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Abstract

In a standard exhaustible resource model, it is known that if, along a competitive path, investment in the augmentable capial good equals the rents on the exhaustible resource (known as Hartwick's rule), then the path is equitable in the sense that the consumption level is constant over time. In this paper, we show the converse of this result: if a competitive path is equitable, then it must satisfy Hartwick's rule.

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1 Introduction

The purpose of this paper is to show that equitable paths in an infinite-horizon exhaustible resource model can be completely characterized in terms of *Hartwick's rule*: invest the rent from the exhaustible resource used at each date in the net accumulation of the produced capital good.

This area of study originates with a paper by Solow (1974), who analyzed a capital accumulation model, with Cobb-Douglas technology, in the presence of an exhaustible resource. He was interested in the possibility of sustainable consumption levels in this context and, eschewing the use of the traditional discounted integral of utilities as a measure of welfare, concentrated attention on the welfare of the least well off generation. His choice of the objective was of the Rawlsian Maximin type, seeking to maximize the least consumption level that can be assured along growth paths from given initial resources. Such a path is efficient as well as equitable, where equity in this context means that the path maintains a constant consumption level at all dates. Subsequently, Hartwick (1977) made the interesting observation that a competitive path, which follows the simple rule of thumb of investing the rents from the exhaustible resources used at each date, in the net accumulation of produced capital goods, is equitable. We shall refer to this investment rule as *Hartwick's rule*. As Solow (1986) has observed, this is an intuitively appealing investment rule of maintaining the consumption potential of society, in a generalized sense, by replacing exhaustible resource stocks, which are used up, with produced capital goods of equal value.

It turns out that Hartwick's Rule has significance in a wider class of models than the special context in which it arose initially. In particular, Dixit, Hammond and Hoel (1980) recognized that Hartwick's Rule is really a statement that the valuation of net investment (including the dis-investment in the exhaustible resource) is zero at each date. They then proceeded to show in a general model of accumulation involving heterogenous capital goods (which could include various non-renewable resource stocks) that if the valuation of net investment is *constant* over time (the constant is not required to be zero) then this would ensure intertemporal equity (in the sense described above, but with "consumption" interpreted now as the utility based on a vector of consumption goods). Furthermore, this investment rule, which might justifiably be called the *Dixit-Hammond-Hoel rule* was also a *necessary* condition for intertemporal equity along competitive paths.

This is an elegant characterization of competitive equitable paths. But it also naturally leads one to re-examine the special significance of Hartwick's rule for intertemporal equity. This question is prompted by the observation that in Solow's original exercise in the context of the exhaustible resource model, the maximin equitable paths do in fact satisfy Hartwick's rule, not just the Dixit-Hammond-Hoel rule. There has been quite a bit of interest in this issue more recently; see Withagen and Asheim (1998) for references to some of the literature that has emerged.

Roughly speaking, this literature might be summarized as showing that for competitive paths which are both equitable *and efficient*, Hartwick's Rule must hold. In the exhaustible resource model (but without the special structure of the Cobb-Douglas technology of Solow (1974)), a result like this was first noted by Dasgupta and Mitra (1983). However, their treatment of equity and efficiency was in the context of a discrete-time model, where Hartwick's rule does not hold in the original form but rather in a modified form. In the continuous time framework of this exhaustible resource model, Hartwick's rule does hold in its original form as a necessary condition along efficient equitable paths (see, for example, the discussion in Hamilton (1995) in the context of CES production functions). In more general models, versions of this result appear in Withagen and Asheim (1998) and Mitra (2000).

The result that we prove in this paper shows that, in the context of the exhaustible resource model in which Hartwick first proposed his rule, Hartwick's rule is both necessary and sufficient for intertemporal equity of competitive paths, provided the exhaustible resource is "important" in production. That is, in contrast to the literature mentioned in the previous paragraph, the issue of (long-run) efficiency of these paths is irrelevant in this particular context. Our result also implies the rather intriguing fact that in the context of this model, competitive paths which satisfy the Dixit-Hammond-Hoel rule (that the value of net investment be *constant*) must also satisfy Hartwick's rule (that the value of net investment be *zero*).

