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Strategy-proofness and efficiency are incompatible in production economies

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Abstract

In a production economy where a single private good is produced via a non-linear concave technology, no direct mechanism satisfies strategy-proofness and efficiency if the preference domain contains the class of linear preferences.

Keywords: Strategy-proofness; Efficiency; Production economies

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1. Introduction

The trade-off between *efficiency* and *strategy-proofness* has been studied in great detail in the case of the distribution and exchange of private goods. These concepts together lead to decidedly unfair allocations of resources. Originating from a conjecture in Hurwicz (1972), this result was first proved in the two-agent case (e.g. Kato and Ohseto, 2002; Ju, 2003; Schummer, 1997; Sprumont, 1995; Zhou, 1991) examining various domain restrictions. Recently, Serizawa (2002) formally established this negative result for an arbitrary number of agents.

We study the same trade-off for simple production economies where a single private good is produced via a concave technology. Maniquet and Sprumont (1999) show that *strategy-proofness* and *efficiency* can coexist in a linear production model on the domain of classical economic preferences, even in

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combination with *anonymity*. They call the resulting unique solution the equal budget free choice mechanism: every agent obtains the bundle she would choose if operating the technology alone.

We show that this positive result does not survive if the single technology is concave but not linear (though not necessarily strictly concave). Then *strategy-proofness* and *efficiency* are incompatible on the domain of linear preferences. Because *strategy-proofness* is a stronger property on larger domains, our result extends to any domain containing the class of linear preferences.

Shenker (1992) considers a cost-sharing model that includes ours as a special case. He states that if the technology exhibits decreasing returns to scale, any incentive compatible sharing rule is of the serial type. This would imply our result as serial-like methods are not first-best efficient. Yet, our result improves upon his statement on the *strategy-proofness* and *efficiency* trade-off in three respects. First of all, Shenker imposes smoothness conditions on the technology and on the allocation rule, which we do not. Secondly, his additional conditions on the technology amount in our setting to strict concavity of the production function, which we do not require. Lastly, and more importantly, his incentive compatibility criterion is much stronger than *strategy-proofness*: implementability in Nash equilibrium strategies is in fact even stronger than *group strategy-proofness*.

2. The model and theorem

Let $N=\{1, \ldots, n\}$ be the set of agents. Let F be a strictly increasing, concave (though not necessarily strictly concave) non-linear function of $\mathbb{R}_+$ to itself such that $F(0)=0$. A *bundle* is an element $z_i=(x_i, y_i)\in\mathbb{R}_+\times\mathbb{R}$, and an *allocation* is a list of n bundles, $z=(z_1, \ldots, z_n)$, one for each agent. The set of feasible allocations is denoted by

$$Z=\left\{z\in(\mathbb{R}_+\times\mathbb{R})^N \,\Big|\, \sum_i y_i \le F\left(\sum_i x_i\right)\right\}.$$

For any subset $S\subseteq N$, we write $x_s=\sum_{i\in s} x_i$ and $y_s=\sum_{i\in s} y_i$.

Each agent is endowed with a *preference*, R_i, over $\mathbb{R}_+\times\mathbb{R}$ which is strictly monotonic: strictly increasing in y_i and strictly decreasing in x_i. We denote by $\mathcal{R}_0$ the class of preferences. A preference profile is a list of n preferences, $R=(R_1, \ldots, R_n)$. We sometimes write $R=(R_j, R_{-j})$ for some $j\in N$. Let $\mathcal{L}\subset\mathcal{R}_0$ be the class of linear preferences. Each preference $L\in\mathcal{L}$ can be identified with a number $l\in\mathbb{R}_{++}$ that corresponds to the slope of its indifference curves in the (x, y)-plane. The corresponding utility for agent i is $u_i(x_i, y_i)=y_i-lx_i$.

For any subset $A\subseteq\mathbb{R}_+\times\mathbb{R}$ and any preference relation $R_i\in\mathcal{R}$, we define $m(A, R_i)=\{z_i\in A \mid \forall z'_i \in A \;\; z_i R_i z'_i\}$ to be the set of maximal elements of A according to R_i. For any preference profile $R\in\mathcal{R}^N$, we denote by

$$PE(R)\equiv\{z\in Z|\forall z' \in Z[z'_i\; R_i z_i \quad \forall i\in N \Rightarrow z'_i I_i z_i \quad \forall i\in N]$$

the set of Pareto-efficient allocations.

