On Demand Responsiveness in Additive Cost Sharing

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Abstract. We propose two new axioms of demand responsiveness for additive cost sharing with variable demands. Group Monotonicity requires that if a group of agents increase their demands, not all of them pay less. Solidarity says that if agent i demands more, j should not pay more if k pays less. Both axioms are compatible in the partial responsibility theory postulating Strong Ranking, i.e., the ranking of cost shares should never contradict that of demands. The combination of Strong Ranking , Solidarity and Monotonicity characterizes the quasi-proportional methods, under which cost shares are proportional to 'rescaled' demands.

The alternative full responsibility theory is based on Separability, ruling out cross-subsidization when costs are additively separable. Neither the Aumann-Shapley nor the Shapley-Shubik method is group monotonic. On the other hand, convex combinations of "nearby" fixed-path methods are group-monotonic: the subsidy-free serial method is the main example. No separable method meets Solidarity, yet restricting the axiom to submodular (or supermodular) cost functions leads to a characterization of the fixed-flow methods, containing the Shapley-Shubik and serial methods.

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1. Introduction

The traditional model of cooperative games is well suited to discuss cost-sharing problems where the demands of the participating agents are fixed (inelastic). Shapley (1953) and Weber (1988) showed how the simple axioms of Additivity (with respect to cost) and Dummy (an agent with zero marginal cost pays nothing) determine individual cost shares as a fixed average of marginal costs.

Following Aumann and Shapley's (1974) seminal contribution, the additive theory of cost sharing was extended to variable individual demands: agents $i = 1, ..., n$ want a quantity x_i of good i, and the problem is to split the total cost $C(x_1, ..., x_n)$ using no other information than the cost function C. Applications range from the pricing of utilities such as water, phone or electricity, where the level of individual consumption affects total cost, to the sharing of waiting time at a congested server (Shenker (1995), Haviv (2001)), and the division of highway construction costs between taxpayers (Lee (2002)).

A simple requirement of demand responsiveness has played a key role in the recent literature on axiomatic cost sharing. Monotonicity, introduced in Moulin (1995) and developed in Friedman and Moulin (1999), stipulates that every user's cost share should be nondecreasing in her own demand. On the normative side, it expresses a weak form of responsibility for one's own demand. If goods are freely disposable, Monotonicity also has a positive interpretation: it prevents manipulation by artificial (and wasteful) increases of individual demands.

In this paper we propose a handful of new demand responsiveness axioms, some of them strengthening Monotonicity, and explore their impact in the additive theory of cost sharing. Our model and our three main axioms are presented in Section 2. We have a finite number of users, individual demands are non negative integers, and the cost function is non decreasing, zero for zero demands, and otherwise arbitrary. We always assume that individual cost shares are non negative and depend additively upon the cost function.

Group Monotonicity requires that if a group of agents simultaneously increase their demands, not all of them pay less. Under the same premises, Strong Group Monotonicity states that the total cost share allocated to the group should not decrease. Both properties have dual normative/positive interpretations similar to those of Monotonicity. If side-payments are not feasible, Group Monotonicity is sufficient to prevent manipulations by a coordinated artificial increase of demands; if they are feasible, we need the latter, stronger property.

Our third main axiom is Solidarity. It requires that a demand change by one agent affects the share of all other agents in the same direction: no one pays more if someone pays less. The recent literature on fair division explores the Solidarity idea in a number of allocation problems (Thomson (1999), Sprumont and Zhou (1999)), but this is its first application to cost sharing.

A natural family of methods meeting all demand responsiveness properties above are the quasi-proportional methods computing i's cost share as $\frac{s(x_i)}{\sum s(x_j)}C(x)$, where s is is an arbitrary nondecreasing "scaling function". These simple and familiar methods only take the actual cost into account, and divide it in proportion to rescaled individual demands. Theorem 1 in Section 3 characterizes this family by combining Solidarity, Monotonicity, and one more axiom called Strong Ranking. The latter requires cost shares to be ranked in the same order as the corresponding demands, for any cost function, symmetric or not.

Strong Ranking is the defining axiom of the "partial responsibility" theory of cost sharing (Moulin and Sprumont (2002)). Agents, while responsible for their demands, are not responsible for the asymmetries of the cost function. This justifies some cross-subsidization. We submit that this viewpoint inspires cost sharing rules for many publicly provided services: disabled customers are not charged more for transportation services, the same stamp buys delivery of mail to central or remote areas, and so on.

The alternative view that agents are responsible for cost asymmetries is the basis of the more familiar "full responsibility" theory. This view leads to the principle of no cross subsidization pervading the natural monopoly literature (Baumol, Panzar and Willig (1982), Sharkey (1982)), and to the related Dummy axiom in the model of cooperative games. In our setting the relevant formulation is the Separability axiom: if the cost function is additively separable ($C(z) = \sum_i c_i(z_i)$ for all z), each agent should pay her stand-alone cost $c_i(x_i)$.

The impact of our demand responsiveness conditions under full responsibility is the subject of Sections 4 and 5. Proposition 2 states that neither Solidarity nor Strong Group Monotonicity is compatible with Separability. On the other hand Group Monotonicity is a very demanding yet achievable property. The simple method charging incremental costs according to a fixed ordering of the users is an example. More generally, a fixed-path method charging incremental costs according to a fixed ordering of the different units of demand, is group-monotonic. However Group Monotonicity is typically not preserved by convex combinations of cost sharing methods. For instance the Shapley-Shubik method (Shubik (1962)), applying the Shapley value to the stand-alone cooperative game at the given demand profile, is not group-monotonic. Thus this property yields a powerful critique of both the oldest cost-sharing method in the literature, and of the most popular one, the Aumann-Shapley method (the latter violates even Monotonicity).

Proposition 3 in Section 4 is an important positive result, allowing us to construct "reasonable" group-monotonic methods in the full responsibility approach. Any convex combination of "nearby" fixed-path methods is group-monotonic as well. The most natural method constructed in this fashion is the subsidy-free serial method (Moulin (1995)), adapting serial cost sharing to our discrete model.

If Solidarity is out of reach in the full responsibility theory, a weaker version of this property is feasible when the cost function is submodular. In that case an increase in user i's demand is a positive externality on other users, and our Submodular Solidarity axiom in Section 5 rules out changes of opposite signs in their cost shares. We introduce the rich family of fixed-flow methods, corresponding to convex combinations of fixed-path methods, and containing for instance the Shapley-Shubik and subsidy-free serial methods. Theorem 2 characterizes this family by the combination of Separability, Monotonicity, and Submodular Solidarity.

Section 6 compares our results under partial and full responsibility. While the quasi-proportional methods satisfy Solidarity, the cross effects of a demand shift may not have the expected sign. For instance, if C is supermodular and x_i increases, agent j's cost share, $\frac{s(x_j)}{\sum s(x_k)}C(x)$, may go down. We introduce the Positive Externalities and Negative Externalities axioms: if agent i's demand raises, all agents j other than i benefit when costs are submodular, and suffer when costs are supermodular. We show in Proposition 4 that these properties are essentially incompatible with partial responsibility: no cost-sharing method satisfies Strong Ranking and Positive Externalities, and the only method satisfying Additivity, Strong Ranking and Negative Externalities divides costs equally no matter what. By contrast, the axioms are easy to meet under full responsibility: the fixed-flow methods as well as the Aumann-Shapley method satisfy Positive and Negative Externalities (Proposition 5).

Section 7 takes a second look at the partial responsibility theory. Moulin and Sprumont (2002) introduce Weak Separability, ruling out cross-subsidization when it is not implied by Strong Ranking, namely when the cost function takes the form $C(z) = \sum_i c(z_i)$. While the quasi-proportional methods violate Weak Separability, the axiom is compatible with partial responsibility, i.e., with Strong Ranking. The simplest example is the cross-subsidizing serial method (Sprumont (1998)). This method is also group-monotonic (Proposition 6). Yet Strong Group Monotonicity is incompatible with the combination of Weak Separability and Strong Ranking (Proposition 7).

2. The model and the demand responsiveness axioms

Each agent $i \in N = \{1, ..., n\}$ demands an integer quantity $x_i \in \mathbb{N} = \{0, 1, ...\}$ of a personalized good. The cost of meeting the demand profile $x \in \mathbb{N}^N$ must be split among the members of N. A cost function is a mapping $C : \mathbb{N}^N \to \mathbb{R}_+$ that is nondecreasing and satisfies $C(0) = 0$; the set of such mappings is denoted $\mathcal{C}$. A (cost-sharing) method φ assigns to each problem $(C, x) \in \mathcal{C} \times \mathbb{N}^N$ a vector of nonnegative cost shares $\varphi(C, x) \in \mathbb{R}_+^N$ such that $\sum \varphi_i(C, x) = C(x)$. We call $\varphi(., x)$ a (cost-sharing) method at x.

We use the following notation. Vector inequalities are written $\leq, <, \ll$. For any $x, x' \in \mathbb{N}^N$, $[x, x'] = \{z \in \mathbb{N}^N \mid x \leq z \leq x'\}$, $[x, x'[= [x, x'] \setminus \{x'\}$; $]x, x']$ and $]x, x'[$ are defined similarly. We let $N(x) = \{i \in N \mid x_i > 0\}$ and write $n(x) = |N(x)|$. For any $S \subseteq N$, we denote by $x_S \in \mathbb{N}^S$ the restriction of x to S and we write $x(S) = \sum_{i \in S} x_i$. We define $e^S \in \mathbb{N}^N$ by $e_i^S = 1$ if $i \in S$ and 0 otherwise. We sometimes write i for $\{i\}$, ij for $\{i, j\}$, and $-S$ for $N \setminus S$. If $i \in N$ and $C \in \mathcal{C}$, we define $\partial_i C : \mathbb{N}^N \to \mathbb{R}_+$ by $\partial_i C(z) = C(z + e^i) - C(z)$. Finally, we let $\Delta = \{y \in \mathbb{R}_+^N \mid \sum_{i \in N} y_i = 1\}$.

The requirement that cost shares be nonnegative is part of our definition of a method. Nonnegativity is a minimal form of demand responsiveness. It does rule out well-known procedures such as equal split beyond stand-alone costs,

$$\varphi_i(C, x) = C(x_i, 0_{-i}) + \frac{1}{n}(C(x) - \sum_{j \in N} C(x_j, 0_{-j})),$$

and the marginal cost procedure

$$\varphi_i(C, x) = x_i \partial_i C(x - e^i) + \frac{1}{n}(C(x) - \sum_{j \in N} x_j \partial_j C(x - e^j)),$$

where $\partial_i C(x - e^i)$ is defined to be zero if $x_i = 0$.

Throughout the paper, we restrict our attention to additive methods.

Additivity. For any $C, C' \in \mathcal{C}$ and $x \in \mathbb{N}^N$, $\varphi(C + C', x) = \varphi(C, x) + \varphi(C', x)$.

This powerful mathematical property is devoid of ethical content. Additive cost-sharing methods are very convenient in practice. When production can be decomposed into the sum of several independent processes (like research, production and marketing; or construction and maintenance), applying the method to each subprocess and adding the resulting cost shares is equivalent to applying the method to the consolidated cost function. The proper level of application of the method is not a matter of dispute.

We now define four demand responsiveness conditions. The first one is the familiar Monotonicity (Moulin (1995)) requiring individual cost shares to be monotonic in the corresponding demands.

Monotonicity. For all $C \in \mathcal{C}$, $x, x' \in \mathbb{N}^N$, and $i \in N$, $\{x_i < x'_i$ and $x_j = x'_j$ for all $j \in N \setminus \{i\}\} \Rightarrow \{\varphi_i(C, x) \leq \varphi_i(C, x')\}$.

When communication between agents is easy, demand coordination is a concern. We propose a group version of Monotonicity requiring that, when several agents increase simultaneously their demands, not all of them pay less.

Group Monotonicity. For any $C \in \mathcal{C}$, any $x, x' \in \mathbb{N}^N$, and any nonempty $S \subseteq N$, $\{x_i < x'_i$ for all $i \in S$ and $x_i = x'_i$ for all $i \in N \setminus S\} \Rightarrow \{\varphi_i(C, x) \leq \varphi_i(C, x')$ for at least one $i \in S\}$.

A natural strengthening of Group Monotonicity evaluates the impact of demand increases followed by side-payments. When money transfers between agents are feasible, John may be willing to strategically raise his demand if the corresponding drop in some other agents' shares more than compensates the increase in John's share. Strong Group Monotonicity prevents such manipulations.

Strong Group Monotonicity. For any $C \in \mathcal{C}$, $x, x' \in \mathbb{N}^N$, any nonempty $S \subseteq N$ and $i \in S$, $\{x_i < x'_i$ and $x_j = x'_j$ for all $j \in N \setminus \{i\}\} \Rightarrow \{\sum_{j \in S} \varphi_j(C, x) \leq \sum_{j \in S} \varphi_j(C, x')\}$.

Clearly Strong Group Monotonicity also rules out the profitability of a coordinated increase by several members of S followed by side-payments within S.

Our last demand responsiveness axiom requires that all members of a group should be treated similarly whenever a demand change occurs outside the group.

Solidarity. For any $C \in \mathcal{C}$, any $i \in N$, and any $x, x' \in \mathbb{N}^N$, $\{x_j = x'_j$ for all $j \in N \setminus \{i\}\} \Rightarrow \{\varphi_j(C, x) \leq \varphi_j(C, x')$ for all $j \in N \setminus \{i\}$, or $\varphi_j(C, x) \geq \varphi_j(C, x')$ for all $j \in N \setminus \{i\}\}$.

We conclude this section by noting the following connection between our axioms.

Proposition 1. Every cost-sharing method satisfying Additivity, Solidarity, and Monotonicity satisfies Strong Group Monotonicity.

Proof. The subset $\mathcal{D} = \{C \in \mathcal{C} \mid C(z) \in \{0, 1\}$ for all $z \in \mathbb{N}^N\}$ of cost functions taking the values 0 or 1 plays a central role in this and several subsequent proofs, in particular that of Theorem 1*. Every $C \in \mathcal{C}$ is a nonnegative linear combination of cost functions in $\mathcal{D}$. Thus an additive method φ meets Strong Group Monotonicity if and only if for all $C \in \mathcal{D}$ and $x \in \mathbb{N}^N$,

$$\sum_{j \in S} \varphi_j(C, x) \leq \sum_{j \in S} \varphi_j(C, x + e^i) \text{ for all } S \subseteq N \text{ and all } i \in S. \tag{1}$$

If $C(x) = 0$, (1) is automatically satisfied. If $C(x) = 1$, then $C(x + e^i) = 1$ and (1) is equivalent to

$$\varphi_j(C, x + e^i) \leq \varphi_j(C, x) \text{ for all distinct } i, j \in N,$$

which is clearly implied by the combination of Solidarity and Monotonicity. ■

3. Demand responsiveness and partial responsibility: the quasi-proportional methods

This section is devoted to the quasi-proportional methods discussed in the Introduction.

