Responsibility and cross-subsidization in cost sharing

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Abstract. We propose two axiomatic theories of cost sharing with the common premise that individual demands are comparable, though perhaps different, commodities, and that agents are responsible for their own demand. Under *partial responsibility* the agents are not responsible for the asymmetries of the cost function: two agents consuming the same amount of output always pay the same price; this holds true under *full responsibility* only if the cost function is symmetric in all individual demands. If the cost function is additively separable, each agent pays his/her stand alone cost under full responsibility; this holds true under partial responsibility only if, in addition, the cost function is symmetric.

By generalizing Moulin and Shenker's (1999) Distributivity axiom to cost-sharing methods for heterogeneous goods, we identify in each of our two theories a different serial method. The *subsidy-free serial* method (Moulin, 1995) is essentially the only distributive method meeting Ranking and Dummy. The *cross-subsidizing serial* method (Sprumont, 1998) is the only distributive method satisfying Separability and Strong Ranking. Finally, we propose an alternative characterization of the latter method based on a strengthening of Distributivity.

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1. Two theories of cost sharing

An equitable allocation of joint costs is one where everyone pays the share that they are responsible for. The challenging theoretical and empirical question is to correctly assess individual responsibilities.

Individual demands influence total cost in two different ways: by their size and by their nature. A simple example is mail distribution, where cost obviously depends on volume but also on destination, rural delivery being more expensive than urban delivery. The same is true of other transportation networks –from the internet to a bus system and to water distribution– where the volume of traffic and the delivery route both affect total cost.

The classical theory of fair pricing developed in the natural monopoly literature (Baumol, Panzar and Willig (1982), Sharkey (1982)), rests on the principle of *no cross-subsidization.* For instance, if serving a certain agent requires a specific investment (such as running a cable) of no use to other users, the corresponding cost is deemed *separable* and should be imputed in full to the agent in question. With more complex cost structures, a formal translation of no cross-subsidization is not a simple matter, but the general principle is that an agent's cost share increases with the marginal cost of his own demand. Thus each agent is held responsible both for the size and the nature of his demand, the latter being captured by the asymmetry of the cost function with respect to individual demands. No cross-subsidization is at work when international mail is more expensive than domestic mail, when power companies charge less for off-peak electricity, when airlines apply a surcharge for excess baggage, and so on.

Yet, cross-subsidization is a pervasive feature of pricing rules for many commodities or services. The same price is charged to deliver mail, or water, to a rural or an urban domestic address; the universal service constraint for telephone implies, among other things, that the connecting charge to a residential customer is the same whether the house is pre-wired or not; special transportation services are offered to handicapped persons at the same price as public transportation for non-handicapped persons. In these familiar instances, cost shares reflect differences in demand size, but not asymmetries of the cost function. The underlying ethical principle is that individuals are responsible for their own demand, but not for cost asymmetries, because the latter are beyond their control. The farmer should not pay more for his mail, because he cannot farm in town, the resident is not responsible for the location of the water treatment facility, the handicapped person is unable to use the regular bus but should not be penalized for it, and so on.

Thus, two views on responsibility in the formation of joint costs coexist: one where users are responsible for asymmetries in the cost function, and one where they are not. Within the familiar axiomatic model of cost sharing we develop accordingly two related theories. Both theories assume that each agent consumes an idiosyncratic commodity (e.g., water delivered at her home), yet commodities are interpersonally comparable. They are all measured in a common unit (e.g., cubic feet of water), so that we can compare the demands of any two agents.

Both theories hold each agent responsible for the size of his/her own demand: a higher demand calls for a higher cost share. The *full responsibility theory* also holds agents responsible for asymmetries in the cost function: if it costs more to deliver the same level of service to Jones than to Smith, Jones should pay more. The *partial responsibility theory* takes the opposite view that Jones and Smith must get the same bill for the same level consumption.

The vast literature on axiomatic cost sharing inspired by Shapley's (1953) seminal contribution (briefly reviewed in section 3) takes squarely the full responsibility viewpoint. By contrast, partial responsibility is a fairly recent theme of the distributive justice literature (again, see section 3). Its only previous application to cost sharing appears to be the informal discussion in Fleurbaey and Trannoy (1998), which inspired our work.

2. Overview of the results

We are given a multivariate cost function $C(z_1, ..., z_n)$ and a demand profile $x = (x_1, ..., x_n)$, where x_i is agent i's demand of commodity i. The differences in nature between the commodities are described by the asymmetries of the function C. If C is symmetric in all variables, the goods are deemed "interchangeable" and the only ethically relevant difference between individual demands is their size.

We maintain the cost-sharing interpretation throughout, yet the output-sharing interpretation is equally meaningful. There x_i is agent $i's$ input (e.g., hours worked), C is the production function, and we must share total output (e.g., revenue) $C(x)$. Think of asymmetries in the production function generated by the division of labor within the firm. If the tasks assigned to two workers require different skills, we expect their compensation (share of output) to reflect this difference when the skills are not firm-specific (e.g., a professional degree). This is the full responsibility viewpoint. On the other hand, in the spirit of partial responsibility, differences in firm-specific skills typically have no impact on compensation: workers with identical seniority and professional degree get the same pay.

Back to the cost-sharing interpretation once and for all, we introduce the key equity principles on which our two theories are built. Consider first the case of an additively separable cost function, $C(z) = \sum_i c_i(z_i)$. Here the size of $i's$ demand does not affect the marginal cost of $j's$ demand, for all ij. Under full responsibility, i must be charged his *stand alone* cost $c_i(x_i)$. This is normatively compelling, and creates the correct incentives: agent $i's$ optimal demand as she faces her stand alone cost is a dominant strategy, and the resulting equilibrium is efficient. We call Strong

Separability the requirement that when costs are additively separable, each agent pays his or her stand alone cost. A closely related property is the familiar *Dummy* axiom. Consider a cost function C for which the marginal cost of one of the demands, z_i, is zero, irrespective of other demands: Dummy states that agent i should not be charged anything. Under the Additivity axiom discussed below, Dummy is in fact equivalent to Strong Separability.

The defining axiom of the partial responsibility approach is *Strong Ranking*, stating that for *any* cost function, symmetric or not, agent i should not pay less than j if she demands no less than j. Strong Ranking is clearly incompatible with Strong Separability (and Dummy). Suppose costs are additively separable, $C(z) = \sum_i c_i(z_i)$, and $i's$ stand alone cost exceeds $j's$ at every level, $c_i(t) > c_j(t)$ for all t: if $x_i = x_j$ Strong Ranking sets equal cost shares for i and j whereby j subsidizes i.

On the other hand, both theories hold agents responsible for the size of their own demand. Two properties common to both theories weaken Strong Ranking and Strong Separability, respectively, by restricting their application to those cost functions symmetric in all variables z_i. *Ranking* requires then that agent i should pay no less than j if she demands no less than j. If costs are not only symmetric but also additively separable, $C(z) = \sum_i c_i(z_i)$, *Separability* insists that we charge his stand alone cost to each agent. The former property is a compelling fairness principle, the latter induces the "correct" incentives as explained above. Separability rules out the simple proportional method often used in practice, and charging $(x_i/\sum_j x_j)C(x)$ to agent i. The latter method satisfies Strong Ranking, however.

Our two theories rely on identical invariance properties, namely the familiar *Additivity* axiom and the less known *Distributivity* axiom (both discussed below), but postulate two different pairs of equity properties.

These powerful invariance properties both state that the computation of cost shares commutes with a certain operation on cost functions: addition in the former axiom, and composition in the latter. Additivity is motivated by the observation that production often can be decomposed in several largely independent processes and in that case the axiom allows one to compute the cost shares separately in each subprocess. Examples include the costs of research, production and marketing if a new product; the costs of construction and maintenance of highways and other communication networks, see Castano-Pardo and Garcia-Diaz (1995) and Lee (2002). Distributivity is the same kind of invariance property when the production process can be decomposed in several sequential processes. But if the addition of cost functions is always well defined, the same is not true of their composition. Distributivity is therefore only defined for *one-output* cost functions, taking the form $C(z) = c(\sum_i z_i)$ and interpreted as the case where individuals demand the "same" commodity. See Section 8 for details.

From this handful of axioms two-cost sharing methods emerge forcefully. Both are extensions of Moulin and Shenker's (1992) serial mechanism for one-output cost functions; they coincide also when the cost function is symmetrical in all its variables. We call these two methods the *subsidy-free serial method* and the *cross-subsidizing serial method.* The former method is essentially the only additive and distributive method satisfying Ranking and Dummy, the two equity requirements of the full responsibility approach. Our results, stated in Theorem 1 and its Corollary, actually require two additional properties: Demand Monotonicity (increasing my demand cannot lower my cost share), in the spirit of demand responsibility, and Dummy Independence (changing the demand of a dummy agent has not effect on cost shares), in the spirit of cost responsibility.

The second method is the only additive and distributive method meeting Separability and Strong Ranking, the two requirements of the partial responsibility approach: see Theorem 2.

Finally, we propose a strong version of Distributivity that generalizes the original property to the composition of an arbitrary cost function with a one-output one. In conjunction with Additivity, this property is so powerful that adding only Ranking and Separability –two properties consistent with both approaches to responsibility– suffices to pin down the cross-subsidizing serial method: see Theorem 3.

Section 3 relates our work to the literature, and Section 4 introduces our cost-sharing model, where goods come in indivisible units. The equity axioms reflecting the two views of responsibility are the subject of Section 5, whereas Section 6 illustrates that, if we do not impose Distributivity, either view is compatible with a large number of cost-sharing methods. The two serial methods, subsidy-free and cross-subsidizing, are defined in Section 7. The Distributivity axiom is the subject of Section 8. Our main results, an axiomatic characterization of the subsidy-free serial method and two characterizations of the cross-subsidizing serial method, are presented respectively in Sections 9 and 10. Section 11 briefly evokes possible extensions of our results, and Section 12 contains the proofs.

3. Related literature

3.1. There is a vast body of axiomatic research on the full responsibility approach to cost sharing. Most of it focuses on the case of perfectly divisible goods (to which we refer as the *continuous* model), so that each demand x_i is a nonnegative real number, whereas in the *discrete* model considered here, x_i is an integer. The difference is mostly technical; the discrete model avoids the many topological difficulties of the continuous model.

In the continuous model, Dummy and Additivity are generally assumed. In the rich class characterized by these two axioms (Friedman (1998), Haimanko (2000)), the three main methods of interest are: the *Aumann-Shapley* method (Aumann and Shapley (1974), Billera and Heath (1982), Mirman and Tauman (1982), Samet and

Tauman (1982), Young (1985)); the *Shapley-Shubik* method (Shapley (1953), Shubik (1962), Sprumont (1998), Friedman and Moulin (1999)); and the *serial* method (Moulin and Shenker (1994), and Friedman and Moulin (1999)). The latter is the continuous version of the subsidy-free serial method.

In the discrete model, the class of methods characterized by the Dummy and Additivity axioms is easier to describe (Wang (1999)), and the same three methods play the central role: the *Aumann-Shapley* method (Moulin (1995), van den Nouweland, Potters, Tijs, and Zarzuelo (1995)); the *Shapley-Shubik* method (Sprumont (2000)); and the *serial* method introduced in Moulin (1995), which coincides with our subsidy-free serial method.

For a detailed survey and further references, we refer the reader to Moulin (2002).

3.2. The partial responsibility approach to cost sharing is the subject of only one non-technical paper by Fleurbaey and Trannoy (1998). However, the general idea that a proper definition of fairness depends on a correct assessment of the scope of individual responsibility is an important new theme in the distributive justice literature. Following Fleurbaey (1994, 1995) and Roemer (1993, 1994, 1996), the literature assumes that individuals are characterized by parameters for which they are responsible, and others for which they are not. The challenge is to define notions of fairness that offset (resp. preserve) inequalities resulting from differences in parameters of the latter (resp. former) type.

Most closely related to the current work are a number of surplus-sharing methods restricted to the case of additively separable production functions: see for instance Bossert (1995), Bossert and Fleurbaey (1996), Sprumont (1997), and Tungodden (2000). By contrast with those contributions, our paper handles full-fledged externalities: the marginal cost of serving a particular agent varies in arbitrary fashion with the demand profile of the others.

3.3. Characterizations of several methods of the serial family are found in the literature. Moulin and Shenker (1994) characterize the serial formula for one-output cost functions with the aid of an axiom placing upper bounds on cost shares. Using a similar axiom, Moulin (1995) and Friedman and Moulin (1999) characterize the subsidy-free serial method in the discrete and continuous contexts respectively. We submit that the Upper Bound axiom is intuitively reminiscent of the very serial formula, and is met by no other familiar method. By contrast, none of the axioms used in the three current theorems bears any prima facie relation to a serial-type formula, and each axiom is satisfied by several completely different methods.

The cross-subsidizing serial method is introduced in Sprumont (1998), along with other nonadditive serial methods (see also Koster, Tijs, and Borm (1998)). They are justified by means of a natural "serial principle". But that principle is too close to the very definition of the rules to deliver genuine axiomatizations.

3.4. Distributivity, a key axiom in all our results, is introduced in Moulin and

Shenker (1999). In the continuous model restricted to one-output cost functions, they show that the serial method is an extreme point of the class of additive and distributive methods, but fail to characterize it in that class by any elementary principle. Extending the Distributivity axiom to methods defined for arbitrary cost functions (and combining it with other simple axioms), the current paper obtains genuine characterizations of the subsidy-free and cross-subsidizing serial methods.

4. The background: additive cost-sharing

Each agent i in a finite set $N = \{1, ..., n\}$ demands an integer quantity $x_i \in \mathsf{N} = \{0, 1, \ ...\}$ of a personalized good. The cost of meeting the demand profile $x \in \mathsf{N}^N$ must be split among the members of N. A *cost function* is a mapping $C : \mathsf{N}^N \to \mathsf{R}_+$ that is nondecreasing and satisfies $C(0) = 0$; the set of such mappings is denoted $\mathcal{C}$. For any $S \subseteq N$ and $z \in \mathsf{R}_+^S$, we let $z_S = \sum_{i \in S} z_i$. When convenient, we write i or ij instead of $\{i\}$ or $\{i, j\}$.

Definition 1. A (*cost-sharing*) *method* φ assigns to each *problem* $(C, x) \in \mathcal{C} \times \mathsf{N}^N$ a vector of nonnegative cost shares $\varphi(C, x) = y \in \mathsf{R}_+^N$ such that $y_N = C(x)$.

A cost function $C \in \mathcal{C}$ is *one-output* if there exists a mapping $\Gamma : \mathsf{N} \to \mathsf{R}_+$ such that

$$C(z) = \Gamma(z_N) \text{ for all } z \in \mathsf{N}^N. \tag{1}$$

With a slight abuse of terminology, we call every nondecreasing mapping $\Gamma : \mathsf{N} \to \mathsf{R}_+$ such that $\Gamma(0) = 0$ a one-output cost function as well; the set of such functions is denoted by $\mathcal{G}$. The usual interpretation is that C is one-output when the goods demanded by the various agents are perfect substitutes.

Definition 2. A (*cost-sharing*) *mechanism* ψ assigns to every one-output problem $(\Gamma, x) \in \mathcal{G} \times \mathsf{N}^N$ a vector of nonnegative cost shares $\psi(\Gamma, x) = y \in \mathsf{R}_+^N$ such that $y_N = \Gamma(x_N)$. We say that the mechanism ψ is *induced* by the method φ if, for all $(\Gamma, x) \in \mathcal{G} \times \mathsf{N}^N$, $\psi(\Gamma, x) = \varphi(C, x)$, where C is defined in (1). Conversely, we call φ an *extension* of ψ.