Let $\mathcal{R} \subseteq \mathcal{R}_0$, a direct allocation mechanism (or *mechanism*) $\mu : \mathcal{R}^N \to Z$ associates with each preference profile a feasible allocation. We are interested in the following axioms to be verified by a mechanism μ:

Pareto efficiency (PE) $\forall R \in \mathcal{R}^N,\ \mu(R) \in PE(R)$.
Strategy-proofness (SP) $\forall R \in \mathcal{R}^N \forall i \in N \forall R'_i \in \mathcal{R},\ \mu_i(R) R_i \mu_i(R'_i, R_{-i})$.

Theorem. *Let $\mathcal{L} \subseteq \mathcal{R} \subseteq \mathcal{R}_0$. No mechanism $\mu : R^N \to Z$ satisfies SP and PE.*

As in Maniquet and Sprumont (1999) we determine the shape of the agents option sets, i.e. the sets of attainable bundles given the reports of others. We proceed by contradiction, assuming that a mechanism μ satisfies SP and PE to later show that the shapes of the option sets generated by μ are not feasible; namely, budget balance is violated.

We start the proof with two lemmas. Lemma 1 states a general property of option sets and can be found in Maniquet and Sprumont (1999). Loosely speaking, Lemma 2 states that if a function f is concave and if a function g has the same slope as f for every value of t in its domain, then g coincides with f up to a positive constant. First, a definition: a *strictly increasing subset of* $\mathbb{R}_+ \times \mathbb{R}$ is a set $h \subset \mathbb{R}_+ \times \mathbb{R}$ such that for all $(x_i, y_i),\ (x'_i, y'_i) \in h,\ x_i > x'_i \Leftrightarrow y_i > y'_i$.

Lemma 1. *Let $\mathcal{R} \subseteq \mathcal{R}_0$. A mechanism $\mu : \mathcal{R}^N \to Z$ satisfies SP if and only if for every $i \in N$, there exists a correspondence $\mathcal{O}_i : \mathcal{R}^{N \setminus \{i\}} \to \mathbb{R}_+ \times \mathbb{R}$, such that for every $R \in \mathcal{R}^N$,*
(i) $\mu_i(R) \in m(\mathcal{O}_i(R_{-i}), R_i)$,
(ii) $\mathcal{O}_i(R_{-i})$ *is a strictly increasing subset of* $\mathbb{R}_+ \times \mathbb{R}$.

Lemma 2. *Let f: $\mathbb{R}_+ \to \mathbb{R}$ be an increasing concave function and let g: $\mathcal{D} \to \mathbb{R}$ with $\mathcal{D} \subseteq \mathbb{R}_+$. Let $l_m \geq 0$ and $x_m = sup(arg\,max_{t \in \mathcal{D}}\, g(t) - l_m t)$. Then, if*

$$\forall l \in]0, l_m[, \quad arg\ \max_{t \in \mathbb{R}_+} (f(t) - lt) \supseteq arg\ \max_{t \in \mathcal{D}} (g(t) - lt) \neq \emptyset, \tag{1}$$

(*with the convention that arg max* $_{t \in \mathbb{R}_+} (f(t) - lt) = \{+\infty\} \neq \emptyset$ *if* $l < \lim_{t \to \infty} f'(t)$), *there exists $\alpha \in \mathbb{R}$ such that g and $f + \alpha$ coincide on $\mathcal{D} \cap [x_m, +\infty]$.*

Proof. Define, for any $l \in]0, l_m[$,

$$\psi(l) = \max_{t \geq 0} f(t) - lt \quad \text{and} \quad \theta(l) = \max_{t \in \mathcal{D}} g(t) - lt$$

Letting $x(l)$ be a solution of $\max_{t \geq 0} f(t) - lt$, writing $\psi(l) = f(x(l)) - lx(l)$ yields that the derivative of ψ at l equals $\psi'(l) = x'(l)(f'(x(l)) - l) - x(l)$ with either $f'(x(l)) = l$ (in general) or $x'(l) = 0$ (at a kink in the graph of f). I.e., $\psi'(l) = -x$ for some $x \in \arg\max_{t \geq 0} f(t) - lt$. From the concavity of f, $\arg\max_{t \geq 0} f(t) - lt$ is single-valued everywhere except on a countable subset of values of l, corresponding to the linear parts of f (if any). Similarly, the derivative of θ at l equals $\theta'(l) = -x$ for some $x \in \arg\max_{t \in \mathcal{D}} g(t) - lt$ for any $l \in]0, l_m[$.