Definition 1. A method φ is quasi-proportional if $\varphi_i(C,x) = \frac{s(x_i)}{\sum_{j\in N} s(x_j)} C(x)$ for all $i, C,$ and $x,$ where $s : \mathbb{N} \to \mathbb{R}_+$ is nondecreasing and $s(1) > 0$ (with the convention that $\varphi(C,0) = 0$). Examples include the egalitarian method $\varphi_i(C,x) = \frac{1}{n}C(x)$ and the proportional method $\varphi_i(C,x) = \frac{x_i}{x(N)}C(x)$.

These methods meet Additivity, Monotonicity, and Solidarity, hence, by Proposition 1, satisfy all the demand responsiveness axioms of Section 2. No information about "counter-factual" costs is used in computing the cost shares, which are therefore completely insensitive to the asymmetries in the cost structure. In particular, agents who ask more pay more, regardless of the cost function.

Strong Ranking. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, and $i,j \in N$, $\{x_i \le x_j\} \Rightarrow \{\varphi_i(C,x) \le \varphi_j(C,x)\}$.

This defines the "partial responsibility" theory of cost sharing in which agents are held responsible for the size of their own demand but not for the idiosyncrasies of the cost function. Note that the egalitarian method barely satisfies Strong Ranking: it violates the strict version requiring that the ranking of cost shares be identical to that of demands when the cost function is strictly monotonic. The proportional method is more responsive to demands; its simplicity makes it very popular in practice.

We now show that Additivity, Solidarity, Monotonicity and Strong Ranking essentially characterize the quasi-proportional methods. The precise statement requires a couple of additional definitions. Notice first that the egalitarian method possesses a property slightly stronger than Solidarity,

Strict Solidarity. For any $C \in \mathcal{C}$, any $i \in N$, and any $x, x' \in \mathbb{N}^N$, $\{x_j = x'_j$ for all $j \in N\backslash\{i\}\} \Rightarrow \{$ i) $\varphi_j(C,x) < \varphi_j(C,x')$ for all $j \in N\backslash\{i\}$ or ii) $\varphi_j(C,x) = \varphi_j(C,x')$ for all $j \in N\backslash\{i\}$ or iii) $\varphi_j(C,x) > \varphi_j(C,x')$ for all $j \in N\backslash\{i\}\}$,

which the proportional method fails because agents demanding zero are never affected by changes in others' demands.

On the other hand, the proportional method meets

Zero Cost for Zero Demand. For any $C \in \mathcal{C}$, any $i \in N$, and any $x \in \mathbb{N}^N$, $\{x_i = 0\} \Rightarrow \{\varphi_i(C,x) = 0\}$,

which the egalitarian method obviously fails. This axiom is compelling if agents are not responsible for the demands of others[1].

[1] While automatically satisfied in the more familiar "full responsibility" theory (where it follows

Finally, we observe that both the egalitarian and the proportional method meet

Positive Cost for Positive Demand. For any $C \in \mathcal{C}$, any $i \in N$, and any $x \in \mathbb{N}^N$,
$\{x_i > 0$ and $C(x) > 0\} \Rightarrow \{\varphi_i(C, x) > 0\}$,

which is a strong but simple interpretation of responsibility for one's demand[2]. We are now ready to state our first main result.

Theorem 1.

i) Assume $n \geq 4$. A cost-sharing method φ satisfies Additivity, Strong Ranking, Strict Solidarity, and Monotonicity if and only if there exists a nondecreasing function $s : \mathbb{N} \to \mathbb{R}$ such that $s(0) > 0$ and

$$\varphi_i(C, x) = \frac{s(x_i)}{\sum_{j \in N} s(x_j)} C(x) \tag{2}$$

for all $C \in \mathcal{C}$, $i \in N$, and $x \in \mathbb{N}^N$.

ii) Assume $n \geq 3$. A cost-sharing method φ satisfies Additivity, Strong Ranking, Solidarity, Monotonicity, Zero Cost for Zero Demand and Positive Cost for Positive Demand if and only if there exists a nondecreasing function $s : \mathbb{N} \to \mathbb{R}$ such that $s(0) = 0$, $s(1) > 0$, and (2) holds for all $C \in \mathcal{C}$, $i \in N$, and $x \in \mathbb{N}^N$ (with the convention $\varphi(C, 0) = 0$).

Choosing a positive constant a and setting $s(t) = t + a$ for all t, equation (2) defines a hybrid method approaching the egalitarian method when $a \to \infty$ and the proportional one when $a \to 0$. Any such method meets Strict Solidarity.

Four observations are in order.

(1) Theorem 1 is a corollary to Theorem 1* and Lemma 2, stated and proved in the Appendix, which dispense with Strong Ranking. Dropping Strong Ranking allows for a family including more complicated methods, of which two interesting subclasses are easily described.

Consider first the methods $\varphi(C, x) = \theta(x)C(x)$, where $\theta(x)$ is an arbitrary mapping from $\mathbb{N}^N \setminus \{0\}$ to ¢. We call such methods simple: they completely ignore the shape of the cost function and reduce the cost-sharing problem to a "general claim arbitration" problem (where the amount to be divided may exceed or fall short of the sum of the claims) as in Herrero, Maschler and Villar (1999) and Naumova (2002).

from Additivity and Separability: see Section 4), Zero Cost for Zero Demand must be explicitly required in the partial responsibility theory.

[2] This axiom is incompatible with the main axiom of the full responsibility theory, Separability, because a dummy agent pays nothing even if his demand is positive. By contrast, Positive Cost for Positive Demand is easily met in the partial responsibility approach.

It follows from the proof of Theorem 1* that a simple method meets Additivity, Solidarity and Monotonicity if and only if it is an asymmetric quasi-proportional method, i.e.,

$$\theta_i(x) = \frac{s_i(x_i)}{\sum_{j \in N} s_j(x_j)}, \tag{3}$$

for all $i \in N$, where $s_i : \mathbb{N} \to \mathbb{R}_+$ is nondecreasing and $s_i(1) > 0$.

A second interesting subclass meeting our three axioms consists of the methods

$$\varphi_i(C, x) = \frac{\lambda}{n-1} C(x_i, 0_{-i}) + \frac{1}{n}\Big(C(x) - \sum_{j \in N} \frac{\lambda}{n-1} C(x_j, 0_{-j})\Big) \tag{4}$$

for all i, C, and x, where $\lambda \in [0, 1]$. For $\lambda = 0$, this is just the egalitarian method. Setting $\lambda = n - 1$ yields "equal split beyond stand-alone costs" introduced in Section 2, which however does not guarantee nonnegative cost shares. The nonnegativity constraints force λ between 0 and 1. These methods split equally the balance above a fraction of stand-alone costs. The entire family of methods meeting Additivity, Monotonicity and Solidarity combines features of these methods and the quasi-proportional methods: see the Appendix for details.

(2) Replacing the combination of Solidarity and Monotonicity by Strong Group Monotonicity (which is implied: recall Proposition 1) allows for many more methods. For instance, the simple methods $\varphi(C, x) = \theta(x)C(x)$ satisfy Strong Group Monotonicity if and only if θ_i is non-increasing in x_j for any two distinct i, j (which is much less restrictive than the asymmetric quasi-proportional form (3)) and Strong Ranking if and only if $\theta_i(x) \leq \theta_j(x) \Leftrightarrow x_i \leq x_j$ for any i, j and x. Characterizing the entire class of methods meeting Additivity, Strong Group Monotonicity and Strong Ranking is a challenging open problem.

(3) All methods in Theorem 1 satisfy a requirement stronger than Solidarity. When an arbitrary subset of agents change their demands, the impact on the cost shares of the others goes in the same direction: for any $C \in \mathcal{C}$, any $S \subseteq N$, and any $x, x' \in \mathbb{N}^N$, $\{x_j = x'_j$ for all $j \in N \setminus S\} \Rightarrow \{\varphi_j(C, x) \leq \varphi_j(C, x')$ for all $j \in N \setminus S$ or $\varphi_j(C, x) \geq \varphi_j(C, x')$ for all $j \in N \setminus S\}$.

(4) The egalitarian method occupies a special place within the quasi-proportional methods, because it meets a much stronger version of the Solidarity axiom:

Full Solidarity. For any $C \in \mathcal{C}$, any $i \in N$, and any $x, x' \in \mathbb{N}^N$, $\{x_i < x'_i$ and $x_j = x'_j$ for all $j \in N \setminus \{i\}\} \Rightarrow \{\varphi(C, x) \leq \varphi(C, x')\}$.

This axiom strengthens both Solidarity and Monotonicity. It characterizes the fixed-proportions methods: a cost-sharing method φ satisfies Additivity and Full Solidarity if and only if there exists $\lambda \in \Delta$ such that $\varphi(C,x) = C(x)\lambda$ for all $C \in \mathcal{C}$ and $x \in \mathbb{N}^N$. We omit the easy proof[3].

4. Demand Responsiveness and full responsibility

We turn to demand responsiveness properties within the familiar "full responsibility" approach, where agents are responsible for the idiosyncrasies of the cost function. The defining axiom says that everyone should pay the cost of their own demand whenever that cost does not depend on others agents' demands.

Separability. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, $\{C(z) = \sum_{i \in N} c_i(z_i)$ for all $z \in \mathbb{N}^N$, where $c_i(0) = 0$ for all $i\} \Rightarrow \{\varphi_i(C,x) = c_i(x_i)$ for all $i \in N\}$.

All quasi-proportional methods violate Separability. More fundamentally, the axiom is incompatible with Strong Ranking even in the absence of Additivity. Suppose costs are additively separable ($C(z) = \sum c_i(x_i)$) and i's stand-alone cost exceeds j's at all levels ($c_i(t) > c_j(t)$ for all $t > 0$). If $x_i = x_j$, Strong Ranking imposes equal cost shares for i and j while Separability requires that i pays more than j.

Under Additivity, Separability is equivalent to the more familiar Dummy axiom requiring that dummy agents pay nothing: for all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, and $i \in N$, $\{\partial_i C = 0\} \Rightarrow \{\varphi_i(C,x) = 0\}$[4].

The combination of Additivity and Separability is well understood. Moulin and Vohra (2003) offer a representation in terms of flows. A conservative unit flow - or simply, a flow - to a demand profile $x \in \mathbb{N}^N$ is a mapping $f(.,x) : [0,x[\to \mathbb{R}_+^N$ satisfying the convention that $f_i(z,x) = 0$ whenever $z_i = x_i$, the normalization $\sum_{i \in N} f_i(0,x) = 1$, and the conservation constraints $\sum_{i \in N} f_i(z,x) = \sum_{i \in N(z)} f_i(z - e^i, x)$ for all $z \in]0,x[$. Note that this implies $\sum_{i \in N(x)} f_i(x - e^i, x) = 1$.

Lemma 1 (Moulin and Vohra (2003)). A method φ satisfies Additivity and Separability if and only if, for every $x \in N^N$, there is a (necessarily unique) flow $f(.,x)$ to x such that

$$\varphi_i(C,x) = \sum_{z \in [0,x[} f_i(z,x)\partial_i C(z) \tag{5}$$

for all $C \in C$ and all $i \in N$; we call f the flow representation of φ.

An important consequence of this representation is the following property.

Independence of Irrelevant Costs. For all $x \in \mathbb{N}^N$ and $C^1, C^2 \in C$, $\{C^1(z) = C^2(z)$ for all $z \in [0,x]\} \Rightarrow \{\varphi(C^1,x) = \varphi(C^2,x)\}$.

[3] Note that if $D \in \mathcal{D}$ and $D(x) = 1$, Full Solidarity implies $\varphi(D,x) = \varphi(D, x + e^i)$ for all i. This implies first that $\varphi(\delta^x, x)$ is independent of x, then $\varphi(D,x)$ is independent of both D and x.

[4] See Moulin and Vohra (2003), Corollary 3, and notice that under Additivity, their Non-Dummy axiom is equivalent to Separability.

The simplest separable additive methods correspond to flows $f(.,x)$ mapping $[0,x[$ into $\{0,1\}^N$. In this case the entire flow runs along a monotone path from 0 to x in the integer grid. Fix such a path. At each step of the path, the coordinate of exactly one agent increases by one. Charging to this agent the corresponding cost increase and summing over the entire path defines a cost-sharing method at the demand profile x. Formally, any sequence of agents $\pi : \{1,...,x(N)\} \to N$ such that $|\pi^{-1}(i)| = x_i$ for each $i \in N$, defines a monotone path z^π to x by setting $z^\pi(0) = 0$ and $z^\pi(t) = z^\pi(t-1) + e^{\pi(t)}$ for $t = 1,...,x(N)$. This path, in turn, generates the flow $f_i(z,x) = 1$ if for some t, $z^\pi(t-1) = z$, $z^\pi(t) = z + e^i$, and $f_i(z,x) = 0$ otherwise. The corresponding cost shares at x are

$$\varphi_i^\pi(C,x) = \sum_{t\in\pi^{-1}(i)} \partial_i C(z^\pi(t-1)) \qquad (6)$$

for all $i \in N$ and $C \in \mathcal{C}$. If, for every demand profile x, $\varphi(.,x)$ is generated by a monotone path to x, we say that φ is path-generated. Wang (1999) shows that a method satisfies Additivity and Separability if and only if it is the pointwise limit of convex combinations of path-generated methods. Note that if we restrict attention to a bounded subset of demand profiles, the two axioms characterize simply the (fixed) convex combinations of path-generated methods.

Example 1. The Aumann-Shapley[5] method φ^{as} solves every problem (C,x) by averaging the cost-share vectors $\varphi^\pi(C,x)$ generated along all paths π to x. In the corresponding flow representation, $f_i^{as}(z,x)$ equals the proportion of paths to x which go through z and $z+e^i$: straightforward computations yield $f_i^{as}(z,x) = \frac{\alpha(z)\alpha(x-z-e^i)}{\alpha(x)}$ for all $i \in N$ and $z \in [0,x-e^i]$, where $\alpha(z) = z(N)!/\prod_{j\in N} z_j!$.

Example 2. The Shapley-Shubik method φ^{ss} (Shubik (1962)) averages the cost shares computed along the paths that follow the edges of the cube $[0,x[$. Letting $E_i(x) = \{z \in [0,x[\mid \forall j \in N\setminus i, z_j \in \{0,x_j\}\}$, the flow associated with $\varphi^{ss}(.,x)$ is given by $f_i^{ss}(z,x) = \frac{(n(x)-n(z,x))!(n(z,x)-1)!}{n(x)!}$ if $z, z+e^i \in E_i(x)$ and $f_i^{ss}(z,x) = 0$ otherwise, where $n(z,x) = |\{j \in N \mid z_j < x_j\}|$.