Throughout the paper, we restrict our attention to additive cost-sharing methods. A cost-sharing method φ is *additive* if $\varphi(C^1 + C^2, .) = \varphi(C^1, .) + \varphi(C^2, .)$ for all $C^1, C^2 \in \mathcal{C}$. Likewise, a cost-sharing mechanism ψ is additive if $\psi(\Gamma^1 + \Gamma^2, .) = \psi(\Gamma^1, .) + \psi(\Gamma^2, .)$ for all $\Gamma^1, \Gamma^2 \in \mathcal{G}$. The sets of additive cost-sharing methods and mechanisms are respectively denoted by Φ and Ψ.

Additivity is a powerful mathematical axiom with no equity content. Its motivation is essentially pragmatic: an additive method is easy to compute. Under Additivity, computing cost shares for each one of several parallel cost functions and adding them up gives the same result as computing the cost shares in one shot, for the aggregate cost function.

5. The two views on responsibility and the corresponding axioms of fairness

The common premise in both views is that each agent is responsible for the size of her own demand. One approach holds agents responsible as well for their impact on the cost function, the other does not.

Our first two equity axioms follow naturally from the common premise. First, *if* all goods have the same impact on the cost function, agents consuming more should pay more.

Ranking. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, and $i, j \in N$, $\{C$ is a symmetric function of all its variables and $x_i \leq x_j\} \Rightarrow \{\varphi_i(C, x) \leq \varphi_j(C, x)\}$.

Clearly, Ranking implies *Anonymity*: agents with equal demands pay equal cost shares if the cost function is symmetric. The symmetry proviso is essential. Under a symmetric cost function, any difference in cost shares must originate in differences in demands. Since agents are responsible for those, higher demands command higher cost shares.

If the cost function is not only symmetric but also additively separable, the cost of meeting an agent's demand is independent of other demands, and demand responsibility requires that agent to pay precisely that cost.

Separability. For all $c \in \mathcal{G}$ and $x \in \mathbb{N}^N$, $\{C(z) = \sum_{i \in N} c(z_i)$ for all $z \in \mathbb{N}^N\} \Rightarrow \{\varphi_i(C, x) = c(x_i)$ for all $i \in N\}$.

We turn to the two axioms driving a wedge between the two views of responsibility. If agents are responsible for their individual impact on the cost function, "dummies" should pay nothing. If $S \subseteq N$, we denote by e^S the demand profile in $\mathbb{N}^N$ defined by $e_i^S = 1$ if $i \in S$ and 0 otherwise. For any $i \in N$ and $C \in \mathcal{C}$, we define i's marginal cost function $\partial_i C : \mathbb{N}^N \to \mathbb{R}_+$ by $\partial_i C(z) = C(z + e^{\{i\}}) - C(z)$.

Dummy. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, and $i \in N$, $\{\partial_i C = 0\} \Rightarrow \{\varphi_i(C, x) = 0\}$.

Originally formulated in the cooperative game model by Shapley (1953), the Dummy axiom was later extended to the cost-sharing model with continuous demands by Aumann and Shapley (1974), Billera and Heath (1982) and Mirman and Tauman (1982), among others, and to the model with discrete demands by Moulin (1995), Wang (1999), and Sprumont (2000).

Under Additivity, the Dummy axiom implies a stronger form of Separability applying to any additively separable cost function, symmetric or not.

Strong Separability. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, $\{C(z) = \sum_{i \in N} c_i(z_i)$ for all $z \in \mathbb{N}^N\} \Rightarrow \{\varphi_i(C, x) = c_i(x_i)$ for all $i \in N\}$.

The converse is true as well: an additive method satisfies Strong Separability *if and only if* it satisfies Dummy (Moulin and Vohra (2002)).

The alternative theory relies on a completely different equity axiom. If agents are not responsible for their impact on the cost function, their cost shares should not be sensitive to the asymmetries it displays. Agents who ask more should pay more, *regardless* of the cost function.

Strong Ranking. For all $C \in \mathcal{C}$, $x \in \mathsf{N}^N$, and $i, j \in N$, $\{x_i \leq x_j\} \Rightarrow \{\varphi_i(C,x) \leq \varphi_j(C,x)\}$.

This axiom implies *Strong Anonymity:* agents with equal demands pay equal cost shares.

As explained in Section 2, Dummy and Strong Ranking are incompatible. We submit that Dummy is the key axiom of the *full responsibility* theory, Strong Ranking that of the *partial responsibility* theory. Ranking and Separability are meaningful requirements in both theories.

Ranking is implied by Strong Ranking and is therefore redundant in the partial responsibility approach. We explained above that Dummy and Additivity together imply Separability: the latter axiom is therefore redundant in the full responsibility approach. Summing up, our full responsibility theory of cost sharing is centered around the combination of *Dummy* (or *Strong Separability*) and *Ranking* while our partial responsibility theory is built on *Strong Ranking* and *Separability.* We show in the next section that either combination of equity axioms allows for a wide variety of methods.

6. A brief look at the two classes of methods

The class of additive methods satisfying Dummy is conveniently described using the concept of path-generated method. In what follows, vector inequalities are denoted $\leq, <, \ll$ and if $x, x' \in \mathsf{N}^N$ and $x \leq x'$, $[x, x']$ is the interval $\{z \in \mathsf{N}^N : x \leq z \leq x'\}$.

Definition 3. A *path* to $x \in \mathsf{N}^N$ is a mapping $\pi : \{0, 1, ..., x_N\} \to [0, x]$ such that $\pi(0) = 0$, $\pi(x_N) = x$, and for all $t \geq 1$ there is some $i \in N$ such that $\pi(t) - \pi(t-1) = e^{\{i\}}$. Equivalently, any sequence of agents $\{i_1, i_2, ..., i_{x_N}\}$ where agent i appears exactly x_i times defines a path π to x by letting $\pi(t) - \pi(t-1) = e^{\{i_t\}}$ for each $t \geq 1$. We denote by $\Pi(x)$ the set of paths to x.

Important examples are the so-called *priority paths* associated with the $n!$ sequences in which all occurrences of any given agent are consecutive. For instance, the priority path to x corresponding to the natural ordering of the agents is described by the sequence $\{1, ..., 1, 2, ..., 2, ..., n, ..., n\}$ in which each i appears x_i times.

Definition 4. A cost-sharing method φ is *path-generated* if for every $x \in \mathsf{N}^N$ there is a path π to x such that, for every $C \in \mathcal{C}$,

$$\varphi(C,x) = \varphi^{\pi}(C,x) := \sum_{t=1}^{x_N} [C(\pi(t)) - C(\pi(t-1))][\pi(t) - \pi(t-1)].$$

The simplest path-generated methods are the *ordered contributions methods,* generated by the priority paths. Each such method uses a single fixed ordering of the agents: for every demand profile x, cost shares are computed along the priority path to x corresponding to the given ordering. For instance, the ordered contributions method $\varphi^{\leq}$ corresponding to the natural ordering of the agents yields the cost shares $\varphi_i^{\leq}(C,x) = C(\sum_{j\leq i} x_j e^{\{j\}}) - C(\sum_{j\leq i-1} x_j e^{\{j\}})$ for all $i \in N$ and every problem (C,x).

It is clear that every path-generated method satisfies Additivity and Dummy. Conversely, Wang (1999) showed that every method $\varphi \in \Phi$ satisfying Dummy is a convex combination of path-generated methods: for each $x \in \mathbb{N}^N$ there is a probability distribution $\mu(.,x)$ on $\Pi(x)$ such that

$$\varphi(C,x) = \sum_{\pi\in\Pi(x)} \mu(\pi,x)\varphi^{\pi}(C,x) \text{ for all } C \in \mathcal{C}. \tag{2}$$

Note that no relation is imposed on the probability distributions used for different demand profiles. Thus the subset of Φ circumscribed by Dummy is quite large: it is convex and its dimension is countably infinite. These properties remain true if we add Ranking.

Two important examples are the *Aumann-Shapley method* φ^{as}, which uses for all x the uniform distribution over *all* paths to x, and the *Shapley-Shubik method* φ^{ss}, which uses the uniform distribution putting weight only on the priority paths to x.

Turning now to the partial responsibility approach, we note that the class of methods in Φ satisfying Strong Ranking is also very large. Basic examples include the *egalitarian method*

$$\varphi(C,x) = \frac{C(x)}{n} \cdot e^N$$

and the *proportional method*

$$\varphi(C,x) = \frac{C(x)}{x_N} \cdot x,$$

but both methods clearly violate Separability.

A large family of methods meeting Strong Ranking and Separability are the *equi-incremental methods*. We describe the construction of such a method for a demand profile whose coordinates are distinct, say, $x_1 < x_2 < ... < x_n$. The definition is then extended by symmetry to any profile. Choose a path π to the vector $(1,2,...,n)$: in the associated sequence $\{i_1, i_2, ..., i_{n(n+1)/2}\}$, agent i appears i times. For $t = 1, ..., n(n+1)/2$, define $z(t)$ by $z_i(t) = x_{\pi_i(t)}$ for all $i \in N$, with the convention that $x_0 = 0$, and let $S(t) = \{i \in N : \pi_{i_t}(t) \leq i\}$. Agent i's cost share is given by

$$\varphi_i(C,x) = \sum_{t:i\in S(t)} \frac{1}{\#S(t)}[C(z(t)) - C(z(t-1))].$$

Note that all coordinates of the sequence $z(t)$ are in $\{0, x_1, ..., x_n\}$ for all t, and between $z(t)$ and $z(t+1)$ exactly one coordinate z_i increases from its current level x_j to x_{j+1}; the corresponding incremental cost is equally shared among the agents $j+1, ..., n$.

As an illustration, consider the case $n = 2$ and fix a demand profile x, $x_1 < x_2$. There are three paths to $(1, 2)$ corresponding to the sequences of agents $\{1, 2, 2\}$, $\{2, 1, 2\}$, and $\{2, 2, 1\}$. Let us compute the cost shares corresponding to the third path. By definition, $\pi(1) = (0, 1), \pi(2) = (0, 2), \pi(3) = (1, 2)$ and $z(1) = (0, x_1), z(2) = (0, x_2), z(3) = (x_1, x_2)$. Next, $S(1) = \{1, 2\}, S(2) = \{2\}, S(3) = \{1, 2\}$ and therefore

$$\varphi_1(C, x) = \frac{1}{2}[C(x_1, x_2) + C(0, x_1) - C(0, x_2)]. \tag{3}$$

The other two paths yield the familiar serial formula:

$$\varphi_1(C, x) = \frac{1}{2}C(x_1, x_1).$$

Thus there are two equi-incremental methods in this case. For $n = 3$ and a given demand profile x with $x_1 < x_2 < x_3$, there are already 25 different methods. For instance the sequence {3,2,3,2,1,3} yields the cost shares $\varphi_1(C, x) = \frac{1}{3}C(0, x_1, x_1) + \frac{1}{3}[C(x_1, x_2, x_2) - C(0, x_2, x_2)]$ and $\varphi_2(C, x) = \varphi_1(C, x) + \frac{1}{2}[C(0, x_2, x_2) - C(0, x_1, x_1)]$.

Just like in the discussion of path-generated methods, the sequences used for different demand profiles need not be related in any particular way. Hence the set of equi-incremental methods is very large too. Any converse combination of equi-incremental methods, where the weights are independent of C, is an element of Φ meeting Strong Ranking and Separability. This family of methods does not exhaust the subset of Φ circumscribed by these two axioms.

7. Two serial cost-sharing methods

We define in this section the "subsidy-free serial" method and the "cross-subsidizing serial" method. We argue in Sections 9 and 10 that, in view of the Distributivity property, these two methods play a central role in, respectively, the full and the partial responsibility approaches. Both extend Moulin and Shenker's (1992) serial mechanism to a full-fledged method (recall the distinction between methods and mechanisms introduced in Definitions 1 and 2). We recall the definition of the serial mechanism.

Definition 5. Let $\mathsf{N}^N_* = \{x \in \mathsf{N}^N : x_1 \leq ... \leq x_n\}$. For any $x \in \mathsf{N}^N_*$, define $x^1 = x_1 e^N$, $x^2 = x_1 e^{\{1\}} + x_2 e^{N \setminus \{1\}}$, ..., $x^{n-1} = x_1 e^{\{1\}} + ... + x_{n-1} e^{\{n-1,n\}}$, $x^n = x$. The *serial mechanism* ψ^s assigns to every one-output problem $(\Gamma, x) \in \mathcal{G} \times \mathsf{N}^N_*$ the vector of cost shares $\psi^s(\Gamma, x) = \frac{1}{n}\Gamma(x^1_N)e^N + \frac{1}{n-1}[\Gamma(x^2_N) - \Gamma(x^1_N)]e^{N \setminus \{1\}} + ... + [\Gamma(x^n_N) - \Gamma(x^{n-1}_N)]e^{\{n\}}$.

The cost shares for an arbitrary one-output problem $(\Gamma, x) \in \mathcal{G}\times \mathbb{N}^N$ obtain by applying the formula after reordering the coordinates of the demand profile x in nondecreasing order.

Looking first at the full responsibility approach, we define a cost-sharing method extending the serial mechanism and satisfying Dummy and Ranking. The intuition for our method is simpler in the continuous model (Friedman and Moulin (1999)), where we compute an agent's cost share by integrating his marginal cost along the "constrained egalitarian path" to the demand profile. That path is defined by the property that the quantity of each good whose demand is not met increases at an equal rate. In the discrete model, no path treats agents symmetrically, so the egalitarian path can only be approximated. For the sake of fairness, we average the cost shares computed along all the approximating paths.

Definition 6. A path π to $x \in \mathbb{N}_*^N$ with associated sequence $\{i_1, i_2, ..., i_{x_\mathsf{N}}\}$ is called *egalitarian* if for all $i, j \in N$,

$$x_i \leq x_j \Rightarrow |\pi_i(t) - \pi_j(t)| \leq 1 \text{ for } t = 1, ..., \max\{t' : i_{t'} = i\}.$$

Let $\Pi^e(x)$ be the set of egalitarian paths. The *subsidy-free serial method* φ^{fs} assigns to every problem $(C, x) \in \mathcal{C}\times \mathbb{N}_*^N$ the arithmetic average of the vectors of cost shares generated by all the egalitarian paths to x :

$$\varphi^{fs}(C, x) = \frac{1}{\#\Pi^e(x)} \sum_{\pi\in\Pi^e(x)} \varphi^{\pi}(C, x).$$

Again, the cost shares for an arbitrary problem $(C, x) \in \mathcal{C}\times \mathbb{N}^N$ obtain by applying the formula after reordering the coordinates of the demand profile in nondecreasing order. This method was introduced in Moulin (1995).

As an example, suppose $n = 2$ and $x = (2, 3)$. There are four egalitarian paths, π^1 to π^4, corresponding respectively to the sequences $\{1, 2, 1, 2, 2\}, \{2, 1, 2, 1, 2\}$, $\{1, 2, 2, 1, 2\}$, and $\{2, 1, 1, 2, 2\}$. The subsidy-free serial method averages over all four with weight 1/4. Notice, however, that the paths π^3 and π^4 are redundant: the subsidy-free serial cost shares can be computed by averaging over π^1 and π^2 with weight 1/2. The latter are examples of simple paths. In general, we call an egalitarian path π to x *simple* if for all i, j such that $x_i \leq x_j$, the sign of $\pi_i(t) - \pi_j(t)$ is the same for all $t = 1, ..., \max\{t' : i_{t'} = i\}$. While there are $(n!)^{x_1} \cdot ((n-1)!)^{x_2 - x_1} \cdot ... \cdot 2^{x_{n-1} - x_{n-2}}$ egalitarian paths to x, only $n!$ of them are simple, and we need only average over these to compute the subsidy-free serial cost shares.