Our analysis also reveals a richer set of equivalence results, which may be described as follows. Consider the following three conditions that a feasible path may satisfy : (i) it is competitive; (ii) it is equitable; (iii) it satisfies Hartwick's rule. It turns out that if the path satisfies any two of these three conditions, it must also satisfy the third. In particular, this indicates that along equitable paths, Hartwick's rule ensures "myopic efficiency", which is quite different from the role for which it was originally introduced in the literature.

2 The Framework

2.1 An Exhaustible Resource Model

This is a model with one produced good, which serves as both the capital as well as the consumption good, and an exhaustible resource. Labor is assumed to be constant over time. The model described below is a standard one employed in the literature on allocation of resources over time in the presence of an exhaustible resource (see for example Dasgupta and Heal (1974, 1979), Solow (1974)).

Denote by k the stock of the augmentable capital good and by S the stock of the exhaustible resource. A number δ, satisfying $0 \leq \delta < \infty$, denotes the constant exponential depreciation rate of augmentable capital. Let $G : \mathbb{R}_+^2 \to \mathbb{R}_+$ denote the gross production function for the capital cum consumption good, using the capital input stock k and the flow of exhaustible resource used, r. We define the net production function, $F : \mathbb{R}_+^2 \to \mathbb{R}$ by: $F(k, r) = G(k, r) - \delta k$ for all $(k, r) \in \mathbb{R}_+^2$. It is assumed that the flow of resource use cannot exceed a maximum level denoted by $R > 0$. The output $G(k, r)$ can be used to replace worn out capital (if any), δk, to augment the capital stock through net investment, $z = \dot{k}$, or to provide consumption, c.

The following assumptions are made on G and w.

(A.1) $G(0, r) = G(k, 0) = 0$ for $k \in \mathbb{R}_+$, $r \in \mathbb{R}_+$.

(A.2) G is continuous, concave and nondecreasing on $\mathbb{R}_+^2$, and continuously differentiable on $\mathbb{R}_{++}^2$, with $G_1(k, r) > 0$, $G_2(k, r) > 0$.

(A.3) $\alpha \equiv \inf_{(k,r) \gg 0}[rG_2(k, r)/G(k, r)] > 0$.

While (A.1) and (A.2) are standard assumptions in this context, (A.3) conveys the restriction that the exhaustible resource is "important" in production [see Mitra (1978)]. The Cobb-Douglas production function (with capital coefficient $\beta > 0$, resource coefficient $\alpha > 0$, and $\alpha + \beta \leq 1$) satisfies (A.1)-(A.3).

2.2 Competitive Paths

A *path* from initial stock (K, S) in $\mathbb{R}_+^2$ is a triplet of functions $(k(t), r(t), c(t))$, where $k(\cdot) : [0, \infty) \to \mathbb{R}_+$, $r(\cdot) : [0, \infty) \to \mathbb{R}_+$, and $c(\cdot) : [0, \infty) \to \mathbb{R}_+$, such that $k(t), r(t), c(t)$ are differentiable functions of t, and satisfy:

$$c(t) = F(k(t), r(t)) - \dot{k}(t),\ r(t) \leq R \ and \ \dot{k}(t) \geq -\delta k(t) \ for \ t \geq 0;$$
$$\int_0^\infty r(t)dt \leq S;\ and\ k(0) = K \tag{2.1}$$

A path $(k(t), r(t), c(t))$ from (K, S) in $\mathbb{R}^2_+$ is called *interior* if $k(t) > 0, r(t) > 0$ *and* $c(t) > 0$ for $t \geq 0$.

A path $(k(t), r(t),\ c(t))$ from (K, S) is called *equitable* if $c(t)$ is constant over time. It is called *inefficient* if there is another path $(k'(t),\ r'(t), c'(t))$ from (K, S), such that $c'(t) \geq c(t)$ for $t \geq 0$, and denoting Lebesgue measure on the reals by λ,

$$\lambda\{t : c'(t) > c(t)\} > 0 \tag{2.2}$$

It is called *efficient* if it is not inefficient.