Example 3. The subsidy-free serial method φ^{sf}(Moulin (1995)) averages the cost shares computed along the paths that follow the "constrained diagonal" to x. Let $G(x) = \{z \in [0,x[\mid \forall i,j \in N, (x_i \le x_j) \Rightarrow (|z_i - z_j| \le 1 \text{ or } z_i = x_i < z_j)\}$ and $G^*(x) = \{z \in [0,x[\mid \forall i,j \in N, (x_i \le x_j) \Rightarrow (z_i - z_j = 0 \text{ or } z_i = x_i < z_j)\}$. For each $z \in G(x)$, denote by $\overline{z}(x)$ the smallest $z^* \in G^*(x)$ such that $z^* > z$ and let $\underline{z}(x)$ be the largest $z^* \in G^*(x)$ such that $z^* \le z$. The flow $f^{sf}(.,x)$ associated with

[5] The original definition and characterizations of this method are in the model with continuous demands: Billera and Heath (1982), Mirman and Tauman (1982). We present here its counterpart in the discrete model. For an axiomatization of this discrete version, see Sprumont (2004).

the serial method $\varphi^{sf}(.,x)$ at x is given by $f_i^{sf}(z,x) = f_i^{ss}(z - \underline{z}(x), \overline{z}(x) - \underline{z}(x))$ if $z, z + e^i \in G(x)$ and $f_i^{sf}(z,x) = 0$ otherwise.

We are now ready to describe the consequences of our four demand responsiveness conditions under full responsibility. First, we note that the two strongest conditions are incompatible with Additivity and Separability.

Proposition 2. Suppose $n \geq 3$. Then
i) no cost-sharing method satisfies Additivity, Separability, and Solidarity;
ii) no cost-sharing method satisfies Additivity, Separability, and Strong Group Monotonicity.

Proof. Statement i) We establish the incompatibility for $n = 3$; the extension to $n \geq 3$ follows immediately by considering cost functions where all but three agents are dummies. Let φ satisfy the three stated axioms and let f be its flow representation as in formula (5). Let $x = e^{12}$, $x' = e^{123}$. Define $C^1 \in \mathcal{C}$ by $C^1(z) = 1$ if $z \geq e^1$ or $z \geq e^{23}$, and $C^1(z) = 0$ otherwise. Similarly, define $C^2 \in \mathcal{C}$ by $C^2(z) = 1$ if $z \geq e^2$ or $z \geq e^{13}$, and $C^2(z) = 0$ otherwise. Note that $C^1 = D^1$ on $[0,x]$, where D^1 is the additively separable function $D^1(z) = 1$ if $z_1 \geq 1, D^1(z) = 0$ otherwise. Thus by Independence of Irrelevant Costs and Separability $\varphi(C^1,x) = e^1$. On the other hand $\varphi_2(C^1,x') = f_2(e^3,x')$. A similar argument applied to C^2 gives $\varphi(C^2,x) = e^2$ and $\varphi_1(C^2,x') = f_1(e^3,x')$.

If $\varphi_1(C^1,x') < \varphi_1(C^1,x) = 1$, Solidarity implies $\varphi_2(C^1,x') \cdot \varphi_2(C^1,x) = 0$, hence $\varphi_2(C^1,x') = 0$. The latter equality holds as well if $\varphi_1(C^1,x') = \varphi_1(C^1,x) = 1$. Thus $f_2(e^3,x') = 0$. A similar argument involving C^2 instead of C^1 gives $f_1(e^3,x') = 0$. Now flow conservation yields $f_3(0,x') = 0$. By exchanging the roles of the agents, we get similarly $f_i(0,x') = 0$ for $i = 1,2,$ a contradiction to the definition of a conservative unit flow.

Statement ii) The proof mimics that of statement i): the first paragraph is unchanged; then apply Strong Group Monotonicity to C^1 (with $i = 3$ and $S = \{1,3\}$) to get $\varphi_2(C^1,x') \cdot \varphi_2(C^1,x) = 0$, hence $f_2(e^3,x') = 0$, and to C^2 (with $i = 3$ and $S = \{2,3\}$) to get $f_1(e^3,x') = 0$. The rest is unchanged. ∎

By contrast to Proposition 2, Monotonicity and Group Monotonicity are compatible with Additivity and Separability. Both have a lot of bite. As mentioned in the Introduction, the Aumann-Shapley method is not monotonic[6]. The Shapley-Shubik method is monotonic, but not group-monotonic. We only prove the latter statement.

[6] A simple counterexample has two agents and the cost function $\delta^{(1,1)}(z) = 1$ for all $z \geq (1,1)$ and $\delta^{(1,1)}(z) = 0$ otherwise. Then $\varphi^{as}(\delta^{(1,1)},(1,1)) = (\frac{1}{2},\frac{1}{2})$ and $\varphi^{as}(\delta^{(1,1)},(2,1)) = (\frac{1}{3},\frac{2}{3})$, a contradiction to Monotonicity.

Let $N = \{1, 2, 3\}$ and C be the cost function:

$$C(z) = 1 \text{ if } z \geq (2,0,1) \text{ or } z \geq (1,1,1) \text{ or } z \geq (0,2,1), \quad 0 \text{ otherwise.}$$

Check that $\varphi_1^{ss}(C, (1,1,1)) = \varphi_2^{ss}(C, (1,1,1)) = \frac{1}{3} > \frac{1}{6} = \varphi_1^{ss}(C, (2,2,1)) = \varphi_2^{ss}(C, (2,2,1))$, which contravenes Group Monotonicity.

The characterization of all methods meeting Additivity, Separability and Group Monotonicity remains an open problem. In the rest of this section, we construct a fairly rich family of such methods.

We start with the fixed-path methods. Any fixed sequence $\pi : \mathbb{N} \setminus \{0\} \to N$ generates an infinite fixed path $z^\pi : \mathbb{N} \to \mathbb{N}^N$ through the induction $z^\pi(0) = 0$, $z^\pi(t) = z^\pi(t-1) + e^{\pi(t)}$ for $t \in \mathbb{N} \setminus \{0\}$. If $\pi^{-1}(i)$ is infinite for each $i \in N$, we call the sequence π unbounded; the corresponding fixed path is unbounded in each coordinate (i.e., $z_i^\pi(t) = |\{s \in \pi^{-1}(i) \mid s \leq t\}|$ goes to infinity as t grows). To any demand profile $x \in \mathbb{N}^N$, we associate a sequence $\pi^x : \{1, ..., x(N)\} \to N$ by keeping the first x_i occurrences of each i : for instance, if $n = 3$, $x = (1, 4, 2)$ and $\pi = 1, 2, 3, 1, 2, 3, 1, 2, 3, ...,$ then $\pi^x = 1, 2, 3, 2, 3, 2, 2$. The sequence π^x in turn generates the monotone path z^{π^x} to x and the associated cost-sharing method $\varphi^{\pi^x}(., x)$ at x via formula (6). The fixed-path method φ^π based on π is defined by $\varphi^\pi(., x) = \varphi^{\pi^x}(., x)$ for all x.

To see why such methods satisfy Group Monotonicity, fix π and x. For each $i \in N$ let $t_i(\pi, x)$ be the smallest integer t such that $z_i^\pi(t) = x_i$. Let $S \subseteq N$ and consider a demand profile x^0 such that $x_i < x_i^0$ for all $i \in S$ and $x_i = x_i^0$ for all $i \in N \setminus S$. Find the (necessarily unique) agent $i_0 \in S$ such that $t_{i_0}(\pi, x) \leq t_i(\pi, x)$ for all $i \in S$. Because π is a fixed sequence, $\varphi_{i_0}^\pi(C, x^0)$ is computed along a path $z^{\pi^{x^0}}$ to x^0 which coincides with z^{π^x} for all $t \leq t_{i_0}(\pi, x) \leq t_{i_0}(\pi, x^0)$. Therefore $\varphi_{i_0}^\pi(C, x) \leq \varphi_{i_0}^\pi(C, x^0)$, proving Group Monotonicity.

Fixed-path methods are asymmetric. To restore symmetry, we could take convex combinations of such methods, but the example of the Shapley-Shubik method demonstrates that in general this operation will not preserve Group Monotonicity. Interestingly, taking convex combinations of "sufficiently close" fixed-path methods does preserve Group Monotonicity. Formally, let π be a fixed unbounded sequence and let $i, j \in N$. Construct $\pi^{ij} : \mathbb{N} \setminus \{0\} \to \{i, j\}$ by deleting from π all occurrences of agents in $N \setminus \{i, j\}$. This sequence generates the projection of z^π on $\mathbb{N}^{\{i,j\}}$, denoted $z^{\pi^{ij}} : \mathbb{N} \setminus \{0\} \to \mathbb{N}^{\{i,j\}}$, in the usual way. We say that two fixed unbounded sequences π, σ are nearby if for all $i, j \in N$ and all $t \in \mathbb{N}$,

$$\{z^{\sigma^{\{i,j\}}}(t) \neq z^{\pi^{\{i,j\}}}(t)\} \Rightarrow \{z^{\sigma^{\{i,j\}}}(t+1) = z^{\pi^{\{i,j\}}}(t+1)\}.$$

Proposition 3. If $\{\pi_1, ..., \pi_K\}$ is a family of unbounded sequences which are pairwise nearby, and $\lambda_1, ..., \lambda_K$ are nonnegative numbers summing up to 1, then $\sum_{k=1}^{K} \lambda_k \varphi^{\pi_k}$ satisfies Additivity, Separability, and Group Monotonicity. In particular, the subsidy-free serial method (Example 3) satisfies these three axioms.

Proof. By Lemma 1, $\sum_{k=1}^{K} \lambda_k \varphi^{\pi_k}$ meets Additivity and Separability. To check Group Monotonicity, fix $C \in \mathcal{C}$, $S \subseteq N$, and $x, x' \in \mathbb{N}^N$ such that $x_i < x'_i$ for all $i \in S$ and $x_i = x'_i$ for all $i \in N \backslash S$. Let $i_1 \in S$ be the agent in S such that $t_{i_1}(\pi_1, x) \leq t_i(\pi_1, x)$ for all $i \in S$. As explained previously, $\varphi^{\pi_1}_{i_1}(C, x) \leq \varphi^{\pi_1}_{i_1}(C, x')$.

Next we show that $\varphi^{\pi_k}_{i_1}(C, x) \leq \varphi^{\pi_k}_{i_1}(C, x')$ for all $k \in \{2, ..., K\}$. Fix k and recall that π_k is nearby π_1. This implies that for all $i \in N$, either $t_{i_1}(\pi_k, x) \leq t_i(\pi_k, x)$ or, if the opposite strict inequality is true,

$$t_i(\pi_k, x) < s < t_{i_1}(\pi_k, x) \Rightarrow s \notin \pi_k^{-1}(i).$$

If $i \in S$, the above property implies $t_{i_1}(\pi_k, x) < t_i(\pi_k, x + e^i) \leq t_i(\pi_k, x')$. It follows that $t_{i_1}(\pi_k, x) < t_i(\pi_k, x')$ for all $i \in S$. Therefore $\varphi^{\pi_k}_{i_1}(C, x')$ is computed along a path to x' which coincides, for all $t \leq t_{i_1}(\pi_k, x)$, with the path to x along which $\varphi^{\pi_k}_{i_1}(C, x)$ is computed.

It follows that the subsidy-free serial method meets the three axioms because φ^{sf} is simply the arithmetic average of the $n!$ methods generated by the fixed unbounded sequences $p(1)$, $p(2), ...,$ $p(n)$, $p(1)$, $p(2), ...,$ $p(n), ...$ corresponding to the possible permutations p on N. ■

Our companion paper, Moulin and Sprumont (2002), offers a characterization of the subsidy-free serial method based on the property called Distributivity, and expressing for the composition of cost functions (with perfect substitute outputs) the same invariance as Additivity does for their addition.

5. The fixed-flow methods

Within the full responsibility theory, we explore now two axioms weakening, respectively, Strong Ranking and Solidarity. Separability is incompatible with Strong Ranking but is clearly compatible with the following weaker condition:

Ranking. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, and $i, j \in N$, $\{C$ is a symmetric function of all its variables and $x_i \leq x_j\} \Rightarrow \{\varphi_i(C, x) \leq \varphi_j(C, x)\}$.

The restriction to symmetric cost functions is very natural: when C is symmetric, any difference in cost shares must originate in differences in demands, hence the ranking of cost shares should follow that of individual demands. Examples 1, 2, 3 all satisfy Ranking.

Next, Separability is incompatible with Solidarity (Proposition 2), but it is consistent with a restricted version of that axiom. Solidarity is very demanding when all types of production externalities are allowed, thus it is natural to limit its application to those cases where externalities are clearly signed. For any $C \in \mathcal{C}$, $i, j \in N$, and $z \in \mathbb{N}^N$, define $\partial_{ij}C(z) = \partial_i C(z + e^j) - \partial_i C(z)$. The subsets of submodular and supermodular cost functions, respectively, are $\mathcal{C}_{sub} = \{C \in \mathcal{C} \mid \partial_{ij}C \leq 0$ for all distinct $i, j \in N\}$ and $\mathcal{C}_{sup} = \{C \in \mathcal{C} \mid \partial_{ij}C \geq 0$ for all distinct $i, j \in N\}$.

Submodular Solidarity. For any $C \in \mathcal{C}_{sub}$, any $i \in N$, and any $x, x' \in \mathbb{N}^N$, $\{x_j = x'_j$ for all $j \in N \setminus \{i\}\} \Rightarrow \{\varphi_j(C, x) \leq \varphi_j(C, x')$ for all $j \in N \setminus \{i\}$ or $\varphi_j(C, x) \geq \varphi_j(C, x')$ for all $j \in N \setminus \{i\}\}$. **Supermodular Solidarity** is defined by replacing $\mathcal{C}_{sub}$ with $\mathcal{C}_{sup}$ in the previous statement.

Examples 1, 2, 3 all satisfy the restricted solidarity properties. Combining either of them with Monotonicity and Separability circumscribes a very natural family of methods. For any flow $f(., x)$ on $[0, x[$ and any $x' \in [0, x]$, the projection of $f(., x)$ on $[0, x'[$, denoted $p_{x'}f(., x)$, is defined as follows: for any $i \in N$ and $z \in [0, x'[$ write $K = \{j \in N \mid z_j = x'_j\}$ and let

$$\begin{aligned} p_{x'}f_i(z, x) &= 0 \text{ if } i \in K, \\ &= \sum_{w_K \in [x'_K, x_K]} f_i((w_K, z_{N \setminus K}), x) \text{ otherwise,} \end{aligned}$$

with the convention that the sum is simply $f_i(z, x)$ if $K = \emptyset$. Note that $p_{x'}f(., x)$ is a flow to x'.

For simplicity, we state our characterization result for bounded cost-sharing problems: we fix $\overline{x} \in (\mathbb{N} \setminus \{0\})^N$ and speak of a cost-sharing method restricted to $[0, \overline{x}]$ if x varies in $[0, \overline{x}]$. The translation of all our axioms into this framework is straightforward: one merely needs to restrict their application to the demand profiles in $[0, \overline{x}]$.

Definition 2. A cost-sharing method φ restricted to $[0, \overline{x}]$ is a fixed-flow method if there is a flow $f(., \overline{x})$ to $\overline{x}$ such that, for every $x \in [0, \overline{x}]$, the flow $p_x f(., \overline{x})$ represents $\varphi(., x)$.

Examples include the fixed-path, Shapley-Shubik, and subsidy-free serial methods. One can check directly $f(., x) = p_x f(., x')$ for any $x, x', x \leq x'$, or apply our next result. On the other hand, the Aumann-Shapley method is not a fixed-flow method. The proof of the following result is in the Appendix.