Turning to the partial responsibility approach, a straightforward modification of the formula in Definition 5 extends the serial mechanism to a method meeting Strong Ranking and Separability.

Definition 7. The *cross-subsidizing serial method* φ^{cs} assigns to every problem $(C,x) \in \mathcal{C} \times \mathsf{N}_*^N$ the vector of cost shares $\varphi^{cs}(C,x) = \frac{1}{n}C(x^1)e^N + \frac{1}{n-1}[C(x^2) - C(x^1)]e^{N\setminus\{1\}} + ... + [C(x^n) - C(x^{n-1})]e^{\{n\}}$. The cost shares for an arbitrary problem obtain by applying the formula after reordering the coordinates of the demand profile in nondecreasing order. This method was introduced in Sprumont (1998).

The cross-subsidizing serial is one of the equi-incremental methods described in the previous section: when $x_1 < ... < x_n$, it is generated by the sequence of agents $\{1, 2, 2, 3, 3, 3, ...\}$.

We let the reader check that both serial methods coincide not only for one-output problems, but also whenever the cost function is symmetrical in all its variables.

8. Distributivity and Strong Distributivity

The subsidy-free and cross-subsidizing serial methods share a natural and powerful property, known as Distributivity, that lends them a very central status in the full and partial responsibility theories, respectively.

The axiom bears on the sequential decomposition of the production of a single output. Suppose that an input is first transformed into an intermediate good, next used to produce the final product or service. Meeting the final demand profile x requires $z = \Gamma^2(x_N)$ units of the intermediate good, the production of which necessitates $y = \Gamma^1(z)$ units of input.

Given a mechanism ψ, we can allocate costs step by step: the shares of intermediate good are $\psi(\Gamma^2, x)$; viewing those shares as demands for the intermediate good, the input shares are $\psi(\Gamma^1, \psi(\Gamma^2, x))$. Alternatively, we could apply the mechanism directly to the composed cost function: the cost shares are then $\psi(\Gamma^1 \circ \Gamma^2, x)$. Distributivity requires that the two computations give the same result:

$$\psi(\Gamma^1, \psi(\Gamma^2, x)) = \psi(\Gamma^1 \circ \Gamma^2, x). \tag{4}$$

Like Additivity, Distributivity is an invariance axiom with no equity content. In the continuous model, (4) is a well-defined requirement. Notice that it implies

$$\{\psi(\Gamma^1, x) = \psi(\Gamma^2, x)\} \Rightarrow \{\psi(\Gamma \circ \Gamma^1, x) = \psi(\Gamma \circ \Gamma^2, x)\}$$

for any one-output cost function Γ. However, in the discrete model (4) is not well defined because the coordinates of $\psi(\Gamma^2, x)$ need not be integers *and* because $\Gamma^1 \circ \Gamma^2$ is not defined. For a proper definition, we let $\overline{\mathcal{G}}$ be the set of nondecreasing functions $\overline{\Gamma} : \mathsf{R}_+ \to \mathsf{R}_+$ satisfying $\overline{\Gamma}(0) = 0$. If $\overline{\Gamma} \in \overline{\mathcal{G}}$, denote its restriction to N by Γ.

Definition 8. A cost-sharing mechanism ψ is *distributive* if it satisfies the following two properties:

(A) for all $\overline{\Gamma^1} \in \overline{\mathcal{G}}$, all $\Gamma^2 \in \mathcal{G}$ and $x \in \mathsf{N}^N$ such that $\psi(\Gamma^2, x) \in \mathsf{N}^N$, $\psi(\Gamma^1, \psi(\Gamma^2, x)) = \psi(\overline{\Gamma^1} \circ \Gamma^2, x)$;
(B) for all $\overline{\Gamma} \in \overline{\mathcal{G}}$, all $\Gamma^1, \Gamma^2 \in \mathcal{G}$, and all $x \in \mathsf{N}^N$, $\{\psi(\Gamma^1, x) = \psi(\Gamma^2, x)\} \Rightarrow \{\psi(\overline{\Gamma} \circ \Gamma^1, x) = \psi(\overline{\Gamma} \circ \Gamma^2, x)\}$.
A cost-sharing method φ is distributive if the cost-sharing mechanism it induces is distributive.

The structure of the distributive mechanisms in Ψ is described in detail in Subsection 12.2. The class of such mechanisms is quite rich. The main examples are the ordered contributions, egalitarian, proportional, and serial mechanisms.

The ordered contributions methods are distributive, but they fail Ranking. If we restore Ranking by taking the uniform average over all orderings of the agents (thus obtaining the Shapley-Shubik method: see Section 6), we lose Distributivity. For a simple proof of this claim, suppose $n = 2$, $x = (1, 2)$, and choose a one-output cost function Γ^2 such that $\Gamma^2(1) = 0$, $\Gamma^2(2) = 1$, $\Gamma^2(3) = 3$. The Shapley-Shubik mechanism yields $\psi_1^{ss}(\Gamma^2, x) = \frac{1}{2}[\Gamma^2(1) + \Gamma^2(3) - \Gamma^2(2)] = 1 = x_1$. Therefore, for any $\overline{\Gamma}^1$, $\psi_1^{ss}(\Gamma^1, \psi(\Gamma^2, x)) = \psi_1^{ss}(\Gamma^1, x) = \frac{1}{2}[\Gamma^1(1) + \Gamma^1(3) - \Gamma^1(2)]$ while $\psi_1^{ss}(\overline{\Gamma}^1 \circ \Gamma^2, x) = \frac{1}{2}[\Gamma^1(\Gamma^2(1)) + \Gamma^1(\Gamma^2(3)) - \Gamma^1(\Gamma^2(2))] = \frac{1}{2}[\Gamma^1(0) + \Gamma^1(3) - \Gamma^1(1)]$.

Two central methods of the full responsibility approach are the Aumann-Shapley (Section 6) and subsidy-free serial methods. As they extend respectively the proportional and serial mechanisms, both are distributive.

In the partial responsibility approach, the cross-subsidizing serial method satisfies Distributivity as well (since it is another extension of the serial mechanism). It turns out that no other equi-incremental method passes this test. For instance, the two-agent method given by (3) if $x_1 < x_2$ and its symmetric counterpart if $x_1 \geq x_2$ is not distributive: indeed, it induces the Shapley-Shubik mechanism on the one-output problems.

Distributivity only restricts the solution of one-output problems. Yet a similar property is easily defined for general cost functions. Suppose that producing the final demand profile x requires $z = C^2(x)$ units of an intermediate good, the production of which necessitates $y = \Gamma^1(z)$ units of input. Using a given cost-sharing method φ with induced mechanism ψ, we could again allocate costs step by step or directly, and Distributivity requires

$$\psi(\Gamma^1, \varphi(C^2, x)) = \varphi(\Gamma^1 \circ C^2, x).$$

Again, this property is not well defined in our discrete model because $\varphi(C^2, x)$ may have non-integer coordinates and, moreover, the composition on the right-hand side is not defined. We adapt definition 8 as follows.

Definition 9. A cost-sharing method φ is *strongly distributive* if it satisfies the following two properties:

(A*) for all $\overline{\Gamma} \in \overline{\mathcal{G}}$, all $C \in \mathcal{C}$ and $x \in \mathsf{N}^N$ such that $\varphi(C,x) \in \mathsf{N}^N$, $\psi(\Gamma, \varphi(C,x)) = \psi(\overline{\Gamma} \circ C, x)$;

(B*) for all $\overline{\Gamma} \in \overline{\mathcal{G}}$, all $C^1, C^2 \in \mathcal{C}$, and all $x \in \mathsf{N}^N$, $\{\varphi(C^1,x) = \varphi(C^2,x)\} \Rightarrow \{\varphi(\overline{\Gamma} \circ C^1, x) = \varphi(\overline{\Gamma} \circ C^2, x)\}$.

Strong Distributivity is much more demanding than Distributivity. The set of strongly distributive methods in Φ is rather small, as explained in Subsection 12.5. Simple examples include the ordered contributions, egalitarian, and proportional methods. More interestingly, the cross-subsidizing serial method, which satisfies Strong Ranking and Separability, is strongly distributive: see Theorem 3 in Section 10. Thus Strong Distributivity is compatible with the partial responsibility approach.

On the other hand, the axiom is incompatible with the full responsibility approach. Notice first that both the Aumann-Shapley and the subsidy-free serial methods fail the Strong Distributivity test. To check this claim, let $x = (1,2)$, and C be a cost function such that $C(1,0) = 1$, $C(0,1) = C(0,2) = 2$, and $C(1,1) = C(1,2) = 3$. Averaging marginal costs over the three paths to x, the Aumann-Shapley method yields $\varphi^{as}(C,x) = x$. Therefore, for any $\overline{\Gamma}$, $\psi_1^{as}(\Gamma, \varphi^{as}(C,x)) = \psi_1^{as}(\Gamma, x) = \frac{1}{3}\Gamma(x_1 + x_2) = \frac{1}{3}\Gamma(3)$. On the other hand, $\varphi_1^{as}(\overline{\Gamma} \circ C, x) = \frac{1}{3}\Gamma(C(1,0)) + \frac{1}{3}[\Gamma(C(1,1)) - \Gamma(C(0,1))] + \frac{1}{3}[\Gamma(C(1,2)) - \Gamma(C(0,2))] = \frac{1}{3}\Gamma(1) + \frac{2}{3}\Gamma(3) - \frac{2}{3}\Gamma(2)$. Similarly, the subsidy-free serial method yields $\psi_1^{fs}(\Gamma, \varphi^{fs}(C,x)) = \frac{1}{2}\Gamma(2)$ and $\psi_1^{fs}(\overline{\Gamma} \circ C, x) = \frac{1}{2}\Gamma(1) + \frac{1}{2}\Gamma(3) - \frac{1}{2}\Gamma(2)$.

In fact, no additive cost-sharing method satisfying Dummy and Ranking is strongly distributive:

Proposition 1. *A cost-sharing method $\varphi \in \Phi$ satisfies Dummy and Strong Distributivity if and only if it is an ordered contributions method.*

There exist nonadditive methods meeting Dummy, Ranking and Strong Distributivity: splitting the cost equally among all agents who are not dummies is an example.

9. A characterization of the subsidy-free serial method

To characterize the subsidy-free serial method within the set of additive methods satisfying Dummy and Distributivity, we introduce two additional properties.

Dummy Independence. For all $C \in \mathcal{C}$, $x, x' \in \mathsf{N}^N$, and $i \in N$, $\{\partial_i C = 0$ and $x_j = x'_j$ for all $j \in N \backslash \{i\}\} \Rightarrow \{\varphi(C,x) = \varphi(C,x')\}$.

Dummy Independence states that a change in a dummy agent's demand has no effect on cost shares. Very natural in the full responsibility approach to cost-sharing, this axiom is a mild complement of the Dummy axiom: all the methods discussed so far that satisfy Dummy also satisfy Dummy Independence. The latter axiom is implied by the former in the two-agent case, but for larger populations there is no logical relation between them. Examples of methods that satisfy Dummy but violate Dummy Independence include methods using different priority paths for different demand profiles. The egalitarian method (Section 6) is an example of a method satisfying Dummy Independence and violating Dummy[1].

Demand Monotonicity. For all $C \in \mathcal{C}$, $x, x' \in \mathsf{N}^N$, and $i \in N$, $\{x_i \leq x'_i$ and $x_j = x'_j$ for all $j \in N \backslash \{i\}\} \Rightarrow \{\varphi_i(C, x) \leq \varphi_i(C, x')\}$.

Demand Monotonicity is a natural ethical requirement in any cost-sharing theory holding agents responsible for their demand; it is meaningful in both the full and partial responsibility approaches. Alternatively, it may be defended on strategic grounds: a demand monotonic method is not vulnerable to artificial inflation of individual demands. In conjunction with Additivity and Dummy, Demand Monotonicity has a lot of bite: while the Shapley-Shubik and subsidy-free serial methods satisfy it, the Aumann-Shapley method does not (see Moulin (1995)).

Before characterizing the subsidy-free serial method, we describe the entire class of cost-sharing methods in Φ satisfying Distributivity, Dummy, Dummy Independence, and Demand Monotonicity. It contains the ordered contribution methods, the subsidy-free serial method, and a finite number of hybrid methods combining both types of methods as follows.

Definition 10. Let $\preccurlyeq$ be a preordering (that is, a complete and transitive relation) on N, and let $\{N_1, N_2, ..., N_K\}$, be the ordered partition it generates (where $k \leq l$ if and only if $i \preccurlyeq j$ for all $i \in N_k$ and all $j \in N_l$). Write $M_k = \cup_{l=1}^{k} N_l$ for $k = 1, ..., K$. For each cost-sharing problem (C, x), the $\preccurlyeq$ *– ordered composition of subsidy-free serial methods,* $\varphi^{\preccurlyeq}$, computes the cost shares in two steps: it determines the incremental cost of serving each group N_k if all members of all preceding groups have been served, then splits that incremental cost between the members of N_k according to the subsidy-free serial method. Formally, for each $k = 1, ..., K$, define $C_k : \mathsf{N}^{N_k} \to \mathsf{R}_+$ by

$$C_k(z) = C(x(M_{k-1}), z, 0(N \backslash M_k)) - C(x(M_{k-1}), 0(N \backslash M_{k-1}))$$

for all $z \in \mathsf{N}^{N_k}$, where $x(S)$ denotes the projection of x on S. The mapping C_k is a valid cost function for the agent set N_k : it is nondecreasing and $C_k(0) = 0$. Assign to each agent $i \in N_k$ the cost share

$$\varphi_i^{\preccurlyeq}(C, x) = \varphi_i^{fs}(C_k, x(N_k)),$$

[1] Under responsibility for one's demand, it is natural to require that an agent demanding nothing pays nothing. Given this property (formally defined after the Corollary to Theorem 1 below), Dummy Independence implies Dummy at once.

where, with some abuse of notation, φ^{fs} denotes the subsidy-free serial method for N_k.

If 4 is an ordering, every group N_k contains a single agent, and φ^{4} is the corresponding ordered contributions method. At the other extreme, if 4 puts all agents in one indifference class, φ^{4} is the subsidy-free serial method. There are as many ordered compositions of serial methods as there are preorderings on the set of agents. This is a large number. In the 3-agent case we have 13 such methods: the 6 ordered contributions methods, 3 methods based on preorderings of the type $i \prec j \sim k$, 3 based on preorderings of the type $i \sim j \prec k$, and the subsidy-free serial method. With 5 agents, there are 541 methods, and 47,293 with 7 agents: see Maassen and Bezembinder (2002) for a general formula.

Theorem 1. *A cost-sharing method $\varphi \in \Phi$ satisfies Distributivity, Dummy, Dummy Independence, and Demand Monotonicity if and only if it is an ordered composition of subsidy-free serial methods: there exists a preordering 4 on N such that $\varphi = \varphi^{4}$.*

The subsidy-free serial method stands out in that class because it treats all agents alike. In particular, we obtain the following characterization.

Corollary to Theorem 1. *The only cost-sharing method in Φ satisfying Distributivity, Dummy, Dummy Independence, Demand Monotonicity, and Ranking is the subsidy-free serial method φ^{fs}.*

In the corollary, Ranking may be replaced with the property of Anonymity defined in Section 5, or even the following much weaker symmetry requirement on the mechanism ψ associated with φ:

Weak Anonymity. For all $\Gamma \in \mathcal{G}$, $x \in \mathsf{N}^N$, and $i, j \in N$, $\{x_i = x_j\} \Rightarrow \{\psi_i(\Gamma, x) = \psi_j(\Gamma, x)\}$.