An interior path $(k(t), r(t), c(t))$ from (K, S) in $\mathbb{R}^2_+$ is called *competitive* if it satisfies *Hotelling's Rule* equating the returns on the capital good and the exhaustible resource:

$$\dot{F}_2(k(t), r(t))/F_2(k(t), r(t)) = F_1(k(t), r(t)) \tag{2.3}$$

We can associate with $(k(t), r(t), c(t))$ a path of prices $(p(t))$ as follows:

$$p(t) = 1/F_2(k(t), r(t)) \tag{2.4}$$

Then, given the concavity of F, one can verify that Hotelling's Rule (2.3) implies *intertemporal profit maximization*; that is, for all $t \geq 0$, and all $(k, r) \in \mathbb{R}^2_+$, we have:

$$p(t)F(k(t), r(t)) - (-\dot{p}(t))k(t) - r(t) \geq p(t)F(k, r) - (-\dot{p}(t))k - r \tag{2.5}$$

where $(-\dot{p}(t))$ is to be interpreted as the rental rate on capital.

Conversely, if $p(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is any differentiable function of time, such that (2.5) is satisfied by the interior path $(k(t), r(t), c(t))$ for all $t \geq 0$, and all $(k, r) \in \mathbb{R}^2_+$, then it also satisfies:

$$p(t)F_1(k(t), r(t)) + \dot{p}(t) = 0;\ p(t)F_2(k(t), r(t)) = 1 \ for \ t \geq 0 \tag{2.6}$$

so that Hotelling's Rule (2.3) must hold.

In view of this, given an interior path $(k(t), r(t), c(t))$ from (K, S) in $\mathbb{R}^2_+$ which is competitive (that is, which satisfies Hotelling's rule (2.3)), we will always associate with it the price path $(p(t))$, defined by (2.4), and refer to the path $(k(t), r(t), c(t))$ as *competitive at the prices* $(p(t))$.

3 Hartwick's Rule

3.1 Hartwick's Rule implies Equity

Hartwick's rule is a prescription to invest resource rents in the accumulation of the (augmentable) capital good; that is:

$$\dot{k}(t) = r(t)F_2(k(t), r(t)) \quad for \ t \geq 0 \tag{3.1}$$

Hartwick (1977) showed that if an interior competitive path $(k(t), r(t), c(t))$ satisfies (3.1), then it is equitable.

Hartwick's result may be seen as part of the following observation: if an interior path $(k(t), r(t), c(t))$ satisfies Harwtick's rule (3.1) then, it is equitable *if and only if* it is competitive.

To see this, differentiate the feasibility condition:

$$c(t) = F(k(t), r(t)) - \dot{k}(t)$$

to get:

$$\dot{c}(t) = F_1(k(t), r(t))\dot{k}(t) + F_2(k(t), r(t))\dot{r}(t) - \frac{d}{dt}\dot{k}(t) \tag{3.2}$$

Use Hartwick's rule (3.1), to substitute for $\dot{k}(t)$ and $\frac{d}{dt}\dot{k}(t)$ in (3.2), and get

$$\begin{aligned} \dot{c}(t) &= F_1(k(t), r(t))F_2(k(t), r(t))r(t) + F_2(k(t), r(t))\dot{r}(t) \\ &\quad -\dot{F}_2(k(t), r(t))r(t) - \dot{r}(t)F_2(k(t), r(t)) \\ &= [F_1(k(t), r(t))F_2(k(t), r(t)) - \dot{F}_2(k(t), r(t))]r(t) \end{aligned}$$

Since $r(t) > 0$ for an interior path, this yields that $\dot{c}(t) = 0$ if and only if Hotelling's rule (2.3) holds. This establishes that the path $(k(t), r(t), c(t))$ is equitable if and only if the path is competitive.

3.2 Equity implies Hartwick's Rule

We show in this subsection that if an interior competitive path $(k(t), r(t), c(t))$ is equitable, then it satisfies Hartwick's Rule (3.1). Establishing this converse theorem is considerably more involved than the result discussed in the previous section. However, the proof can be conveniently split up into several steps, which are of some independent interest.