Theorem 2. Let $\overline{x} \in (\mathbb{N} \setminus \{0\})^N$. For any cost-sharing method φ restricted to $[0, \overline{x}]$, the following statements are equivalent:
i) φ is a fixed-flow method;
ii) φ satisfies Additivity, Separability, Monotonicity, and Submodular Solidarity;
iii) φ satisfies Additivity, Separability, Monotonicity, and Supermodular Solidarity.

We conclude this section with three comments.

1) Theorem 2 is tight. For a method satisfying all axioms but Additivity, let $\varphi_i(C,x) = \frac{C(x_i,0_{-i})+1}{\sum_{j\in N}(C(x_j,0_{-j})+1)}C(x)$ unless C is additively separable, in which case we compute cost shares by applying Separability. A method violating only Separability is the proportional method $\varphi_i(C,x) = \frac{x_i}{x(N)}C(x)$. An example violating only Monotonicity is the Aumann-Shapley method: we prove in Section 6 that it actually satisfies properties stronger than Submodular Solidarity and Supermodular Solidarity. Finally, we describe a method violating only the latter two axioms. Let $n = 3$ and $\overline{x} = (2,2,2)$. Consider the two monotone paths to $\overline{x}$ defined by the sequences

$$\begin{aligned} \pi_1 &= 1,2,3,1,2,3, \\ \pi_2 &= 1,2,3,2,1,3. \end{aligned}$$

Let f^1, f^2 be the flows to $\overline{x}$ corresponding to π_1, π_2. For each $x \in]0,\overline{x}]$, define the flow $f(.,x)$ to x by $f(.,x) = p_x f^1$ if $x = (2,2,1)$, and $f(.,x) = p_x f^2$ otherwise. This generates a method restricted to $[0,\overline{x}]$ with the desired properties.

2) Fixed flow methods satisfy a stronger property than Dummy, called Strong Dummy: for all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, and $i \in N$, $\{\partial_i C = 0\} \Rightarrow \{\varphi_i(C,x) = 0$ and $\varphi_j(C,x) = \varphi_j(C,(0_i,x_{-i}))$ for all $j \in N \setminus i\}$. This follows by applying $f(.,x') = p_{x'}f(.,x)$ to $x' = (0_i,x_{-i})$.However the combination of Additivity, Strong Dummy and Monotonicity is not enough to characterize the fixed flow methods.

3) Theorem 2 suggests interesting open questions. Many fixed-flow methods, including the Shapley-Shubik and subsidy-free serial methods, meet Ranking. If $n = 2$, all fixed-flow methods on $[0,\overline{x}]$ where $\overline{x_1} = \overline{x_2}$ and the flow f is symmetric ($f_1(z,\overline{x}) = f_2((z_2,z_1),\overline{x})$ for all z) meet Ranking. Surprisingly, this property does not generalize to $n \geq 3$. Here is an example of a method based on a symmetric flow to $\overline{x} = \overline{x_1}e^N$ that violates Ranking. Let $n = 3$, $\overline{x} = (3,3,3)$, and consider the six monotone paths to $\overline{x}$ defined by the sequence

$$1,2,1,2,1,3,2,3,3$$

and the five sequences obtained from it by permuting agents. Let f be the (fully symmetric) flow on $[0,\overline{x}]$ obtained by putting an equal weight on these six paths, and let φ be the resulting fixed-flow method. Consider the cost function

$$\begin{aligned} C(z) &= 1 \text{ if } z \geq z^* \text{ for some } z^* \in Z^*, \\ &\quad 0 \text{ otherwise,} \end{aligned}$$

where $Z^* = \{(1,2,2),(2,1,2),(2,2,1),(3,1,1),(1,3,1),(1,1,3)\}$. The function C is symmetric in all its variables, and one computes $\varphi(C,(3,2,1)) = (\frac{1}{2},\frac{1}{6},\frac{1}{3})$, in contradiction of Ranking. This raises the problem of determining which fixed-flow methods meet Ranking.

Characterizing the class of fixed-flow methods meeting Group Monotonicity is another challenging open problem.

6. Comparing the partial and full responsibility approaches

Under Submodular or Supermodular Solidarity, the impact of an increase in agent i's demand on j's cost share must go in the same direction for all j other than i but this direction is a priori unrestricted. It is natural to require more: that all j other than i benefit when costs are submodular, and suffer when costs are supermodular.

Positive Externalities. For any $C \in \mathcal{C}_{sub}$, any $i \in N$, and any $x, x' \in \mathbb{N}^N$, $\{x_i < x'_i$ and $x_j = x'_j$ for all $j \in N \setminus \{i\}\} \Rightarrow \{\varphi_j(C, x) \geq \varphi_j(C, x')$ for all $j \in N \setminus \{i\}\}$. **Negative Externalities** is defined by replacing the subset $\mathcal{C}_{sub}$ with $\mathcal{C}_{sup}$ and the inequality sign $\geq$ with $\leq$ in the previous statement.

All quasi-proportional methods violate Positive Externalities and all but one violate Negative Externalities. The tension between these two axioms and Strong Ranking is systematic.

Proposition 4.
i) No cost-sharing method satisfies Strong Ranking and Positive Externalities.
ii) The only cost-sharing method satisfying Additivity, Strong Ranking and Negative Externalities is the egalitarian method[7].

Proof. i) Consider the cost function $C(z) = 0$ if $z \leq 2e^1$, $= 1$ otherwise. Check it is submodular (it is a particular case of the function defined by (26) in the Appendix). By nonnegativity of cost shares and Positive Externalities, $\varphi(C, z) = 0$ and $\varphi(C, z + e^2) = e^2$, violating Strong Ranking. Note that Additivity is not invoked.

ii) We focus on the case $n = 2$ and leave the extension to the reader. Let $z \neq 0$ and consider the supermodular cost function δ^z defined by (28) in Step 3 of the proof of Theorem 2. Let x be an arbitrary demand profile. If $\delta^z(x) = 0$, the nonnegativity of cost shares forces $\varphi(\delta^z, x) = (0, 0)$. If $\delta^z(x) = 1$, distinguish three cases. If $x_1 = x_2$, Strong Ranking forces $\varphi(\delta^z, x) = (\frac{1}{2}, \frac{1}{2})$. If $x_1 < x_2$, Negative Externalities and the fact that $\varphi(\delta^z, (x_2, x_2)) = (\frac{1}{2}, \frac{1}{2})$ imply $\varphi_2(\delta^z, x) \leq \frac{1}{2}$. Strong Ranking then forces $\varphi(\delta^z, x) = (\frac{1}{2}, \frac{1}{2})$. A symmetrical argument handles the case $x_1 > x_2$. This shows that φ is the egalitarian method on the cost functions δ^z. As every cost function coincides on every $[0, x]$ with a linear combination of such functions, the claim follows by Additivity. ■

By contrast, the fixed-flow methods satisfy Positive and Negative Externalities. Many more separable methods meet these axioms: an important example is the Aumann-Shapley method. The proof of the following result is in the Appendix.

[7] Note that Negative Externalities is implied by Full Solidarity and compare the current result with observation (4) following Theorem 1.

Proposition 5. The fixed-flow methods and the Aumann-Shapley method satisfy Positive Externalities and Negative Externalities.

To sum up, Strong Ranking is compatible with Solidarity but the direction of the cross-effects may be counter-intuitive; Separability is consistent with a limited form of solidarity only but the cross-effects may be guaranteed to have the expected sign.

7. Weak separability

This section takes a second look at the partial responsibility theory. In Moulin and Sprumont (2002) we argue that a sound theory of partial responsibility should perform cross-subsidization to correct for cost asymmetries, and for that purpose only. Subsidization is not justified when the cost function is symmetric: in such cases, the separability principle still applies. Thus if the cost function is not only symmetric but also additively separable, each agent will pay her "own" separable cost:

Weak Separability. For all $C \in \mathcal{C}$, $x \in \mathbb{N}^N$, $\{C(z) = \sum_{i \in N} c(z_i)$ for all $z \in \mathbb{N}^N\} \Rightarrow \{\varphi_i(C,x) = c(x_i)$ for all $i \in N\}$.

All quasi-proportional methods violate Weak Separability. For instance, if $n = 2$, $x_1 < x_2$, and $C(z) = c(z_1) + c(z_2)$, the proportional method cross-subsidizes agent 2 at 1's expense when c is convex, and vice-versa when c is concave.

Yet, Weak Separability is compatible with Strong Ranking. A complete description of the methods meeting Additivity, Weak Separability, and Strong Ranking is offered in Lemma 3 in the Appendix. A key member of that class is the cross-subsidizing serial method defined in Sprumont (1998)[8].

Example 5. For any $x \in \mathbb{N}^N_* = \{x \in \mathbb{N}^N : x_1 \leq \dots \leq x_n\}$ and all $i \in N$, define $x^i = x_i e^N \wedge x$. The cross-subsidizing serial method φ^{cs} assigns to every problem $(C,x) \in \mathcal{C} \times \mathbb{N}^N_*$ the vector of cost shares $\varphi^{cs}(C,x) = \frac{1}{n}C(x^1)e^N + \frac{1}{n-1}[C(x^2) - C(x^1)]e^{N \setminus 1} + \dots + [C(x^n) - C(x^{n-1})]e^n$. The cost shares for an arbitrary problem obtain by applying the formula after reordering the coordinates of the demand profile in nondecreasing order.

Proposition 6. The cross-subsidizing serial method satisfies Strong Ranking, Weak Separability, and Group Monotonicity. It fails Solidarity and Strong Group Monotonicity.

We omit the easy proof of these two statements. The violation of Strong Group Monotonicity is a consequence of a general incompatibility recorded in the next proposition; see the Appendix for a proof.

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[8] See Moulin and Sprumont (2002) for a characterization based on the Distributivity axiom mentioned at the end of Section 4.

Proposition 7. If $n \geq 3$, no cost-sharing method satisfies Additivity, Strong Ranking, Weak Separability, and Strong Group Monotonicity.

We conjecture that a similar incompatibility holds when Strong Group Monotonicity is replaced with Solidarity.

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8. Appendix

8.1. Shared-flow representation of additive methods. Define a sharing rule for $x \in \mathbb{N}^N$ to be a mapping $s(.,.,x) : [0,x[\times N \to \mathbb{R}^N$. The vector $r(z,j,x)$ specifies how the flow $f_j(z,x)$ between z and $z+e^j$ is shared among the agents. Choosing for each demand profile x a flow $f(.,x)$ and a sharing rule $r(.,.,x)$, the formula

$$\varphi(C,x) = \sum_{z\in[0,x[}\sum_{j\in N} \partial_j C(z) f_j(z,x) r(z,j,x) \qquad (7)$$

for all $C \in \mathcal{C}$ defines an additive cost-sharing method. Keep in mind that r is vector-valued while $\partial_j C$ and f_j are scalar-valued.

Conversely, Moulin and Vohra (2003) show that all additive methods possess at least one such shared-flow representation (f,r). Observe that these methods meet Independence of Irrelevant Costs.

8.2. Statement and proof of Theorem 1*. Theorem 1* and Lemma 2 below constitute a more general characterization result than Theorem 1, in which we drop the Strong Ranking assumption. Denote by $\mathbb{R}_+$ and $\mathbb{R}_{++}$ the sets of nonnegative and strictly positive real numbers, respectively. For each $i \in N$, choose a function $s_i : \mathbb{N} \to \mathbb{R}_+$ such that $s_i(x_i) > 0$ whenever $x_i > 0$. If $x > 0$, write $F_i(x) = \frac{s_i(x_i)}{\sum_N s_j(x_j)}$ and $F(x) = (F_1(x), ..., F_n(x))$ Next, choose a function μ_i assigning a real number to every pair $(x_i, t_i) \in (\mathbb{N}\setminus\{0\})^2$ such that $t_i \leq x_i$. For every cost-sharing problem $(C, x) \in \mathcal{C} \times \mathbb{N}^N$ and each $i \in N$, define

$$sa^{\mu_i}(C, x_i) = \sum_{t_i=1}^{x_i} \mu_i(x_i, t_i)\partial_i C((t_i - 1)e^i), \tag{8}$$

which may be viewed as a generalized stand-alone cost for agent i. The formula

$$\varphi_i^{s,\mu}(C, x) = sa^{\mu_i}(C, x_i) + F_i(x)\Big(C(x) - \sum_{j \in N} sa^{\mu_j}(C, x_j)\Big) \tag{9}$$

defines an additive cost-sharing method provided cost shares are always non negative. Setting $\alpha_j^+(x_j) = \max\{0, \mu_j(x_j, t_j) \mid t_j \in [1, x_j]\}$ and $\alpha_j^-(x_j) = \min\{0, \mu_j(x_j, t_j) \mid t_j \in [1, x_j]\}$, this amounts to imposing the following restrictions on f_j, μ_j for all i and x:

$$F_i(x)\Big(1 - \sum_{N\setminus i} \alpha_j^+(x_j)\Big) + (1 - F_i(x))\alpha_i^-(x_i) \geq 0. \tag{10}$$

This claim follows easily from the inequalities $\alpha_i^-(x_i)C(x_i, 0_{-i}) \leq sa^{\mu_i}(C, x_i) \leq \alpha_i^+(x_i)C(x_i, 0_{-i})$; we omit the straightforward details. By construction, we have

Lemma 2. Every method $\varphi^{s,\mu}$ constructed as in (9) and (10) satisfies Solidarity. If $s_i(0) > 0$ for all $i \in N$, $\varphi^{s,\mu}$ also satisfies Strict Solidarity.

Monotonicity imposes further restrictions on s, μ. One of them is that each s_i is nondecreasing (to see this, consider in (9) a function C such that $C(x_i, 0_{-i}) = 0$ for all i). Necessary and sufficient conditions seem difficult to formulate in a simple way; in particular, the inequalities $\mu_i \geq 0$ are not implied. In the particular case where μ_i is independent of both t_i and x_i, however, one checks that $\varphi^{s,\mu}$ meets Monotonicity if and only if s_i is nondecreasing and $\mu_i \geq 0$ for all i. Then the inequalities (10) reduce to $\sum_{j \in N\setminus i} \mu_j \leq 1$ for all i. Adding the requirement of symmetry, we obtain the family of methods

$$\varphi_i(C, x) = \frac{\lambda}{n-1} C(x_i, 0_{-i}) + \frac{s(x_i)}{\sum_N s(x_j)}\Big(C(x) - \sum_{j \in N} \frac{\lambda}{n-1} C(x_j, 0_{-j})\Big)$$

where $0 \le \lambda \le 1$, and $s : \mathbb{N} \to \mathbb{R}_+$. When $s(x_i) = 1$ for all x_i, this formula reduces to (4) in Section 3.