To conclude this section, we comment on the tightness of Theorem 1. Among the four stated properties, the Shapley-Shubik method violates only Distributivity, the egalitarian method violates only Dummy, and the Aumann-Shapley method violates only Demand Monotonicity. For a nonadditive method meeting the four properties, split the cost equally among the non-dummy agents. We do not know whether there exists a method in Φ satisfying all properties but Dummy Independence.

In view of Footnote 1, we can replace Dummy in Theorem 1 and its corollary with

Zero Charge for Zero Demand. For all $C \in \mathcal{C}$, $x \in \mathsf{N}^N$, and $i \in N$, $\{x_i = 0\} \Rightarrow \{\varphi_i(C, x) = 0\}$.

The corresponding statement is then tight. The proportional method violates Dummy Independence but meets Distributivity, Demand Monotonicity and Zero Charge for Zero Demand.

10. Two characterizations of the cross-subsidizing serial method

In the partial responsibility approach, we present two characterizations of the cross-subsidizing serial method. Our first result relies heavily on Strong Ranking.

Theorem 2. *The only cost-sharing method in Φ satisfying Distributivity, Strong Ranking, and Separability is the cross-subsidizing serial method φ^{cs}.*

This is a tight result. The egalitarian and proportional methods (Section 6) satisfy Distributivity and Strong Ranking, but violate Separability. The subsidy-free serial method is distributive and separable but violates Strong Ranking. The equi-incremental methods (Section 6) other than φ^{cs} meet Strong Ranking and Separability, but not Distributivity. For a nonadditive method satisfying the three properties in the theorem, use the serial mechanism for one-output problems and the proportional method otherwise.

Our next theorem is formally similar to Theorem 2. Yet, from an ethical viewpoint, it is much more neutral: instead of Strong Ranking, we only impose Ranking, which is compatible with the full responsibility approach. As it turns out, the invariance property of Strong Distributivity is so powerful that, in combination with Ranking and Separability, it delivers a second characterization of the cross-subsidizing serial method.

Theorem 3. *The only cost-sharing method in Φ satisfying Strong Distributivity, Ranking, and Separability is the cross-subsidizing serial method φ^{cs}.*

This is again a tight statement. The egalitarian and proportional methods satisfy Strong Distributivity and Ranking, but violate Separability. The ordered contributions methods are strongly distributive and separable, but violate Ranking. Ranking and Separability allow for a wide variety of methods violating Strong Distributivity, including the Shapley-Shubik, Aumann-Shapley, and subsidy-free serial methods. For a nonadditive method satisfying the three axioms in the theorem.

11. Directions for future research

Our results leave open several natural questions. Does Theorem 1 survive if we omit Dummy Independence? What subset of Φ is characterized by the combination of Separability and Strong Ranking? By these two axioms and Demand Monotonicity? An ordered contribution method meets Strong Distributivity and Separability, and so do all ordered compositions of cross-subsidizing serial methods (defined by mimicking Definition 10): does this exhaust the set of additive, separable and strongly distributive methods?

All axioms and methods discussed here are easily translated to the continuous model, where demand profiles vary in $\mathbb{R}_+^N$. Distributivity is now defined directly by property (4). The definit ion of the cross-subsidizing method is identical, that of the subsidy-free method is simpler, as it is generated by a single path (see the discussion preceding Definition 6). Whether or not our three theorems have a counterpart in the continuous model is a question that we find technically challenging. We suspect that the answer will be easier in the case of the full responsibility approach, because the structure of additive methods meeting Dummy is well understood (Friedman (1998), Haimanko (2000)), and so is that of distributive mechanisms (Moulin and Shenker (1999)).

12. Proofs

12.1. Independence of irrelevant costs. We state a well known result about additive cost-sharing methods.

Lemma 0. *If* $\varphi \in \Phi$, $x \in \mathsf{N}^N$, *and* $C^1, C^2 \in \mathcal{C}$, *then* $\{C^1(z) = C^2(z)$ *for all* $z \in [0,x]\} \Rightarrow \{\varphi(C^1,x) = \varphi(C^2,x)\}$.

A proof is in Moulin (1995); the argument establishing statement i) of Lemma 1 there does not use Dummy.

12.2. The structure of the separable distributive mechanisms in Ψ. This subsection analyzes the main implications of Distributivity: more precisely, we describe the structure of a cost-sharing mechanism $\psi \in \Psi$ satisfying Distributivity and the following property:

$$\psi(id,x) = x \text{ for all } x \in \mathsf{N}^N, \tag{5}$$

where $id : \mathsf{N} \to \mathsf{R}_+$ is the identity function. This property is a weak version of the Separability axiom of Section 4 obtained by restricting it to $\Gamma = id$. The mechanism ψ is fixed throughout the subsection.

We introduce some notations first. Define $\delta_t \in \mathcal{G}$ by

$$\delta_t(z) = 1 \text{ if } z \geq t \text{ and } \delta_t(z) = 0 \text{ otherwise.} \tag{6}$$

Next we write $\gamma_\theta(z) = \min\{\theta, z\}$ for all $z, \theta \in \mathsf{R}_+$, and use this notation also for the restriction of this function to N, an element of $\mathcal{G}$. Finally, for all $\Gamma \in \mathcal{G}$, we write $\gamma_\theta \circ \Gamma = \theta \wedge \Gamma \in \mathcal{G}$, namely the function $(\theta \wedge \Gamma)(z) = \min\{\theta, \Gamma(z)\}$.

From now on, we fix a demand profile $x \in \mathsf{N}^N$. Associated with x is a sequence $\{y^1, ..., y^{x_\mathsf{N}}\}$ of vectors in the simplex of R^N defined by $y^t = \psi(\delta_t, x)$ for $t = 1, ..., x_N$. Keeping only one vector from each interval of consecutive identical vectors in that sequence, we obtain a sequence

$$K(x) = \{y_*^1, ..., y_*^K\}. \tag{7}$$

For instance, if $x_N = 5$ and $y^1 = y^2 \neq y^3 \neq y^4 = y^5$, we get $K(x) = \{y^2, y^3, y^5\}$, in which y^2 and y^5 may or may not be different.

Lemma 1. *The vectors in $K(x)$ are linearly independent.*

Proof. We show first that all vectors in $K(x)$ are distinct. By definition, $y_*^k \neq y_*^{k+1}$ for $k = 1, ..., K-1$. Fix $k, k' \in \{1, ..., K\}$, $k' - k \geq 2$, and suppose that $y_*^k = y_*^{k'}$. By definition of $K(x)$, there is a strictly increasing sequence $t_0, t_1, ..., t_K$ with $t_0 = 0$ and $t_K = x_N$ such that

$$y_*^k = y^t \text{ for } t_{k-1} < t \leq t_k \text{ and } k = 1, ..., K. \tag{8}$$

Define the cost functions $\Gamma = \delta_{t_k} + \delta_{t_{k+1}}$ and $\Gamma' = \delta_{t_{k+1}} + \delta_{t_{k'}}$. By additivity of ψ and definition of $K(x)$, $\psi(\Gamma, x) = y_*^k + y_*^{k+1} = y_*^{k+1} + y_*^{k'} = \psi(\Gamma', x)$. Combining that equality and property (B) in Definition 8, with γ_θ defined on $\mathbb{R}_+$, gives $\psi(\theta \wedge \Gamma, x) = \psi(\theta \wedge \Gamma', x)$ for all $\theta \in \mathbb{R}_+$. For $0 < \theta \leq 1$, we compute

$$\begin{aligned} \psi(\theta \wedge \Gamma, x) &= \psi(\theta \delta_{t_k}, x) = \theta y_*^k, \\ \psi(\theta \wedge \Gamma', x) &= \psi(\theta \delta_{t_{k+1}}, x) = \theta y_*^{k+1}. \end{aligned}$$

As $y_*^k \neq y_*^{k+1}$, we obtain a contradiction. Therefore our assumption $y_*^k = y_*^{k'}$ cannot be true.

If the vectors in $K(x)$, all in the simplex of $\mathbb{R}^N$, are linearly dependent, there exist two nonempty disjoint subsets K_1, K_2 of $\{1, ..., K\}$ and strictly positive coefficients λ_{k_i} for all $k_i \in K_i$, $i = 1, 2$, such that

$$\sum_{k_1 \in K_1} \lambda_{k_1} y_*^{k_1} = \sum_{k_2 \in K_2} \lambda_{k_2} y_*^{k_2}.$$

Choosing $\Gamma_i = \sum_{k_i \in K_i} \lambda_{k_i} \delta_{t_{k_i}}$ for $i = 1, 2$, it follows that $\psi(\Gamma_1, x) = \psi(\Gamma_2, x)$ and thus, by property (B) in Definition 8, $\psi(\theta \wedge \Gamma_1, x) = \psi(\theta \wedge \Gamma_2, x)$ for all $\theta \in \mathbb{R}_+$. On the other hand, if k_i^0 denotes the smallest element of K_i and $0 < \theta \leq \lambda_{k_i^0}$, we have $\theta \wedge \Gamma_i = \theta \delta_{t_{k_i^0}}$, hence $\psi(\theta \wedge \Gamma_i, x) = \theta y_*^{k_i^0}$ for $i = 1, 2$. Since all vectors in $K(x)$ are distinct and $k_1^0 \neq k_2^0$, we obtain $\psi(\theta \wedge \Gamma_1, x) \neq \psi(\theta \wedge \Gamma_2, x)$, a contradiction. ■

Denote by $H(x)$ the positive cone generated by $K(x)$, that is, $H(x) = \{z \in \mathbb{R}_+^N : \exists \lambda_1, ..., \lambda_K \in \mathbb{R}_+ \text{ such that } z = \sum_{k=1}^K \lambda_k y_*^k\}$. Note that the identity function coincides with $\sum_{t=1}^{x_N} \delta_t$ up to x_N. Therefore, by Lemma 0,

$$x = \psi(id, x) = \sum_{t=1}^{x_N} y^t = \sum_{k=1}^{K} (t_k - t_{k-1}) y_*^k, \tag{9}$$

where $t_1, ..., t_K$ are the indices defined in (8). This means that $x \in H(x)$. Next, any function $\Gamma \in \mathcal{G}$ coincides with $\sum_{t=1}^{x_N} (\Gamma(t) - \Gamma(t-1)) \delta_t$ up to x_N. Therefore,

$$\psi(\Gamma, x) = \sum_{t=1}^{x_N} (\Gamma(t) - \Gamma(t-1)) y^t = \sum_{k=1}^{K} (\Gamma(t_k) - \Gamma(t_{k-1})) y_*^k,$$

meaning that $\psi(\Gamma, x) \in H(x)$. Our next lemma generalizes these observations.

Lemma 2. *Let $z \in H(x) \cap \mathbb{N}^N$, with coordinates $\lambda_1, ..., \lambda_K \in \mathbb{R}_+$, that is $z = \sum_{k=1}^K \lambda_k y_*^k$. Then i) $\lambda_1, ..., \lambda_K \in \mathbb{N}$, ii) $K(z)$ obtains from $K(x)$ by deleting those vectors y_*^k for which $\lambda_k = 0$, and iii) for all $\Gamma \in \mathcal{G}$,*

$$\psi(\Gamma, z) = \sum_{k=1}^{K} (\Gamma(\Lambda_k) - \Gamma(\Lambda_{k-1}))y_*^k \tag{10}$$

where $\Lambda_k = \sum_{l=1}^k \lambda_l$ and $\Lambda_0 = 0$.

Proof. Let z satisfy the assumptions of the lemma. Define the cost function $\Gamma^\lambda = \sum_{k=1}^K \lambda_k \delta_{t_k}$ and observe that $\psi(\Gamma^\lambda, x) = z$. For $\theta \in \mathbb{R}_+$, consider the function $\theta \wedge \Gamma^\lambda = \sum_{k=1}^K \{(\theta - \Lambda_{k-1})_+ \wedge \lambda_k\}\delta_{t_k}$, where $a_+ = \max\{a, 0\}$. We have

$$\psi(\theta \wedge \Gamma^\lambda, x) = \sum_{k=1}^{K} \{(\theta - \Lambda_{k-1})_+ \wedge \lambda_k\} y_*^k. \tag{11}$$

Because $\psi(\Gamma^\lambda, x) = z \in \mathbb{N}^N$, we can apply property (A) in Definition 8, with γ_θ defined on $\mathbb{R}_+$, and get $\psi(\theta \wedge \Gamma^\lambda, x) = \psi(\gamma_\theta, z)$. Next we note that the function $\theta \to \psi(\gamma_\theta, z)$ is affine between any two consecutive integers $t, t + 1$. This follows from additivity of ψ by computing for $t \leq \theta < \theta' \leq t + 1$, $\psi(\gamma_{\theta'}, z) - \psi(\gamma_\theta, z) = \psi(\gamma_{\theta'} - \gamma_\theta, z) = \psi((\theta' - \theta)\delta_{t+1}, z) = (\theta' - \theta)\psi(\delta_{t+1}, z)$. Therefore the sum in (11) is affine in θ between any two integers. Since the vectors $y_*^1, ..., y_*^K$ are all different, it follows that the coefficients $\lambda_1, ..., \lambda_K$ are all integers, proving statement *i)*.

In view of $\psi(\theta \wedge \Gamma^\lambda, x) = \psi(\gamma_\theta, z)$, we may rewrite (11) in the following form:

$$\psi(\gamma_\theta, z) = \sum_{k=1}^{K} \{\gamma_\theta(\Lambda_k) - \gamma_\theta(\Lambda_{k-1})\} y_*^k,$$

which establishes (10) for $\Gamma = \gamma_\theta$. A cost function $\Gamma \in \mathcal{G}$ which is constant after x_N is a linear combination of γ_θ functions, therefore formula (10) holds for every such function. By Lemma 0, it holds for every $\Gamma \in \mathcal{G}$, proving statement *iii)*.

Applying the equation (10) to δ_t gives $\psi(\delta_t, z) = y_*^k$ if and only if $\Lambda_{k-1} < t \leq \Lambda_k$, proving statement *ii)*. ■

Our last lemma gives a complete description of $K(x)$ in the two-agent case. Since $K(x)$ is obviously nonempty, Lemma 1 implies that it contains either one or two vectors.

Lemma 3. *The vectors in $K(x)$ have rational coordinates. If $K(x) = \{y_*^1, y_*^2\}$, then there exist $a, b \in \mathbb{N}$ with $0 \leq a \leq b - 1$ and k, k' with $\{k, k'\} = \{1, 2\}$ such that*

$$y_*^k = (\frac{a}{b}, \frac{b-a}{b}), \; y_*^{k'} = (\frac{a+1}{b}, \frac{b-a-1}{b}).$$

Proof. If $K(x) = \{y_*^1\}$, recall that $x \in H(x)$: as y_*^1 is in the simplex of $\mathbb{R}_+^2$, we have

$$x = (x_1 + x_2)y_*^1. \tag{12}$$

Since x has integer coordinates, y_*^1 has rational coordinates.