First, one establishes (one-half of) the result of Dixit,Hammond and Hoel (1980) [which we mentioned in Section 1] that for an interior competitive path which is equitable, the value of net investment (in both the augmentable capital good and the exhaustible resource) be constant. Second, one notes that equity implies that the rate of depreciation, δ, must equal zero[1]. Third, the second step is used to establish that the constant obtained in the first step must be zero.

Step 1: Let $(k(t), r(t), c(t))$ be an interior equitable competitive path from $(K, S) \in \mathbb{R}^2_+$, with associated prices $(p(t))$, defined as in (2.4). Denote the value of net investment $p(t)z(t)$ by $n(t)$ for $t \geq 0$. Then, we have for $t \geq 0$:

$$p(t)F(k(t), r(t)) = p(t)c(t) + n(t) \tag{3.3}$$

Differentiating (3.3) with respect to t, we obtain:

$$\begin{aligned} & p(t)[F_1(k(t), r(t))\dot{k}(t) + F_2(k(t), r(t))\dot{r}(t)] + \dot{p}(t)F(k(t), r(t)) \\ & = \dot{p}(t)c(t) + \dot{n}(t) \end{aligned} \tag{3.4}$$

Using (2.4) and the fact that $\dot{n}(t) = \dot{p}(t)z(t) + p(t)\dot{z}(t)$, we have:

$$\begin{aligned} & p(t)F_1(k(t), r(t))\dot{k}(t) + \dot{r}(t) + \dot{p}(t)F(k(t), r(t)) \\ = \;& \dot{p}(t)c(t) + \dot{p}(t)z(t) + p(t)\dot{z}(t) \end{aligned} \tag{3.5}$$

Noting that $F(k(t), r(t)) = c(t) + z(t)$ for $t \geq 0$, we obtain:

$$p(t)F_1(k(t), r(t))\dot{k}(t) + \dot{r}(t) = p(t)\dot{z}(t) \tag{3.6}$$

Using (2.3) and (2.4), we can rewrite (3.6) as:

$$p(t)[-\dot{p}(t)/p(t)]\dot{k}(t) + \dot{r}(t) = p(t)\dot{z}(t)$$

which yields:

$$\dot{r}(t) = p(t)\dot{z}(t) + \dot{p}(t)z(t) \tag{3.7}$$

Clearly, (3.7) implies that:

$$\frac{d}{dt}[p(t)z(t) - r(t)] = 0 \tag{3.8}$$

[1] In establishing *Step 2*, we employ some of the arguments, which we have used previously in Mitra (1978) and Dasgupta and Mitra (1999).

This shows that the value of net investment in both the augmentable capital good and the exhaustible resource, $m(t) \equiv p(t)z(t) - r(t)$ is constant over time.

Step 2: We establish this step by contradiction. Let $(k(t), r(t), c(t))$ be an interior equitable competitive path from $(K, S) \in \mathbb{R}^2_+$. Then there is some number $c > 0$ such that $c(t) = c$ for $t \geq 0$. Suppose that $\delta > 0$. Then, it is shown below that (a) $k(t)$ must be bounded above; and (b) in order to maintain the constant consumption level $c > 0$, the path $(k(t), r(t), c(t))$ violates the feasibility condition (2.1).

We establish (a) as follows. Using (A.1) and (A.2), we can find $a > 0$, such that $G(1, a) < (\delta/2)$. Define $B = \max\{1, (R/a), K\}$, where R is given as in (2.1). Then, we have the following property:

$$If\ k(t) \geq B\ for\ some\ t = s,\ then\ \dot{k}(s) < 0 \tag{3.9}$$

To establish (3.9), we write the following string of inequalities when $k(t) \geq B$ for $t = s$,

$$\begin{aligned} \dot{k}(s) &\leq F(k(s), r(s)) \leq G(k(s), R) - \delta k(s) \\ &\leq k(s)G(1, a) - \delta k(s) < -(\delta/2)k(s) < 0 \end{aligned} \tag{3.10}$$