Theorem 1*.

i) Assume $n \ge 4$ and let φ be a cost-sharing method satisfying Additivity, Strict Solidarity, and Monotonicity. For all $i \in N$, there exists a nondecreasing function $s_i : \mathbb{N} \to \mathbb{R}_{++}$ and a real-valued function μ_i such that $\varphi = \varphi^{s,\mu}$, where $\varphi^{s,\mu}$ is defined in (9).

ii) Assume $n \ge 3$ and let φ be a cost-sharing method satisfying Additivity, Solidarity, Monotonicity, Zero Cost for Zero Demand, and Positive Cost for Positive Demand. For all $i \in N$ there exists a nondecreasing function $s_i : \mathbb{N} \to \mathbb{R}_+$ with $s_i(0) = 0 < s_i(1)$, and a real-valued function μ_i such that $\varphi = \varphi^{s,\mu}$.

Proof of Theorem 1*, statement i)

We fix throughout the proof a method φ meeting Additivity, Strict Solidarity, and Monotonicity. Recall that $\mathcal{D}$ is the subset of cost functions D such that $D(z) = 0, 1$. Writing $\partial D = \{z \in D \mid z - e^i \notin D \text{ for all } i \in N(z)\}$, Independence of Irrelevant Costs yields $\varphi(D, x) = \varphi(D', x)$ whenever $D, D' \in \mathcal{D}$ and $x \in \partial D \cap \partial D'$. We may therefore define $\theta : \mathbb{N}^N \to \Delta$ by

$$\theta(x) = \varphi(D, x),$$

where D is any cost function in $\mathcal{D}$ such that $x \in \partial D$.

We spend Steps 1 to 5 computing $\varphi(D, x)$ for $D \in \mathcal{D}$. After the preliminary Steps 1 to 3, we construct the desired functions s_i, $i = 1, ..n$, and derive the quasi-proportional form $\theta_i(x) = F_i(x)$ in Step 4. Step 5 computes $\varphi(D, x)$ for any D, x. Using Additivity, Step 6 then derives $\varphi(C, x)$ for all $C \in \mathcal{C}$.

We use the following notation. If $D \in \mathcal{D}$ and $x \in \mathbb{N}^N$, $t_i(D, x_{-i}) = \inf\{x_i \mid (x_i, x_{-i}) \in D\}$, with the convention $t_i = +\infty$ if this set is empty. For any $S \subseteq N$, $\Delta(S) = \{y \in \mathbb{R}^S_+ \mid \sum_{i \in S} y_i = 1\}$.

Step 1. We show that if $D \in \mathcal{D}$, $x \in D$ and there exist two distinct agents $i, j \in N$ such that $t_i(D, x_{-i}) > 0$ and $t_j(D, x_{-j}) > 0$, then $\varphi(D, x) \gg 0$. It follows that for any $x \in \mathbb{N}^N$, $\{n(x) \ge 2\} \Rightarrow \{\theta(x) \gg 0\}$.

To prove the first statement, assume its premises and consider the increase in i's demand from 0 to x_i. By Strict Solidarity, $\varphi_{-i}(D, x) - \varphi_{-i}(D, (x_{-i}, 0_i))$ is either strictly positive or zero. In the latter case, $\varphi(D, x) = e^i$, and in the former case $\varphi_{-i}(D, x) \gg 0$. Repeating this argument with agent j, either $\varphi(D, x) = e^j$ or $\varphi_{-j}(D, x) \gg 0$. The only possibility is therefore that $\varphi_{-i}(D, x) \gg 0$ and $\varphi_{-j}(D, x) \gg 0$, hence $\varphi(D, x) \gg 0$. The second statement follows from the first because $t_i(D, x_{-i}) = x_i$ for all i whenever $x \in \partial D$.

Step 2. We show that for any $D \in \mathcal{D}$, $i \in N$, $x \in D$ such that $x_{-i} \neq 0$, and $x' = (x'_i, x_{-i})$ such that $x'_i > x_i$,

$$\varphi_{-i}(D,x') - \varphi_{-i}(D,x) = -\lambda\theta_{-i}(x') \text{ for some } \lambda \geq 0. \tag{11}$$

Monotonicity implies $\varphi_i(D,x') - \varphi_i(D,x) \geq 0$, whereas $\sum_N (\varphi_j(D,x') - \varphi_j(D,x)) = 0$ by budget balance. Thus, by Solidarity,

$$\varphi_j(D,x') - \varphi_j(D,x) \leq 0 \text{ for all } j \neq i. \tag{12}$$

Choose $D' \in \mathcal{D}$ such that $x' \in \partial D'$ and let $\alpha \in \mathbb{R}_{++}$. Applying Strict Solidarity to $\alpha D + D'$ when x_i increases to x'_i, we obtain that $sign\{\alpha(\varphi_j(D,x') - \varphi_j(D,x)) + (\varphi_j(D',x') - \varphi_j(D',x))\}$ is independent of $j, j \neq i$, where $sign(u)$ is $1, 0$, or -1 if u is respectively positive, zero or negative. From Step 1, $\varphi_j(D',x') - \varphi_j(D',x) = \theta_j(x') > 0$ because $n(x') \geq 2$. On the other hand $\varphi_j(D,x') - \varphi_j(D,x) = u_j \leq 0$. By Strict Solidarity, $u_{-i} \ll 0$ or $u_{-i} = 0$. In the latter case, (11) holds with $\lambda = 0$. In the former, the vector $\alpha u_{-i} + \theta_{-i}(x')$ is either strictly positive, zero, or strictly negative. Since this holds for all $\alpha > 0$, u_{-i} and $\theta_{-i}(x')$ are parallel and of opposite sign, and (11) holds with $\lambda > 0$.

Step 3. We show that for any two distinct $i, j \in N$ and any $x_{-ij} \neq 0$, the direction of $\theta_{-ij}(y_{ij}, x_{-ij})$ is independent of $y_{ij} \in \mathbb{N}^2\setminus\{0\}$. That is, there exists $\rho(x_{-ij}) \in \Delta(N\setminus ij)$ such that, for all $y_{ij} \in \mathbb{N}^2\setminus\{0\}$, $\theta_{-ij}(y_{ij}, x_{-ij}) = \lambda\rho(x_{-ij})$ for some $\lambda > 0$.

Take $i = 1$, $j = 2$, fix $x_{-12} \neq 0$, and write $\theta_{-12}(y_{12}, x_{-12}) = s(y) \in \mathbb{R}_+^{N\setminus 12}$, where $y = (y_1, y_2) \in \mathbb{N}^2\setminus\{0\}$. From now on, we do not repeat the restriction $y \neq 0$. By Step 1, $s(y) \gg 0$.

Pick y, y' non-comparable (i.e., neither $y \leq y'$ nor $y' \leq y$ holds) and D such that ∂D contains y and y'. Applying (11) twice, respectively to the increase from y to $y \vee y'$, and from y' to $y \vee y'$,

$$\begin{aligned} \varphi_{-12}(D, y \vee y') - s(y) &= -\lambda s(y \vee y'), \\ \varphi_{-12}(D, y \vee y') - s(y') &= -\lambda' s(y \vee y'). \end{aligned}$$

Taking the difference of these equalities,

$$s(y) - s(y') = \mu s(y \vee y') \text{ for some } \mu \in \mathbb{R}. \tag{13}$$

This property holds for any non-comparable y, y'.

Recall that our goal is to show that the direction of $s(y)$ is independent of y. To this end, assume first that there exist y, y' with $y_2 = y'_2 > 0$, $y_1 < y'_1$ and $s(y) \neq s(y')$.

Construct z, z' such that $z_1 = z'_1 > y'_1$ and $z_2 < z'_2 = y'_2$ (see Figure 1). Two applications of (13) give

$$\begin{aligned} s(z) - s(y) &= \lambda s(z'), \\ s(z) - s(y') &= \lambda' s(z'). \end{aligned}$$

Taking differences we find that $s(z')$ and $s(y) - s(y')$ have the same direction $\rho, \rho \gg 0$. Let L be the straight line borne by ρ and containing $s(y)$. It contains $s(z)$ and $s(y')$.

Now consider w, w' such that $w_1 = w'_1 > z'_1$ and $w_2 < w'_2 = z'_2$ (see Figure 1). By the above argument, $s(w')$ is borne by ρ, and L contains $s(w)$. Applying (13) to z', w gives

$$s(z') - s(w) = \mu s(w').$$

Thus $s(z') \in L$, implying that L is the line borne by ρ through 0. Now s is the direction of $s(t)$ for the six points t of our construction.

We check next that $s(t)$ is borne by ρ for any t such that $t_2 \leq y_2$ (and $t \neq 0$). We just proved this under the assumption that $t_1 > y'_1$. Next suppose $t_1 \leq y'_1$ and $0 < t_2 \leq y_2$. Set $z_0 = (y'_1 + 1, 0)$ and $t' = t \vee z_0$ (see Figure 2). By (13) applied to t, z_0, $s(t)$ is borne by ρ. The remaining values of t are $t = (t_1, 0)$ with $0 < t_1 \leq y'_1$. There we apply (13) to t and $t' = (t_1 - 1, 1)$.

Summing up, we have shown that if s is not constant on a given horizontal line $\{y \mid y_2 = \varepsilon > 0\}$, then the direction of $s(y)$ is constant on the whole band $\{y \mid y_2 \leq \varepsilon; y \neq 0\}$. If there are such integers ε as large as we want, this establishes the desired statement at once. The remaining case is when there is a number ε such that s is constant on every line $\{y \mid y_2 = \varepsilon'\}$, $\varepsilon' \geq \varepsilon$. For any y such that $y_1 > 0$, apply (13) to y and $y' = (y_1 - 1, \varepsilon')$ where $\varepsilon' \geq \max\{y_2 + 1, \varepsilon\}$: we deduce that $s(y)$ is borne by the value of s on $\{y \mid y_2 = \varepsilon'\}$. The remaining case $y_1 = 0$ is handled similarly.

Step 4. We show that there exist n functions $s_i : \mathbb{N} \to \mathbb{R}_{++}$ such that, for all $x \in \mathbb{N}^N$, $\{n(x) \geq 2\} \Rightarrow \{\theta(x) = F(x)\}$.

We begin by considering the demand profiles x such that $n(x) = n$. In particular, $\theta(x) \gg 0$. By Step 3, the ratio $\frac{\theta_i}{\theta_j}(x)$ is independent of x_k for all $k \neq i, j$ (recall $n \geq 4$). For i, j, k all distinct,

$$\frac{\theta_i}{\theta_j}(x) = s_{ij}(x_i, x_j) \Rightarrow s_{ij}(x_i, x_j) s_{jk}(x_j, x_k) s_{ki}(x_k, x_i) = 1.$$

A standard argument (omitted for brevity), shows the existence of n functions s_i on $\mathbb{N} \setminus \{0\}$ such that $\frac{\theta_i}{\theta_j}(x) = \frac{s_i(x_i)}{s_j(x_j)}$ for all i, j, all x. Thus the desired form $\theta(x) = F(x)$ holds whenever $n(x) = n$.

Next we treat the case $n(x) = n - 1$, defining $s_i(0)$ in the process. Consider some x with $N(x) = N \setminus 1$. For all $i, j \geq 2$, we have

$$\frac{\theta_i}{\theta_j}(x) = \frac{\theta_i}{\theta_j}(1, x_{-1}) = \frac{s_i(x_i)}{s_j(x_j)},$$

where the first equality follows from Step 3. By Step 3 again, $\frac{\theta_1}{\theta_2}(x)$ does not depend upon x_j, for any $j \neq 1, 2$, provided x_j remains positive:

$$\frac{\theta_1}{\theta_2}(x) = \frac{\theta_1}{\theta_2}(0, x_2, 1, \ldots 1) =: \frac{1}{g_2(x_2)}.$$

Defining g_i for $i = 3, \ldots, n$ similarly and combining the two properties above,

$$\frac{s_i(x_i)}{s_j(x_j)} = \frac{g_i(x_i)}{g_j(x_j)} \text{ for all } i, j \neq 1 \text{ and all } x_{-1} \gg 0.$$

Thus $\frac{s_i}{g_i}$ depends neither on i nor on x_i. Calling this ratio $s_1(0)$ we now have $\frac{\theta_i}{\theta_j}(x) = \frac{s_i(x_i)}{s_j(x_j)}$ for all i, j. A similar construction delivers $s_i(0)$ for all i, and proves $\theta(x) = F(x)$ when $n(x) = n - 1$.

We complete the argument by decreasing induction on $n(x)$. Fix q, $2 \leq q \leq n - 2$. Assume $\theta(x) = F(x)$ whenever $n(x) \geq q + 1$ and consider x with $n(x) = q$, say $N(x) = \{1, \ldots, q\}$. For any distinct $i, j \in N \setminus \{1, n\}$, Step 3 and the inductive assumption imply

$$\frac{\theta_i}{\theta_j}(x) = \frac{\theta_i}{\theta_j}(x_{-n}, 1_n) = \frac{s_i(x_i)}{s_j(x_j)}.$$

Note that $q \geq 2$ ensures $x_{-i,j} \neq 0$, as required to apply Step 3. Moreover, $q \leq n - 2$ guarantees that we have at least two choices for agent 1 in $N(x)$, ditto for agent n in $N \setminus N(x)$. The equality $\theta(x) = F(x)$ is now clear.

Step 5. We derive an explicit formula for $\varphi(D, x)$ for all $D \in \mathcal{D}$ and $x \in D$. Using the notation $\tau_i(D) = t_i(D, 0_{-i})$, note that $x_i e^i \in D \iff \tau_i(D) \leq x_i$ and define $K(D, x) = \{i \in N \mid \tau_i(D) \leq x_i\}$. Define $H : \mathbb{R}_+^N \setminus \{0\} \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$H_i(a, b) = \frac{a_i}{\sum_N a_j}\Big(1 + \sum_{N \setminus i} (b_i - b_j) a_j\Big). \tag{14}$$

We show that there exist n functions λ_i such that, for all $D \in \mathcal{D}$ and $x \in D$,

$$\begin{aligned} \varphi(D, x) &= H(a, b), \text{ where for all } i, \qquad (15)\\ a_i &= s_i(x_i), \\ b_i &= \lambda_i(x_i, \tau_i(D)) \text{ if } i \in K(D, x), \\ &\quad\ 0 \text{ otherwise.} \end{aligned}$$

In Step 5.a, we prove (15) when $K(D,x) = \emptyset$. In Step 5.b, we construct the functions λ_i and prove (15) when $|K(D,x)| = 1$. Step 5.c completes the proof by an induction argument on $|K(D,x)|$.

Two simple facts about H will be useful. First $\sum_N H_i(a,b) = 1$. Next, for any $z \in \mathbb{R}^N$,

$$\{\sum_N z_i = 1 \text{ and } \frac{z_i}{a_i} - \frac{z_j}{a_j} = b_i - b_j \text{ for all } i,j \text{ such that } a_i, a_j \neq 0\} \quad (16)$$
$$\Rightarrow \{z = H(a,b)\}.$$

We will use the notation $x^{-i} = (0_i, x_{-i},)$.

Step 5.a. When $K(D,x) = \emptyset$, (15) reduces to $\varphi(D,x) = F(x)$. We prove this equality by induction on $n(x)$.