If $K(x) = \{y_*^1, y_*^2\}$, recall from (9) that $x = t_1 y_*^1 + (t_2 - t_1)y_*^2$ and set $y_*^i = (\alpha_i, 1 - \alpha_i)$, $0 \le \alpha_i \le 1$, $i = 1, 2$. For $z_1, z_2 \in \mathbb{N}$, consider the following system in λ_1, λ_2 :

$$\lambda_1 y_*^1 + \lambda_2 y_*^2 = (z_1, z_2).$$

This system is nonsingular by Lemma 1 and its solution is

$$\lambda_1 = \frac{(1 - \alpha_2)z_1 - \alpha_2 z_2}{\alpha_1 - \alpha_2},\ \lambda_2 = \frac{-(1 - \alpha_1)z_1 + \alpha_1 z_2}{\alpha_1 - \alpha_2}. \tag{13}$$

For any strictly positive $\mu \in \mathbb{N}$, if $z_1 = \mu x_1$ and $z_2 = \mu x_2$, we know from (9) that $\lambda_1 = \mu t_1$ and $\lambda_2 = \mu(t_2 - t_1)$, both strictly positive and (by Lemma 2) integers. It follows that for μ large enough, if $z_1 = \mu x_1 + 1$ and $z_2 = \mu x_2$ (or $z_1 = \mu x_1$ and $z_2 = \mu x_2 + 1$), the values of λ_1 and λ_2 given by (13) are also strictly positive and, by Lemma 2 again, integers. Writing that λ_1 changes but remains integer when $z_2 = \mu x_2$ and z_1 shifts from μx_1 to $\mu x_1 + 1$, and three other similar properties, we conclude that the four numbers

$$\frac{1 - \alpha_2}{|\alpha_1 - \alpha_2|}, \frac{\alpha_2}{|\alpha_1 - \alpha_2|}, \frac{1 - \alpha_1}{|\alpha_1 - \alpha_2|}, \frac{\alpha_1}{|\alpha_1 - \alpha_2|}$$

must be nonnegative integers, possibly zero. Adding the first two of these numbers, $\frac{1}{|\alpha_1 - \alpha_2|}$ is also an integer: call it b. Next, since the fourth and second numbers are integers, there exist $a_1, a_2 \in \mathbb{N}$ such that $\alpha_1 = \frac{a_1}{b}$ and $\alpha_2 = \frac{a_2}{b}$. Now $\frac{1}{|\alpha_1 - \alpha_2|} = \frac{b}{|a_1 - a_2|}$ gives $a_1 - a_2 = 1$ and the formula in the lemma follows. ■

12.3. Proof of Theorem 1.

Let $\varphi \in \Phi$ be a cost-sharing method satisfying the four axioms in Theorem 1.

Step 1. By Wang's (1999) lemma, φ is a convex combination of path-generated methods: for each $x \in \mathbb{N}^N$ there is a probability distribution $\mu(., x)$ on $\Pi(x)$ satisfying (2). The following formulation of Wang's result will be useful. For each $i \in N$ and $z \in [0, x - e^i]$, denote by $\Pi_i(z, x)$ the set of paths π to x "passing through" z and $z + e^i$: there is some $t \in \{1, ..., x_N\}$ such that $\pi(t-1) = z$ and $\pi(t) = z + e^i$. Defining $m_i(z, x) = \sum_{\pi \in \Pi_i(z,x)} \mu(\pi, x)$ (with the convention that a sum over the empty set is zero), (2) and Definition 4 yield

$$\varphi_i(C, x) = \sum_{z \in [0, x - e^i]} m_i(z; x) \partial_i C(z) \tag{14}$$

for all $i \in N$ and $(C, x) \in \mathcal{C} \times \mathbb{N}^N$. Because it is constructed from probability distributions over paths, the *weight system* $(m_1, ..., m_n)$ satisfies the *flow conservation* constraints

$$\sum_{i \in N} m_i(0, x) = 1$$

and

$$\sum_{i \in N : z_i < x_i} m_i(z, x) = \sum_{i \in N : z_i > 0} m_i(z - e^i, x) \text{ for all } z \in]0, x[.$$

As already noted, a consequence of Wang's lemma is that φ satisfies Separability. The mechanism ψ it induces therefore possesses all the properties derived in the previous subsection. In Steps 2 to 5, we prove that φ is an ordered composition of subsidy-free methods if $n = 2$. Step 6 extends the conclusion to an arbitrary number of agents.

Step 2. We show that for all $x \in \mathbb{N}^2$, $i \in \{1, 2\}$, and $t, s \in \mathbb{N}$,

$$\{t \leq x_i\} \Rightarrow \{\psi(\delta_t, x) = \psi(\delta_t, x + se^i)\}. \tag{15}$$

This follows from Step 1 and Demand Monotonicity. Fix x, i, t and s such that $t \leq x_i$. We claim that

$$m_i(z, x) = m_i(z, x + se^i) \text{ for every } z \in [0, x - e^i]. \tag{16}$$

Suppose not. Because $\sum_{z' \in [0, x-e^i] : z'_i = z_i} m_i(z', x) = 1 = \sum_{z' \in [0, x-e^i] : z'_i = z_i} m_i(z', x + se^i)$ for every $z \in [0, x - e^i]$ (as follows from the flow conservation constraints or directly by applying (14) to the problems $(C, x), (C, x + se^i)$ where $C(z') = 1$ if $z'_i \geq z_i$ and 0 otherwise), there exists some $z \in [0, x - e^i]$ such that $m_i(z, x) > m_i(z, x + e^i)$. Choosing a cost function C for which $\partial_i C$ is positive only at z, this last inequality and (14) yield $\varphi_i(C, x) > \varphi_i(C, x + se^i)$, violating Demand Monotonicity. This proves (16). Because $t \leq x_i, C = \delta_t$ has $\partial_i C(z) = 0$ for all $z \in [0, x + (s-1)e^i] \setminus [0, x - e^i]$, therefore (14) implies $\psi_i(\delta_t, x) = \psi_i(\delta_t, x + se^i)$, and (15) because $n = 2$.

Step 3. Let $x \in \mathbb{N}^2$ and assume that $K(x) = \{y^1_*, y^2_*\}$. We show that $y^2_* = e^1$ or e^2.

Suppose $y^2_* \neq e^2$. Recall from Lemma 3 that y^1_*, y^2_* have rational coordinates and choose positive integers t_1, t_2 such that

$$z := t_1 y^1_* + t_2 y^2_* \in \mathbb{N}^2 \text{ and } t_1 < z_1.$$

This can be done (by choosing t_2 large enough) because $y^2_* \neq e^2$. By Lemma 2, $K(z) = \{y^1_*, y^2_*\}$ and

$$\begin{aligned} \psi(\delta_t, z) &= y^1_* \text{ for } 1 \leq t \leq t_1, \\ \psi(\delta_t, z) &= y^2_* \text{ for } t_1 < t \leq t_1 + t_2 = z_1 + z_2. \end{aligned}$$

Next, (15) implies $\psi(\delta_t, z + se^1) = \psi(\delta_t, z)$ for $t = 1, ..., z_1$ and any $s \in \mathsf{N}$. Because $t_1 < z_1$, $\psi(\delta_t, z + se^1) = y_*^1$ for $1 \le t \le t_1$ and $\psi(\delta_t, z + se^1) = y_*^2$ for $t_1 + 1 \le t \le z_1$. As $\psi(\delta_t, z + e^1)$ takes only two values when t varies, it remains equal to y_*^2 when $z_1 \le t \le z_1 + z_2 + s$. Thus $z + se^1 = \psi(id, z + se^1) = t_1 y_*^1 + (z_1 + z_2 + s - t_1) y_*^2$. Since this holds for all $s \in \mathsf{N}$, we conclude that $y_*^2 = e^1$.

Step 4. We show that if $n = 2$, ψ is either the serial mechanism ψ^s (from Definition 5) or one of the two ordered contributions mechanisms $\psi^1(\Gamma, x) = (\Gamma(x_1), \Gamma(x_1 + x_2) - \Gamma(x_1))$ or $\psi^2(\Gamma, x) = (\Gamma(x_1 + x_2) - \Gamma(x_2), \Gamma(x_2))$.

Case 1. There exists $x \in \mathsf{N}^2$, $x \gg 0$, such that $K(x) = \{y_*^1\}$.

Recall from (12) that $x = (x_1 + x_2) y_*^1$, so that $y_*^1 \neq e^1, e^2$. Let $i \in \{1, 2\}$. By (15), $\psi(\delta_t, x + e^i) = \psi(\delta_t, x) = y_*^1$ for $1 \le t \le x_i$, hence, $y_*^1 \in K(x + e^i)$. But since $x + e^i$ is not a multiple of x, $K(x + e^i) \neq \{y_*^1\}$ (otherwise, applying (12) to $x + e^i$ brings a contradiction). Hence, by Step 3, $K(x + e^i) = \{y_*^1, e^k\}$ for $k = 1, 2$, and because $x + e^i \in H(x + e^i)$, $K(x + e^i) = \{y_*^1, e^i\}$.

Applying Lemma 3 to $K(x + e^1)$ yields $y_*^1 = (\frac{b-1}{b}, \frac{1}{b})$; applying it to $K(x + e^2)$ gives $y_*^1 = (\frac{1}{b}, \frac{b-1}{b})$ Thus $y_*^1 = (\frac{1}{2}, \frac{1}{2})$. Now, pick any $z \in \mathsf{N}^2$. If $z_1 \le z_2$, then $z \in H(x + e^2)$ and since $z = 2z_1(\frac{1}{2}, \frac{1}{2}) + (z_2 - z_1)e^2$, formula (10) in Lemma 2 gives $\psi(\Gamma, z) = \frac{1}{2}\Gamma(2z_1)e^{\{1,2\}} + [\Gamma(z_1 + z_2) - \Gamma(2z_1)]e^{\{2\}}$, precisely the cost shares $\psi^s(\Gamma, z)$ recommended by the serial mechanism of Definition 5. A symmetric argument applies if $z_2 \le z_1$.

Case 2. For all $x \in \mathsf{N}^2$ such that $x \gg 0$, $K(x) = \{y_*^1, y_*^2\}$.

Let $x \in \mathsf{N}^2$, $x \gg 0$. By Step 3, $K(x) = \{y_*^1, e^2\}$ or $K(x) = \{y_*^1, e^1\}$. Consider the first possibility. By Lemma 3, y_*^1 has rational coordinates. By statement *ii)* in Lemma 2, for every integer λ_1 such that $\lambda_1 y_*^1 = z \in \mathsf{N}^2$ we have $K(z) = \{y_*^1\}$. Thus $z \gg 0$ is impossible and, since $y_*^1 \neq e^2$, we must have $z_2 = 0$. This shows that $K(x) = \{e^1, e^2\}$, and (10) now implies that ψ is the ordered contributions mechanism ψ^1. Similarly, ψ must be the ordered contributions mechanism ψ^2 if $K(x) = \{y_*^1, e^1\}$.

Step 5. We show that if $n = 2$, φ is either the subsidy-free serial method φ^{fs} (from Definition 6) or one of the two ordered contributions methods $\varphi^1(C, x) = (C(x_1, 0), C(x) - C(x_1, 0))$ or $\varphi^2(C, x) = (C(x) - C(0, x_2), C(0, x_2))$.

Consider the representation of φ given by (14). To avoid notational complications, we extend each mapping $m_i(., x)$ to $[0, x]$ by setting $m_i(z; x) = 0$ whenever $z_i = x_i$. We note that (16) and $n = 2$ imply

$$m(z, x) = m(z, x') \text{ whenever } x \le x' \text{ and } z \le x - (1, 1). \tag{17}$$

Because of Step 4, ψ is either one of the ordered contributions mechanisms or the serial mechanism.

Suppose $\psi = \psi^s$. For the subsidy-free serial method (and, say, $x_1 \le x_2$), $m^{fs}(z,x) = (\frac{1}{2},\frac{1}{2})$ if $z_1 = z_2 < x_1$, $m^{fs}(z,x) = (0,\frac{1}{2})$ if $z_1 = z_2 + 1 \le x_1$, $m^{fs}(z,x) = (\frac{1}{2},0)$ if $z_2 = z_1 + 1 \le x_1$, $m^{fs}(z,x) = (0,1)$ if $z_2 \ge z_1 = x_1$, and $m^{fs}(z,x) = 0$ otherwise. We show by induction on $x_1 + x_2$ that $m(\cdot,x) = m^{fs}(\cdot,x)$, which guarantees $\varphi = \varphi^{fs}$ by (14). This is clear for $x_1 + x_2 \le 2$, where $m(\cdot,(1,1))$ is the only case in need of a proof : we have $m(0,(1,1)) = \psi(\delta_1,(1,1)) = (\frac{1}{2},\frac{1}{2})$, and the rest follows by flow conservation. For the induction step, consider first a demand profile x with $x_1 = x_2 = k$. Then the induction hypothesis and (17) imply $m(z,x) = m^{fs}(z,x)$ for $z \le x - (1,2)$ and $z \le x - (2,1)$. Combining this with $\psi(\delta_{2k},x) = m(x - e^1,x) + m(x - e^2,x) = (\frac{1}{2},\frac{1}{2})$ and the flow conservation constraints gives the desired conclusion. Next for a demand profile x such that $x_1 > x_2$ we have by (17) and the induction hypothesis $m(z,x) = m^{fs}(z,x)$ for $z \le (x_2 - 1, x_2 - 1)$ and $\psi(\delta_{2x_2},x) = m((x_2-1,x_2),x) + m((x_2,x_2-1),x) = (\frac{1}{2},\frac{1}{2})$. These two properties and flow conservation imply $m((x_2-1,x_2),x) = (\frac{1}{2},0)$ and the desired conclusion.

Next suppose $\psi = \psi^1$, the ordered contributions mechanism where agent 1 comes first. Then for all $t = 1,...,x_1$, $\psi(\delta_t,x) = \sum_{z::z_1+z_2=t-1} m(z,x) = (1,0)$. By repeated application of flow conservation, we get

$$\begin{aligned} t &= 1 \Rightarrow m(0,x) = (1,0) \Rightarrow m((0,z_2),x) = 0 \text{ if } z_2 > 0, \\ t &= 2 \Rightarrow m((1,0),x) = (1,0) \Rightarrow m((1,z_2),x) = 0 \text{ if } z_2 > 0, \end{aligned}$$

and by induction $m((z_1,0),x) = (1,0)$ for $z_1 = 1,...,x_1 - 1$. Therefore $m(\cdot,x)$ is the flow of the ordered contributions method φ^1 and the proof of Step 5 is complete.

Step 6. We proceed by induction on n, the size of N. In this step we denote by $\varphi(S)$ a cost-sharing method for the set of agents S, see Definition 1. Step 5 establishes Theorem 1 when $n = 2$. We fix N, $n \ge 3$, and a method $\varphi(N)$ satisfying the four axioms in Theorem 1. We assume Theorem 1 holds for any method $\varphi(S)$, $|S| \le n-1$.

We identify the space $\mathcal{C}(S)$ of cost functions on $\mathbb{N}^S$ with the subset of $\mathcal{C}(N)$ containing the functions independent of x_i, for all $i \in N \backslash S$. Dummy Independence allows us to define the projection of $\varphi(N)$ on S, namely the method $\varphi(S)(C,x(S)) = \varphi(N)(C,x)$, for all $C \in \mathcal{C}(S)$ and all $x \in \mathbb{N}^N$. One checks easily that $\varphi(S)$ meets the four axioms in Theorem 1. Therefore $\varphi(S) = \varphi^{\preccurlyeq_S}$ for some preordering $\preccurlyeq_S$ on S.

Fix a representation of $\varphi(N)$ as a family of probability distributions $\mu(\cdot,x)$ on $\Pi(N,x)$, one for each $x \in \mathbb{N}^N$, as in (2), and write $\Pi^*(N,x)$ the support of such a distribution. We denote by $\pi(S) \in \Pi(S,x(S))$ the projection on $[0,x(S)]$ of a path $\pi \in \Pi(N,x)$. Observe that the projection of $\mu(\cdot,x)$ on $\Pi(S,x(S))$ is a representation of $\varphi(S)(\cdot,x)$ via (2). In particular, its support $\Pi^*(S,x(S))$ is the projection of $\Pi^*(N,x)$ on S.