From (1), ψ' and θ' must coincide almost everywhere on $]0, l_m[$. Hence,ψ and θ coincide up to a constant on $]0, l_m[$. Therefore f and g coincide up to a constant on $\mathcal{D}\cap[x_m, +\infty]$. ■

We now tackle the proof of the theorem. For the sake of contradiction let $\mu:\mathcal{L}^N \rightarrow Z$ satisfy SP and PE. For any $L\in\mathcal{L}^N$ denote the corresponding vector of slopes $(l_1, \ldots, l_n)$ and write $\mu(L)=(x_i, y_i)_{i\in \mathrm{N}}$. Define $\underline{l}=\min_{j\in N} l_j$ and $J=\arg\min_{j\in N} l_j$. Denote by F' (resp. F'_-) the derivative (resp. left-derivative) of F. Because F is non-linear, we can assume

$$F'(0) > \underline{l} > \lim_{t\rightarrow+\infty} F'_-(t). \tag{2}$$

Step 1

$$\begin{cases} x_i > 0 \text{ only if } i\in J, & (\alpha) \\ x_N = x_J\in \arg\max(F(t) - \underline{l}t) & (\beta) \text{ and,} \\ y_N = F(x_N). & (\gamma) \end{cases} \tag{3}$$

Condition (α) follows from trade efficiency: any agent i for which $l_i>\underline{l}$ and $x_i>0$ would gladly pay $\frac{l_i+\underline{l}}{2}$ units of output in order to provide one less unit of input; any agent $j\in J$ would accept to trade with i. (β) follows from (α) and production efficiency. Condition (γ) states that all the output is allocated.

Fix $i\in N$ and $L_{-i}\in\mathcal{L}^{N\setminus\{i\}}$ until Step 5. We apply Lemma 1 and write $\mathcal{O}_i(L_{-i})$ the option set of agent i. Define $l_i^-=\min_{j\neq i} l_j$ and $\hat{x}_i=\max(\arg\max(F(t)-l_i^- t))$. Notice that $\hat{x}_i$ is finite because $l_i^->\lim_{t\rightarrow\infty} F'(t)$ (from (2)). For any $L_i\in\mathcal{L}$, denote by $l_i\in\mathbb{R}_{++}$ the corresponding slope. We write $\mu(L_i; L_{-i})=(x_i(l_i), y_i(l_i))_{i\in N}$.

Steps 2–4 are devoted to the description of the shape of $\mathcal{O}_i(L_{-i})$.

Step 2

$$\mathcal{O}i(L_{-i}) \cap ([0, \hat{x}_i] \times \mathbb{R}) \subseteq \{(x, y)\in[0, \hat{x}_i] \times \mathbb{R} | y = \alpha_i + l_i^- x\} \qquad \text{for some } \alpha_i\in\mathbb{R}.$$

Let $l_i>l_i^-$, condition (3.α) requires $x_i(l_i)=0$; denote $\alpha_i=y_i(l_i)$. We claim that $z_i(l'_i)=(0, \alpha_i)$ for all $l'_i>l_i^-$ because of SP. Indeed, assume there existed some $l'_i>l_i^-$ such that $z_i(l'_i)=(0, \alpha'_i)$ with $\alpha'_i\neq \alpha_i$. If $\alpha'_i>\alpha_i$, agent i could benefit from reporting l'_i at (L_i, L_{-i}); if the inequality were reversed, agent i could benefit from reporting l_i at (L'_i, L_{-i}).

Now if $l_i=l_i^-$, $x_i(l_i^-)\in[0, \hat{x}_i]$ by condition (3.β); we show that SP requires $y_i(l_i^-)=\alpha_i+l_i^- x_i(l_i^-)$. Assume $y_i(l_i^-)>\alpha_i+l_i^- x_i(l_i^-)=0$, then $y_i(l_i^-)>\alpha_i$ and agent i can benefit from reporting l_i^- when her true preference is in fact $l_i'>l_i^-$. If $x_i(l_i^-)>0$, let $l_i^* = \frac{y_i(l_i^-)-\alpha_i}{x_i(l_i^-)} > l_i^-$; agent i could benefit from reporting l_i^- instead of $l_i'\in]l_i^-, l_i^*[$. Therefore $y_i(l_i^-) \leq \alpha_i+l_i^- x_i(l_i^-)$. Similarly, $y_i(l_i^-)\geq\alpha_i+l_i^- x_i(l_i^-)$.

Also, for any $l_i<l_i^-$, $x_i(l_i)\geq\hat{x}_i$; and if there exists some $l_i'<l_i^-$ for which $x_i(l_i') = \hat{x}_i$, the same reasoning as above yields $y_i(l_i') = \alpha_i + l_i^-\hat{x}_i$. Step 2 has been proved.

For the next step, we use the following notation: for any $k\in\mathbb{R}$, define $B(k)=\{(x_i, y_i)\in\mathbb{R}_+\times\mathbb{R} | y_i=F(x_i)+k\}$.