The inequalities in (3.10) are all self-evident, except for the third one. The third inequality in (3.10) follows from (A.1) and (A.2) and the fact that $B \geq \max\{1, (R/a)\}$. To see this, note that, since $k(s) \geq 1$,and G is concave, we have:

$$G(1, \frac{R}{k(s)}) \geq \frac{1}{k(s)}G\left(k\left(s\right), R\right) + [1 - \frac{1}{k\left(s\right)}]G\left(0, 0\right) = \frac{1}{k(s)}G\left(k\left(s\right), R\right)$$

This yields:

$$G\left(k\left(s\right), R\right) \leq k\left(s\right) G(1, \frac{R}{k(s)}) \leq k(s)G(1, a)$$

since $(R/k(s)) \leq a$.

Having established (3.9), we now claim that :

$$k(t) \leq B\ for\ t \geq 0 \tag{3.11}$$

Clearly, $k(t) \leq B$ for $t = 0$. So, if (3.11) is violated, there exists $t' > 0$ such that $k(t') > B$. Denote $\{[k(t') - B]/2\}$ by b; then $b > 0$. Then $k(t') > B + b$, while $k(0) < B + b$. Let $A = \{t \in [0, t'] : k(t) \leq B + b\}$. Clearly A is non-empty and bounded. Furthermore, if (t^s) is a sequence of elements in A which converges to some $\hat{t} \in \mathbb{R}$, then $\hat{t} \in [0, t']$, and $k(t^s) \leq B + b$ for all s. Thus, by continuity of $k(\cdot)$, we have $k(\hat{t}) \leq B + b$. Thus, $\hat{t} \in A$, and A is closed. Define $\tau = \max A$; note that τ is well-defined, and $\tau < t'$.

Now, clearly, from the definition of τ, it follows that:

$$k(t) > B + b \; for \; all \; t \in (\tau, t') \tag{3.12}$$

By (3.9), we must then have $\dot{k}(t) < 0$ for all $t \in (\tau, t')$. This produces the contradiction:

$$B + b < k(t') < k(\tau) \leq B + b$$

and establishes (3.11) and hence (a).

To establish part (b) of *Step 2*, we note that by (2.1), $G(k(t), r(t)) = c(t) + \dot{k}(t) + \delta k(t) \geq c(t) = c$ for $t \geq 0$. Thus, we have, using (3.11) :

$$G(B, r(t)) \geq c \; for \; all \; t \geq 0 \tag{3.13}$$

Clearly, (A.1),(A.2), and (3.13) imply that there is some $b' > 0$ such that $r(t) \geq b'$ for $t \geq 0$. But, this clearly violates the feasibility condition:

$$\int_0^\infty r(t)dt \leq S \tag{3.14}$$

This completes *Step 2*.

Step 3: We showed in *Step 1* that $[p(t)\dot{k}(t) - r(t)]$ is constant over time. Denote this constant by E. We will now show that $E = 0$.

Suppose $E < 0$. Since $\dot{k}(t) \geq -\delta k(t) = 0$ by *Step 2*, we have $[-r(t)] = E - p(t)\dot{k}(t) \leq E$, and so $r(t) \geq (-E) > 0$ for $t \geq 0$. But this violates the feasibility condition (3.14).

Suppose $E > 0$. Then, $p(t)\dot{k}(t) = r(t) + E \geq E$ for $t \geq 0$. But, then, using *Step 2*, we get for $t \geq 0$:

$$\begin{aligned} [1/\alpha]r(t) &\geq [G(k(t), r(t))/r(t)G_2(k(t), r(t))]r(t) \\ &= p(t)G(k(t), r(t)) \geq p(t)\dot{k}(t) \geq E \end{aligned} \tag{3.15}$$

the first inequality in (3.15) following from assumption (A.3). Thus, for $t \geq 0$, we have $r(t) \geq \alpha E > 0$, which again violates the feasibility condition (3.14). This completes *Step 3*, and shows that $[p(t)\dot{k}(t) - r(t)] = 0$ for all $t \geq 0$. Thus, using (2.4), we have $\dot{k}(t) = G_2(k(t), r(t))r(t)$ for $t \geq 0$, which is Hartwick's rule.

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