws (does not withdraw). Depositor 2 withdraws if he is impatient and/or his updated belief is below $\hat{p}_2(x_1)$.

6.3.2 Equilibrium given $c^1 < 1$

When $c^1 < 1$ is provided ($c^1(1) = c^1(2) = c^1$), depositors 1 and 2 play a simultaneous-move game at stage 2 if depositor 1 is still active (patient). Depositor 1 knows the productivity status but does not know depositor 2's type. Depositor 2 does not know the productivity status but knows depositor 1 is patient. In this game at stage 2, there exist Bayesian Nash equilibria. There are four possible equilibrium outcomes, depending on the parameters and contract.

1. $\alpha u\left(c^2\left(1,\underline{R}\right)\right) + (1-\alpha)\,u\left(c^2\left(0,\underline{R}\right)\right) < u\left(c^1\right)$ and $p_0 u\left(c^2\left(0,\overline{R}\right)\right) + (1-p_0)\,u\left(c^2\left(1,\underline{R}\right)\right) \geq u\left(c^1\right)$: Depositor 1 withdraws if he has received a low signal, and does not otherwise. Depositor 2 withdraws if he is impatient and does not withdraw otherwise.

2. $u\left(c^2\left(1,\underline{R}\right)\right) < u\left(c^1\right)$ and $p_0 u\left(c^2\left(0,\overline{R}\right)\right) + (1-p_0)\,u\left(c^2\left(1,\underline{R}\right)\right) < u\left(c^1\right)$: Depositor 1 withdraws if he has received a low signal and does not withdraw otherwise. Depositor 2 withdraws.

3. $u\left(c^2\left(1,\underline{R}\right)\right) \geq u\left(c^1\right)$ and $p_0 u\left(c^2\left(0,\overline{R}\right)\right) + (1-p_0)\,u\left(c^2\left(0,\underline{R}\right)\right) < u\left(c^1\right)$: Depositor 1 does not withdraw. Depositor 2 withdraws.

4. $\alpha u\left(c^2\left(1,\underline{R}\right)\right) + (1-\alpha)\,u\left(c^2\left(0,\underline{R}\right)\right) \geq u\left(c^1\right)$ and $p_0 u\left(c^2\left(0,\overline{R}\right)\right) + (1-p_0)\,u\left(c^2\left(0,\underline{R}\right)\right) \geq u\left(c^1\right)$: Depositor 1 does not withdraw. Depositor 2 withdraws if he is impatient and does not withdraw otherwise.

Note that there exists multiple equilibria given some parameter values. Also note that depositor 1 always has incentive to wait if he has received a high signal as $c^1 < 1$ and $c^2\left(1,\overline{R}\right) > 1$.

At stage 1, depositor 1 withdraws if he is impatient. If depositor 1 has withdrawn, depositor 2 withdraws at stage 2 if $p_0 u\left(c^2\left(1,\overline{R}\right)\right) + (1-p_0)\,u\left(c^2\left(1,\underline{R}\right)\right) < u\left(c^1\right)$ and/or he is impatient, and he does not otherwise.

6.3.3 A Numerical Example

In this example, I will employ the following utility function and parameters: $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.001$, $\gamma = 1.01$; $\overline{R} = 1.25$, $\underline{R} = 0.95$, $p_0 = 0.95$; $q = 1$; $\alpha = 0.05$.

Table 3: Optimal Contract - two-depositor, two-stage

	c^1	λ	$w_0\,(p_0)$
Best contract that provides $c^1 > 1$ or $\lambda \neq 2$	1	0	7.3439*
Contract that provides $c^1 = 1$ and $\lambda = 2$ (Autarky)	1	2	7.1529
Best contract that provides $c^1 < 1$	1.0000	0	7.3439

The contract that provides $c^1 = 1$ and $\lambda = 2$ (equivalent to autarky) yields the *ex-ante* expected utility of 7.1529. The optimal contract in this example requires $c^1 = 1$ and $\lambda = 0$. Because the liquidity demand is small (α is small) and the production has high probability to be successful, the bank invests all resources in production. The *ex-ante* expected utility is 7.3439. Given $c^1 = 1$ and $\lambda = 0$, depositor 1 withdraws at stage 1 if and only if a low signal is received or he is impatient, depositor 2 withdraws at stage 2 if depositor 1 has withdrawn at stage 1 or he is impatient, and does not otherwise. (If depositor 1 has withdrawn, depositor 2 is indifferent between withdrawing immediately at stage 2 and waiting until $t = 2$.) When productivity is low, depositor 1's withdrawal forces the bank to liquidate all its assets so depositor 2 also benefits from depositor 1's private information. Of course, if either of the depositors is impatient, the bank has to interrupt production. However, the probability of having a liquidity shock is small enough to be tolerated. The best contract in the category of $c^1 < 1$ provides c^1 very close to 1, and the bank also invests all resources in production. The ex-ante expected utility is very close to 7.3439. Given this contract, there exists an unique equilibrium in which depositor 1 withdraws at stage 1 if and only if he is impatient, he withdraws at stage 2 if and only if he has received a low signal; depositor 2 does not withdraw at stage 1, he withdraws at stage 2 if and only if he is impatient.