Step 3

$\mathcal{O}_i(L_{-i}) \cap ([\hat{x}_i, +\infty[\times \mathrm{R}) \subset B(\beta_i)$ for some $\beta_i \in \mathbb{R}$.

Let ω be the real-valued function whose graph is $\mathcal{O}_i(L_{-i})$ and $\mathcal{D}$ its domain. We wish to apply Lemma 2 where F plays the role of f, ω the role of g and where $l_m = l_i^-$. We need to check that for any $0 < l_i < l_i^-$,

$$\emptyset \neq \arg\max_{t \in \mathcal{D}} \omega(t) - l_i t \subseteq \arg\max_{t \geq 0} F(t) - l_i t.$$

Let $0 < l_i < l_i^-$ write $\mu_i(L)=(x_i, y_i)$. From Lemma 1 $(x_i, y_i) \in m\ (\mathcal{O}_i(L_{-i}), L_i)$; therefore, by the definition of ω, $x_i \in \arg\max_{t \in \mathcal{D}} \omega(t) - l_i t$. However, $J=\{i\}$ because $l_i < l_i^-$; condition (3.β) yields $x_i \in \arg\max_{t \geq 0} F(t) - l_i t$. We can apply Lemma 2 and conclude that ω and F coincide up to a constant on $[\hat{x}_i, +\infty[$.

Step 4

$$\beta_i = \alpha_i + l_i^- \hat{x}_i - F(\hat{x}_i)$$

By concavity of F and by definition of $\hat{x}_i$, F is strictly concave at $\hat{x}_i$. Therefore, $\lim_{l \uparrow l_i^-}$ (arg max $(F(t)$ $lt)) = \hat{x}_i$, and hence $\lim_{l \uparrow l_i^-} x_i(l) = \hat{x}_i$. Steps 2 and 3 yield the result.

Now that the shape of $\mathcal{O}_i(L_{-i})$ has been determined, we show that it is an implausible one. Note that α_i is actually a function of L_{-i}; from now on we write $\alpha_i(L_{-i})$. For any $L \in \mathcal{L}^N$ let l_{n-1}^* denote the second smallest entry of the corresponding vector of slopes $(l_1, \ldots, l_n) \in \mathbb{R}_{++}^n$. Also, for any $l \in \mathbb{R}_{++}$, define $h(l) = \max(F(t) - lt)$.

Step 5

For any $L \in \mathcal{L}^N$, $\sum \alpha_i(L_{-i}) = h(l_{n-1}^*)$.

Consider $L \in \mathcal{L}^N$ and the corresponding $(l_1, \ldots, l_n) \in \mathbb{R}_+^N$. Recall that $J = \arg\min_{j \in N} l_j$; clearly $J \neq \emptyset$. Step 1 requires $x_i = 0$ and $y_i = \alpha_i(L_{-i})$ for all $i \notin J$. If $|J| = 1$, say $J=\{j\}$, then $y_i = \alpha_j(L_{-j}) + l_{n-1}^* \hat{x}_j + F(x_j) - F(\hat{x}_j)$. By efficiency:

$$y_N = F(x_N),$$

$$\text{i.e.} \quad \sum \alpha_i(L_{-i}) + l_{n-1}^* \hat{x}_j + F(x_j) - F(\hat{x}_j) = F(x_j),$$

$$\text{i.e.} \quad \sum \alpha_i(L_{-i}) + h(l_{n-1}^*) \text{ by construction of } \hat{x}_j.$$

Now suppose $|J| \geq 2$, then $l_{n-1}^* = \underline{l}$. Moreover, for any $j \in J$, $x_j \geq 0$, $y_j = \alpha_j(L_{-j}) + l_{n-1}^* x_j$ and $x_J \in \arg\max F(t) - l_{n-1}^* t)$. Once again the result follows from efficiency:

$$y_N = F(x_N),$$

$$\text{i.e.} \quad \sum \alpha_i(L_{-i}) + l_{n-1}^* x_J = F(x_J),$$

$$\text{i.e.} \quad \sum \alpha_i(L_{-i}) = h(l_{n-1}^*).$$

Upon noticing that h is strictly decreasing on $]\underline{l}, F'(0)[$, a slight variation of a standard argument in the literature on Clarke–Grove mechanisms (omitted for brevity but available upon request) yields a contradiction: $(l_1, \ldots, l_n) \mapsto h(l^*_{n-1})$ cannot be decomposed into n functions depending only on $n-1$ variables.

Remark. We strongly suspect that our proof technique successfully applies to the many-inputs-one-output case without many conceptual modifications.

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