We show first that all preorderings $\preccurlyeq_S$, $S \subset N$, are compatible, and define $\preccurlyeq_N$ on N as follows: $i \preccurlyeq_N j \Leftrightarrow i \preccurlyeq_S j$ for some S. Notice that for $S \subset T \subset N$, $\preccurlyeq_S$

is the restriction of $\mathsf{4}_T$ to S, so $\mathsf{4}_N$ is well defined and complete. Transitivity is clear if $n \geq 4$, because $\mathsf{4}_{\{i,j,k\}}$ is transitive for any distinct i, j, k. If $n = 3$ we check transitivity by distinguishing 4 cases.

Case 1. $1 \prec_{\{1,2\}} 2$ and $2\ \mathsf{4}_{\{2,3\}}\ 3$. The only path in $\Pi(N, x)$ projecting on $\{1, 2\}$ to the $(1 \prec 2)$-priority path, and on $\{2, 3\}$ to the $(2 \prec 3)$-priority path, is the $(1 \prec 2 \prec 3)$-priority path. Thus $1 \prec_N 2 \prec_N 3$.

Case 2. $1 \prec_{\{1,2\}} 2$ and $2\ \mathsf{v}_{\{2,3\}}\ 3$. For any $x \geq (2, 2, 2)$, a path in $\Pi(N, x)$ projecting on $\{1, 2\}$ to the $(1 \prec 2)$-priority path, and on $\{2, 3\}$ to an egalitarian path (Definition 6), cannot project on $\{1, 3\}$ to the $(3 \prec 1)$-priority path or to an egalitarian path. Thus $\varphi(\{1, 3\})$ is the $(1 \prec 3)$-priority method and $\mathsf{4}_N$ is transitive.

Case 3. $1\ \mathsf{v}_{\{1,2\}}\ 2$ and $2 \prec_{\{2,3\}} 3$. This case is similar to case 2.

Case 4. $1\ \mathsf{v}_{\{1,2\}}\ 2$ and $2\ \mathsf{v}_{\{2,3\}}\ 3$. A path in $\Pi(N, x)$ projecting to egalitarian paths on $\{1, 2\}$ and on $\{2, 3\}$ cannot project to a priority path on $\{1, 3\}$ (provided x is large enough), therefore $\varphi(\{1, 3\})$ is the subsidy-free serial method and $1\ \mathsf{v}_N\ 2\ \mathsf{v}_N\ 3$.

Having established that $\mathsf{4}_N$ is a preordering, suppose that it has at least two indifference classes. We can partition $N = N_1 \cup N_2$, such that $N_1 \prec N_2$. Take any path $\pi \in \Pi^*(N, x)$. Comparing the representation of $\varphi(N)$ given by (2) and Definition 10, we note that in the sequence associated with π (Definition 3), all occurrences of agents in N_1 precede those of agents in N_2, or equivalently, π is the ordered composition of $\pi(N_1)$ with $\pi(N_2)$. It follows that $\varphi(N)$ is the ordered composition of $\varphi(N_1)$ with $\varphi(N_2)$ (we omit the straightforward details). Hence, $\varphi(N) = \varphi^{\mathsf{4}_\mathsf{N}}$.

We are left with the case where $\mathsf{4}$ is full indifference, and $\varphi(S)$ is the subsidy-free serial method for all $S \subset N$. Fix a demand profile $x = (x_1, ..., x_n)$ in N_*^N. Any path $\pi \in \Pi^*(N, x)$ projects to an egalitarian path on $N \backslash \{i\}$ for all i, therefore π itself is egalitarian. Such a path goes through x^1 (Definition 5) and is parallel to $\pi(N \backslash \{1\})$ afterwards:

$$\text{for all } t \geq nx_1,\ \pi(t, x) = (x_1, \pi(N \backslash \{1\})(t - (n-1)x_1, x(N \backslash \{1\}))).$$

This implies that the probability distribution $\mu(\cdot, [x^1, x])$ induced by $\mu(\cdot, x)$ on $\Pi^e([x^1, x])$ (the set of egalitarian paths from x^1 to x), gives the same flow $m(\cdot, x)$ (defined in Step 1) between x^1 and x as the subsidy-free serial method (because $\varphi(N \backslash \{1\})$ is subsidy-free serial). It remains to be shown that $\mu(\cdot, [0, x^1])$ gives the same flow between 0 and x^1 as the uniform distribution on $\Pi^e(x^1)$. To this end we must check first that the mechanism ψ induced by φ is serial.

Because the flow $m(\cdot, x)$ is the subsidy-free serial flow between x^1 and x, we have $\psi(\delta_t, x) = \psi^s(\delta_t, x)$ for all $t \geq nx_1 + 1$. If all coordinates of x are distinct, $\psi(\delta_t, x)$ takes the successive values $\frac{1}{n-1} e^{N\hat{A}\{1\}}, ..., e^{\{n\}}$ as t goes from $nx_1 + 1$ to x_N. By Lemma 1 , $\psi(\delta_t, x)$ takes at most n distinct values, therefore it must be constant for $t = 1, ..., nx_1$. Next $\psi(id, x) = x$ implies $\psi(\delta_t, x) = \frac{1}{n} e^N$ for these t, and $\psi(\cdot, x)$ is indeed serial. If not all coordinates of x are distinct, we choose x^* with distinct coordinates and such that $x \in H(x^*)$: property *iii)* in Lemma 2 implies at once that $\psi(\cdot, x)$ is serial as well.

Observe now that any path in $\Pi^e(x^1)$ goes through $e^N, 2e^N, ..., x^1 = x_1e^N$. Identifying the $n!$ paths in $\Pi([te^N, (t+1)e^N])$ with the $n!$ orderings of N, $\mu(\cdot, x)$ induces a distribution such that

i) for all $i \in N$, the induced probability on the $(n-1)!$ orderings of $N \backslash \{i\}$ is uniform, and

ii) for all $i, j \in N$, the probability of i being ranked jth is $1/n$.

The former property holds because $\varphi(N \backslash \{i\})$ is subsidy-free serial, the latter because ψ is serial. We leave it to the reader to check that properties i) and ii) imply a uniform probability on all $n!$ orderings of N, which in turn establishes that $\varphi(\cdot, x)$ is subsidy-free serial, and concludes the proof of Theorem 1. ■

12.4. Proof of Theorem 2. We leave it to the reader to check that φ^{cs} satisfies Distributivity, Strong Ranking and Separability. Next, we fix a method $\varphi \in \Phi$ satisfying these three axioms and prove that $\varphi = \varphi^{cs}$.

Step 1. Notation and preliminary observations.

Let $\mathcal{D} = \{D \in \mathcal{C} : D(z) \in \{0, 1\}$ for all $z \in \mathsf{N}^N\}$. As this will cause no confusion, we identify $D \in \mathcal{D}$ with $D^{-1}(1)$, an upper-comprehensive subset of $\mathsf{N}^N \backslash \{0\}$. We denote by ∂D the lower frontier of D, that is, $\partial D = \{z \in D$: for all $z' \in \mathsf{N}^N$, $z' < z \Rightarrow z' \notin D\}$. Note that the mapping $D \to \partial D$ is one-to-one. For all $i \in N$ and $t \in \mathsf{N} \backslash \{0\}$, define the function $\delta^i_t \in \mathcal{D}$ by $\delta^i_t(z) = 1$ if and only if $z_i \geq t$ (or, equivalently, by $\partial \delta^i_t = \{te^i\}$). Finally, for all $x \in \mathsf{N}^N \backslash \{0\}$, define $\mathcal{D}(x) = \{D \in \mathcal{D} : D(x) = 1\}$ (equivalently, $D \in \mathcal{D}(x)$ if and only if $\partial D \cap]0, x] \neq \emptyset$).

For any $z \in \mathsf{N}^N$, let $N(z) = \{i \in N : z_i > 0\}$. If $x \in \mathsf{N}^N \backslash \{0\}$ and $D \in \mathcal{D}(x)$, define $D^x = \vee_{z \in \partial D \cap]0,x]} (\wedge_{i \in N(z)} \delta^i_{z_i})$, where $\vee$ and $\wedge$ denote the supremum and infimum operations on functions (or equivalently, the union and intersection operations on sets). We will use the following fact:

$$\forall x \in \mathsf{N}^N \backslash \{0\}, \forall D \in \mathcal{D}(x), D = D^x \text{ on } [0, x]. \tag{18}$$

The straightforward proof of this fact is omitted.

Step 2. Recalling the notation introduced in Definition 5, we define, for any $x \in \mathsf{N}^N_*$ and $k = 1, ..., n$, the set $\mathcal{D}(k, x) = \{D \in \mathcal{D}(x) : \partial D \cap [0, x] \subseteq [0, x^k]\}$, and we write $a^k = \frac{1}{n-k+1} e^{\{k,...,n\}}$. We denote by Σ the unit simplex of R^N and let $\Sigma_* = \{a \in \Sigma : a_1 \leq ... \leq a_n\}$. We claim that

$$\forall x \in \mathsf{N}^N_*, \ \forall k = 1, ..., n, \ \forall D \in \mathcal{D}(k, x), \ \varphi(D, x) \in co\{a^1, ..., a^k\}. \tag{19}$$

To prove (19), fix $x \in \mathsf{N}_*^N$ and $k \in \{1, ..., n\}$. We use the convention $x_0 = 0$. The argument is divided in three substeps.

1) For any integers t, l such that $x_{l-1} < t \leq x_l$, if any, Separability, Strong Ranking, and the nonnegativity of the cost shares imply, respectively

$$\begin{aligned} \varphi(\sum_{i \in N} \delta_t^i, x) &= (n - l + 1)a^l, \\ \varphi(\delta_t^i, x) &\in \Sigma_* \text{ for } i = l, ..., n, \\ \varphi(\delta_t^i, x) &= 0 \text{ for } i = 1, ..., l - 1. \end{aligned}$$

As a^l is an extreme point of Σ_*, we conclude that $\varphi(\delta_t^i, x) = a^l$ for $i = l, ..., n$.

2) The set $M(k, x) = \{D \in \mathcal{D} : \varphi(D, x) \in co\{a^1, ..., a^k\}\}$ is closed under $\vee$ and $\wedge$.

To check this, note that the additivity of φ implies that the mapping $D \to \varphi(D, x)$ is modular: for all $D, D' \in \mathcal{D}$, $\varphi(D, x) + \varphi(D', x) = \varphi(D \wedge D', x) + \varphi(D \vee D', x)$. Strong Ranking implies that each of the four terms in this equality belongs to Σ_*. As $co\{a^1, ..., a^k\}$ is a face of Σ_*, the claim follows.

3) To complete the proof of (19), fix $D \in \mathcal{D}(k, x)$. For any $z \in \partial D \cap [0, x]$ and $i \in N(z)$, we have $1 \leq z_i \leq x_i^k$ by definition of $\mathcal{D}(k, x)$. By substep 1,

$$\begin{aligned} i &\leq k - 1 \Rightarrow z_i \leq x_i \Rightarrow \varphi(\delta_{z_i}^i, x) \in \{a^1, ..., a^i\}, \\ i &\geq k \Rightarrow z_i \leq x_k \Rightarrow \varphi(\delta_{z_i}^i, x) \in \{a^1, ..., a^k\}. \end{aligned}$$

Therefore $\delta_{z_i}^i \in M(k, x)$. By substep 2, this implies $D^x \in M(k, x)$ and, by (18) and Lemma 0, $D \in M(k, x)$.

Step 3. We prove that ψ is the serial mechanism ψ^s (Definition 5). Since φ satisfies Distributivity and Separability, the induced mechanism ψ satisfies all the properties derived in Subsection 12.1. We fix $x \in \mathsf{N}_*^N$ such that $0 < x_1 < ... < x_n$ and proceed in two substeps.

1) Consider the set $K(x)$ in (7). We claim that

$$K(x) = \{a^1, ..., a^n\}. \tag{20}$$

For any positive integer t, the function δ_t in (6) belongs to $\mathcal{D}$: it is identified with the set $\delta_t = \{z \in \mathsf{N}^N : z_N \geq t\}$, with lower frontier $\partial\delta_t = \{z \in \mathsf{N}^N : z_N = t\}$. Let $K(x) = \{y_*^1, ..., y_*^K\}$ be the sequence in Σ_* defined in (7). We prove by induction on $k = 1, ..., n$ the following property

$$P(k) : y_*^l = a^l \text{ for } 1 \leq l \leq k \text{ and } y_*^m \in co\{a^{k+1}, ..., a^n\} \text{ for } k + 1 \leq m \leq n.$$

Note that $P(n)$ is precisely the claim in (20).

We begin by proving $P(1)$. Because $x_1 > 0$, notice that $\delta_1 \in \mathcal{D}(1, x)$. By Step 2, $y_*^1 = \varphi(\delta_1, x) = a^1$. By (9),

$$x = \lambda_1 a^1 + \sum_{k=2}^{K} \lambda_k y_*^k \text{ for some } \lambda_1, ..., \lambda_K \in \mathbb{N}\backslash\{0\}. \tag{21}$$

Set $z = \sum_{k=2}^{K} \lambda_k y_*^k$ and observe that all coordinates of z are rational. We can choose a positive integer μ large enough to ensure that $z' = \mu z \in \mathbb{N}_*^N$. Suppose $z_1' > 0$. Then $\delta_1 \in \mathcal{D}(1, z')$ and by Step 2, $\varphi(\delta_1, z') = a^1 = y_*^1$. But applying statement *ii)* in Lemma 2 to z' yields $K(z') = \{y_*^2, ..., y_*^K\}$ and therefore $\varphi(\delta_1, z') = y_*^2$, a contradiction since $y_*^1 \neq y_*^2$ by definition of $K(x)$. Therefore $z_1' = z_1 = 0$, and z belongs to (the intersection of $\mathbb{N}_*^N$ with) the cone generated by $\{a^2, ..., a^n\}$. That set is a face of $\mathbb{N}_*^N$, hence $y_*^k \in co\{a^2, ..., a^n\}$ for $2 \leq k \leq n$. This proves $P(1)$.

Next, we assume $P(k)$ for $1 \leq k \leq n-1$, and prove $P(k+1)$. By (9),

$$x = \sum_{l=1}^{k} \lambda_l a^l + \sum_{m=k+1}^{K} \lambda_m y_*^m. \tag{22}$$

Because $x_k < x_{k+1}$, we must have $k < K$. Set $w = \sum_{m=k+1}^{K} \lambda_m y_*^m$, observe that all coordinates of w are rational, and find an integer μ such that $w' = \mu w \in \mathbb{N}^N$. By $P(k)$ and (22), we have

$$w_1' = ... = w_k' = 0 < w_{k+1}' < ... < w_n'. \tag{23}$$

In particular, $w' \in \mathbb{N}_*^N$ and $\delta_1 \in \mathcal{D}(k+1, w')$. By Step 2, $\varphi(\delta_1, w') \in co\{a^1, ..., a^{k+1}\}$. On the other hand, applying Lemma 2 to w' yields $K(w') = \{y_*^{k+1}, ..., y_*^K\}$ and $\varphi(\delta_1, w') = y_*^{k+1}$. Now Step 2 and $P(k)$ give

$$y_*^{k+1} \in co\{a^1, ...a^{k+1}\} \cap co\{a^{k+1}, ..., a^n\} = \{a^{k+1}\},$$

proving the first part of property $P(k+1)$.