The smallest possible size is $n(x) = 2$: if $x = x_1 e^1$, say, then $x \in D$ forces $\tau_1(D) \leq x_1$. So we fix x with $N(x) = \{1,2\}$. Define $x_* = x + e^3$ and choose D^* such that

$$D^* \cap [0,x] = D \cap [0,x] \text{ and } x_*^{-i} \in \partial D^* \text{ for } i = 1,2.$$

This is possible because $x_*^{-1} - e^3 = x^{-1} = x_2 e^2 \notin D$ and similarly $x_*^{-2} - e^3 \notin D$.

By construction of D^* and Step 4, $\varphi(D^*, x_*^{-1}) = \theta(x_*^{-1}) = F(x_*^{-1})$. Applying (11) to D^*, x_* and x_*^{-1},

$$\varphi_{-1}(D^*, x_*) - F_{-1}(x_*^{-1}) = -\lambda F_{-1}(x_*) \Rightarrow \varphi_{-1}(D^*, x_*) = \mu F_{-1}(x_*).$$

This, the symmetric property upon exchanging 1 and 2, and the fact that $F(x_*) \gg 0$, give finally $\varphi(D^*, x_*) = F(x_*)$.

Next we apply (11) to D^*, x_* and $x_*^{-3} = x$:

$$\varphi_{-3}(D^*, x_*) - \varphi_{-3}(D^*, x) = -\lambda F_{-3}(x_*) = -\lambda' F_{-3}(x).$$

Independence of Irrelevant Costs implies $\varphi(D^*, x) = \varphi(D,x)$, so we conclude $\varphi_{-3}(D,x) = \mu F_{-3}(x)$ for some $\mu \geq 0$. Repeating this construction with coordinate 4 instead of 3 yields $\varphi_{-4}(D,x) = \nu F_{-4}(x)$ for some $\nu \geq 0$. The conclusion $\varphi(D,x) = F(x)$ follows as in the previous paragraph.

Now the induction argument. Fix D,x such that $K(D,x) = \emptyset$. Suppose $t_i(D,x_{-i}) = 0 \Leftrightarrow x^{-i} \in D$ for at least two agents, say $i = 1,2$. Because $K(D,x^{-i}) = \emptyset$ for $i = 1,2$, the inductive assumption yields $\varphi(D,x^{-i}) = F(x^{-i})$ and, by (11), $\varphi_{-i}(D,x) = \lambda F_{-i}(x)$ for $i = 1,2$. This implies $\varphi(D,x) = F(x)$ as above.

If $t_i(D,x_{-i}) = 0$ for at most one i, then we can assume $t_j(D,x_{-j}) > 0$ for $j = 1,2$. We now construct $x_* = x + e^3$ and D^* exactly as above. This is possible because

$t_1(D, x_{-1}) > 0$ means $x^{-1} \notin D$, therefore $x_*^{-1} - e^3 \notin D$, and similarly $x_*^{-2} - e^3 \notin D$. The same argument ends the proof.

Step 5.b. We prove (15) when $|K(D, x)| = 1$.

In this and the next substep, we use one more piece of notation: given $z, z' \in \mathbb{R}^N$, $a \in \mathbb{R}^N_{++}$ and two distinct $i, j \in N$,

$$z \sim^a_{ij} z' \Leftrightarrow \frac{z_i}{a_i} - \frac{z_j}{a_j} = \frac{z'_i}{a_i} - \frac{z'_j}{a_j}.$$

We start by choosing x with $n(x) = 3$, say, $N(x) = \{1, 2, 3\}$ and D such that $K(D, x) = \{1\}$ and $x^{-1} \in D$. We have drawn on Figure 3 the traces of D on the three coordinate hyperplanes. We write $z = \varphi(D, x)$ and $z^i = \varphi(D, x^{-i})$, $a_i = s_i(x_i)$ for all i.

Applying (11) to x, x^{-1} gives $z_{-1} - z^1_{-1} = -\lambda F_{-1}(x)$, hence, by Step 5.a,

$$\frac{z_i}{a_i} = \frac{z_j}{a_j} \text{ for all } i, j \neq 1. \tag{17}$$

Similarly, (11) applied to x, x^{-2} and x, x^{-3} gives

$$z \sim^a_{ij} z^2 \text{ for all } i, j \neq 2, \tag{18}$$

$$z \sim^a_{ij} z^3 \text{ for all } i, j \neq 3. \tag{19}$$

These three properties imply

$$\frac{z^2_1}{a_1} - \frac{z^2_3}{a_3} = \frac{z_1}{a_1} - \frac{z_3}{a_3} = \frac{z_1}{a_1} - \frac{z_2}{a_2} = \frac{z^3_1}{a_1} - \frac{z^3_2}{a_2}. \tag{20}$$

By Independence of Irrelevant Costs, $z^2 = \varphi(D, x^{-2})$ only depends upon x^{-2} and the trace of D on $[0, x^{-2}]$. Similarly z^3 depends only upon x^{-3} and the trace of D on $[0, x^{-3}]$. Thus the common quantity in (20) only depends on x_1 and the trace of D on $[0, x_1 e^1]$, namely $\tau_1(D)$. We write this quantity as $\lambda_1(x_1, \tau_1(D))$ or λ_1 for simplicity in the rest of this step.

Combining (17), (20), we get

$$\frac{z_1}{a_1} - \frac{z_i}{a_i} = \lambda_1 \text{ for all } i \neq 1,$$

and by (16),

$$z = H(a, (\lambda_1, 0_{-1})),$$

which is precisely (15) for (D, x).

Next we derive a similar formula for z^2 and for $w^1 = \varphi(D, x_1e^1)$. Applying (11) to x^{-2}, x_1e^1, we obtain

$$z^2 \sim^{a^2}_{ij} w^1 \text{ for all } i, j \neq 3, \text{ with } a^2_2 = s_2(0),\ a^2_i = a_i,\ i \neq 2. \tag{21}$$

Combining (21) and (18), and the fact that $\sim^a_{ij}$ is an equivalence relation, we get for all $i \geq 4$,

$$w^1 \sim^{a^2}_{1i} z^2 \sim^a_{1i} z \Rightarrow w^1 \sim^a_{1i} z \Leftrightarrow \frac{w^1_1}{a_1} - \frac{w^1_i}{a_i} = \lambda_1$$

where $a_i = s_i(0)$. Now w^1 only depends on x_1 and $\tau_1(D)$, as does λ_1. If we repeat the construction of Step 5.b while exchanging the roles of 2 and 4, i.e., for x' such that $N(x') = \{1, 3, 4\}$ and D' such that $K(D', x') = \{1\}$ and $x'^{-1} \in D$, we get

$$\frac{w^1_1}{a_1} - \frac{w^1_2}{s_2(0)} = \lambda_1.$$

Combining these properties of w^1, including a similar equality where 3 replaces 2, and (16), we conclude

$$w^1 = H(a^1, (\lambda_1, 0_{-1})),$$

where $a^1_1 = s_1(x_1)$ and $a^1_i = s_i(0)$ for $i \geq 2$: this is precisely (15) for $(D,\ x_1e^1)$. As $\varphi(D,\ x_1e^1)$ only depends on x_1 and $\tau_1(D)$, this establishes (15) for any (D, x) where $n(x) = 1$.

Next we compute z^2. From (18) and (21), we obtain respectively

$$\frac{z^2_1}{a_1} - \frac{z^2_2}{s_2(0)} = \lambda_1,$$
$$\frac{z^2_1}{a_1} - \frac{z^2_i}{a_i} = \lambda_1 \text{ for all } i \geq 3.$$

Thus (16) implies $z^2 = H(a^2, (\lambda_1, 0_{-1}))$. Again, $\varphi(D, x^{-2})$ only depends upon x^{-2} and the restriction of D to $[0, x^{-2}]$. Our choice of D in Step 5.b places no constraint on the restriction of D to $[0, x^{-2}]$, except $K(D, x^{-2}) = \{1\}$. Therefore we have proven (15) for any (D, x) where $n(x) = 2$ and $|K(D, x)| = 1$.

To complete the proof of (15) for any (D, x) such that $n(x) = 3$ and $|K(D, x)| = 1$, it only remains to take care of the case where $x^{-1} \notin D$ (recall we assumed $x^{-1} \in D$ instead at the beginning of Step 5.b). Consider such a configuration illustrated in Figure 3. We already know that $z^2 = \varphi(D, x^{-2})$ and $z^3 = \varphi(D, x^{-3})$ are given by (15). Applying (11) to $z = \varphi(D, x)$, z^2 and to z, z^3 delivers respectively the direction of z_{-2} and of z_{-3}, hence determines z by (16). We omit the details.

To complete Step 5.b, it remains to establish formula (15) when $|K(D,x)| = 1$ and $n(x) > 3$. We proceed by induction on $n(x)$. The argument is essentially the same as above, once we observe that $K(D,x) = \{1\}$ implies that for all $i \neq 1$, $x_i e^i \notin D$, hence $t_i(D, x_{-i}) = 0$.

Assume the desired conclusion holds whenever $n(x) \leq t - 1$ and let (D,x) be such that $K(D,x) = \{1\}$ and $n(x) = q$. Without loss of generality, assume $N(x)$ contains $2, 3$. As before (see (21)) we define a by $a_i = s_i(x_i)$ for all i, and a^i by $a^i_i = s_i(0)$ and $a^i_j = a_j$ for all $j \neq i$. Clearly $K(D, x^{-2}) = \{1\}$, so the inductive assumption gives

$$z^2 = \varphi(D, x^{-2}) = H(a^2, (\lambda_1, 0_{-1})).$$

Applying (11) to $z = \varphi(D,x)$ and z^2 gives

$$\frac{z_1}{a_1} - \frac{z_i}{a_i} = \lambda_1 = \frac{z_1^2}{a_1} - \frac{z_i^2}{a_i} \text{ for all } i \geq 3.$$

The symmetrical argument involving z and z^3 gives $\frac{z_1}{a_1} - \frac{z_2}{a_2} = \lambda_1$ and the proof of Step 5.b is complete.

Step 5.c. We prove (15) by induction on $|K(D,x)|$.

Fix $k \geq 2$ and assume (15) holds whenever $|K(D,x)| \leq k - 1$. Consider (D,x) with $|K(D,x)| = k$, say, $K(D,x) = \{1, ..., k\}$. Define $z = \varphi(D,x)$, $z^i = \varphi(D, x^{-i})$, a, and a^i exactly as in Step 5.b. Clearly $K(D, x^{-i}) = \{1, ..., k\} \setminus \{i\}$ for $i \leq k$, therefore the inductive assumption implies

$$z^i = H(a^i, (b_{-i}, 0_i)) \text{ for } i = 1, ..., k,$$

where b is defined by $b_j = \lambda_j(x_j, \tau_j(D))$ if $j \leq k$, and $b_j = 0$ otherwise. Applying (11) successively to z, z^2 and z, z^3 gives, for all $i \geq 3$,

$$z \sim^a_{1i} z^2 \Rightarrow \frac{z_1}{a_1} - \frac{z_i}{a_i} = b_1 - b_i,$$

and

$$z \sim^a_{12} z^3 \Rightarrow \frac{z_1}{a_1} - \frac{z_2}{a_2} = b_1 - b_2,$$

and (16) then yields $z = H(a,b)$, as desired.

Step 6. We construct the functions μ_i, $i \in N$, and complete the proof.

Rewrite (14) as

$$H_i(a,b) = \frac{a_i}{\sum_N a_j}\left(1 - \sum_N b_j a_j\right) + a_i b_i.$$

Define μ_i, $i \in N$, by

$$\mu_i(x_i, t_i) = \begin{cases} s_i(x_i)\lambda_i(x_i, t_i) \text{ if } t_i \leq x_i, \\ 0 \text{ otherwise.} \end{cases}$$

Recalling our convention $\tau_i(D) = +\infty$ if D does not intersect the i-axis, formula (15) can be rewritten as follows. For all i, x, D such that $x \in D$,

$$\varphi_i(D, x) = \mu_i(x_i, \tau_i(D)) + F_i(x)(1 - \sum_N \mu_j(x_j, \tau_j(D))).$$

This formula coincides with (9) because (8) reduces to $sa^{\mu_i}(D, x_i) = \mu_i(x_i, \tau_i(D))$ when $D \in \mathcal{D}$. An additive method is entirely determined by its behavior on such pairs (D, x), thus the proof is complete. ■

Proof of Theorem 1*, statement ii)

We fix throughout an additive method φ meeting the four axioms in the statement, denoted for brevity SOL, MON, ZCZD and PCPD. The structure of the proof , and the numbering of the corresponding steps and substeps, are identical to those of the proof of statement i): we compute $\theta(x)$ first, thus constructing F, then $\varphi(D, x)$ by constructing μ Some of the steps are considerably shorter, however.

Step 1. ZCZD and PCPD imply at once for all x, D,

$$x \in D \Rightarrow \{\varphi_i(D, x) > 0 \Leftrightarrow i \in N(x)\}$$

and in particular $\theta_i(x) > 0 \Leftrightarrow x_i > 0$.

Step 2. We show here a stronger property than (11), namely for all x, D, i such that $x_i > 0$ and $x_{-i} \neq 0$,

$$x^{-i} \in D \Rightarrow \varphi_{-i}(D, x) - \varphi_{-i}(D, x^{-i}) = -\lambda\theta_{-i}(x) \text{ for some } \lambda > 0, \tag{22}$$

$$x^{-i} \notin D \Rightarrow \varphi_{-i}(D, x) = \lambda\theta_{-i}(x) \text{ for some } \lambda > 0. \tag{23}$$

Note that by taking the difference of (22) for x_i and x'_i, we get (11).

To prove (22) and (23), we note that $\varphi_i(D, x) < \varphi_i(D, x^{-i})$ (by ZCZD and PCPD), hence $\sum_{N \setminus i} \varphi_j(D, x) > \sum_{N \setminus i} \varphi_j(D, x^{-i})$. By SOL, $\varphi_j(D, x) \geq \varphi_j(D, x^{-i})$ for each $j \in N \setminus i$. Now suppose that for some $j, k \in N(x) \setminus i$ we have $\varphi_j(x) < \varphi_j(x^{-i})$ whereas $\varphi_k(x) = \varphi_k(x^{-i})$. Pick D' containing x but not x' and apply SOL to $\alpha D + D'$ between x^{-i} and x :

$$\{\alpha(\varphi_j(D,x) - \varphi_j(D,x^{-i})) + \varphi_j(D',x)\}\varphi_k(D',x) \geq 0,$$

which contradicts PCPD for α large enough. Thus $\varphi_j(D,x) - \varphi_j(D,x^{-i}) < 0$, $j \in N(x) \setminus i$, and for all $\alpha > 0$, all $j,k \in N(x) \setminus \{i\}$,

$$\{\alpha(\varphi_j(D,x) - \varphi_j(D,x^{-i})) + \varphi_j(D',x)\}\{\alpha(\varphi_k(D,x) - \varphi_k(D,x^{-i})) + \varphi_k(D',x)\} \geq 0,$$

implying $\varphi_{-i}(D,x) - \varphi_{-i}(D,x^{-i}) = -\nu\varphi_{-i}(D',x)$ for some $\nu > 0$. If we choose D' so that $x \in \partial D'$ we get (22), and (23) follows.