It remains to be shown that $y_*^m \in co\{a^{k+2}, ..., a^n\}$ for $m = k+2, ..., K$. Because $y_*^m \in \mathbb{N}_*^N$, it is enough to show $(y_*^m)_{k+1} = 0$. Write $w' = \mu\lambda_{k+1}a^{k+1} + z$, where $z = \sum_{m=k+2}^{K} \mu\lambda_m y_*^m$. Property (23) implies $z_{k+1} < ... < z_n$. Thus $z \in \mathbb{N}_*^N$. Suppose $z_{k+1} > 0$. Then $\delta_1 \in \mathcal{D}(k+1, z)$. By Step 2, $\varphi(\delta_1, z) \in co\{a^1, ..., a^{k+1}\} = co\{y_*^1, ...y_*^{k+1}\}$. On the other hand, Lemma 2 yields $\varphi(\delta_1, z) = y_*^{k+2}$, a contradiction since all elements of $\{y_*^1, ..., y_*^K\}$ are distinct. Thus $z_{k+1} = 0$, and therefore $(y_*^m)_{k+1} = 0$, completing the proof of property $P(k+1)$, and (20).

2) Pick now an arbitrary $z \in \mathbb{N}_*^N$. By (20), there exist nonnegative numbers $\lambda_1, ..., \lambda_n$ such that $z = \sum_{k=1}^{n} \lambda_k a^k$. Thus $z \in H(x)$ and we may apply statement *ii)* in Lemma 2. Check now that (10) says precisely that $\psi(., z) = \psi^s(., z)$.

Step 4. We show that $\varphi = \varphi^{cs}$.

Let $x \in \mathsf{N}_*^N \backslash \{0\}$. For $k = 1, ..., n$, write $X_k = [0, x^k] \backslash [0, x^{k-1}]$, with the convention $x^0 = 0$; thus $\{X_1, ..., X_n\}$ is a partition of $]0, x]$. For $z \in \mathsf{N}^N \backslash \{0\}$, define $D_z \in \mathcal{D}$ by $D_z(w) = 1$ if and only if $w \geq z$, or equivalently, $\partial D_z = \{z\}$.

We will show that $\varphi(D_z, x) = \varphi^{cs}(D_z, x)$ for all $z \in \mathsf{N}^N \backslash \{0\}$. Recalling the formula for φ^{cs} in Definition 7, this amounts to proving that

$$\text{for all } k = 1, ..., n \text{ and all } z \in X_k,\ \varphi(D_z, x) = a^k. \tag{24}$$

Clearly every $C \in \mathcal{C}$ coincides on $[0, x]$ with some linear combination of the mappings D_z, $z \in]0, x]$ (a proof of this simple fact is in Moulin (1995)). Therefore, once (24) is established, the equality follows from Additivity and Lemma 0. We prove by descending induction on $t = x_N, x_N - 1, ..., 1$ the following property

$$P(t) : \text{for all } k = 1, ..., n \text{ and } z \in \delta_t \cap X_k,\ \varphi(D_z, x) = a^k.$$

Property $P(1)$ is exactly (24), as δ_1 contains $]0, x]$.

We begin by proving $P(n)$. By Step 3, $\varphi(\delta_{x_\mathsf{N}}, x) = \psi(\delta_{x_\mathsf{N}}, x) = a^n$. Since δ_{x_N} and D_x coincide on $[0, x]$, it follows from Lemma 0 that $\varphi(D_x, x) = a^n$, as desired.

Next, we fix $t, 1 \leq t < x_{N-1}$, assume property $P(t+1)$, and prove $P(t)$. Write the restriction of δ_t to $[0, x]$ as a linear combination of the mappings D_z :

$$\delta_t = \sum_{z \in \delta_t \cap [0,x]} \alpha_z D_z \text{ on } [0, x]. \tag{25}$$

Observe that $\alpha_z \in \mathsf{Z}$ for all $z \in \delta_t \cap [0, x]$, and that $\alpha_z = 1$ if $z \in \partial\delta_t$, i.e., if $z_N = t$. There is a unique integer k, $1 \leq k \leq n$, such that $(x^{k-1})_N < t \leq (x^k)_N$ (even if some of the successive vectors x^k coincide). Because $\delta_t \cap X_l = \emptyset$ for $l = 1, ..., k-1$, (25) can be rewritten as follows:

$$\delta_t = \sum_{m=k}^{n} \sum_{z \in \delta_t \cap X_m} \alpha_z D_z \text{ on } [0, x]. \tag{26}$$

Apply (26) successively to $x^k, ..., x^n$, taking into account that $D_z(x^m) = 1$ if and only if $z \in \cup_{i=k}^m X_i$. We obtain

$$\sum_{z \in \delta_t \cap X_k} \alpha_z = 1 \text{ and } \sum_{z \in \delta_t \cap X_m} \alpha_z = 0 \text{ for } m = k+1, ..., n. \tag{27}$$

Partition $\delta_t \cap X_m$ into $B_m = \partial\delta_t \cap X_m$ and $A_m = (\delta_t \cap X_m) \backslash B_m$: thus, $z_N = t$ and $\alpha_z = 1$ if $z \in B_m$ whereas $z_N \geq t + 1$ if $z \in A_m$. Property $P(t+1)$ implies $\varphi(D_z, x) = a^m$ for all $z \in A_m$. To prove $P(t)$, it is enough to show $\varphi(D_z, x) = a^m$ for all $m = k+1, ..., n$ and all $z \in B_m$.

Applying $\varphi(.,x)$ to both sides of (26) and denoting $\varphi(D_z,x)=y_z$, we get

$$a^k=\varphi(\delta_t,x)=\sum_{m=k}^{n}(\sum_{z\in B_m}y_z+\sum_{z\in A_m}\alpha_z a^m). \tag{28}$$

Write $\beta_m=|B_m|$ and $b^m=\frac{1}{\beta_m}\sum_{z\in B_m}y_z$ whenever $B_m\neq\emptyset$. Note that $B_k\neq\emptyset$, so that b^k is well defined. Moreover, $y_z\in\Sigma_*$ by Strong Ranking, hence b^m belongs to Σ_* whenever it is defined. Taking (27) into account and using the convention that $b^m=0$ if $\beta_m=0$, rewrite (28) as

$$\beta_k(b^k-a^k)+\sum_{m=k+1}^{n}\beta_m(b^m-a^m)=0. \tag{29}$$

We prove next that $b^k=a^k$. This is obvious if $\beta_m=0$ for $m=k+1,...,n$, so we assume $\beta=\sum_{k+1}^{n}\beta_m>0$. Define $b^*=\frac{1}{\beta}\sum_{k+1}^{n}\beta_m b^m$ and rewrite (29) as

$$b^*=\frac{1}{\beta}\sum_{m=k+1}^{n}\beta_m a^m+\frac{\beta_k}{\beta}(b^k-a^k). \tag{30}$$

If $z\in B_k$, then $D\in\mathcal{D}(k,x)$ and therefore, by Step 2, $y_z\in co\{a^1,...,a^k\}$, hence, $b^k\in co\{a^1,...,a^k\}$. On the other hand, $b^*\in\Sigma_*$ because $b^m\in\Sigma_*$ whenever $\beta_m>0$. Because $a^1,...,a^n$ are linearly independent, b^* is uniquely written as a *nonnegative* linear combination of $a^1,...,a^n$. If $b^k-a^k\neq 0$, this vector is a linear combination of $a^1,...,a^k$ with at least one negative coefficient, and (30) brings a contradiction. Therefore $b^k=a^k$.

As a^k is an extreme point of Σ_*, this equality implies that $y_z=a^k$ for all $z\in B_k$. In order to show that this equality holds for all $z\in B_m$ and all $m=k+1,...,n$, we use (30) repeatedly. If m_0 is the smallest integer $m_0\geq k+1$ such that $\beta_{m_0}>0$, we write $\beta'=\sum_{m_0+1}^{n}\beta_m>0$, and rewrite (30) as

$$\frac{1}{\beta'}\sum_{m=m_0+1}^{n}\beta_m b^m=\frac{1}{\beta}\sum_{m=m_0+1}^{n}\beta_m a^m+\frac{\beta_{m_0}}{\beta'}(b^{m_0}-a^{m_0}).$$

Observe that $b^{m_0}\in co\{a^1,...,a^{m_0}\}$ because $D_z\in\mathcal{D}(m_0,x)$ whenever $z\in B_{m_0}$. Mimicking the argument in the previous paragraph, we conclude that $b^{m_0}=a^{m_0}$, and $y_z=a^{m_0}$ for all $z\in B_{m_0}$. An obvious induction argument completes the proof. ■

12.5. The structure of the strongly distributive methods in Φ. In preparation for the proofs of Proposition 1 and Theorem 3, this subsection describes the structure of the methods in Φ satisfying Strong Distributivity. Let φ be such a method, and fix a demand profile $x\in\mathbb{N}^N$. For any $D\in\mathcal{D}(x)$ (recall the notation in

Step 1 of the proof of Theorem 2), we write $\varphi(D, x) = y_D$, an element of Σ. Recall from Step 2 of that proof that the mapping $D \to y_D$ is modular.

Step 1. The chain lemma.

A chain in $\mathcal{D}(x)$ is a decreasing sequence of distinct sets $D_1 \supset D_2 \supset ... \supset D_L$. Associated with such a chain is a sequence $\{y^1, ..., y^K\}$ in Σ obtained from the sequence $\{y_{D_1}, ..., y_{D_L}\}$ by keeping only one from each interval of consecutive identical vectors.

Lemma 4. *Given two chains* $\{D_1^l\}, \{D_2^l\}$ *in* $\mathcal{D}(x)$ *(of possibly different lengths) with associated sequences* $\{y_1^1, ..., y_1^{K_1}\}, \{y_2^1, ..., y_2^{K_2}\}$*, and given two sequences of positive real numbers* $\{\lambda_1^1, ..., \lambda_1^{K_1}\}, \{\lambda_2^1, ..., \lambda_2^{K_2}\}$,

$$\{\sum_{k=1}^{K_1} \lambda_1^k y_1^k = \sum_{k'=1}^{K_2} \lambda_2^{k'} y_2^{k'}\} \Rightarrow \{K_1 = K_2 = K \, and \; \forall k = 1, ..., K, \lambda_1^k = \lambda_2^k \, and \; y_1^k = y_2^k\}. \tag{31}$$

Proof. For $i = 1, 2$, define $C_i = \sum_{k=1}^{K_i} \lambda_i^k D_i^{l_k}$, where $D_i^{l_k}$ is anyone of the consecutive identical elements in the chain corresponding to y_i^k (so that, in particular, $y_{D_i^{l_k}} = y_i^k$). Rewrite the premise of (31) as $\varphi(C_1, x) = \varphi(C_2, x)$. By property (B*) in Definition 9 applied to the mapping γ_θ defined on $\mathbb{R}_+$,

$$\varphi(\theta \wedge C_1, x) = \varphi(\theta \wedge C_2, x) \text{ whenever } 0 \leq \theta \leq C_1(x) = C_2(x).$$

Next we compute

$$\theta \wedge C_i = \sum_{k=1}^{k_i^*} \lambda_i^k D^{k_i} + (\theta - \sum_{k=1}^{k_i^*} \lambda_i^k) D_i^{k_i^*+1},$$

where k_i^* is such that $\sum_{k=1}^{k_i^*} \lambda_i^k \leq \theta \leq \sum_{k=1}^{k_i^*+1} \lambda_i^k$. Therefore,

$$\sum_{k=1}^{k_1^*} \lambda_1^k y_1^k + (\theta - \sum_{k=1}^{k_1^*} \lambda_1^k)\; y_1^{k_1^*+1} = \sum_{k'=1}^{k_2^*} \lambda_2^{k'} y_2^{k'} + (\theta - \sum_{k'=1}^{k_2^*} \lambda_2^{k'})\; y_2^{k_2^*+1}$$

for all θ and where k_1^*, k_2^* depend upon θ. Taking $\theta \leq \lambda_i^1$ for $i = 1, 2$ gives $y_1^1 = y_2^1$. If $\lambda_1^1 \neq \lambda_2^1$, say, $\lambda_1^1 < \lambda_2^1$, we choose θ such that $\lambda_1^1 \leq \theta \leq \lambda_2^1$ and get $\lambda_1^1 y_1^1 + (\theta - \lambda_1^1) y_1^2 = \theta y_2^1$, contradicting $y_1^1 \neq y_1^2$. Thus $\lambda_1^1 = \lambda_2^1$. An obvious induction argument completes the proof. ■

A consequence of Lemma 4 is that for any sequence $\{y^1, ..., y^K\}$ associated with a chain $\{D^l\}$ in $\mathcal{D}(x)$, the vectors $y^1, ..., y^K$ are linearly independent (and, in particular, all distinct). Indeed if these vectors are linearly dependent, we have an equality of the type $\sum_{k=1}^{K_1} \lambda_1^k y_1^k = \sum_{k'=1}^{K_2} \lambda_2^{k'} y_2^{k'}$, where all λ_i^k are positive and vectors appearing on the left-hand side are distinct from those on the right-hand side, in contradiction to Lemma 4.

Step 2. Define $Y = \{y \in \Sigma : \exists D \in \mathcal{D}(x)$ such that $y = y_D\}$ and, for all $y \in Y$, let $\Delta(y) = \{D \in \mathcal{D}(x) : y_D = y\}$. The sets $\Delta(y)$, $y \in Y$, form a partition of $\mathcal{D}(x)$. We claim that for each $y \in Y$, $\Delta(y)$ is an *interval*: there exists (possibly equal) functions $D_+(y)$, $D_-(y) \in \Delta(y)$ such that

$$\text{for all } D \in \mathcal{D}(x),\ \{y_D = y\} \Leftrightarrow \{D_-(y) \supseteq D \supseteq D_+(y)\}. \tag{32}$$

For any $D_1, D_2 \in \Delta(x)$, $y_{D_1 \vee D_2} + y_{D_1 \wedge D_2} = 2y$ by modularity of the map $D \to y_D$. If $y_{D_1 \vee D_2} \neq y_{D_1 \wedge D_2}$, the two chains $\{D_1 \vee D_2, D_1 \wedge D_2\}$ and $\{D\}$ with the corresponding sequences of positive real numbers $\{1, 1\}$ and $\{2\}$ contradict Lemma 4. Therefore $y_{D_1 \vee D_2} = y_{D_1 \wedge D_2} = y$, and $\Delta(y)$ is closed under $\vee$ and $\wedge$. Thus $\Delta(y)$ has a largest element, which we denote $D_-(y)$ and a smallest element, $D_+(y)$.

Let $D \in \mathcal{D}(x)$ be such that $D_-(y) \supseteq D \supseteq D_+(y)$. If $y_D \neq y$, the two chains $\{D_-(y), D\}$ and $\{D, D_+(y)\}$ with corresponding sequences $\{1, 1\}$ and $\{1, 1\}$ contradict Lemma 4. This proves (32).

Step 3. Recalling the definition of D_z in Step 4 of the proof of Theorem 2, we say that $z \in]0, x]$ is *pivotal* (for the method φ at x, which are kept fixed) if and only if $y_{D_z} \neq y_{D_z \setminus \{z\}}$. We denote by P the set of pivotal vectors. Note that $x \in P$. We claim that

$$\text{for all } D, D' \in \mathcal{D}(x),\ D \cap P = D' \cap P \text{ if and only if } y_D = y_{D'}. \tag{33}$$

To prove the "only if" part of this statement, fix $D, D' \in \mathcal{D}(x)$ such that $D \cap P = D' \cap P$. If D and D' coincide on $[0, x]$, $y_D = y_{D'}$ by Lemma 0. Assume from now on that D, D' do not coincide on $[0, x]$, say, $(D \backslash D') \cap]0, x] \neq \emptyset$. It is easily seen that there exists $z \in (D \backslash D') \cap]0, x]$ such that $z \in \partial D$. Applying modularity to $D \backslash \{z\}$ and D_z,

$$y_{D \backslash \{z\}} + y_{D_z} = y_D + y_{D_z \backslash \{z\}}.$$

Since $z \notin P$, we have $y_{D_z} = y_{D_z \backslash \{z\}}$, hence, $y_{D \backslash \{z\}} = y_D$. If $((D \backslash \{z\}) \backslash D') \cap]0, x]$ is nonempty, repeat this argument to remove $z' \in ((D \backslash \{z\}) \backslash D') \cap \partial (D \backslash \{z\})$ and obtain $y_{D \backslash \{z, z'\}} = y_{D \backslash \{z\}} = y_D$. After finitely many steps we are left with a set D_0 such that $D_0 \cap]0, x] = (D \cap D') \cap]0, x]$ and $y_{D_0} = y_D$. If $D_0 \cap]0, x] = D' \cap]0, x]$, Lemma 0 implies $y_{D_0} = y_{D'}$, hence, $y_D = y_{D'}$, as desired. Otherwise, a symmetric argument shows that $y_{D'} = y_{D^0}$, where $D^0$and D_0 coincide on $[0, x]$, so that $y_D = y_{D'}$ holds in all cases.

To prove the "if" statement, we show that for all $y \in Y$, $(D_-(y) \backslash D_+(y)) \cap P = \emptyset$. Together with (32), this implies that $D \cap P = D' \cap P$ for all $D, D' \in \Delta(y)$, as desired. Thus, pick $z \in D_-(y) \backslash D_+(y)$ and let $D = D_+(y) \vee (D_z \backslash \{z\})$ and $D' = D_+(y) \vee D_z$. By (32), $D, D' \in \Delta(y)$. Applying modularity to D, D_z, we get $y_D + y_{D_z} = y_{D'} + y_{D_z \backslash \{z\}}$, hence, $y_{D_z} = y_{D_z \backslash \{z\}}$. This means $z \notin P$, as desired.

Step 4. Denote by $\mathcal{S}(P)$ the set of nonempty upper-comprehensive subsets of P, that is, $S \in \mathcal{S}(P)$ if and only if $\emptyset \neq S \subseteq P$ and, for all $z, z' \in P$, $\{z \in S$ and $z \leq z'\} \Rightarrow z' \in S$. For any $y \in Y$, let $\sigma(y) = D \cap P$, where D is any element of $\Delta(y)$. The set $\sigma(y)$ is well defined by Step 3, and it belongs to $\mathcal{S}(P)$. We claim that σ is a bijection from Y into $\mathcal{S}(P)$.

Property (33) implies that σ is one-to-one. To see that it is onto, fix $S \in \mathcal{S}(P)$ and let D_S be its upper-comprehensive envelope in $\mathbb{N}^N$: $D_S = S + \mathbb{N}^N$. Note that $D_S \in \mathcal{D}(x)$ as $P \subseteq]0, x]$. Because $\Delta(y_{D_S})$ contains D_S, $\sigma(y_{D_S}) = D_S \cap P = S$.

In what follows, we write $y_{D_S} = y_S$ for simplicity, and $\sigma^{-1}(S) = y_S$ is the inverse mapping of σ.

12.6. Proof of Proposition 1.

Let $\varphi \in \Phi$ satisfy Strong Distributivity and Dummy.

Step 1. Let $x \in \mathbb{N}^N, x \gg 0$. We show that $\varphi(., x)$ is an ordered contributions method at x: with the notation of Definition 10, there is an ordering $\preccurlyeq$ on N such that, for all $C \in \mathcal{C}$, $\varphi(C, x) = \varphi^{\preccurlyeq}(C, x)$.

Let Y and P be the sets associated with φ and x as in Steps 2 and 3 of Subsection 12.5. Recalling the definition of δ_t^i from Step 1 in Subsection 12.4, Dummy implies that

$$\varphi(\delta_t^i, x) = e^i \text{ for all } i \in N \text{ and } t = 1, ..., x_i.$$

Set $S_i = \sigma(e^i)$, namely, $S_i = \delta_t^i \cap P$.

We claim that the sets S_i, $i \in N$, form a chain. Pick two distinct agents in N, say, 1 and 2. Because σ is one-to-one, $S_1 \neq S_2$. By modularity, $y_{S_1 \cup S_2} + y_{S_1 \cap S_2} = y_{S_1} + y_{S_2} = e^1 + e^2$. Therefore y_{S_1} and y_{S_2} belong to the interval $[e^1, e^2]$. If both are in $]e^1, e^2[$, the vectors in $\{y_{S_1 \cup S_2}, e^1, y_{S_1 \cap S_2}\}$ are not linearly independent and, by Lemma 4, the sequence $\{D_{S_1 \cup S_2}, D_{S_1}, D_{S_1 \cup S_2}\}$, where $D_S = S + \mathbb{N}^N$ (see Step 4 of Subsection 12.5), cannot be a chain. As S_1, S_2 are upper-comprehensive in P, it follows that $S_1 \supset S_2$ or $S_1 \subset S_2$. Since the choice of 1 and 2 was arbitrary, the claim is proved.

Without loss of generality, assume $S_1 \supset ... \supset S_n$. Suppose Y contains a vector y different from e^i for all $i \in N$, say, $y = \sum_{i \in M} \lambda_i e^i$, where $\lambda_i > 0$ for all $i \in M$ and $M = \{i_1, ..., i_m\}$ is not a singleton. The two chains $\{D_{S_{i_1}}, ..., D_{S_{i_m}}\}$ and $\{D_{\sigma(y)}\}$ with corresponding sequences $\{\lambda_{i_1}, ..., \lambda_{i_m}\}$ and $\{1\}$ contradict Lemma 4. This proves $Y = \{e^1, ..., e^n\}$.

As σ is a bijection, $\mathcal{S}(P) \subseteq \{S_1, ..., S_n\}$ and since the latter is a chain, any two distinct vectors $z, z' \in P$ are ordered: $z < z'$ or $z > z'$. Thus $P = \{z^1, ..., z^n\}$ for some increasing sequence $z^1 < ... < z^n = x$. We claim that,

$$\varphi(C, x) = \sum_{k=1}^{n} (C(z^k) - C(z^{k-1}))\, e^k \text{ for every } C \in \mathcal{C}, \tag{34}$$

where $z^0 = 0$. For $C = D \in \mathcal{D}(x)$, $D \cap P = \{z^k, ..., z^n\} = S_k$ for some k, and $y_D = \sigma(S_k) = e^k$. This last equation is precisely (34) for $D \in \mathcal{D}(x)$, and these functions form a basis of $\mathcal{C}$. Hence the claim. To complete the proof, we determine the vectors $z^1, ..., z^n$. Applying (34) to δ_t^i, we obtain $z_i^i \geq t > z_i^{i-1}$ for all i and all $t = 1, ..., x_i$ (recall $x \gg 0$). As $\{z^k\}$ is an increasing sequence in $]0, x]$, this implies $z^k = \sum_{l=k}^n x_l e^l$ for $k = 1, ..., n$. This shows that $\varphi(., x) = \varphi^{\leq}(., x)$, the ordered contributions method at x based on the ordering $\leq$

Step 2. It is a simple matter to adapt the argument of Step 1 to the case where the set M of positive coordinates of $x \in \mathsf{N}^N$ is smaller than N. We apply Dummy to δ_t^i only for $i \in M$, and show that $S_i = \sigma(e^i), i \in M$, form a chain, next that $Y = \{e^i, i \in M\}$. The conclusion is that $\varphi(., x)$ is an ordered contributions method for some ordering on M.

Step 3. We check that the orderings identified at each $x \in \mathsf{N}^N$ in Steps 1 and 2 are all equal. Let $\preccurlyeq$ be the ordering associated with $x = e^N$, and assume without loss of generality that $\preccurlyeq$ is equal to $\leq$, that is, $1 \prec ... \prec n$. With the notations of Lemmas 1 and 2, we have $K(e^N) = \{e^1, e^2, .., e^n\}$. Therefore $H(e^N) = \mathsf{N}^N$ and statement *iii)* in Lemma 2 implies that ψ is the ordered contributions mechanism based on the ordering $\leq$. Thus the ordering associated with $\varphi(., x)$ in Step 2 cannot contradict $\leq$ and the proof of Proposition 1 is complete. ■

12.7. Proof of Theorem 3. We leave it to the reader to check that φ^{cs} satisfies Strong Distributivity. Next, we fix a method $\varphi \in \Phi$ satisfying Strong Distributivity, Ranking and Separability, and we prove that $\varphi = \varphi^{cs}$.

Fix $x \in \mathsf{N}_*^N$. In Steps 1 and 2, we prove that $\varphi(., x) = \varphi^{cs}(., x)$ under the assumption that $0 < x_1 < ... < x_n$. Step 3 drops that assumption. Let Y and P be the sets associated with φ and x as in Steps 2 and 3 of Subsection 12.5. For any two integers k, $1 \leq k \leq n$, and t, $t \geq 1$, define $D(k, t) = \{z \in \mathsf{N}^N : |\{i \in N : z_i \geq t\}| \geq k\}$, the set of vectors with at least k coordinates not smaller than t. Note that $D(k, t)$ is a symmetric element of $\mathcal{D}$ and that $D(k, t) \in \mathcal{D}(x) \Leftrightarrow t \leq x_{n-k+1}$. If the latter inequality holds, Separability implies that $\varphi(D(k, t), x) \in \Sigma_*$. Set $S(k, t) = D(k, t) \cap P$; note that $S(k, t) \neq \emptyset \Leftrightarrow t \leq x_{n-k+1}$, because P contains x.

Step 1. We prove by induction on $k = 0, 1, ..., n-1$, the property $P(k)$ consisting of the following three statements:

i) $\forall z \in P, \forall i \in N$, $z_i \in [0, x_{n-k-1}] \cup \{x_{n-k}, x_{n-k+1}, ..., x_n\}$,

ii) $S(1, x_{n-k}) = S(k+1, x_{n-k})$,

iii) $y_{S_{n-k}} = a^{n-k}$,

where, by convention, $x_0 = 0$. The set in ii) will be denoted by S_{n-k}.

We start by proving $P(0)$. For $k = 0$, statement ii) is vacuously true; write $S(1, x_n) = S_n$. Let $\Delta_t = \sum_{i=1}^n \delta_t^i$; this is a symmetric element of $\mathcal{C}$. Choose an integer t, $x_{n-1} < t \leq x_n$. Notice that Δ_t and $D(1,t)$ coincide on $[0,x]$. Therefore, by Separability and Lemma 0,

$$a^n = \varphi(\Delta_t, x) = y_{S(1,t)} \text{ for all } t \text{ such that } x_{n-1} < t \leq x_n.$$

As σ^{-1} is one-to-one, the set $S(1,t)$ is independent of t, and equal to $S(1,x_n) = S_n$. If $z \in P$ is such that $x_{n-1} < z_n < x_n$, then $S(1,z_n) \supsetneq S(1,z_n+1)$. This establishes statement i) for $k=0$, and $P(0)$.

Next, we fix k, $1 \leq k \leq n-1$, assume $P(0),...,P(k-1)$, and prove $P(k)$. We let the reader check that for any integer t, $x_{n-k-1} < t \leq x_{n-k}$, the functions Δ_t and $\sum_{l=1}^{k+1} D(l,t)$ coincide on $[0,x]$. Therefore, by Separability and Lemma 0,

$$(k+1)a^{n-k} = \sum_{l=1}^{k+1} y_{S(l,t)}.$$

By Ranking, $y_{S(l,t)} \in \Sigma_*$ for $l = 1,...,k+1$. As a^{n-k} is an extreme point of Σ_*, this gives $y_{S(l,t)} = a^{n-k}$ for all l and all t. Property $P(k)$ follows as above; we omit the details.

Step 2. We have proved $P(0),...,P(n-1)$. Statement i) in $P(n-1)$ says that for any $z \in P$ and $i \in N$, $z_i \in \{0, x_1, ..., x_n\}$. Notice that $t \leq t' \Rightarrow S(1,t') \subseteq S(1,t)$. Therefore statements ii) and iii) say that we have a chain $S_1 \supset\supset S_n$ and that for all $k = 1,...,n$,

$$y_{S_k} = a^k \text{ and } S(1,x_k) = S(n-k+1,x_k) = S_k.$$

Consider $z \in S_n$: for all $k = 1,...,n$, $z \in S_k \subseteq D(n-k+1,x_k)$, that is, at least $n-k+1$ coordinates of z are not smaller than x_k. Since $z \leq x$, this implies $z = x^n$ and therefore $S_n = \{x^n\}$. Next, we claim that

$$S_k = \{x^k, ..., x^n\} \text{ for } k = n, ..., 1.$$

The proof is by descending induction on k. Fix $k \leq n-1$, assume the induction hypothesis $S_{k+1} = \{x^{k+1},...,x^n\}$, and consider $z \in S_k \backslash S_{k+1}$. We have $z \notin S_{k+1}$, implying $z \notin D(1,x_{k+1})$ and $z_i \notin]x_k, x_{k+1}[$ for all $i \in N$, by property i) in $P(n-1)$. Hence $z \leq x^k$. Now the properties $z \in S_l \subseteq D(n-l+1,x_l)$ for $l = 1,...,k$ and $P(n-1)$ imply, as above, $z = x^k$, and the induction hypothesis completes the proof of the claim.

Next consider z in $P \backslash S_1$. We have $z \notin D(1,x_1)$, hence, property i) of $P(n-1)$ implies $z = 0$, contradicting the definition of P. Therefore, $P = S_1 = \{x^1,...,x^n\}$. By Step 3 in 12.5 and property iii) in $P(0),...,P(n-1)$, the vectors $\varphi(D,x)$ are now determined for all $D \in \mathcal{D}(x)$, namely, $\varphi(D,x) = a^k$ for the unique $k \in \{1,...,n\}$ such that $D \cap P = \{x^k,...,x^n\}$. Thus $\varphi(D,x) = \varphi^{cs}(D,x)$ for every $D \in \mathcal{D}(x)$ and, by Additivity, $\varphi(.,x) = \varphi^{cs}(.,x)$.

Step 3. Let now x be an arbitrary element of N_*^N. The proof that $\varphi(.,x) = \varphi^{cs}(.,x)$ is entirely similar to the one given under the assumption that all coordinates of x differ. For brevity, we only illustrate the argument with the following example.

Suppose $0 < x_1 = x_2 < x_3 < x_4 = x_5 = x_6$. From propositions $P(2), P(3)$, and $P(5)$, we derive successively

$$\begin{aligned} S(1,x_4) &= S(3,x_4) = S_4 \text{ and } y_{S_4} = a^4, \\ S(1,x_3) &= S(4,x_3) = S_3 \text{ and } y_{S_3} = a^3, \\ S(1,x_1) &= S(6,x_1) = S_1 \text{ and } y_{S_1} = a^1. \end{aligned}$$

These three sets form a chain $S_1 \supset S_3 \supset S_4$ because $t \rightarrow S(1,t)$ is inclusion-monotonic. Since $D(3,x_4) \cap D(4,x_3) \cap D(6,x_1) \cap [0,x] = \{x\}$, we conclude that $S_4 = \{x^4\}$. Next consider $z \in S_3 \backslash S_4$. Since $z \notin D(1,x_4)$ and $z_i \notin]x_3,x_4[$ for all i, we have $z \leq x^3$ and, since $z \in D(4,x_3) \cap D(6,x_1)$, $z = x^3$. A similar argument shows that $S_1 \backslash S_3 = \{x^1\}$. Finally, any $z \in P \backslash S_1$ must be zero because $z \notin D(1,x_1)$ and $z_i \in \{0, x_1, x_3, x_4\}$ for all i, so that $P = S_1 = \{x^1, x^3, x^4\}$. The final argument is the same as in Step 2. ∎

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