Step 3. The direction of the vector $\theta_{-i}(x)$ is independent of $x_i \in \mathbb{N} \setminus \{0\}$, when $x_{-i} \neq 0$.

Indeed, pick x_i, x_i' such that $0 < x_i < x_i'$, and D such that $(x_i, x_{-i}) \in \partial D$. By (23), $\varphi_{-i}(D,(x_i',x_{-i})) = \lambda\theta_{-i}((x_i',x_{-i}))$ for some $\lambda > 0$ and by (11) $\varphi_{-i}(D,(x_i',x_{-i})) - \varphi_{-i}(D,(x_i,x_{-i})) = \varphi_{-i}(D,(x_i',x_{-i})) - \theta_{-i}((x_i,x_{-i})) = -\mu\theta_{-i}(x)$ for some $\mu \geq 0$.

Step 4. We construct the nondecreasing functions s_i such that $s_i(0) = 0 < s_i(1)$ and $\theta(x) = F(x)$ for all x such that $n(x) \geq 3$.

Setting $N_0 = N(x)$ and $\theta_0(x)$ to be the projection of $\theta(x)$ on $\mathbb{R}^{N_0}$, we note that $\theta_0(x)$ is strictly positive and in $\Delta(N_0)$. By Step 3, $\frac{\theta_{0i}}{\theta_{0j}}$ depends on x_i, x_j only, and a standard argument gives then the functions s_i on $\mathbb{N} \setminus \{0\}$ such that $\theta_0(x) = F(x)$ for all x such that $N(x) = N_0$. The argument is essentially the same as in Step 4 of the previous proof, except that we only need N_0 to contain 3 or more agents. This is why we only need to assume $n \geq 3$ in statement ii) of Theorem 1*.

In view of Step1, it is now routine to show that the functions s_i do not depend on N_0, and to extend the equality $\theta(x) = F(x)$ to all coordinates by setting $s_i(0) = 0$.

Step 5. We show, as in the previous proof, the existence of functions λ_i allowing the representation of $\varphi(D,x)$ by (15).

Step 5.a. For any (D,x) with $x \in D$ and $K(D,x) = \emptyset$, $\varphi(D,x) = F(x)$.

Prove this first for x such that $N(x) = \{1,2\}$. Set $x_* = x + e^3$, pick D^* such that $D^* = D$ on $[0,x]$ and $x_*^{-1}, x_*^{-2} \notin D^*$. By (23) $\varphi_{-i}(D^*,x_*)$ is borne by $F_{-i}(x_*)$ for $i = 1,2$. In view of Step 4 this implies $\varphi(D^*,x_*) = F(x_*)$. Invoke next (22): $\varphi_{-3}(D^*,x_*) - \varphi_{-3}(D^*,x) = -\lambda F_{-3}(x_*)$. Therefore $\frac{s_1(x_1)}{s_2(x_2)} = \frac{\varphi_1}{\varphi_2}(D_*,x) = \frac{\varphi_1}{\varphi_2}(D,x)$, where the latter equality follows from Independence of Irrelevant Costs.

The ascending induction on $n(x)$ is now immediate by (22) and Step 1.

Step 5.b. We assume now $|K(D,x)| = 1$, and of course $x \in D$. If $n(x) = 1$ there is nothing to prove (by ZCZD), contrary to the previous proof. If $n(x) = 2$, we simply copy Step 5.b to define the functions λ_i and compute z, z^2, z^3. By Step 1 we need not worry about whether or not $x^{-1} \in D$. The ascending induction on $n(x)$ is unchanged.

Step 5.c. The ascending induction on $|K(D,x)|$ is unchanged.

Step 6. is unchanged. ■

8.3. Proof of Theorem 1. The "if" part of statements i) and ii) is clear. To prove the "only if" part in either statement, we assume that a method given by (9) satisfies Strong Ranking, and prove that μ_i, hence sa^{μ_i} is identically zero for all i.

We check first $s_i = s_j$ for all i, j. Pick $x = te^N, t \geq 1$, and $C, C(z) = 1$ if $n(z) \geq 2, C(z) = 0$ otherwise. Then $\varphi(C,x) = F(x)$, so Strong Ranking implies $s_i(t) = s_j(t)$ for $t \geq 1$. Under Zero Cost for Zero Demand, this is enough. On the other hand if $n \geq 4$, we choose $x' = te^{N \setminus \{1,2\}}$ and the same cost function C to get $s_1(0) = s_2(0)$, and we are done.

Assume now there is an agent i and x_1, t_1 such that $\mu_1(x_1, t_1) \neq 0$. Choose $C = \delta^{(t_1, 0_{-1})}$ as in (28) so that $sa^{\mu_i}(C, x_i) = 0$ for $i \geq 2$, $sa^{\mu_1}(C, x_1) = \mu_1(x_1, t_1)$. Then for $i \geq 2$,

$$\varphi_1(C, x_1 e^N) = \mu_1 + \frac{1}{n}(1 - \mu_1) \neq \frac{1}{n}(1 - \mu_1) = \varphi_i(C, x_1 e^N),$$

contradicting Strong Ranking. ■

8.4. Proof of Theorem 2. Step 1. i) implies ii) and iii). Let φ be a fixed-flow method. It is easy to check that φ satisfies Additivity, Separability and Monotonicity; we prove here that it also meets Submodular Solidarity and Supermodular Solidarity. Let $f(.,\overline{x})$ be the fixed-flow associated with φ and write $f(.,x) = p_x f(.,\overline{x})$ for all $x \in [0,\overline{x}]$. Observe that $p_x f(.,x+e^i) = f(.,x)$ for all i, all $x \in [\overline{x} - e^i]$. We fix i, j, $i \neq j$, $x \in [\overline{x} - e^i]$, and $z \in [x - e^j]$.

First we compare $f_j(z,x)$ and $f_j(z,x+e^i)$. If $z_i < x_i$, then K contains neither i nor j, hence $(x+e^i)_K = x_K$, implying $f_j(z,x) = f_j(z,x+e^i)$. If $z_i = x_i$, then K contains i but not j, hence $[x_K, (x+e^i)_K] = \{x_K, x_K + e^i_K\}$ so that $f_j(z,x) = f_j(z,x+e^i) + f_j(z+e^i, x+e^i)$.

Now we compute

$$\begin{aligned}
&\varphi_j(C, x+e^i) - \varphi_j(C,x) \\
&= \sum_{z\in[0,x]: z_i = x_i, z_j < x_j} \left(\partial_j C(z+e^i) f_j(z+e^i, x+e^i) + \partial_j C(z)(f_j(z,x+e^i) - f(z,x))\right) \\
&= \sum_{z\in[0,x]: z_i = x_i, z_j < x_j} (\partial_j C(z+e^i) - \partial_j C(z)) f_j(z+e^i, x+e^i),
\end{aligned}$$

implying Submodular Solidarity and Supermodular Solidarity at once.

Step 2. ii) implies i). Fix a method φ satisfying Additivity, Separability, Monotonicity, and Submodular Solidarity. Let f be its flow representation as in (5). We show $f(.,x) = p_x f(.,\overline{x})$ for any $x \in [0,\overline{x}]$.

Step 2.a. For all $i \in N$, $x \in [0,\overline{x} - e^i]$, and $z \in [0,x - e^i]$,

$$f_i(z,x) = f_i(z,x+e^i). \tag{24}$$

This is a well-known consequence of Monotonicity: see Moulin (1995) or Sprumont (2000) for a proof. Equation (24) does not imply the property $f_j(z,x) = f_j(z,x+e^i)$ for $j \neq i$ (except when $n = 2$, where it follows from flow conservation); that property is derived in the next step using Submodular Solidarity.

Step 2.b. For all $i \in N$, $x \in [0,\overline{x} - e^i]$, and $z \in [0,x - e^i]$,

$$f(z,x) = f(z,x+e^i). \tag{25}$$

The proof of (25) is by induction on $z(N)$, the sum of the coordinates of z. Let $z = 0$. By Step 2.a, $f_i(0,x) = f_i(0,x+e^i)$. If (25) fails, as total outflow from 0 is 1 under $f(.,x)$ and $f(.,x+e^i)$, there exist $j,k \in N \backslash i$ such that $f_j(0,x) < f_j(0,x+e^i)$ and $f_k(0,x) > f_k(0,x+e^i)$. Consider the cost function

$$\begin{aligned} \delta_0(w) &= 0 \text{ if } w = 0, \\ &\quad 1 \text{ otherwise.} \end{aligned}$$

Note that $\delta_0 \in C_{sub}$. But $\varphi_j(\delta_0,x) = f_j(0,x) < f_j(0,x+e^i) = \varphi_j(\delta_0,x+e^i)$ and $\varphi_k(\delta_0,x) = f_k(0,x) > f_k(0,x+e^i) = \varphi_k(\delta_0,x+e^i)$, contradicting Submodular Solidarity.

Next, fix $k > 0$ and assume that (25) is true for all $z \in [0,x - e^i[$ such that $z(N) \leq k - 1$. Fix $z \in [0,x - e^i[$ such that $z(N) = k$. By the induction hypothesis $f_j(z - e^j,x) = f_j(z - e^j,x+e^i)$ for all $j \in N$ such that $z_j > 0$, hence the total incoming flow at z is the same under $f(.,x)$ and $f(.,x+e^i)$:

$$\sum_{j \in N(z)} f_j(z - e^j,x) = \sum_{j \in N(z)} f_j(z - e^j,x+e^i).$$

Conservation of flows and (24) imply now $\sum_{j \in N \backslash i} f_j(z,x) = \sum_{j \in N \backslash i} f_j(z,x+e^i)$. If $z_j < x_j$ for at most one $j \in N \backslash i$, we are done. Otherwise suppose, by contradiction, that $f(z,x) \neq f(z,x+e^i)$: there exist $j,k \in N \backslash i$ such that $f_j(z,x) < f_j(z,x+e^i)$ and $f_k(z,x) > f_k(z,x+e^i)$. Consider the cost function

$$\begin{aligned} \delta_z(w) &= 0 \text{ if } w \leq z, \\ &\quad 1 \text{ otherwise.} \end{aligned} \tag{26}$$

Again, $\delta_z \in C_{sub}$. Compute

$$\begin{aligned} & \varphi_j(\delta_z, x + e^i) - \varphi_j(\delta_z, x). \\ = & \sum_{w_{N\setminus j} \in [0, z_{N\setminus j}]} (f_j((z_j, w_{N\setminus j}), x + e^i) - f_j((z_j, w_{N\setminus j}), x)) \\ = & f_j(z, x + e^i) - f_j(z, x) > 0 \end{aligned}$$

and, symmetrically, $\varphi_k(\delta_z, x + e^i) - \varphi_k(\delta_z, x) = f_k(z, x + e^i) - f_k(z, x) < 0$, contradicting Submodular Solidarity.

Step 2.c. For all $i \in N$, $x \in [0, \overline{x} - e^i]$, and $z \in [0, x[$,

$$f(z, x) = p_x f(z, x + e^i). \tag{27}$$

Since $p_{x'} p_x f(., \overline{x}) = p_{x'} f(., \overline{x})$ for $x' \leq x \leq \overline{x}$, this will complete the proof of Step 2.

By Step 2.b, (27) is true for $z \in [0, x - e^i[$ as for such a z we have $i \notin K$ and $x_K = (x + e^i)_K$. It remains to extend it to all $z \in [x_i e^i, x[$. Consider first $z = x_i e^i$ and suppose, by way of contradiction, that $f(x_i e^i, x) \neq p_x f(x_i e^i, x + e^i)$: there exist $j, k \in N \setminus i$ such that $f_j(x_i e^i, x) > f_j(x_i e^i, x + e^i) + f_j((x_i + 1)e^i, x + e^i)$ and $f_k(x_i e^i, x) < f_k(x_i e^i, x + e^i) + f_k((x_i + 1)e^i, x + e^i)$. Recalling Step 2.b, this yields a violation of Submodular Solidarity for the cost function $\delta_{(x_i+1)e^i}$. Thus $f(z, x) = p_x f(z, x + e^i)$ for $z = x_i e^i$. An induction argument mimicking that in Step 2.b completes Step 2.c, and the proof that φ is a fixed-flow method.

Step 3. iii) implies i). The argument in Step 2 is easily adapted. The submodular cost functions δ_z are merely replaced with the supermodular functions δ^z defined by

$$\begin{aligned} \delta^z(w) = \; & 1 \text{ if } w \geq z, \\ & 0 \text{ otherwise,} \end{aligned} \tag{28}$$

and the induction argument is carried on $x(N) - z(N)$ rather than on $z(N)$. ■

8.5. Proof of Proposition 5. Step 1 of the proof of Theorem 2 shows that the fixed-flow methods meet Positive Externalities and Negative Externalities.

Next, we show that the Aumann-Shapley method (Example 1) also does. To shorten notation, we denote φ the Aumann-Shapley method, and f its flow representation. Fix a demand profile x, two arbitrary agents 1 and 2, and $z_{-1} \in [0_{-1}, x_{-1}[$. For $z_1 = 0, ..., x_1 + 1$, we write $f(z_1, x)$ instead of $f((z_1, z_{-1}), x)$ and $f(z_1, x + e^1)$ instead of $f((z_1, z_{-1}), x + e^1)$. We claim that

$$\sum_{z_1=0}^{x_1} f_2(z_1, x) = \sum_{z_1=0}^{x_1+1} f_2(z_1, x + e^1) \tag{29}$$

and, for $k = 0, \ldots, x_1 - 1$,

$$\sum_{z_1=0} f_2(z_1, x) \geq \sum_{z_1=0} f_2(z_1, x + e^1). \tag{30}$$

The claim implies that the sequence $(f_2(z_1, x); z_1 = 0, .., x_1)$, augmented by 0 as the last term, stochastically dominates $(f_2(z_1, x + e^1); z_1 = 0, .., x_1 + 1)$. Using (5),this implies $\varphi_2(C, x) \geq \varphi_2(C, x + e^1)$ if C is submodular and the reverse weak inequality holds if C is supermodular.

Equation (29) follows immediately from the fact that the Aumann-Shapley method satisfies Strong Dummy (see comment 2 after Theorem 2). We omit the easy proof.

To prove (30), recall the notation and formula in Example 1 and define $\beta_k = \alpha((k, z_{-1}))$, $\gamma_k = \alpha(x - (k, z_{-1}) - e^2)$ for $k = 0, \ldots, x_1 - 1$, and $\gamma_{-1} = \alpha(x - (0, z_{-1}) + e^1 - e^2)$. Note that

$$\begin{aligned}
\frac{\alpha(x + e^1)}{\alpha(x)} &= \frac{x(N) + 1}{x_1 + 1}, \\
\beta_k &= \frac{\prod_{t=1}^{k} (z(N\setminus 1) + t)}{k!} \beta_0, \\
\gamma_k &= \frac{\prod_{t=1}^{k} (x_1 - t + 1)}{\prod_{t=1}^{k} (x(N) - z(N\setminus 1) - t)} \gamma_0.
\end{aligned}$$

Writing $a_k = z(N\setminus 1) + k$ and $b_k = x(N) - z(N\setminus 1) - k$, (30) reads

$$\begin{aligned}
&\frac{x(N) + 1}{x_1 + 1}\left(1 + \frac{a_1 x_1}{b_1} + \frac{a_1 a_2 x_1 (x_1 - 1)}{2 b_1 b_2} + \ldots + \frac{a_1 \ldots a_k x_1 \ldots (x_1 - k + 1)}{k! b_1 \ldots b_k}\right) \\
\geq\ & \frac{b_0}{x_1 + 1} + a_1 + \frac{a_1 a_2 x_1}{2 b_1} + \ldots + \frac{a_1 \ldots a_k x_1 \ldots (x_1 - k + 2)}{k! b_1 \ldots b_{k-1}}.
\end{aligned}$$

Using $x(N) + 1 = a_1 + b_0$, subtract $\frac{b_0}{x_1+1}$ from both sides of this inequality and divide each side by a_1 to obtain the equivalent inequality

$$\begin{aligned}
&\frac{1}{x_1 + 1} + \frac{x(N) + 1}{x_1 + 1}\left(\frac{x_1}{b_1} + \frac{a_2 x_1 (x_1 - 1)}{2 b_1 b_2} + \ldots + \frac{a_2 \ldots a_k x_1 \ldots (x_1 - k + 1)}{k! b_1 \ldots b_k}\right) \\
\geq\ & 1 + \frac{a_2 x_1}{2 b_1} + \ldots + \frac{a_2 \ldots a_k x_1 \ldots (x_1 - k + 2)}{k! b_1 \ldots b_{k-1}}.
\end{aligned}$$

Subtract $\frac{1}{x_1+1}$ from both sides and divide by x_1 to obtain

$$\frac{x(N)+1}{x_1+1}\left(\frac{1}{b_1}+\frac{a_2(x_1-1)}{2b_1b_2}+...+\frac{a_2...a_k(x_1-1)...(x_1-k+1)}{2...kb_1...b_k}\right)$$
$$\geq \frac{1}{x_1+1}+\frac{a_2}{2b_1}+...+\frac{a_2...a_k(x_1-1)...(x_1-k+2)}{2...kb_1...b_{k-1}},$$

in which a_1 and b_0 no longer appear.

Next, using $x(N)+1=a_2+b_1$, a similar two-step argument reduces that inequality to the equivalent

$$\frac{x(N)+1}{x_1+1}\left(\frac{1}{b_2}+\frac{a_3(x_1-2)}{3b_2b_3}+...+\frac{a_3...a_k(x_1-2)...(x_1-k+1)}{3...kb_2...b_k}\right)$$
$$\geq \frac{1}{x_1+1}+\frac{a_3}{3b_2}+...+\frac{a_3...a_k(x_1-2)...(x_1-k+2)}{3...kb_2...b_{k-1}},$$

where a_2 and b_1 no longer appear.

Repeating this procedure $k-1$ times establishes that the original inequality(30) is equivalent to

$$\frac{x(N)+1}{x_1+1}\left(\frac{1}{b_{k-1}}+\frac{a_k(x_1-k+1)}{kb_{k-1}b_k}\right) \geq \frac{1}{x_1+1}+\frac{a_k}{kb_{k-1}}.$$

Using $x(N)+1=a_k+b_{k-1}$, subtracting $\frac{1}{x_1+1}$ from both sides and multiplying by $\frac{b_{k-1}}{a_k}$, we get $\frac{1}{x_1+1}+\frac{(x(N)+1)(x_1-k+1)}{(x_1+1)kb_k} \geq \frac{1}{k}$, which reduces to $x(N)+1 \geq b_k$. ∎

8.6. Statement and proof of Lemma 3. We work out the restrictions imposed by Strong Ranking and Weak Separability on formula (7). For any $x \in \mathbb{N}^N$, let $\Delta(x) = \{y \in \Delta \mid \text{for all } i,j \in N,\ x_i \leq x_j \Rightarrow y_i \leq y_j\}$. Write $\{x_i \mid i \in N(x)\} = \{\xi_1,...,\xi_K\}$, where $0 < \xi_1 < ... < \xi_K$, and set $\xi_0 = 0$. For $k = 1,...,K$, let $N_k = \{i \in N \mid x_i \geq \xi_k\}$. Using this notation, the extreme points of $\Delta(x)$ are the vectors $b^k = \frac{1}{|N_k|}e^{N_k}$, $k = 1,...,K$. We sometimes write $K(x)$ to emphasize that K depends upon x.

Lemma 3. An additive cost-sharing method φ satisfies Strong Ranking and Weak Separability if and only if it possesses a shared-flow representation (f,r) such that, for all $x \in \mathbb{N}^N$, $z \in [0,x[$, $j \in N$, and $k \in \{1,...,K(x)\}$,

$$\{\xi_{k-1} \leq z_j < \xi_k\} \Rightarrow \{r(z,j,x) = b^k\}. \tag{31}$$

Proof. Step 1. We show that an additive cost-sharing method φ satisfies Strong Ranking if and only if it possesses a shared-flow representation (f, r) such that, for all $x \in \mathbb{N}^N$, $z \in [0, x[$, and $j \in N$, $r(z, j, x) \in \Delta(x)$.

The "if" statement is clear from the representation in (7). Conversely, assume φ meets Additivity and Strong Ranking, and fix $x \in \mathbb{N}^N$ such that $x_1 \leq \ldots \leq x_n$. Define the linear isomorphism $h : \mathbb{R}^N \to \mathbb{R}^N$ by

$$h(y) = (ny_1, (n-1)(y_2 - y_1), \ldots, 2(y_{n-1} - y_{n-2}), y_n - y_{n-1})$$

and observe that it maps $\Delta(x)$ into a face $\Omega(x)$ of Δ. For instance, if $x_1, \ldots, x_n$ are all distinct, h is an isomorphism from $\Delta(x)$ into Δ; if $N = \{1, 2, 3, 4, 5\}$ and $x_1 = x_2 < x_3 < x_4 = x_5$, then $\Omega(x)$ is the face $y_2 = y_5 = 0$.

Define $\psi(C, x) = h(\varphi(C, x))$ for all $C \in \mathcal{C}$. By Strong Ranking, $\varphi(C, x)$ is borne by a vector in $\Delta(x)$, hence $\psi(C, x) \geq 0$. Therefore $\psi(., x)$ is a bona fide additive cost-sharing method at x. Thus it has a shared-flow representation

$$\psi(C, x) = \sum_{z \in [0, x[} \sum_{j \in N} \partial_j C(z) g_j(z, x) r(z, j, x) \tag{32}$$

for all $C \in \mathcal{C}$.

If $y_i = 0$ for all $y \in \Omega(x)$, then $\psi_i(C, x) = 0$ for all $C \in \mathcal{C}$, implying that $r_i(z, j, x) = 0$ for all $z \in [0, x[$ and $j \in N$. Thus $r(z, j, x) \in \Omega(x)$ for all z and j. Applying h^{-1} to both sides of (32), we obtain

$$\varphi(C, x) = \sum_{z \in [0, x[} \sum_{j \in N} \partial_j C(z) g_j(z, x) h^{-1}(r(z, j, x))$$

for all $C \in \mathcal{C}$. Since $h^{-1}(r(z, j, x)) \in \Delta(x)$, we are done.

Step 2. We complete the proof. Let φ be a cost-sharing method satisfying Additivity, Strong Ranking, and Weak Separability, and fix $x \in \mathbb{N}^N$. For $a \in \mathbb{N} \setminus \{0\}$ and $i \in N$, consider the function δ^{ae^i} defined in (28): $\delta^{ae^i}(z) = 1$ if $z_i \geq a$ and $\delta^{ae^i}(z) = 0$ otherwise. By Weak Separability, $\varphi(\sum_{i \in N} \delta^{ae^i}, x) = e^{\{i \in N | x_i \geq a\}}$ for all $a \in \mathbb{N} \setminus \{0\}$. Equivalently,

$$\text{for all } k \in \{1, \ldots, K\} \text{ and } \xi_{k-1} < a \leq \xi_k, \quad \sum_{i \in N_k} \varphi(\delta^{ae^i}, x) = e^{N_k}. \tag{33}$$

By Strong Ranking $\varphi(\delta^{ae^i}, x) \in \Delta(x)$ for all i. By (33) and the fact that b^k is an extreme point of $\Delta(x)$, $\varphi(\delta^{ae^i}, x) = b^k$ whenever $\xi_{k-1} < a \leq \xi_k$ and $i \in N_k$. If φ is represented by (f, r), $\varphi(\delta^{ae^i}, x) = \sum_{z : z_i = a-1} f_i(z, x) r(z, i, x)$, therefore the same extremality argument yields $r(z,i,x) = b^k$ for all $z \in [0,x[$ such that $z_i = a - 1$ and $f_i(z,x) > 0$. Note that z_i varies from ξ_{k-1} to $\xi_k - 1$. If $f_i(z,x) = 0$, the choice of $r(z,i,x)$ is irrelevant and we may set $r(z,i,x) = b^k$ in that case too. This proves the "only if" part of the lemma. The proof of the converse statement is similar because every symmetric separable cost function $C(z) = \sum_{i \in N} c(z_i)$ may be written as a linear combination of cost functions δ^{ae^i}. ∎

It is interesting to compare the shared-flow representation in Lemma 3 with that of separable methods given in formula (5) in Section 4. In both cases the share system r in the general formula (7) is entirely determined and the flow f is arbitrary. Thus the two families of methods -separable on one hand, weakly separable and meeting Strong Ranking on the other hand- are comparably rich. In the former case, however, every flow f yields a different method, whereas in the latter case different flows may lead to the same method. Consider for instance the cross-subsidizing serial method φ^{cs} (Example 5) and fix $x \in \mathbb{N}_*^N$. Every flow $f(.,x)$ that goes entirely through $x^1, x^2, .., x^n$, together with the sharing rule $r(.,.,x)$ defined by formula (31), determines a valid representation of $\varphi^{cs}(.,x)$.

8.7. Proof of Proposition 7. Assume $n \geq 3$ and let φ be a cost-sharing method satisfying all four stated axioms. By Lemma 3, φ has a shared-flow representation (f,r) such that for all $x \in \mathbb{N}^N$, $z \in [0,x[$, $j \in N$, and $k \in \{1,...,K(x)\}$, equation (31) holds. In particular, this implies Zero Cost for Zero Demand because $x_i = 0$ implies $b_i^k = 0$ for all $k = 1,...,K(x)$. As a consequence, any contradiction derived for $n = 3$ extends to more agents by considering demand profiles where all agents but three demand zero.

Fix thus $n = 3$ and let x be such that $2 \leq x_1 < x_2 < x_3$. Write f for $f(.,x)$. As already noted in the proof of Proposition 1, Strong Group Monotonicity is equivalent to the property that for all $D \in \mathcal{D}$, all distinct agents $i,j \in N$, and all $x \in \mathbb{N}^N$,

$$D(x) = 1 \Rightarrow \varphi_j(D, x + e^i) \leq \varphi_j(D,x). \tag{34}$$

In the following argument, we write $D \vee D'$ for the supremum of two cost functions $D, D' \in \mathcal{D}$, that is, $(D \vee D')^{-1} = D^{-1}(1) \cup D'^{-1}(1)$. Choose $\tilde{x}_1 \in [0,x_1[$ and set

$$D = \delta^{(0,0,x_3)} \vee \delta^{(\tilde{x}_1+1,0,0)},$$

where δ^z is the cost function defined in (28). By (31) and Independence of Irrelevant Costs, $\varphi(D,(\tilde{x}_1,x_2,x_3)) = e^3$ because D coincides with $\delta^{(0,0,x_3)}$ on $[0,(\tilde{x}_1,x_2,x_3)]$. By (34), $\varphi_2(D,x) \leq \varphi_2(D,(\tilde{x}_1,x_2,x_3))$, hence $\varphi_2(D,x) = 0$. By (31) again, this gives $f_1(z - e^1) = 0$ for all z such that $z_1 = \tilde{x}_1 + 1$ and $z_3 < x_3$. Since this is true for all $\tilde{x}_1 \in [0,x_1[$, the entire flow must be contained in $A = \{z \in [0,x[\mid z_1 = 0 \text{ or } z_3 = x_3\}$, that is to say $f_i(z) > 0$ requires $z_3 = x_3$ or $z_3 < x_3, z_1 = 0$, and $i \neq 1$.

Next choose $x_1^0 \in [1, x_1]$ and set

$$D = \delta^{(0,x_2,x_3-1)} \vee \delta^{(0,x_1^0,x_3)}.$$

Now D coincides with $\delta^{(0,x_2,x_3-1)}$ on $[0, x - e^3]$, (31) and Independence of Irrelevant Costs imply that $\varphi(D, x - e^3)$ is a convex combination of e^3 and $\frac{1}{2}e^{23}$, hence $\varphi_1(D, x - e^3) = 0$. By (34), $\varphi_1(D, x) = 0$. This implies that $f_2(z - e^2) = 0$ for all z such that $0 \le z_1 \le x_1$, $1 \le z_2 \le x_1$, and $z_3 = x_3$.

Next pick $x_2^0 \in [x_1, x_2[$ and set

$$D = \delta^{(x_1,x_1,x_3)} \vee \delta^{(0,x_2^0+1,x_3)}.$$

As D coincides with $\delta^{(x_1,x_1,x_3)}$ on $[0, (x_1, x_2^0, x_3)]$, (31), Independence of Irrelevant Costs and the fact that $f(., (x_1, x_2^0, x_3))$ is contained in A together imply that $\varphi(D,$
The cross-subsidizing serial method occupies a central position among the methods described by Lemma 2: see our companion papers Moulin and Sprumont (2002), (2003) for two characterizations of it, based respectively on Distributivity and Coordinality.$(x_1, x_2^0, x_3)) = \frac{1}{3}e^{123}$. Now (34) yields $\varphi_3(D, x) \le \frac{1}{3}$ By Strong Ranking, this forces $\varphi(D, x) = \frac{1}{3}e^{123}$. Thus $f_2(z - e^2) = 0$ for all z such that $0 \le z_1 \le x_1 - 1$, $x_1 + 1 \le z_2 \le x_2$, and $z_3 = x_3$. Moreover, $f_3(z - e^3) = 0$ for all z such that $z_1 = 0$, $x_1 + 1 \le z_2 \le x_2$, and $z_3 = x_3$.

Gathering our results, we conclude that i) the entire flow must go through the edge between $(0, x_1, x_3 - 1)$ and $(0, x_1, x_3)$, i.e., $f_3(0, x_1, x_3 - 1) = 1$, and ii) within $\{z \in [0, x[\mid z_3 = x_3\}$, the flow follows the path $(0, x_1, x_3) \to (x_1, x_1, x_3) \to (x_1, x_2, x_3)$.

Finally consider

$$D = \delta^{(x_1-1,0,x_3)} \vee \delta^{(0,x_1,x_3)}.$$

By the shape of the flow $f(., x - e^1)$ and (31), $\varphi(D, x - e^1) = \frac{1}{3}e^{123}$. By the shape of $f(., x)$ and (31), $\varphi(D, x) = e^3$, a contradiction to Monotonicity. ■